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GA_2 INDEX OF SOME GRAPH OPERATIONS

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Abstract

Let $G = (V, E)$ be a graph. For $e = uv \in E(G)$, $n_u(e)$ is the number of vertices of G lying closer to u than to v and $n_v(e)$ is the number of vertices of G lying closer to v than u. The GA_2 index of G is defined as $\sum_{uv\in E(G)} \frac{2\sqrt{n_u(e)n_v(e)}}{n_u(e)+n_v(e)}$. We explore here some mathematical properties and present explicit formulas for this new index under several graph operations.

1 Introduction

In this paper, we only consider simple connected graphs. As usual, the distance between the vertices u and v of G is denoted by $d_G(u, v)$ ($d(u, v)$ for short). It is defined as the length of a minimum path connecting them and $d_G(u)(d(u)$ for short) denotes the degree of u in G. The Wiener index of a graph G is defined as $W(G) = \sum_{\{u,v\}} d(u, v)$[7, 17, 20, 23]. GA_2 index of the graph of G is defined by $GA_2(G) = \sum_{uv\in E(G)} \frac{2\sqrt{n_u(e|G)n_v(e|G)}}{n_u(e|G)+n_v(e|G)}$[4] that $n_u(e|G)(n_u(e)$ for short) is the number of vertices of G lying closer to u and $n_v(e|G)$ is the number of vertices of G lying closer to v. Notice that vertices equidistance from u and v are not taken into account.

The Cartesian product $G \times H$ of graphs G and H is a graph such that $V(G \times H) = V(G) \times V(H)$, and any two vertices (a, b) and (u, v) are adjacent in $G \times H$ if and only if either $a = u$ and b is adjacent with v, or $b = v$ and a is adjacent with u, see [10] for details. The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2) or ($u_1 = u_2$ and v_1 is adjacent with v_2),[10, p. 185]. For given graphs G_1 and G_2 we define their corona product $G_1 \circ G_2$ as the

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graph obtained by taking $|V(G_1)|$ copies of G_2 and joining each vertex of the i-th copy with vertex $v_i \in V(G_1)$. Obviously, $|V(G_1 \circ G_2)| = |V(G_1)|(1 + |V(G_2)|)$ and $|E(G_1 \circ G_2)| = |E(G_1)| + |V(G_1)|(|V(G_2)| + |E(G_2)|)$.

The Szeged index was originally defined as $Sz(G) = \sum_{e=uv\in E(G)}[n_u(e)n_v(e)]$[5, 13, 16, 17] where $n_u(e)$ and $n_v(e)$ are the same as the definition of GA_2. Now, we define $GA_1(G) = GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$[22] where $d(u)$ is the degree of vertex u. Throughout this paper, C_n, P_n, K_n and W_n denote the cycle, path, complete graphs and wheel on n vertices. Also, $K_{m,n}$ denotes the complete bipartite graph. Our other notations are standard and taken mainly from [3, 8, 21].

2 Some properties of $\mathbf{GA_2}$ index

The Geometric-Arithmetic inequality $\sqrt{n_u(e)n_v(e)} \leq \frac{n_u(e)+n_v(e)}{2}$, implies that $GA_2(G) \leq |E(G)|$, with equality if and only if for all $e \in E(G)$, $n_u(e) = n_v(e)$.

A k-regular graph G on n vertices is called strongly regular with parameters (n,k;a,c) if and only if each pair of adjacent vertices have a common neighbors and any two distinct non-adjacent vertices have c common neighbors ([6], p.177). We also say that G is (n,k;a,c)-strongly regular. A strongly regular graph is primitive if both G and its complement $\bar{G}$ connected; otherwise it is imprimitive or trivial. We restrict our attention to primitive strongly regular graphs, since an imprimitive strongly regular graph is either a complete multipartite graph or its complement, i.e., the disjoint union of some copies of K_m, for some m. This restriction allows us to assume $c = 0$ and $c = k$. The simplest non-trivial examples of strongly regular graphs are c and the Petersen graph, with 5 the parameter vectors (5,2,0,1) and (10,3,0,1), respectively. It is easy to see that a non-trivial strongly regular graph has diameter 2.

Proposition 1. If G is a strongly $k-$regular graph then $GA_2(G) = \frac{1}{2}k|V(G)|$.

Proof. We assume $c \neq 0$ and $c \neq k$. Let us consider an edge $e = uv$ of G. Its endvertices have a common neighbors and all of them are equidistant to u and v. The vertex u has another $k-1-a$ neighbors and all of them are closer to u than to v. Together with u itself, this gives us $n(e) = k-a$. We need not bother to consider other u vertices: those at the distance 2 from u are either adjacent to v, or are at the distance 2 from v, since the diameter of G is equal to 2. Hence they cannot contribute to $n(e)$. By the same reasoning, $n(e) = k - a$. Therefore, by definition $GA_2(G) = |E(G)| = \frac{1}{2}k|V(G)|$. This if the end of the proof. $\square$

Proposition 2.[4, Theorem 3] For any connected graph G with m edges,

$$GA_2(G) \leq \sqrt{mSz(G)},$$

with equality if and only if $G \cong K_n$.

Proposition 3.[4, Theorem 4] For any connected graph G with m edges,

$$GA_2(G) \leq \sqrt{Sz(G) + m(m-1)},$$

with equality if and only if $G \cong K_n$.

Proposition 4.[4, Theorem 6] Let G be a connected graph with n vertices and $m \geq 1$ edges. Then

$$GA_2(G) \geq \frac{2}{n}\sqrt{Sz(G) + m(m-1)}.$$

The equality is attained if and only if $G \cong K_2$.

Proposition 5.[17] If T is a tree then $Sz(T) = W(T)$.

Corollary 6. If T is a n-vertex tree then $GA_2(T) \leq \sqrt{(n-1)W(T)}$, $GA_2(T) \leq \sqrt{W(T) + (n-1)(n-2)}$ and $GA_2(T) \geq \frac{2}{n}\sqrt{W(T) + (n-1)(n-2)}$.

Proposition 7. Suppose G is a connected graph. Then $GA_2(G) \leq \lceil \frac{|E(G)|-1}{2} \rceil + \sqrt{\lceil \frac{|E(G)|-1}{2} \rceil^2 + Sz(G)}$, with equality if and only if G is a union of the odd number of K_2.

Proof. By definition,

$$\begin{aligned}
[GA_2(G)]^2 &= \sum_{uv \in E(G)} \frac{4n_u(e)n_v(e)}{[n_u(e)+n_v(e)]^2} + 2\sum_{uv \neq xy \in E(G)} \frac{2\sqrt{n_u(e)n_v(e)}}{n_u(e)+n_v(e)} \cdot \frac{2\sqrt{n_x(e)n_y(e)}}{n_x(e)+n_y(e)} \\
&\leq \sum_{uv \in E(G)} n_u(e)n_v(e) + 2\lceil \frac{|E(G)|-1}{2} \rceil \cdot GA_2(G) \\
&= Sz(G) + 2\lceil \frac{|E(G)|-1}{2} \rceil \cdot GA_2(G) \\
&\Rightarrow [GA_2(G) - \lceil \frac{|E(G)|-1}{2} \rceil]^2 \leq \lceil \frac{|E(G)|-1}{2} \rceil^2 + Sz(G).
\end{aligned}$$

Therefore,

$$GA_2(G) \leq \lceil \frac{|E(G)|-1}{2} \rceil + \sqrt{\lceil \frac{|E(G)|-1}{2} \rceil^2 + Sz(G)}$$

and equality holds if and only if G is a union of the odd number of K_2. □

3 Main Results

In this section, some exact formulas for the GA_2 index of the Cartesian product, composition, join and corona of graphs are presented.

The Wiener index of the Cartesian product of graphs was studied in [7, 20]. In [17], Klavžar, Rajapakse and Gutman computed the Szeged index of the Cartesian product graphs. The recent authors, [1, 2, 9, 11, 12, 13, 14, 15, 16, 18, 24], computed some exact formulas for the hyper-Wiener, vertex PI, edge PI, the first Zagreb, the second Zagreb, the edge Wiener and the edge Szeged indices of some graph operations. The aim of this section is to continue this program for computing the GA_2 index of these graph operations.

Proposition 8. Let G_1 and G_2 be connected graphs. Then $GA_2(G_1 \times G_2) = GA_2(G_2)|V(G_1)| + GA_2(G_1)|V(G_2)|$.

Proof. If $e = uv$, $e' = (u,x)(v,x)$ then $n_{(u,x)}(e') = |V(G_2)|n_u(e)$ and $n_{(v,x)}(e') = |V(G_2)|n_v(e)$. Thus $\frac{2\sqrt{n_{(u,x)}(e')n_{(v,x)}(e')}}{n_{(u,x)}(e')+n_{(v,x)}(e')} = \frac{2\sqrt{n_u(e)n_v(e)}}{n_u(e)+n_v(e)}$ and by definition,

$$
\begin{aligned}
GA_2(G_1 \times G_2) &= \sum_{e_1=uv\in E(G_1\times G_2)} \frac{2\sqrt{n_u(e_1)n_v(e_1)}}{n_u(e_1)+n_v(e_1)} \\
&= \sum_{e'=(u,x)(v,x)} \frac{2\sqrt{n_{(u,x)}(e')n_{(v,x)}(e')}}{n_{(u,x)}(e')+n_{(v,x)}(e')} + \sum_{e'=(u,x)(u,y)} \frac{2\sqrt{n_{(u,x)}(e')n_{(u,y)}(e')}}{n_{(u,x)}(e')+n_{(u,y)}(e')} \\
&= |V(G_2)| \sum_{e=uv\in E(G_1)} \frac{2\sqrt{n_u(e)n_v(e)}}{n_u(e)+n_v(e)} + |V(G_1)| \sum_{e=xy\in E(G_2)} \frac{2\sqrt{n_x(e)n_y(e)}}{n_x(e)+n_y(e)} \\
&= GA_2(G_2)|V(G_1)| + GA_2(G_1)|V(G_2)|.
\end{aligned}
$$

This completes our argument. □

Corollary 9. Suppose G_1, G_2, ..., G_n are graphs. Then

$$GA_2(\prod_{i=1}^{k} G_i) = (\prod_{i=1}^{k} |V(G_i)|) \sum_{i=1}^{k} \frac{GA_2(G_i)}{|V(G_i)|}.$$

Corollary 10. Suppose G is a graph. Then $GA_2(G^n) = nGA_2(G)|V(G)|^{n-1}$. In particular, $GA_2(Q_n) = n2^{n-1}$.

Corollary 11. If $G_1 = P_m \times P_n$, $G_2 = P_m \times C_n$ and $G_3 = C_m \times C_n$ are C_4-net, C_4-nanotube and C_4-nanotorus, respectively. Then
$GA_2(G_1) = \frac{4|E(G_1)|}{|V(G_1)|} \sum_{i=1}^{|V(G_1)|-1} \sqrt{i(|V(G_1)| - i)} + \frac{4|V(G_1)|}{|E(G_1)|} \sum_{i=1}^{|E(G_1)|-1} \sqrt{i(|E(G_1)| - i)}$,
$GA_2(G_2) = \frac{4|V(G_2)|}{|E(G_2)|} \sum_{i=1}^{|E(G_2)|-1} \sqrt{i(|E(G_2)| - i)} + |E(G_2)||V(G_2)|$,
$GA_2(G_3) = 2|E(G_3)||V(G_3)|$.

Proof. We notice that if $e = uv$ is an arbitrary edge of P_n or C_n then $n_u(e) = n_v(e)$. Thus $\frac{2\sqrt{n_u(e)n_v(e)}}{n_u(e)+n_v(e)} = 1$ for each edge of P_n or C_n. Therefore, $GA_2(P_n) = \frac{4}{n} \sum_{i=1}^{n-1} \sqrt{i(n-i)}$ and $GA_2(C_n) = n$. Now, Proposition 8 completes the proof. □

Proposition 12. Let $G = G_1 + G_2$, where $G_i^{'}$s are r_i-regular, $i = 1, 2$. Then $GA_2(G) = GA_2(G_1) + GA_2(G_2) + 2|V(G_1)||V(G_2)| \frac{\sqrt{(|V(G_1)|-r_1)(|V(G_2)|-r_2)}}{|V(G_1)|+|V(G_2)|-(r_1+r_2)}$.

Proof. Suppose $G = G_1 + G_2$. We can partition the edges of $G = G_1 + G_2$ into three subsets E_1, E_2 and E_3, as follows:

$$\begin{aligned} E_i &= \{e \in E(G_1 + G_2) | e \in E(G_i)\ \}, i = 1, 2 \\ E_3 &= \{e \in E(G_1 + G_2) | e = uv, u \in V(G_1)\ and\ v \in V(G_2)\}. \end{aligned}$$

By [13, Theorem 2], if $e = u_1v_1 \in E_i$ then $n_{u_1}(e|G) = n_{u_1}(e|G_i)$ and $n_{v_1}(e|G) = n_{v_1}(e|G_i)$. If $e = uv \in E_3$ then $n_u(e|G) = |V(G_2)| - d_{G_2}(v)$ and $n_v(e|G) = |V(G_1)| - d_{G_1}(u)$. Therefore,

$$\begin{aligned} GA_2(G) &= \sum_{uv \in E(G_1)} \frac{2\sqrt{n_u(e|G_1)n_v(e|G_1)}}{n_u(e|G_1) + n_v(e|G_1)} + \sum_{uv \in E(G_2)} \frac{2\sqrt{n_u(e|G_2)n_v(e|G_2)}}{n_u(e|G_2) + n_v(e|G_2)} \\ &+ \sum_{\substack{u \in V(G_1) \\ v \in V(G_2)}} \frac{2\sqrt{(|V(G_2)| - d_{G_2}(v))(|V(G_1)| - d_{G_1}(u))}}{(|V(G_2)| - d_{G_2}(v)) + (|V(G_1)| - d_{G_1}(u))} \\ &= GA_2(G_1) + GA_2(G_2) + 2|V(G_1)||V(G_2)| \frac{\sqrt{(|V(G_1)| - r_1)(|V(G_2)| - r_2)}}{|V(G_1)| + |V(G_2)| - (r_1 + r_2)}. \end{aligned}$$

This is the end of our proof. □

Corollary 13. If G is r-regular graph then

$$GA_2(nG) = nGA_2(G) + 2\sum_{i=2}^{n} |V(G)|^i \frac{\sqrt{(|V(G)| - r)(|V(G)|^{i-1} - r)}}{|V(G)|^i - 2r}.$$

Corollary 14. $GA_2(K_{m,n}) = 2\frac{(mn)^{\frac{3}{2}}}{m+n}$, $GA_2(K\underbrace{n, n, \ldots, n}_{t\ times}) = 2\sum_{i=2}^{t} \sqrt{n^i}$ and $GA_2(W_n) = n - 1 + 2(n-1)\frac{\sqrt{n-3}}{n-2}$.

We present formula for GA_2 index of open fence, $P_n[K_2]$.

Example 15. $GA_2(P_n[K_2]) = n + \frac{4}{n-1}\sum_{i=1}^{n-1} \sqrt{(2i-1)(2n-2i-1)}$.

Proposition 16. If G_2 is triangle-free and r-regular graph then

$$GA_2(G_1[G_2]) < |V(G_2)|^2 |E(G_1)| \frac{|V(G_2)|(|V(G_1)| - 1)}{|V(G_2)| - 2r} + |V(G_1)| GA_1(G_2).$$

Proof. Suppose $G = G_1[G_2]$ and $t_G(e)$ denotes the number of triangles containing e of the graph G. Let

$$\begin{aligned} A_u &= \{(u, v) | v \in V(G_2)\}, \\ B_u &= \{(u, v_1)(u, v_2) | v_1v_2 \in E(G_2)\}, \\ T(u_1, u_2) &= \{(x, y)(a, b) | ((x, y), (a, b)) \in A_{u_1} \times A_{u_2}\}, \\ E(G) &= (\cup_{u_1u_2 \in E(G_1)} T(u_1, u_2)) \cup (\cup_{v \in V(G_1)} B_v). \end{aligned}$$

By [13, Theorem 3], if $e = (u_1, v_1)(u_2, v_2) \in T(u_1, u_2)$ then

$$n_{(u_1,v_1)}(e|G) = |V(G_2)|n_{u_1}(u_1u_2|G_1) - d_{G_2}(v_2)$$

and $n_{(u_2,v_2)}(e|G) = |V(G_2)|n_{u_1}(u_1u_2|G_1) - d_{G_2}(v_1)$ and if $e = (u, v_1)(u, v_2) \in B_u$ then $n_{(u,v_1)}(e|G) = d_{G_2}(v_1)$ and $n_{(u,v_2)}(e|G) = d_{G_2}(v_2)$. Therefore

$$\begin{aligned} GA_2(G) &= \sum_{e=uv\in\cup_{u_1u_2\in E(G_1)}T(u_1,u_2)} \frac{2\sqrt{n_{(u_1,v_1)}(e|G)n_{(u_2,v_2)}(e|G)}}{n_{(u_1,v_1)}(e|G) + n_{(u_2,v_2)}(e|G)} \\ &+ \sum_{e=(u,v_1)(u,v_2)\in B_u} \frac{2\sqrt{d_{G_2}(v_1)d_{G_2}(v_2)}}{d_{G_2}(v_1) + d_{G_2}(v_2)} \\ &= \sum_{e=uv\in\cup T(u_1,u_2)} \frac{|V(G_2)|n_{u_1}(u_1u_2|G_1) - r}{|V(G_2)|n_{u_1}(u_1u_2|G_1) - 2r} + |V(G_1)|GA_1(G_2) \\ &< |V(G_2)|^2|E(G_1)|\frac{|V(G_2)|(|V(G_1)| - 1)}{|V(G_2)| - 2r} + |V(G_1)|GA_1(G_2), \end{aligned}$$

which completes our proof. □

Proposition 17. If H is triangle-free and r-regular graph then

$$GA_2(G \circ H) = GA_1(H) + GA_2(G) + |V(G)||V(H)|\frac{2\sqrt{|V(G)| + |V(G)||V(H)| - r - 1}}{|V(G)| + |V(G)||V(H)| - r}.$$

Proof. The edges of $G \circ H$ are partitioned into three subsets E_1, E_2 and E_3 as follows:

$$\begin{aligned} E_1 &= \{e \in E(G \circ H)|\ e \in E(H_i)\ i = 1, 2 \ldots, n\}, \\ E_2 &= \{e \in E(G \circ H)|\ e \in E(G)\}, \\ E_3 &= \{e \in E(G \circ H)|\ e = uv, u \in V(H_i), i = 1, 2 \ldots, n \textit{ and } v \in V(G)\}. \end{aligned}$$

Suppose $e = uv \in E(H)$. If there exists $w \in V(H)$ such that $uw \neq E(H)$ and $vw \neq E(H)$ then $d_{G\circ H}(u, w) = d_{G\circ H}(v, w) = 2$. Also, if there is $w \in V(H)$ such that $uw \in E(H)$ and $vw \in E(H)$ then $d_{G\circ H}(u, w) = d_{G\circ H}(v, w) = 1$. Moreover, if $e = uv \in E_1$ then

$$n_u(e|G \circ H) = d_H(u) - t_H(uv), \quad n_v(e|G \circ H) = d_H(v) - t_H(uv),$$

and if $e = uv$ in E_2 then $n_u(e|G \circ H) = (|V(H)| + 1)n_u(e|G)$, $n_v(e|G \circ H) = (|V(H)| + 1)n_v(e|G)$, $n_u(e|G \circ H)n_v(e|G \circ H) = |V(G \circ H)| - (d_H(u) + 1)$ and $n_u(e|G \circ H) + n_v(e|G \circ H) = |V(G \circ H)| - d_H(u)$. If $e = uv \in E_3$ then by above calculations,

$$GA_2(G \circ H) = \sum_{uv\in E(G\circ H)} \frac{2\sqrt{n_u(e|G \circ H)n_v(e|G \circ H)}}{n_u(e|G \circ H) + n_v(e|G \circ H)}$$

$$
\begin{aligned}
&= \sum_{uv\in E_1} \frac{2\sqrt{(d_H(u)-t_H(uv))(d_H(v)-t_H(uv))}}{d_H(u)+d_H(v)-2t_H(uv)} \\
&+ \sum_{uv\in E_2} \frac{2\sqrt{(|V(H)|+1)^2 n_u(e|G)n_v(e|G)}}{(|V(H)|+1)(n_u(e|G)+n_v(e|G))} \\
&+ \sum_{uv\in E_3} \frac{2\sqrt{|V(G\circ H)|-(d_H(u)+1)}}{|V(G\circ H)|-d_H(u)} \\
&= GA_1(H)+GA_2(G)+|V(G)||V(H)|\frac{2\sqrt{|V(G)|+|V(G)||V(H)|-r-1}}{|V(G)|+|V(G)||V(H)|-r},
\end{aligned}
$$

as desired. □

As an application of this result, we present the formulae for GA_2 index of thorny cycle $C_n \circ \bar{K}_m$.

Corollary 18. $GA_2(C_n \circ \bar{K}_m) = n + nm\frac{2\sqrt{nm+n-1}}{n(m+1)}$.

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