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G_1-STRUCTURES OF SECOND ORDER

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Abstract

We introduce a generalization to the second order of the notion of the G_1-structure, the so called generalized almost tangent structure. For this purpose, the concepts of the second order frame bundle $H^2(V_m)$, its structural group L_m^2 and its associated tangent bundle of second order $T^2(V_m)$ of a differentiable manifold V_m are described from the point of view that is used. Then, a G_1-structure of second order -called G_1^2-structure- is constructed on V_m by an endomorphism J acting on $T^2(V_m)$, satisfying the relation $J^2 = 0$ and some hypotheses on its rank. Its connection and characteristic cohomology class are defined.

Some of the G-structures of the first order are those defined by nilpotent operators of degree $r+1$ ($r \geq 1$) that is, the G_r-structures, defined by J. Lehman-Lejeune ([15]) and studied by H.A. Eliopoulos ([11]).

The G_1-structure of the first order, briefly G_1-structure, is defined ([15]) on an m-dimensional differentiable manifold V_m of class C^∞ by means of an 1-form J, of constant rank p, with values in the tangent bundle, such that at each point $x \in V_m$,

$$J_x^2 = 0. \tag{1}$$

That is,

> dim Im $J_x = p \geq 1$, dim ker $J_x = q$, $m = p + q$ and q independent of the point x of V_m.

The G_1-structure is also studied by [1]; it is called generalized almost tangent structure.

Our objective in the present paper is to find a prolongation of this structure, that is, there is defined a G-structure of second order on V_m, called the G_1-structure of order 2, briefly a G_1^2-structure, by means of an 1-form of second order J, of constant rank $p + \binom{p+1}{2}$, with values in the tangent bundle of second order, satisfying at each point $x \in V_m$ the relation (1).

At first, a brief discussion of the notions of the frame bundle of second order $H^2(V_m)(V_m, L^2_m)$ and its associated ([**13**]) tangent bundle $T^2(V_m)(V_m, L^2_m, F^2)$ of order two, will be given from the standpoint used in the generalization of the real almost product structure to the second order ([**6**]).

Then, the G^2_1-structure, its adapted basis, connection and characteristic cohomology class will be defined.

1. The fibre bundles $H^2(V_m)$ and $T^2(V_m)$

We recall from [**6**] the following:

Let V_m be an m-dimensional differentiable manifold of class C^∞ and $H^2(V_m) = \bigcup_{x \in V_m} H^2_x(V_m)$ the fibre bundle of all 2-frames of the manifold V_m, where H^2_x is ([**7**]) the set of all invertible 2-jets of R^m into V_m with source $0 \in R^m$ and target $x \in V_m$. This bundle is ([**8**]) a principal fibre bundle with basis V_m and structural group L^2_m; it is called the principal prolongation of order 2 of the manifold V_m.

The structural group L^2_m is ([**10**]) the set $j^2 \circ f$ of all invertible 2-jets with source and target $0 \in R^m$ of a 2-mapping f at the point $0 \in R^m$. Hence, each $a \in L^2_m$ can be written in the form,

$$\text{(1.1)} \quad a = (a^i_{j_1}, a^i_{j_1 j_2}), \quad i, \quad j_1, \quad j_2 = 1, 2, \dots, m, \quad \det(a^i_{j_1}) \neq 0$$
$$\text{and } a^i_{j_1 j_2} \text{ is symmetric with respect to } j_1, j_2.$$

Also,

$$\dim L^2_m = m\binom{m+2}{2} - m = m^2 + m\binom{m+1}{2} = m^2 + m^2\left(\frac{m+1}{2}\right).$$

If

$$\beta = (\beta^{j_1}_{k_1}, \beta^{j_1}_{k_1 k_2}) \in L^2_m,$$

then, from the composition of 2-jets ([**8**]), it follows that the product of the two elements a and β of L^2_m,

$$a\beta = c = (c^{i_1}_{k_1}, c^{i_1}_{k_1 k_2}),$$

can be defined by the relations

$$\begin{cases} c^{i_1}_{k_1} = a^{i_1}_{j_1}\beta^{j_1}_{k_1}, \\ c^{i_1}_{k_1k_2} = a^{i_1}_{j_1}\beta^{j_1}_{k_1k_2} + a^{i_1}_{j_1j_2}\beta^{j_1}_{k_1}\beta^{j_2}_{k_2}. \end{cases}$$

Remark 1.1. We assume, now, that to the element $a \in L^2_m$, corresponds the matrix A of the form,

$$A = \begin{bmatrix} a^{i_1}_{j_1} & 0 \\ a^{i_1}_{j_1j_2} & a^{i_1}_{j_1}a^{i_1}_{j_2} \end{bmatrix}. \tag{1.2}$$

If B is a matrix of the same form, corresponding to the element β of L^2_m, then to $a\beta \in L^2_m$ corresponds the product of matrices AB,

$$AB = C = \begin{bmatrix} c^{i_1}_{k_1} & 0 \\ c^{i_1}_{k_1k_2} & c^{i_1}_{k_1}c^{i_2}_{k_2} \end{bmatrix} = \begin{bmatrix} \beta^{j_1}_{k_1} & 0 \\ \beta^{j_1}_{k_1k_2} & \beta^{j_1}_{k_1}\beta^{j_2}_{k_2} \end{bmatrix} \begin{bmatrix} a^{i_1}_{j_1} & 0 \\ a^{i_1}_{j_1j_2} & a^{i_1}_{j_1}a^{i_2}_{j_2} \end{bmatrix} =$$
$$= \begin{bmatrix} a^{i_1}_{j_1}\beta^{j_1}_{k_1} & 0 \\ a^{i_1}_{j_1}\beta^{j_1}_{k_1k_2} + a^{i_1}_{j_1j_2}\beta^{j_1}_{k_1}\beta^{j_2}_{k_2} & a^{i_1}_{j_1}a^{i_2}_{j_2}\beta^{j_1}_{k_1}\beta^{j_2}_{k_2} \end{bmatrix},$$

with the notation that first is written the matrix B and then A, but after the multiplication (each line of B with every column of A) first is written the elements of A and then of B.

It can be verified that to each element $a \in L^2_m$ corresponds a matrix A of the form (1.2) and conversely.

Hence, the group of matrices of the form A, subgroup of the group of matrices $Gl(N,R)$, $N = m + \frac{m(m+1)}{2}$, can be identified with the group L^2_m.

The Lie algebra $\underline{L}^2_m$ of the Lie group L^2_m is ([**17**]) defined by,

$$\underline{L}^2_m = \{\lambda/\lambda = (\lambda^i_{j_1}, \lambda^i_{j_1j_2}),\ i,\ j_1,\ j_2 = 1,2,\dots,m,$$
$$\lambda^i_{j_1} \in R^m \otimes R^{m^*},\ \lambda^i_{j_1j_2} \in R^m \otimes S^2(R^{m^*})\},$$

where $S^2(R^{m^*})$ is the set of the 2-linear symmetric forms on R^m.

We consider, also, $T^2_{1,x} * (V_m)$ the set of all 2-jets of the functions on V_m with source $x \in V_m$ and target 0. The set $T^{2*}_1(V_m) = \bigcup_{x\in V_m} T^{2*}_{1,x}(V_m)$ has ([**8**], [**10**]) the structure of a vector bundle with basis V_m structural group L^2_m and fibre $L^2_{1,m}$; it is associated with $H^2(V_m)$.

$L^2_{1,m}$ is ([**7**], [**10**]) the set $j^2 \circ g$ of all 2-jets with source $0 \in R^m$ and target $0 \in R$ of a 2-mapping g at $0 \in R^m$. Therefore, each $y \in L^2_{1,m}$ can be written in the form,

$$y = (y_{i_1}, y_{i_1i_2}),\ i_1,\ i_2 = 1,2,\dots,m,\ y_{i_1i_2} \tag{1.3}$$
is symmetric with respect to i_1, i_2.

Also,

$$\dim L^2_{1,m} = \binom{m+2}{2} - 1 = m + \binom{m+1}{2} = m + \frac{m(m+1)}{2}.$$

Let $\{x^i\}_{i=1,2,\dots,m}$ be a system of local coordinates at $x \in V_m$ for a given chart. Then ([**8**]), the element $\omega \in T^2_1 * (V_m)$ can be expressed in the form,

$$\text{(1.4)} \qquad \omega = (x^i, y_{i_1}, y_{i_1 i_2}),\ i,\ i_1,\ i_2 = 1, 2, \dots, m,\ y_{i_1 i_2} \text{ is symmetric with respect to } i_1,\ i_2.$$

If

$$(x^{j'}, y_{j'_1}, y_{j'_1 j'_2}),$$

is the expression of ω in a new coordinate system $\{x^{j'}\}_{j=1,2,\dots,m}$ at $x \in V_m$, then ([**10**]) the transformation law for the local coordinates of ω is given by the equations,

$$\text{(1.5)} \qquad \begin{cases} x^i = \varphi^i(x^{j'}), \\ y_{j'_1} = y_{i_1} a^{i_1}_{j'_1}, \\ y_{j'_1 j'_2} = y_{i_1} a^{i_1}_{j'_1 j'_2} + y_{i_1 i_2} a^{i_1}_{j'_1} a^{i_2}_{j'_2}\ , \end{cases}$$

where $a^i_{j'_1} = \frac{\partial \varphi^i}{\partial x^{j'_1}}$, $a^i_{j'_1 j'_2} = \frac{\partial^2 \varphi^i}{\partial x^{j'_1} \partial x^{j'_2}}$, so that $(a^i_{j'_1}, a^i_{j'_1 j'_2}) \in L^2_m$.

The dual vector bundle $T^2(V_m)$ of the vector bundle $T^2_1 * (V_m)$, has ([**4**]) the basis V_m, the structural group L^2_m and the fibre $F^2 = (L^2_{1,m})^*$. Hence ([**2**]) $T^2(V_m)(V_m, L^2_m, F^2)$ is the vector bundle of all tangent vectors of order 2 and $v \in T^2_x$ is a tangent vector of order 2 (or 2-tangent vector) at the point $x \in V_m$.

Remark 1.2. However, there are other, different, notions of the tangent bundle of higher order ([**16**], [**19**]) using another point of view.

From [**6**] follows that $T^2(V_m)(V_m, L^2_m, F^2)$ is also associated with the principal fibre bundle $H^2(V_m)(V_m, L^2_m)$, which can be identified with the space of bases of the vector spaces T^2_x at $x \in V_m$.

Let now,

(1.6)

$e = (e_{i_1}, e_{i_1 i_2})$, i_1, $i_2 = 1, 2, \dots, m$, $e_{i_1 i_2}$ symmetric in the indices i_1, i_2,

be the natural basis of T^2_x defined by the local chart $\{x^i\}_{i=1,2,\dots,m}$ at the point $x \in V_m$. Then, every $v \in T^2_x$ can be expressed uniquely in the form,

$$\text{(1.7)} \qquad v = \sum_{i_1=1}^{m} v^{i_1} e_{i_1} + \sum_{i \le i_1 \le i_2 \le m} v^{i_1 i_2} e_{i_1 i_2},$$

where $(v^{i_1}, v^{i_1 i_2})$ are some constants and $v^{i_1 i_2}$ is symmetric in the indices i_1, i_2.

For another local chart $\{x^{j'}\}_{j=1,2,\dots,m}$ at $x \in V_m$ the corresponding transformation law for the local coordinates of $v \in T_x^2$ will be,

$$\tag{1.8} \begin{cases} v^{j_1'} = a_{i_1}^{j_1'} v^{i_1} + a_{i_1 i_2}^{j_1'} v^{i_1 i_2}, \\ v^{j_1' j_2'} = a_{i_1}^{j_1'} a_{i_2}^{j_2'} v^{i_1 i_2}, \end{cases}$$

where $a = (a_{i_1}^{j_1'}, a_{i_1 i_2}^{j_1'}) \in L_m^2$.

For convenience in calculation, from now on, we will keep using matrices.

Thus, the relation (1.8) for the element v of T_x^2 in the overlap of two local charts (U, x^i) and $(V, x^{j'})$, can be written in the form,

$$\tag{1.9} [\, v^{j_1'} \quad v^{j_1' j_2'} \,] = [\, v^{i_1} \quad v^{i_1 i_2} \,] \begin{bmatrix} a_{i_1}^{j_1'} & 0 \\ a_{i_1 i_2}^{j_1'} & a_{i_1}^{j_1'} a_{i_2}^{j_2'} \end{bmatrix},$$

or briefly,

$$V_x^v = A_u^v(x) V_x^u,$$

where

$$\tag{1.10} \begin{cases} A_u^v(x) = \begin{bmatrix} a_{i_1}^{j_1'} & 0 \\ a_{i_1 i_2}^{j_1'} & a_{i_1}^{j_1'} a_{i_2}^{j_2'} \end{bmatrix}, \; V_x^u = [\, v^{i_1} \quad v^{i_1 i_2} \,], \\ V_x^v = [\, v^{j_1'} \quad v^{j_1' j_2'} \,]. \end{cases}$$

The element $a = (a_{i_1}^{j_1'}, a_{i_1 i_2}^{j_1'})$ of L_m^2 is identified with the matrix A defined by (1.10).

Similarly, according to the relation (1.5), the transformation law for each element ω of T_x^{2*} (that is of $T_{1,x}^2 * (V_m)$) by means of matrices, can be written in the form,

$$\tag{1.11} \begin{bmatrix} \omega_{j_1'} \\ \omega_{j_1' j_2'} \end{bmatrix} = \begin{bmatrix} a_{j_1'}^{i_1} & 0 \\ a_{j_1' j_2'}^{i_1} & a_{j_1'}^{i_1} a_{j_2'}^{i_2} \end{bmatrix} \begin{bmatrix} \omega_{i_1} \\ \omega_{i_1 i_2} \end{bmatrix},$$

where $(a_{j_1'}^{i_1}, a_{j_1' j_2'}^{i_1}) \in L_m^2$ is the inverse element of $a = (a_{i_1}^{j_1'}, a_{i_1 i_2}^{j_1'}) \in L_m^2$. Briefly,

$$\omega_v^x = \omega_u^x A_v^u(x), \qquad A_v^u(x) = [A_u^v(x)]^{-1}.$$

2. G_1^2-structures

Definition 2.1. A G_1-structure of second order, briefly G_1^2-structure, can be defined on an m-dimensional differentiable manifold V_m of class C^∞, by means of an 1-form of second order J, with values in the second order tangent bundle $T^2(V_m)$, such that at each point $x \in V_m$,

$$J_x \text{ is of rank } p + \binom{p+1}{2} \text{ everywhere in } V_m \tag{2.1}$$

and

$$J_x^2 = 0. \tag{2.2}$$

From the condition (2.2) follows that $J_x(T_x^2)$ is composed of the eigenvectors of J. On the other hand $J_x(T_x^2) \subset \ker J_x$.

If S_x is the complementary space of $\ker J_x$ with respect to $T_x^2(V_m)$ then,

$$T_x^2(V_m) = \ker J_x \oplus S_x$$

and J induces an isomorphism between S_x and $J(S_x)$.

Thus,

$$\dim T_x^2(V_m) = m + \binom{m+1}{2} > 2\left[p + \binom{p+1}{2}\right] \text{ and } m \geq 2p.$$

We have,

$$\dim S_x = p + \binom{p+1}{2}$$

and

$$\dim \ker J_x = q + \binom{q+1}{2} + pq, \qquad p \leq q, \qquad m = p + q$$

and the dimension of $\ker J_x$ is independent of the point $x \in V_m$.

Obviously,

$$\dim T_x^2 = p + \binom{p+1}{2} + q + \binom{q+1}{2} + pq.$$

Also,

$$\begin{aligned} &\dim \ker J_x = \\ &= p + (q-p) + \binom{p+1}{2} + p(q-p) + p(m-q) + \binom{q-p+1}{2} + (q-p)(m-q). \end{aligned} \tag{2.3}$$

Let $\{e_{i_1}, e_{i_1 i_2}\}$, $i_1, i_2 = 1, 2, \ldots, m$ be a basis of the second order tangent space T_x^2. Using the indices,

$$\begin{aligned}
&\alpha(1) = 1, 2, \ldots, p,\ \alpha(2) = p+1, \ldots, q,\\
&A(1) = 1, 2, \ldots, q \text{ (that is, } A(1) = (\alpha(1), \alpha(2))) \text{ and}\\
&A(2) = q+1, \ldots, q+p = m \text{ (that is, } A(2) = q + \alpha(1)),
\end{aligned}$$

it follows, that the second order tangent vectors

$$\{e_{A_1(1)},\ e_{A_1(1)A_2(1)},\ e_{\alpha_1(1)A_2(1)}\} \tag{2.4}$$

and

$$\{e_{A_1(2)},\ e_{A_1(2)A_2(2)}\} \tag{2.5}$$

define a basis on $\ker J_x$ and S_x respectively.

Also, by (2.3) the basis (2.4) can be written in the form,

$$\begin{aligned}
\{&(e_{\alpha_1(1)},\ e_{\alpha_1(2)}),\\
&(e_{\alpha_1(1)\alpha_2(1)},\ e_{\alpha_1(1)\alpha_2(2)},\ e_{\alpha_1(1)A_2(2)},\ e_{\alpha_1(2)\alpha_2(2)},\ e_{\alpha_1(2)A_2(2)})\}.
\end{aligned} \tag{2.6}$$

Definition 2.2. The basis $\{e_{i_1}, e_{i_1 i_2}\}$ of T_x^2 will be called adapted to the G_1^2-structure with respect to x or simply G_1^2-adapted basis if

$$\begin{cases} J_x e_{A_1(2)} = e_{\alpha_1(1)},\\ J_x e_{A_1(2)A_2(2)} = e_{\alpha_1(1)\alpha_2(1)}. \end{cases} \tag{2.7}$$

To the operator J_x corresponds the element F of the tensor product $T_x^2 \otimes (T_x^2)*$,

$$\begin{cases} F = (F_{i_1}^{j_1}, F_{i_1}^{j_1 j_2}, F_{i_1 i_2}^{j_1}, F_{i_1 i_2}^{j_1 j_2})\ i_1, i_2, j_1, j_2 = 1, 2, \ldots, m,\\ F_{i_1}^{j_1 j_2} \text{ symmetric with respect to } j_1, j_2, F_{i_1 i_2}^{j_1} \text{ symmetric}\\ \text{ with respect to } i_1, i_2 \text{ and } F_{i_1 i_2}^{j_1 j_2} \text{ symmetric in indices } i_1,\\ i_2 \text{ and } j_1, j_2. \end{cases} \tag{2.8}$$

Then, J is given by

$$\begin{cases} (J_x v)^{j_1} = F_{i_1}^{j_1} v^{i_1} + F_{i_1 i_2}^{j_1} v^{i_1 i_2},\\ (J_x v)^{j_1 j_2} = F_{i_1}^{j_1 j_2} v^{i_1} + F_{i_1 i_2}^{j_1 j_2} v^{i_1 i_2}, \end{cases} \tag{2.9}$$

where $v = (v^{i_1}, v^{i_1 i_2})$ is a 2-tangent vector at $x \in V_m$.

The tensor F of the relation (2.8) can be represented by the following matrix,

$$F = \begin{bmatrix} F_{i_1}^{j_1} & F_{i_1}^{j_1 j_2} \\ F_{i_1 i_2}^{j_1} & F_{i_1 i_2}^{j_1 j_2} \end{bmatrix}. \tag{2.10}$$

From (2.2) and (2.9) one verifies easily the following equations,

$$\begin{cases} F_{i_1}^{j_1} F_{j_1}^{k_1} + F_{i_1}^{j_1 j_2} F_{j_1 j_2}^{k_1} = 0, \\ F_{i_1}^{j_1} F_{j_1}^{k_1 k_2} + F_{i_1}^{j_1 j_2} F_{j_1 j_2}^{k_1 k_2} = 0, \\ F_{i_1 i_2}^{j_1} F_{j_1}^{k_1} + F_{i_1 i_2}^{j_1 j_2} F_{j_1 j_2}^{k_1} = 0, \\ F_{i_1 i_2}^{j_1} F_{j_1}^{k_1 k_2} + F_{i_1 i_2}^{j_1 j_2} F_{j_1 j_2}^{k_1 k_2} = 0. \end{cases} \tag{2.11}$$

Using, now, an G_1^2-adapted basis on T_x^2 and the relations:

$$\begin{array}{lll} J_x e_{\alpha_1(1)} = 0, & J_x e_{\alpha_1(2)} = 0, & J_x e_{A_1(2)} = e_{\alpha_1(1)}, \\ J_x e_{\alpha_1(1)\alpha_2(1)} = 0, & J_x e_{\alpha_1(1)\alpha_2(2)} = 0, & J_x e_{\alpha_1(1)A_2(2)} = 0, \\ J_x e_{\alpha_1(2)\alpha_2(2)} = 0, & J_x e_{\alpha_1(2)A_2(2)} = 0, & J_x e_{A_1(2)A_2(2)} = e_{\alpha_1(1)\alpha_2(1)}, \end{array}$$

it follows that the tensor F associated to the operator J can be represented by the matrix:

$$F = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta_{\alpha_1(1)}^{\beta_1(1)} & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_{\alpha_1(1)}^{\beta_1(1)} \delta_{\alpha_2(1)}^{\beta_2(1)} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}. \tag{2.12}$$

Thus we have,

Proposition 2.1. *There is always a G_1^2-adapted basis of $T_x^2(V_m)$ at each point $x \in V_m$, in which F has the constant components given by the matrix (2.12).*

Let $\{e_{j_1'}, e_{j_1' j_2'}\}$ be another G_1^2-adapted basis, then the transformation law between the G_1^2-adapted bases may be written,

$$\begin{aligned} e_{i_1} &= e_{j_1'} l_{i_1}^{j_1'}, \\ e_{i_1 i_2} &= e_{j_1'} l_{i_1 i_2}^{j_1'} + e_{j_1' j_2'} l_{i_1}^{j_1'} l_{i_2}^{j_2'}, \end{aligned}$$

where the element $l = (l^{j_1'}_{i_1}, l^{j_1'}_{i_1 i_2}) \in L^2_m$. Explicitly,

$$(2.13)\qquad \begin{cases} l^{j_1'}_{i_1} = \left(l^{\beta_1'(1)}_{\alpha_1(1)}, l^{\beta_1'(1)}_{\alpha_1(2)}, l^{\beta_1'(1)}_{A_1(2)}, l^{\beta_1'(2)}_{\alpha_1(2)}, l^{\beta_1'(2)}_{A_1(2)}\right) \in G_1 \\ l^{j_1'}_{i_1 i_2} = \left(l^{\beta_1'(1)}_{\alpha_1(1)\alpha_2(1)}, l^{\beta_1'(1)}_{\alpha_1(1)\alpha_2(2)}, l^{\beta_1'(1)}_{\alpha_1(1)A_2(2)}, \right. \\ \qquad l^{\beta_1'(1)}_{\alpha_1(2)\alpha_2(2)}, l^{\beta_1'(1)}_{\alpha_1(2)A_2(2)}, l^{\beta_1'(1)}_{A_1(2)A_2(2)}, \\ \qquad l^{\beta_1'(2)}_{\alpha_1(1)\alpha_2(2)}, l^{\beta_1'(2)}_{\alpha_1(1)A_2(2)}, l^{\beta_1'(2)}_{\alpha_1(2)\alpha_2(2)}, \\ \qquad \left. l^{\beta_1'(2)}_{\alpha_1(2)A_2(2)}, l^{\beta_1'(2)}_{A_1(2)A_2(2)}\right), \end{cases}$$

where G_1 is ([**15**]) the structural group of the G_1-structure of the first order. It is consisting of the matrices of the form

$$\begin{bmatrix} A & 0 & 0 \\ B & \Gamma & 0 \\ \Delta & E & A \end{bmatrix},$$

with $A \in L_p$, $B \in \mathrm{End}(R^p, R^{q-p})$, $\Gamma \in L_{q-p}$, $\Delta \in \mathrm{End}(R^p, R^{m-q})$ and $E \in \mathrm{End}(R^{q-p}, R^{m-q})$.

Using matrices, l can be written by the matrix

$$(2.14)\qquad L = \begin{bmatrix} l^{j_1'}_{i_1} & 0 \\ l^{j_1'}_{i_1 i_2} & l^{j_1'}_{i_1} l^{j_2'}_{i_2} \end{bmatrix},$$

where

$$(2.15)\qquad l^{j_1'}_{i_1} = \begin{bmatrix} l^{\beta_1'(1)}_{\alpha_1(1)} & 0 & 0 \\ l^{\beta_1'(1)}_{\alpha_1(2)} & l^{\beta_1'(2)}_{\alpha_1(2)} & 0 \\ l^{\beta_1'(1)}_{A_1(2)} & l^{\beta_1'(2)}_{A_1(2)} & l^{\beta_1'(1)}_{\alpha_1(1)} \end{bmatrix} \in G_1,$$

$$(2.16)\qquad l^{j_1'}_{i_1 i_2} = \begin{bmatrix} l^{\beta_1'(1)}_{\alpha_1(1)\alpha_2(1)} & 0 & 0 \\ l^{\beta_1'(1)}_{\alpha_1(1)\alpha_2(2)} & l^{\beta_1'(2)}_{\alpha_1(1)\alpha_2(2)} & 0 \\ l^{\beta_1'(1)}_{\alpha_1(1)A_2(2)} & l^{\beta_1'(2)}_{\alpha_1(1)A_2(2)} & 0 \\ l^{\beta_1'(1)}_{\alpha_1(2)\alpha_2(2)} & l^{\beta_1'(2)}_{\alpha_1(2)\alpha_2(2)} & 0 \\ l^{\beta_1'(1)}_{\alpha_1(2)A_2(2)} & l^{\beta_1'(2)}_{\alpha_1(2)A_2(2)} & 0 \\ l^{\beta_1'(1)}_{A_1(2)A_2(2)} & l^{\beta_1'(2)}_{A_1(2)A_2(2)} & l^{\beta_1'(1)}_{\alpha_1(1)\alpha_2(2)} \end{bmatrix}.$$

Let G^2_1 be the subgroup of L^2_m, consisting of all elements of the form (2.13) with corresponding matrix of the form (2.14) with (2.15) and (2.16). It can be verified that,

Proposition 2.2. *The group G_1^2 can be characterized as the subgroup of L_m^2 defined by all elements of L_m^2 which commute with F.*

Let $E_1^2(V_m)$ be the set of all the adapted bases at the different points of V_m and p the canonical mapping,

$$p : E_1^2(V_m) \longrightarrow V_m,$$

which associates with an adapted basis at x the point x itself. $E_1^2(V_m)$ is equipped with a structure of principal fibre bundle of basis V_m and structural group G_1^2.

Conversely, we assume that the differentiable manifold V_m admits a G_1^2-structure, where G_1^2 is the group of matrices of the form (2.14) with (2.15) and (2.16). Then, it can be defined on V_m, a tensor field F of type $(1,1)$ and of rank $p + \binom{p+1}{2}$. F has (2.12) as components with respect to the adapted basis and satisfies the condition (2.2).

Thus, we have,

Theorem 2.1. *A necessary and sufficient condition for a differentiable manifold V_m to admit a G_1^2-structure is that the structural group of the second order frame bundle $H^2(V_m)$ be reduced to the group G_1^2.*

3. G_1^2-Connections

Definition 3.1. Any infinitesimal connection ([**18**], [**5**]) defined on the principal bundle $E_1^2(V_m, G_m^2)$ is called a G_1^2-connection.

We consider a covering of V_m by open neighborhoods endowed with local cross sections of $E_1^2(V_m)$. Any G_1^2-connection may be defined in each neighborhood U by a local form π with values in the Lie algebra $\underline{G}_1^2$ of the group G_1^2.

Hence, a G_1^2-connection is represented by the element of the Lie algebra $\underline{G}_1^2$,

$$\pi_u = (\pi^{i_1}_{j_1}, \pi^{i_1}_{j_1 j_2}), \qquad i_1, j_1, j_2 = 1, 2, \ldots, m, \tag{3.1}$$

where the linear differential forms on U, $(\pi^{i_1}_{j_1}) \in R^m \otimes R^{m*}$ and $(\pi^{i_1}_{j_1 j_2}) \in R^m \otimes S^2(R^{m*})$ satisfy the relations,

$$\begin{cases} \pi^{\beta_1(2)}_{\alpha_1(1)} = \pi^{B_1(2)}_{\alpha_1(1)} = \pi^{B_1(2)}_{\alpha_1(2)} = 0,\ \pi^{\beta_1(1)}_{\alpha_1(1)} = \pi^{B_1(2)}_{A_1(2)}, \\ \pi^{\beta_1(2)}_{\alpha_1(1)\alpha_2(1)} = \pi^{B_1(2)}_{\alpha_1(1)\alpha_2(1)} = \pi^{B_1(2)}_{\alpha_1(1)\alpha_2(2)} = \pi^{B_1(2)}_{\alpha_1(1)A_2(2)} = \\ = \pi^{B_1(2)}_{\alpha_1(2)\alpha_2(2)} = \pi^{B_1(2)}_{\alpha_1(2)A_2(2)} = 0,\ \pi^{\beta_1(1)}_{\alpha_1(1)\alpha_2(1)} = \pi^{B_1(2)}_{A_1(2)A_2(2)}. \end{cases} \tag{3.2}$$

It can be verified that,

Proposition 3.1. *With respect to a G_1^2-connection, the absolute differential of the tensor F is zero.*

$E_1^2(V_m)$ may be considered as a sub-bundle of the fibre bundle $H_1^2(V_m)$ of 2-frames that is of bases of vector spaces $\{T_x^2\}_{x\in V_m}$ ([**6**]).

A G_1^2-connection defines canonically a connection of order 2 ([**9**], [**14**]) on V_m with which it may be identified.

Conversely, let us consider a connection of order 2 and a covering of V_m by open neighborhoods equipped with local cross sections of $E_1^2(V_m)$. This connection may be defined on each neighborhood by a local form ω with values in the Lie algebra of L_m^2,

$$\begin{gathered} \omega = (\omega_{j_1}^{i_1}, \omega_{j_1 j_2}^{i_1}), \quad i_1, j_1, j_2 = 1, 2, \ldots, m, \quad \omega_{j_1}^{i_1} \in R^m \otimes R^{m*}, \\ \omega_{j_1 j_2}^{i_1} \in R^m \otimes S^2(R^{m*}) \text{ and } (\omega_{j_1}^{i_1}),\ (\omega_{j_1 j_2}^{i_1}) \\ \text{are local linear differential forms.} \end{gathered} \tag{3.3}$$

In order that the given connection may be identified with a G_1^2-connection it is necessary and sufficient that the form (3.3) belongs in the Lie algebra of the structural group G_1^2 of $E_1^2(V_m)$. That is, comparing with (3.2),

Proposition 3.2. *In order that a connection of order 2 may be identified with a G_1^2-connection, it is necessary and sufficient that the absolute differential of the tensor F is zero with respect to this connection.*

Given a G_1^2-connection Y, the curvature form of this connection is the tensor 2-form, of adjoint type.

$$\Omega = \nabla \pi = d\pi + \pi \wedge \pi, \tag{3.4}$$

which is defined on $E_1^2(V_m)$ with values to $\underline{G}_1^2$.

If we consider a covering of V_m by neighborhoods equipped with local cross sections of $E_1^2(V_m)$, then, Ω may be defined in each neighborhood U by a local form with values in the Lie algebra $\underline{G}_1^2$

$$\Omega_u = (\Omega_{j_1}^{i_1}, \Omega_{j_1 j_2}^{i_1}), \qquad i, j_1, j_2 = 1, 2, \ldots, m, \tag{3.5}$$

where $\Omega_{j_1}^{i_1} \in R^m \otimes R^{m*}$, $\Omega_{j_1 j_2}^{i_1} \in R^m \otimes S^2(R^{m*})$ and $(\Omega_{j_1}^{i_1})$, $(\Omega_{j_1 j_2}^{i_1})$ are linear differential forms on U.

It may be seen from (3.4),

$$\begin{cases} \Omega_{j_1}^{i_1} = d\pi_{j_1}^{i_1} + \pi_{k_1}^{i_1} \wedge \pi_{j_1}^{k_1}, \\ \Omega_{j_1 j_2}^{i_1} = d\pi_{j_1 j_2}^{i_1} + \pi_{k_1}^{i_1} \wedge \pi_{j_1 j_2}^{k_1} + \pi_{k_1 k_2}^{i_1} \wedge \pi_{j_1}^{k_1} \pi_{j_2}^{k_2}. \end{cases} \tag{3.6}$$

In particular, it can be verified that,

$$\Omega^{\alpha(1)}_{\alpha(1)} = d\pi^{\alpha(1)}_{\alpha(1)},\ \Omega^{\alpha(2)}_{\alpha(2)} = d\pi^{\alpha(2)}_{\alpha(2)},\ \Omega^{A(2)}_{A(2)} = d\pi^{A(2)}_{A(2)},$$
$$\Omega^{\alpha(1)}_{\alpha(1)\beta(1)} = d\pi^{\alpha(1)}_{\alpha(1)\beta(1)},\ \Omega^{\alpha(1)}_{\alpha(1)\beta(2)} = d\pi^{\alpha(1)}_{\alpha(1)\beta(2)},\ \Omega^{\alpha(1)}_{\alpha(1)B(2)} = d\pi^{\alpha(1)}_{\alpha(1)B(2)},$$
$$\Omega^{\alpha(2)}_{\alpha(2)\beta(2)} = d\pi^{\alpha(2)}_{\alpha(2)\beta(2)},\ \Omega^{\alpha(2)}_{\alpha(2)B(2)} = d\pi^{\alpha(2)}_{\alpha(2)B(2)},\ \Omega^{A(2)}_{A(2)B(2)} = d\pi^{A(2)}_{A(2)B(2)}.$$

Then,

$$\Psi = (\Omega^{\alpha(1)}_{\alpha(1)} = \Omega^{A(2)}_{A(2)}, \Omega^{\alpha(2)}_{\alpha(2)}, \Omega^{\alpha(1)}_{\alpha(1)\beta(1)} = \Omega^{A(2)}_{A(2)B(2)}, \Omega^{\alpha(1)}_{\alpha(1)\beta(2)},$$
$$\Omega^{\alpha(1)}_{\alpha(1)B(2)}, \Omega^{\alpha(2)}_{\alpha(2)\beta(2)}, \Omega^{\alpha(2)}_{\alpha(2)B(2)})$$

is a closed 2-form on $E^2_1(V_m)$.

Definition 3.2. We call Ψ the characteristic form of the G^2_1-connection Y.

Proposition 3.3. *The characteristic 2-forms of all the G^2_1-connections have the same cohomology class of degree 2 (characteristic cohomology class of the G^2_1-structure).*

4. G-structures of second order defined by linear operators satisfying algebraic relations

Using the way discussed previously (sections 1, 2) a generalization to the second order of the real almost product structure is given already in [6].

On the other hand, the definition (2.1) for a differentiable manifold V_{2m}, with rank $J = m + \binom{m+1}{2}$ gives a generalization of the almost tangent structure to the second order.

In this case, matrix (2.12) reduces to the form,

$$\tag{4.1} F = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ \delta^{\beta_1}_{\alpha_1} & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} & 0 & 0 \end{bmatrix} \end{bmatrix},$$

and the matrix (2.14) to the form,

$$L = \begin{bmatrix} \begin{bmatrix} l^{\beta'_1}_{\alpha_1} & 0 \\ l^{\beta'_1}_{A_1} & l^{\beta'_1}_{\alpha_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} l^{\beta'_1}_{\alpha_1\alpha_2} & 0 \\ l^{\beta'_1}_{\alpha_1 A_2} & 0 \\ l^{\beta'_1}_{A_1 A_2} & l^{\beta'_1}_{\alpha_1\alpha_2} \end{bmatrix} & \begin{bmatrix} l^{\beta'_1}_{\alpha_1} l^{\beta'_2}_{\alpha_2} & 0 & 0 \\ l^{\beta'_1}_{\alpha_1} l^{\beta'_2}_{A_2} & l^{\beta'_1}_{\alpha_1} l^{\beta'_2}_{\alpha_2} & 0 \\ l^{\beta'_1}_{A_1} l^{\beta'_2}_{A_2} & l^{\beta'_1}_{A_1} l^{\beta'_2}_{\alpha_2} & l^{\beta'_1}_{\alpha_1} l^{\beta'_2}_{\alpha_2} \end{bmatrix} \end{bmatrix}$$

with the indices $\alpha = \alpha(1) = 1, 2, \ldots, m$, $A = A(2) = m+1, \ldots, 2m = m+\alpha$, and

$$\begin{bmatrix} l^{\beta'_1}_{\alpha_1} & 0 \\ l^{\beta'_1}_{A_1} & l^{\beta'_1}_{\alpha_1} \end{bmatrix} \in G^m_{mm},$$

where G^m_{mm} is ([**3**], [**12**]) the structural group of the almost tangent structure with $l^{\beta'_1}_{\alpha_1} \in L_m, l^{\beta'_1}_{A_1} \in \mathrm{End}(R^m, R^m)$.

Thus, G-structures on V_m of the first order defined by linear operators and satisfying some algebraic relations can be generalized to G-structure of the second order, defined by endomorphism,

$$J : T^2(V_m) \longrightarrow T^2(V_m)$$

and satisfying the same algebraic relations.

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