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G-SPACES WITH PRESCRIBED EQUIVARIANT COHOMOLOGY

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ABSTRACT. Let G be a finite group. In this note we study the question of realizing a collection of graded commutative algebras over $\mathbf{Q}$ as the cohomology algebras with rational coefficients of the fixed point sets X^H ($H < G$) of a G-space X.

Let $\mathcal{A}$ be a graded commutative algebra over $\mathbf{Q}$. The question of realizing $\mathcal{A}$ as the cohomology with rational coefficients of a space X is answered by Quillen **[4]** and more directly by Sullivan **[5]**. In particular, Sullivan constructs a space X of finite type, i.e. $\pi_i(X)$ is a finitely generated abelian group for every i, which realizes $\mathcal{A}$. Now let G be a finite group which acts on $\mathcal{A}$ from the left by algebra isomorphisms. Because of the functoriality of the constructions in **[4]** and **[5]** one can construct a G-space X such that $H^*(X;\mathbf{Q}) \cong \mathcal{A}$, where the isomorphism is G-equivariant. The space X in both cases is a rational space.

In this note we consider a more general question. Let $\mathcal{O}_G$ be the category of canonical orbits of a finite group G **[1]**. The objects of $\mathcal{O}_G$ are the quotient spaces G/H, where H is a subgroup of G ($H < G$), and the morphisms are the G-maps between them, where G acts on G/H by left multiplication.

DEFINITION 1. A system of graded commutative algebras (GA's) for G is a covariant functor from $\mathcal{O}_G$ into the category of graded commutative connected algebras over $\mathbf{Q}$.

We recall that a GA $\mathcal{A}$ is said to be connected if $\mathcal{A}^0 = \mathbf{Q}$ and is said to be of finite type if $\mathcal{A}^n$ is a finite-dimensional vector space over $\mathbf{Q}$.

Let X be a G-space such that each fixed point set X^H, $H < G$, is nonempty and connected. Given X, a system of GA's $\underline{H}^*(X)$ is defined by

$$\underline{H}^*(X)(G/H) \equiv H^*(X^H;\mathbf{Q})$$

on objects of $\mathcal{O}_G$. If $f\colon G/H \to G/K$ is a G-map, then there exists an element $g \in G$ such that $g^{-1}Hg < K$, and the map f is determined by $H \mapsto gK$. The map f induces a map $\bar{f}\colon X^K \to X^H$ by $x \mapsto gx$ and therefore a unique map $\underline{H}^*(X)(f) \equiv \bar{f}^*\colon H^*(X^H;\mathbf{Q}) \to H^*(X^K;\mathbf{Q})$.

The main result of this paper is the following

THEOREM 2. *Given a system* $\mathbf{H}$ *of connected GA's of finite type, there exists a* G-CW-*complex* X *of finite type such that* $\underline{H}^*(X) \cong \mathbf{H}$.

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We say that a G-space X is of finite type if each X^H $(H < G)$ is of finite type.

For the proof of the theorem we need some additional terminology and the following lemmas.

Definition 3. A coefficient (respectively rational coefficient) system for G is a contravariant functor from $\mathcal{O}_G$ into the category of abelian groups (respectively the category of vector spaces over $\mathbf{Q}$).

For instance, if X is a G-CW-complex [**1**], one can define coefficient systems $\underline{\pi}_n(X)$ and $\underline{C}_n(X)$ by

$$\underline{\pi}_n(X)(G/H) \equiv \pi_n(X^H) \quad \text{and} \quad \underline{C}_n(X)(G/H) \equiv C_n(X^H; \mathbf{Z}),$$

where the latter is the cellular complex of X^H.

Definition 4. Let A be a coefficient system of finitely generated abelian groups and let M be a rational coefficient system. Let f: $A \to M$ be a monomorphism. The system A is called a system of lattices for M if $f(G/H) \otimes \mathrm{id}$:

$$A(G/H) \otimes \mathbf{Q} \xrightarrow{\cong} M(G/H) \quad \text{for all subgroups } H < G.$$

Lemma 5. *Every rational coefficient system M has a system of lattices A.*

Proof. The construction of A is done by induction on the subgroups of G. We first choose a lattice for the vector space $M(G/G)$.

Let $H < G$ be a subgroup and assume inductively that $A(G/K)$ is constructed for all subgroups $K < G$ which contain H properly. $M(G/H)$ is a $\mathbf{Q}(NH/H)$-module and there exists a $\mathbf{Z}(NH/H)$-submodule B such that $B \otimes_{\mathbf{Z}} \mathbf{Q} \cong M(G/H)$ (cf. [**2**, p. 495]). We consider bases for B and $A(G/K_1), \ldots, A(G/K_m)$ regarded as $\mathbf{Z}$-modules, where $K_1, \ldots, K_m$ are representatives of all conjugacy classes of subgroups of G which contain H as a maximal subgroup. Let $A_1, \ldots, A_m$ be the matrices which represent the linear maps $M(G/K_i) \to M(G/H)$ (induced by the projections $G/H \to G/K_i$) with respect to the bases above. Let N be an integer such that $NA_1, \ldots, NA_m$ are matrices with integral entries. We multiply the basis elements of B by $1/N$ and consider the new $\mathbf{Z}$-module generated this way. Obviously, it is a $\mathbf{Z}(NH/H)$-module and it has the property that it contains the images of $A(G/K)$ $(K > H)$ via the maps induced by the projections. Hence, we have constructed $A(G/H)$. This completes the proof of the lemma.

Lemma 6. *Let A' be a coefficient system of finitely generated abelian groups and let M be a rational coefficient system. Let f': $A' \to M$ be a monomorphism. Then M has a system of lattices A which extends A', i.e. there exist monomorphisms i and f such that the diagram*

$$\begin{array}{ccc} A' & \xrightarrow{f'} & M \\ {\scriptstyle i}\searrow & & \nearrow{\scriptstyle f} \\ & A & \end{array}$$

commutes.

PROOF. The same inductive argument works as above. In order to construct $A(G/H) \supset A'(G/H)$ we use the semisimplicity of $\mathbf{Q}(NH/H)$ to split off

$$A'(G/H) \otimes \mathbf{Q} \subset M(G/H).$$

Let $\overline{M}$ be a $\mathbf{Q}(NH/H)$-submodule of $M(G/H)$ which is a complement of $A'(G/H) \otimes \mathbf{Q}$, and let $\overline{B}$ be a $\mathbf{Z}(NH/H)$-submodule of $\overline{M}$ such that $\overline{B} \otimes \mathbf{Q} = \overline{M}$. We choose a basis for $A'(G/H) \oplus \overline{B}$ and proceed as above.

PROOF OF THEOREM 2. We first use D. Sullivan's spatial realization of an algebra [5]. It is a functor F from the category of differential graded commutative algebras (DGA's) into the category of simplicial sets such that $H^*(F(\mathcal{A}); \mathbf{Q}) \cong H^*(\mathcal{A})$ in a canonical way. Composing F with the geometric realization functor $S \mapsto |S|$, we get a functor into the category $\mathcal{T}$ of CW-complexes.

Let $\underline{F}\colon \mathcal{O}_G \to \mathcal{T}$ be the functor defined by $\underline{F}(G/H) \equiv |F(\mathbf{H}(G/H))|$ on objects of $\mathcal{O}_G$, where $\mathbf{H}(G/H)$ is considered as a DGA with differential equal to 0.

Such a functor is called an $\mathcal{O}_G$-space in [3]. Given an $\mathcal{O}_G$-space T, Elmendorf constructs a G-space $C(T)$ in a functorial way such that $C(T)^H$ is naturally homotopy equivalent to $T(G/H)$, for $H < G$. Let $Y \equiv C(\underline{F})$. By construction, Y is a G-CW-complex which realizes $\mathbf{H}$. Moreover, Y (and each Y^H) is a rational space.

Next we construct a G-CW-complex X of finite type and a G-map $f\colon X \to Y$ such that $f^H\colon X^H \to Y^H$ is a rationalization for every subgroup $H < G$. We proceed by induction on the equivariant Postnikov decomposition of Y [6] starting from a point. Let X_n be a G-CW-complex of finite type and let

$$i_n\colon X_n \xrightarrow{i_n} Y_n$$

be a G-map which is a G-rationalization, i.e.

$$(i_n)_z\colon \underline{\pi}_k(X_n) \otimes \mathbf{Q} \xrightarrow{\cong} \underline{\pi}_k(Y_n).$$

Here Y_n is the nth stage of the equivariant Postnikov decomposition of Y. The $(n+1)$st equivariant k-invariant of Y is an equivariant cohomology class represented by a natural transformation

$$k\colon \underline{C}_{n+2}(Y_n) \to \underline{\pi}_{n+1}(Y).$$

We compose k with $i_n\colon \underline{C}_{n+2}(X_n) \to \underline{C}_{n+2}(Y_n)$ and consider the image of $k \circ i_n$ in $\underline{\pi}_{n+1}(Y)$. By Lemma 6, we can extend it to a system of lattices A of $\underline{\pi}_{n+1}(Y)$. Let $\hat{A}$ be any coefficient system of finitely generated abelian groups which surjects onto A,

$$0 \to T \to \hat{A} \to A \to 0,$$

where $T(G/H)$ is finite for every G/H. Consider the diagram:

$$\begin{array}{ccc}
\underline{C}_{n+2}(Y_n) & \xrightarrow{k} & \underline{\pi}_{n+1}(Y) \supset A \\
{\scriptstyle i_n}\uparrow & & \uparrow \\
\underline{C}_{n+2}(X_n) & \xrightarrow{k'} & \hat{A}
\end{array}$$

Since $\underline{C}_{n+2}(X_n)$ is projective in the category of coefficient systems [**1**], there is a lifting k'. Since $k \circ i_n$ is a cocycle, it annihilates the cycles of $\underline{C}_{n+2}(X_n)$. Therefore, k' maps the cycles of $\underline{C}_{n+2}(X_n)$ to torsion elements of $\hat{A}$. Hence a multiple $m \cdot k'$ is a cocycle. Now, we construct X_{n+1} from X_n such that $\underline{\pi}_{n+1}(X_{n+1}) \equiv \hat{A}$, and with $m \cdot k'$ as the $(n+1)$st k-invariant. Note that over the rationals, k and $m \cdot k$ determine the same $(n+1)$st stage Y_{n+1} of the Postnikov tower. So Y_{n+1} is an equivariant rationalization of X_{n+1}.

In exactly the same way as in Theorem 2, we can prove the following statement.

THEOREM 7. *Given a system* $\mathfrak{A}$ *of connected nilpotent differential graded algebras over* $\mathbf{Q}$ *of finite type there exists a* G-CW-*complex* X *of finite type and a morphism* $\mathfrak{A} \to \underline{\mathcal{E}}_X$ *which induces an isomorphism on cohomology for every* G/H $(H < G)$.

The definition of a system of DGA's is analogous to Definition 1. A system of DGA's may not be injective in the sense of [**6**]. The system of DGA's $\underline{\mathcal{E}}_X$ was introduced and studied in [**6**]. By definition, $\underline{\mathcal{E}}_X(G/H) \equiv \mathcal{E}_{X^H}$ on objects, where $\mathcal{E}_X$ is the Sullivan-De Rham complex of PL forms on X [**5**].

A DGA $\mathfrak{A}$ is said to be connected if $H^0(\mathfrak{A}) = \mathbf{Q}$ and it is said to be of finite type if $H^i(\mathfrak{A})$ is a finite-dimensional vector space over $\mathbf{Q}$. For the definition of nilpotency see [**5**].

Theorem 7 generalizes a result by Sullivan [**5**] in the nonequivariant case (trivial G-action).

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