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**Published on:** 20 Sep 2008 - Manuscripta Mathematica (Springer-Verlag)

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► **To cite this version:**

Benoît Claudon. Gamma-reduction for smooth orbifolds. *manuscripta mathematica*, Springer Verlag, 2008, 127 (4), pp.521-532. hal-00226344

**HAL Id: hal-00226344**

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# $\Gamma$ -REDUCTION FOR SMOOTH ORBIFOLDS

BENOÎT CLAUDON

ABSTRACT. The aim of this short note is to show how to construct a rational Remmert reduction (the  $\tilde{\Gamma}$ -reduction) for the universal cover of smooth orbifolds  $(X/\Delta)$ . Doing this, we are led to introduce some singular Kähler metric on  $(X/\Delta)$  adapted to the  $\mathbb{Q}$ -divisor  $\Delta$ .

## INTRODUCTION

Let  $X$  be a compact Kähler manifold and  $\tilde{X}$  be its universal cover. Recall first that the birational structure of the latter is partially described by the following result, in the direction of Shafarevich's Conjecture [Sha74] :

**Theorem 0.1** (th. 3.5, p. 264 [Cam94]).

*There exists a unique meromorphic fibration (i.e. surjective with connected fibers)*

$$\gamma_{\tilde{X}} : \tilde{X} \dashrightarrow \Gamma(\tilde{X})$$

*which is almost holomorphic<sup>1</sup>, proper and satisfying the following condition: if  $Z \subset \tilde{X}$  is a compact irreducible analytic subset of  $\tilde{X}$  passes through a very general point  $x \in \tilde{X}$ , it is contained in the fiber through  $x$*

$$Z \subset \gamma_{\tilde{X}}^{-1}(\gamma_{\tilde{X}}(x)).$$

**Definition 0.1.**

*The fibration  $\gamma_{\tilde{X}}$  is called the  $\tilde{\Gamma}$ -reduction (or Shafarevich map in the terminology of [Kol93]) of  $\tilde{X}$ .*

Here we consider a smooth geometric orbifold  $(X/\Delta)$  given by a  $\mathbb{Q}$ -divisor

$$\Delta = \sum_{j \in J} \left(1 - \frac{1}{m_j}\right) \Delta_j$$

where  $m_j \geq 2$  are positive integers and  $\text{Supp}(\Delta) = \bigcap_{j \in J} \Delta_j$  is of normal crossings. Following the works of Kato and Namba, we can define a suitable notion of ramified cover for  $(X/\Delta)$  and, up to slight modifications for  $\Delta$ , there exists then a universal cover

$$\pi_{\Delta} : \tilde{X}_{\Delta} \longrightarrow (X/\Delta)$$

attached to every smooth orbifold  $(X/\Delta)$ . This paper is devoted to prove the following theorem.

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<sup>1</sup>1991 *Mathematics Subject Classification*. 14J10, 14E20, 32J, 27.

<sup>1</sup>the indetermination locus of  $\gamma_{\tilde{X}}$  does not surject onto  $\Gamma(\tilde{X})$ .

**Theorem 0.2.**

Let  $(X/\Delta)$  be a smooth Kähler orbifold and  $\tilde{X}_\Delta$  its universal cover. There exists a unique almost holomorphic proper fibration

$$\tilde{\gamma}_\Delta : \tilde{X}_\Delta \dashrightarrow \Gamma(\tilde{X}_\Delta)$$

which satisfies the same condition as in the theorem 0.1: every compact irreducible subvariety of  $\tilde{X}_\Delta$  passing through a very general point  $x \in \tilde{X}_\Delta$  is contained in the fiber  $\tilde{\gamma}_\Delta^{-1}(\tilde{\gamma}_\Delta(x))$ .

In [Cam07, th. 11.21], the  $\Gamma$ -reduction is constructed for smooth orbifolds  $(X/\Delta)$  but the fibration is defined on the orbifold itself (and not on the universal cover). Moreover, the singular metrics introduced here (see section 2) seem to be a natural notion for smooth Kähler orbifolds.

Before proving theorem 0.2, we shall start with a brief account of the works of Kato and Namba (ramified covers and fundamental groups for orbifolds). We then introduce some singular Kähler metric adapted to the additional orbifold structure : pulling it back, it induces a uniform Kähler structure on the universal cover  $\tilde{X}_\Delta$  which is sufficient to construct the fibration  $\tilde{\gamma}_\Delta$ .

## 1. UNIVERSAL COVER FOR SMOOTH ORBIFOLDS

**1.1. Orbifold fundamental group.** Let us first recall what smoothness means for a pair  $(X/\Delta)$ .

**Definition 1.1.**

A geometric orbifold  $(X/\Delta)$  is said to be smooth if the underlying variety  $X$  is a smooth manifold and if the  $\mathbb{Q}$ -divisor  $\Delta$  has only normal crossings. If in a coordinate patch, the support of  $\Delta$  can be defined by an equation

$$\prod_{j=1}^r z_j = 0,$$

we will say that these coordinates are adapted to  $\Delta$ .

In the category of (smooth) orbifolds, there is a good notion of fundamental group. It is defined in the following way : if  $\Delta = \sum_{j \in J} (1 - \frac{1}{m_j}) \Delta_j$ , choose a small loop  $\gamma_j$  around each component  $\Delta_j$  of the support of  $\Delta$  (for instance the boundary of a small disc centered on a smooth point of  $\Delta_j$  and transverse to it). Consider now the fundamental group of  $X^* = X \setminus \text{Supp}(\Delta)$  and its normal subgroup<sup>2</sup> generated by the loops  $\gamma_j^{m_j}$ :

$$\langle\langle \gamma_j^{m_j}, j \in J \rangle\rangle \leq \pi_1(X^*).$$

**Definition 1.2.**

The fundamental group of  $(X/\Delta)$  is defined to be:

$$\pi_1(X/\Delta) := \pi_1(X^*) / \langle\langle \gamma_j^{m_j}, j \in J \rangle\rangle.$$

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<sup>2</sup>it does not depend on the choice of a base point.

**Example 1.1.**

To illustrate the definition above, consider different orbifold structures on  $\mathbb{P}^1$ :

- (1) if  $\Delta$  has just one point in its support, then  $\pi_1(\mathbb{P}^1/\Delta) = \{1\}$ .
- (2) for  $\Delta = (1 - 1/m)\{0\} + (1 - 1/n)\{\infty\}$ , we get  $\pi_1(\mathbb{P}^1/\Delta) = \mathbb{Z}/d\mathbb{Z}$  where  $d = \gcd(m, n)$ .
- (3) the double cover of  $E \rightarrow \mathbb{P}^1$  ( $E$  being an elliptic curve) branches over four points  $p_1, p_2, p_3$  and  $p_4$ :  $\pi_1(\mathbb{P}^1/\Delta)$  is thus an extension of  $\mathbb{Z}^2$  by  $\mathbb{Z}/2\mathbb{Z}$  with  $\Delta = \sum_j \frac{1}{2}p_j$ .

**1.2. Branched coverings.** Associated to the fundamental group, there is a notion of ramified cover adapted to an orbifold structure. This is one of the fundamental results of [Kat87] (see also [Nam87]).

**Definition 1.3.**

A covering **branched at most at  $\Delta$**  is a holomorphic map  $\pi : Y \rightarrow X$  with

- (1)  $Y$  normal and connected,  $\pi$  with discrete fibers,
- (2)  $\pi$  induces an étale cover over  $X^*$ ,
- (3) over  $\Delta_j$ , the ramification index of  $\pi$  is  $n_j$ , a divisor of  $m_j$  (for all  $j$ ),
- (4) every point  $x \in X$  admits some connected neighbourhood  $V$  satisfying: every connected component  $U$  of  $\pi^{-1}(V)$  meets the fiber  $\pi^{-1}(x)$  in only one point and the restriction  $\pi|_U : U \rightarrow V$  is a proper (finite) map.

We shall say that  $\pi$  **branches at  $\Delta$  exactly** if  $n_j = m_j$  for all  $j \in J$ .

As in the absolute case (i.e. where  $\Delta = \emptyset$ ), there exists a Galois correspondence between subgroups of  $\pi_1(X/\Delta)$  and coverings of  $X$  branched at most at  $\Delta$ .

**Theorem 1.1** ([Kat87],[Nam87]).

If  $(X/\Delta)$  is a smooth orbifold, there exists a natural one-to-one correspondence between subgroups  $G$  of  $\pi_1(X/\Delta)$  and coverings  $\pi : Y \rightarrow X$  branched at most at  $\Delta$ . If the subgroup  $G$  is normal (resp. of finite index), the corresponding covering is Galois (resp. finite).

**Remark 1.1.**

The smoothness assumption is not essential but, in this situation, the local (orbifold) fundamental groups are finite. This finiteness condition is actually the needed one to achieve finiteness as in definition 1.3 (4) above.

The correspondence in theorem 1.1 goes in the following way. If  $\pi : Y \rightarrow X$  is a branched covering (branching at most at  $\Delta$ ), consider the subgroup obtained as the image of the composite morphism:

$$\pi_1(Y^*) \xrightarrow{\pi_*} \pi_1(X^*) \xrightarrow{\varphi} \pi_1(X/\Delta),$$

where  $Y^* = \pi^{-1}(X^*)$  and  $\varphi$  is the natural projection. In the other way, choose a subgroup  $G \leq \pi_1(X/\Delta)$  and consider  $G' = \varphi^{-1}(G) \leq \pi_1(X^*)$ : it corresponds to an étale covering  $\pi : Y^* \rightarrow X^*$ ; the finiteness of the local fundamental groups can then be used to complete the covering over the

support of  $\Delta$ . The fact that  $\langle\langle \gamma_j^{m_j}, j \in J \rangle\rangle \subset G'$  is then equivalent to the ramification condition in the definition 1.3 (and  $\pi : Y \rightarrow X$  is a branched covering according to this definition).

**Corollary 1.1.**

*Corresponding to the trivial subgroup  $\{1\} \subset \pi_1(X/\Delta)$ , there exists a simply connected normal complex space  $\tilde{X}_\Delta$  and a branched covering*

$$\pi_\Delta : \tilde{X}_\Delta \rightarrow (X/\Delta)$$

*branched at most at  $\Delta$ . It is called the universal covering of  $(X/\Delta)$ .*

**Remark 1.2.**

*From the construction itself, it can be easily shown that  $\tilde{X}_\Delta$  has only quotient singularities (located over the singular locus of  $\Delta$*

$$\text{Sing}(\Delta) = \bigcup_{i \neq j} \Delta_i \cap \Delta_j.$$

*It is then a  $V$ -manifold in the sense of [Sat56]<sup>3</sup>.*

**Example 1.2.**

Let us consider the following smooth orbifold surface  $(\mathbb{P}^2/\Delta)$  with

$$\Delta = \frac{1}{2}\Delta_1 + \frac{1}{2}\Delta_2$$

( $\Delta_1$  and  $\Delta_2$  two distinct lines in  $\mathbb{P}^2$  meeting in one point  $p$ ). An easy computation shows that

$$\pi_1(\mathbb{P}^2/\Delta) = \mathbb{Z}/2\mathbb{Z}$$

and the universal cover of  $(\mathbb{P}^2/\Delta)$  is thus a double cover  $S \rightarrow (\mathbb{P}^2/\Delta)$ . Over the point  $p$ ,  $S$  has a conic singularity locally given by the quotient

$$\mathbb{C}^2 / \langle (x, y) \mapsto (-x, -y) \rangle.$$

The surface  $S$  is actually the cone over the normal rational curve in  $\mathbb{P}^2$ .

**1.3. Regularization of orbifold structures.** To avoid the fact that the universal cover does not necessarily branch exactly at  $\Delta$ , [Cam07, § 11] introduced the notion of regular divisor.

**Definition 1.4.**

*Let  $(X/\Delta)$  be a smooth orbifold. Let us denote  $d_j$  the order of  $\gamma_j$  in the quotient  $\pi_1(X/\Delta)$  ( $d_j$  divides  $m_j$ ) and define*

$$\Delta_{reg} = \sum_{j \in J} \left(1 - \frac{1}{d_j}\right) \Delta_j.$$

*It is called the **regularization** of  $\Delta$ . If  $\Delta_{reg} = \Delta$ , the divisor is said to be **regular**.*

**Example 1.3.**

In the example 1.1, the regularization of  $\Delta$  is given by

$$(1) \quad \Delta_{reg} = \emptyset.$$

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<sup>3</sup>for an obvious reason, we will not use the terminology "orbifold" in the sense of Thurston.

- (2)  $\Delta_{reg} = (1 - 1/d) \{0\} + (1 - 1/d) \{\infty\}$  with  $d = \gcd(m, n)$  (in particular,  $\Delta_{reg} = \emptyset$  if  $d = 1$ ).
- (3)  $\Delta_{reg} = \Delta$ .

By construction, we can see that

$$\pi_1(X/\Delta_{reg}) = \pi_1(X/\Delta).$$

Moreover, the integers  $d_j$  are also the ramification indices of the branched covering  $\pi_\Delta : \tilde{X}_\Delta \rightarrow (X/\Delta)$ .

**Proposition 1.1.**

*If  $(X/\Delta)$  is a smooth orbifold, the orbifold  $(X/\Delta_{reg})$  has the same fundamental group and the same universal cover and the covering map*

$$\pi_\Delta : \tilde{X}_\Delta \rightarrow (X/\Delta_{reg})$$

*branches exactly at  $\Delta_{reg}$ .*

Up to regularization (which does not change fundamental group and universal cover), we can then assume that  $\Delta$  is regular and that the universal covering branches exactly at  $\Delta$ .

**Assumption :** In the rest of the paper, we assume the  $\mathbb{Q}$ -divisor  $\Delta$  to be **regular**.

2.  $\tilde{\Gamma}$ -REDUCTION OF  $\tilde{X}_\Delta$

**2.1. Singular Kähler metrics.** To apply the construction of [Cam94], we only need a sufficiently uniform Kähler metric on the universal cover  $\tilde{X}_\Delta$  of a smooth Kähler<sup>4</sup> orbifold  $(X/\Delta)$ . If  $\pi_\Delta$  is the covering map et  $\omega$  any Kähler metric on  $X$ ,  $\pi_\Delta^* \omega$  is a closed non negative  $(1, 1)$ -form on  $\tilde{X}_\Delta$ . Unfortunately, over  $\Delta$ ,  $\pi_\Delta$  is not a local isomorphism and  $\pi_\Delta^* \omega$  is degenerate. We have to introduce some singularity (concentrated on  $\Delta$ ) to balance the ramification of  $\pi_\Delta$ .

**Proposition 2.1.**

*Let  $(X/\Delta)$  be a smooth Kähler orbifold with  $\Delta = \sum_{j \in J} (1 - 1/m_j) \Delta_j$ . Let  $\omega$  be any Kähler metric on  $X$ , let  $C > 0$  be a real number and  $s_j \in H^0(X, \mathcal{O}_X(\Delta_j))$  be a section defining  $\Delta_j$ . Consider the following expression:*

$$\omega_\Delta = C\omega + \sum_{j \in J} i\partial\bar{\partial} |s_j|^{2/m_j}$$

*where  $|\cdot|$  is any smooth metric on  $\mathcal{O}_X(\Delta_j)$  (for each  $j$ ). If  $C$  is large enough, the above formula defines a closed positive  $(1, 1)$ -current (smooth away from  $\Delta$ ) satisfying moreover*

$$\omega_\Delta \geq \omega$$

*in the sense of currents.*

---

<sup>4</sup>an orbifold is said to be Kähler if the underlying manifold is so.

**Remark 2.1.**

A look at the local model enlightens the previous proposition. Consider  $\mathbb{C}^n$  with the orbifold divisor given by the equation

$$\prod_{j=1}^n z_j^{1-1/m_j} = 0$$

(with eventually  $m_j = 1$  for some  $j$ ). The sections  $s_j$  are simply the coordinates  $z_j$  and a simple computation gives

$$\begin{aligned} \omega_\Delta &= \omega_{eucl} + \sum_{j=1}^n i\partial\bar{\partial} |z_j|^{2/m_j} \\ &= \omega_{eucl} + \sum_{j=1}^n \frac{idz_j \wedge d\bar{z}_j}{m_j^2 |z_j|^{2(1-1/m_j)}} \end{aligned}$$

Moreover, the uniformization is given by

$$\pi : \begin{cases} \mathbb{C}^n & \longrightarrow \mathbb{C}^n \\ (t_1, \dots, t_n) & \longmapsto (z_1, \dots, z_n) = (t_1^{m_1}, \dots, t_r^{m_r}, t_{r+1}, \dots, t_n) \end{cases}$$

and, in this chart, the above expression becomes

$$\begin{aligned} \pi^*(\omega_\Delta) &= \sum_{j=1}^n \frac{id(t_j^{m_j}) \wedge d(\bar{t}_j^{m_j})}{|t_j^{m_j}|^{2(1-1/m_j)}} + \pi^*(\omega_{eucl}) \\ &= \sum_{j=1}^n \frac{im_j^2 t_j^{m_j-1} dt_j \wedge \bar{t}_j^{m_j-1} d\bar{t}_j}{|t_j|^{2(m_j-1)}} + \pi^*(\omega_{eucl}) \\ &= \sum_{j=1}^n im_j^2 (1 + |t_j|^{2(m_j-1)}) dt_j \wedge d\bar{t}_j. \end{aligned}$$

In the uniformization  $\pi$ , the  $(1, 1)$ -form  $\omega_\Delta$  becomes a genuine Kähler metric.

**Proof of proposition 2.1 :**

We only need to check that each  $|s_j|^{2/m_j}$  is a quasi-psh function on  $X$ . In adapted coordinates, the sections  $s_j$  are given by  $z_j$  but we have to take care of the weights of the metrics (on  $\mathcal{O}_X(\Delta_j)$ ):

$$|s_j|^{2/m_j} = f_j |z_j|^{2/m_j}$$

with  $f_j$  a smooth positive function. We then get

$$\begin{aligned} i\partial\bar{\partial} |s_j|^{2/m_j} &= if_j \partial\bar{\partial} |z_j|^{2/m_j} + i\partial f_j \wedge \bar{\partial} |z_j|^{2/m_j} \\ (1) \quad &+ i\partial |z_j|^{2/m_j} \wedge \bar{\partial} f_j + i |z_j|^{2/m_j} \partial\bar{\partial} f_j. \end{aligned}$$

The following identity

$$\begin{aligned} 0 \leq i\partial(|z_j|^{2/m_j} + f_j) \wedge \bar{\partial}(|z_j|^{2/m_j} + f_j) &= i\partial f_j \wedge \bar{\partial} f_j + i\partial f_j \wedge \bar{\partial} |z_j|^{2/m_j} \\ &+ i\partial |z_j|^{2/m_j} \wedge \bar{\partial} f_j + i\partial |z_j|^{2/m_j} \wedge \bar{\partial} |z_j|^{2/m_j}, \end{aligned}$$



gives the inequality (in the sense of currents):

$$\begin{aligned} i\partial f_j \wedge \bar{\partial} |z_j|^{2/m_j} + i\partial |z_j|^{2/m_j} \wedge \bar{\partial} f_j &\geq -i\partial f_j \wedge \bar{\partial} f_j - i\partial |z_j|^{2/m_j} \wedge \bar{\partial} |z_j|^{2/m_j} \\ (2) \qquad \qquad \qquad &\geq -i\partial f_j \wedge \bar{\partial} f_j - \frac{i |z_j|^{2/m_j} dz_j \wedge d\bar{z}_j}{|z_j|^{2(1-1/m_j)}}. \end{aligned}$$

Since  $f_j$  is smooth, there exists a constant  $C_j > 0$  such that (locally)

$$(3) \qquad i |z_j|^{2/m_j} \partial \bar{\partial} f_j - i\partial f_j \wedge \bar{\partial} f_j \geq -C_j \omega$$

Combining (1), (2) and (3) gives

$$\begin{aligned} \omega_\Delta &\geq (C - \sum_j C_j) \omega + \sum_j \frac{(f_j - |z_j|^{2/m_j}) i dz_j \wedge d\bar{z}_j}{|z_j|^{2(1-1/m_j)}} \\ &\geq (C - \sum_j C_j) \omega \text{ on a neighbourhood of } 0. \end{aligned}$$

Since  $X$  is compact, it is covered by a finite number of such small balls and we can choose  $C$  large enough to achieve positivity for  $\omega_\Delta$ .  $\square$

Pulling back this singular metric to the universal cover, we get the needed uniform metric.

**Proposition 2.2.**

Choose  $\omega_\Delta$  as in the proposition 2.1. The pull-back

$$\widetilde{\omega}_\Delta = \pi_\Delta^*(\omega_\Delta)$$

defines a Kähler metric on  $\widetilde{X}_\Delta$  as a  $V$ -manifold. This means that the  $(1,1)$ -form  $\widetilde{\omega}_\Delta$  is continuous on  $\widetilde{X}_\Delta$ ,  $C^\infty$  on the non-singular locus of  $\widetilde{X}_\Delta$  and its lift to any local uniformization extends to a smooth invariant metric<sup>5</sup>.

**Proof :**

The local computation made in the remark 2.1 shows exactly that the singular part of the metric  $\omega_\Delta$  balances the ramification of  $\pi_\Delta$ .  $\square$

**2.2. Proof of theorem 0.2.** We can now prove the existence of the  $\widetilde{\Gamma}$ -reduction for  $\widetilde{X}_\Delta$ . We will need some compactness properties of the cycles space  $\mathcal{C}(\widetilde{X}_\Delta)$  constructed in [Bar75]. To construct the  $\widetilde{\Gamma}$ -reduction, we will apply the fundamental result of [Cam81]. For the convenience of the reader, we restate it here.

**Theorem 2.1** (th. 1, p. 189 [Cam81]).

Let  $Y$  be a normal analytic space,  $T$  an irreducible component of  $\mathcal{C}(Y)$ ,

$$G_T = \{(y, Z_t) \in Y \times T \mid y \in \text{Supp}(Z_t)\}$$

the incidence graph of the family of cycles parametrized by  $T$  and let us denote by

$$q : G_T \longrightarrow Y \text{ et } r : G_T \longrightarrow T$$

---

<sup>5</sup><sub>a</sub>  $V$ -manifold is locally a quotient  $U/G$  where  $U \subset \mathbb{C}^n$  is an open subset and  $G$  a finite group acting on  $U$ ; in this setting, a lift is just the pull-back to  $U$ .

the corresponding projections. If  $q$  is surjective (i.e. the cycles  $(Z_t)_{t \in T}$  cover  $Y$ ) and proper and if the generic fiber of  $r$  is irreducible (i.e.  $Z_t$  is irreducible for  $t \in T$  generic), there exists an almost holomorphic proper fibration

$$g_T : Y \dashrightarrow Q_T$$

whose fiber through a generic point  $y \in Y$  is the equivalence class generated by this family of cycles (two points are said to be equivalent if there is a connected (finite) chain of cycles of the family containing them).

We thus only need to prove the following

**Proposition 2.3.**

If  $T \subset \mathcal{C}(\tilde{X}_\Delta)$  is an irreducible component and  $q : G_T \rightarrow \tilde{X}_\Delta$  (and  $r : G_T \rightarrow T$ ) is the corresponding projection, then  $q$  is proper.

This proposition is a straightforward consequence of the following lemma which restates the fact that the geometry associated to the Kähler form  $\tilde{\omega}_\Delta$  is uniform (see [Bor92] for the metric structure of  $V$ -manifold).

**Lemma 2.1.**

There exist some constants  $r, \delta > 0$  such that: for every irreducible compact subvariety  $Z \subset \tilde{X}_\Delta$  and every  $z \in Z$ ,

$$\text{Vol}_{\tilde{\omega}_\Delta}(Z \cap B(z, r)) := \int_{Z \cap B(z, r)} \tilde{\omega}_\Delta^{\dim(Z)} \geq \delta$$

where  $B(z, r)$  is the ball of radius  $r$  centered at  $z$  for the distance  $d_\Delta$  induced by  $\tilde{\omega}_\Delta$  on  $\tilde{X}_\Delta$ .

Here, the volume of an irreducible subvariety is computed as follows: if  $Z$  is contained in a chart  $p : U \rightarrow V \simeq U/G$ ,

$$\int_Z \tilde{\omega}_\Delta^{\dim(Z)} = \frac{1}{|G|} \int_{p^{-1}(Z)} p^*(\tilde{\omega}_\Delta)^{\dim(Z)}$$

as usually (note that the metric  $p^*(\tilde{\omega}_\Delta)$  is now smooth) and we use a partition of unity to deal with the general case.

**Proof of the proposition 2.3 :**

Let  $T$  be an irreducible component of  $\mathcal{C}(\tilde{X}_\Delta)$  and  $K$  a compact of  $\tilde{X}_\Delta$ . Since  $\tilde{\omega}_\Delta$  is a Kähler metric, the volume of the cycles parametrized by  $T$  is constant; let us denote it by  $v$ . Consider the following compact subset of  $\tilde{X}_\Delta$ :

$$\hat{K} = \left\{ x \in \tilde{X}_\Delta \mid d_\Delta(x, K) \leq M \right\} \text{ with } M > r \left\lceil \frac{v}{\delta} \right\rceil.$$

The lemma 2.1 can then be rephrased in the following way:

$$q^{-1}(K) \subset r^{-1}(r(q^{-1}(\hat{K}))).$$

But Bishop's theorem asserts the compactness of  $r(q^{-1}(\hat{K}))$  (see [Lie78]) and the projection  $r$  is always proper. From this we deduce the compactness of  $q^{-1}(K)$  (and the properness of  $q$ ).  $\square$

We can now finish the

**Proof of the theorem 0.2 :**

Let us  $d$  denote the smallest integer such that there exists an almost holomorphic proper fibration

$$f : \tilde{X}_\Delta \longrightarrow V$$

with  $\dim(V) = d$ . Assume that the maximality property of the fibers of  $f$  (stated in theorem 0.2) is not satisfied: for a very general point  $x \in \tilde{X}_\Delta$ , there exists a compact irreducible subvariety not contained in the fiber through  $x$ . Since  $\mathcal{C}(\tilde{X}_\Delta)$  is a second countable space, there exists an irreducible component  $T$  of  $\mathcal{C}(\tilde{X}_\Delta)$  such that the family  $(U_t)_{t \in T}$  of cycles parametrized by  $T$  satisfy

$$\forall t \in T, \dim(f(U_t)) > 0$$

and  $U_t$  is irreducible for a generic  $t \in T$ . Thanks to proposition 2.3, we can apply theorem 2.1 to the family of *compact* cycles

$$Z_t = f^{-1}(f(U_t))$$

parametrized (meromorphically) by  $T$ . The corresponding quotient

$$g_T : \tilde{X}_\Delta \dashrightarrow Q_T$$

is an almost holomorphic proper fibration whose fibers are strictly contained in the one of  $f$ ; from this we deduce

$$\dim(Q_T) < d = \dim(V),$$

which contradicts the minimality of  $d$ .

The uniqueness of the  $\tilde{\Gamma}$ -reduction follows in the same way: if  $f$  and  $g$  are two such fibrations and  $x$  a sufficiently general point, the fiber of  $f$  through  $x$  is contained in the fiber of  $g$  (maximality of the fibers of  $g$ ). Reversing the order of  $f$  and  $g$ , we get the other inclusion and the two fibrations have the same fibers.  $\square$

**Remark 2.2.**

*The preceding construction can be made with an arbitrary non compact branched covering  $Y \longrightarrow (X/\Delta)$  (branched at most at  $\Delta$ ). The regularization  $\Delta_Y$  must be adapted to the previous map and it yields*

$$\Delta \geq \Delta_Y \geq \Delta_{reg}$$

*(the fundamental group is thus unchanged). The pull-back of the singular metric  $\omega_{\Delta_Y}$  to  $Y$  gives the right object to apply the construction described above.*

To conclude, we would like to recall the results of [Cam07, th. 11.21] and to compare both fibrations.

**Theorem 2.2.**

*Let  $(X/\Delta)$  be a smooth Kähler orbifold. There exists a unique almost holomorphic fibration*

$$\gamma_{(X/\Delta)} : (X/\Delta) \dashrightarrow \Gamma(X/\Delta)$$

*satisfying both following properties:*

- (1)  $\pi_1(X_y/\Delta_{X_y})_{(X/\Delta)} := \text{Im}(\pi_1(X_y/\Delta_{X_y}) \longrightarrow \pi_1(X/\Delta))$  is finite for  $y \in \Gamma(X/\Delta)$  general,
- (2) if  $g : (V/\Delta_V) \longrightarrow (X/\Delta)$  is a divisible orbifold morphism from a smooth compact orbifold  $(V/\Delta_V)$  such that  $g(V)$  meets  $X_y$  (for  $y \in \Gamma(X/\Delta)$  generic) and if

$$\text{Im}\left(\pi_1(V/\Delta_V) \xrightarrow{g_*} \pi_1(X/\Delta)\right)$$

is finite, then  $g(V)$  is contained in  $X_y$ .

Recall that a divisible orbifold morphism

$$g : (V/\Delta_V) \longrightarrow (X/\Delta)$$

induces a well-defined morphism at the level of fundamental groups:

$$g_* : \pi_1(V/\Delta_V) \longrightarrow \pi_1(X/\Delta).$$

See [Cam07, § 2.2 and § 11.1] for the notions involved.

In the tame situation of a smooth subvariety meeting  $\Delta$  transversally<sup>6</sup> (which is the case of the general fiber of a fibration), the inclusion

$$i : (V/\Delta|_V) \hookrightarrow (X/\Delta)$$

is clearly a divisible orbifold morphism. The following lemma shows us that the fibers of  $\tilde{\gamma}_\Delta$  are the connected components of the inverse images by the covering map  $\pi_\Delta$  of the fibers of  $\gamma_{(X/\Delta)}$ .

**Lemma 2.2.**

Let  $V$  be a smooth subvariety of  $X$  meeting  $\Delta$  transversally and  $\pi_\Delta : \tilde{X}_\Delta \longrightarrow (X/\Delta)$  the universal cover of  $(X/\Delta)$ . Both conditions are equivalent :

- (1)  $\pi_1(V/\Delta_V)_{(X/\Delta)}$  is a finite group.
- (2) each connected component of  $\pi_\Delta^{-1}(V)$  is compact.

**Proof :**

Let  $Z$  be such a connected component. Restricting  $\pi_\Delta$  to  $Z$  yields a branched covering

$$p : Z \longrightarrow (V/\Delta_V)$$

which corresponds to the subgroup

$$G = \text{Ker}(\pi_1(V/\Delta_V) \longrightarrow \pi_1(X/\Delta)).$$

According to theorem 1.1,  $p$  is finite if and only if  $G$  is of finite index in  $\pi_1(V/\Delta_V)$ . Equivalently,  $Z$  is compact if and only if  $\pi_1(V/\Delta_V)_{(X/\Delta)}$  is finite.  $\square$

**Acknowledgements.** I am very grateful to Frédéric Campana for introducing me to this subject. For this and also for many interesting discussions, I would like to thank him.

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<sup>6</sup>by this we mean that  $(V/i^*\Delta)$  should be a smooth orbifold.

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