Gabor meets Littlewood–Paley:
Gabor expansions in $L^p(\mathbb{R}^d)$

by

Karlheinz Gröchenig (Storrs, CT) and
Christopher Heil (Atlanta, GA)

Abstract. It is known that Gabor expansions do not converge unconditionally in L^p and that L^p cannot be characterized in terms of the magnitudes of Gabor coefficients. By using a combination of Littlewood–Paley and Gabor theory, we show that L^p can nevertheless be characterized in terms of Gabor expansions, and that the partial sums of Gabor expansions converge in L^p-norm.

1. Introduction. The analysis of L^p-spaces on $\mathbb{R}^d$ traditionally employs Littlewood–Paley theory and, more recently, wavelet theory (which is the discrete manifestation of Littlewood–Paley theory). In this context, $L^p(\mathbb{R}^d)$ can be characterized in terms of an appropriate function-space norm of the absolute value of the Haar wavelet transform. More generally, wavelet bases are actually unconditional bases for $L^p(\mathbb{R}^d)$ (see [15]).

By contrast, phase-space methods (appearing under the labels of time-frequency analysis, Gabor analysis, coherent state transforms, etc.) are typically deemed to be inappropriate tools for analyzing L^p (see, e.g., [3]). This opinion is supported by the fact that the orthonormal bases associated with phase-space methods (the Wilson bases) are not unconditional bases for $L^p(\mathbb{R}^d)$. As a consequence of this fact, $L^p(\mathbb{R}^d)$ cannot be characterized in terms of the absolute value of the short-time Fourier transform or the absolute values of the coefficients in a Gabor expansion (to be precise, this is proved in [7] for the specific case of twice-redundant Gabor frames, but we would be extremely surprised if it was not valid for all Gabor frames).

Our aim in this paper is to show that $L^p(\mathbb{R}^d)$ can, in fact, be characterized by the coefficients in a Gabor expansion, though not solely in terms of

the absolute values of those coefficients. This result is in the spirit of the Littlewood–Paley characterization of Fourier series in $L^p(\mathbb{T}^d)$ in terms of their Fourier coefficients. Indeed, we will make use of this characterization, and it is exactly the point where the theory of Gabor expansions meets the classical Littlewood–Paley theory. As an unexpected outcome of this analysis, we show that the partial sums of a Gabor expansion converge in L^p. The ordering in the Gabor series is, of course, important, because it can be inferred from known results that these series can at best converge conditionally in L^p. However, the fact that Gabor series converge at all in L^p has not been observed before.

For the proof of this characterization of L^p, we develop a theory of Gabor analysis on the amalgam space $W^{\sim} = (L^1, \ell^{\infty})$, which is of independent interest. This seems to be the largest Banach *function* space isometrically invariant under translations on which Gabor analysis is possible. Moreover, Gabor analysis on this space is possible using the largest feasible window class, the amalgam space $W = (L^{\infty}, \ell^1)$. In particular, we show that the Walnut representation of the Gabor frame operator is valid in a weak sense on $W^{\sim}$.

In the course of writing this paper we learned that L. Grafakos and C. Lennard have simultaneously obtained results on the L^p-properties of Gabor expansions that are similar to some of ours in certain respects [10]. However, their approach is radically different, relying on an impressive arsenal of inequalities from "hard analysis" rather than on Littlewood–Paley theory. Additionally, their results require some restrictive conditions on the allowable windows.

2. Background and discussion of results

2.1. *The STFT and window classes.* The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is $\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \, e^{-2\pi i t \cdot \omega} \, dt$. The Fourier transform is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ onto itself, and extends to the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions by duality.

Translation and modulation of a function f are defined, respectively, by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_y f(t) = e^{2\pi i y \cdot t} \, f(t).$$

The *short-time Fourier transform* (STFT) of a function f with respect to a window g is

$$S_g f(x, y) = \langle f, M_y T_x g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i y \cdot t} \, \overline{g(t - x)} \, f(t) \, dt$$

whenever the integral makes sense. Analogously to the Fourier transform, the STFT extends in a distributional sense to $f, g \in \mathcal{S}'(\mathbb{R}^d)$ (cf. [8, Prop. 1.42]).

For our purposes, we will usually require that the window g lies in the Wiener algebra $W = W(\mathbb{R}^d) = (L^\infty, \ell^1)$, which is the amalgam space defined by the norm

$$\|f\|_W = \sum_{k \in \mathbb{Z}^d} \operatorname*{ess\,sup}_{x \in [0,1]^d} |f(x+k)| = \sum_{k \in \mathbb{Z}^d} \|f \cdot T_k \chi_{[0,1]^d}\|_\infty.$$

Since $W(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d)$ for all $1 \le p \le \infty$, the STFT $S_g f$ with respect to $g \in W$ is well defined for all $f \in L^p$. It was shown in [18] that the Wiener algebra is a convenient and general class of windows for Gabor analysis.

If we let Q_α denote the cube

$$Q_\alpha = [0, \alpha]^d,$$

and write

$$\|m\|_{p,E} = \|m \cdot \chi_E\|_p = \left(\int_E |m(x)|^p \, dx \right)^{1/p}$$

for the local L^p-norm over a set E, then it is easy to see that

$$(1) \qquad \|f\|_{W,\alpha} = \sum_{k \in \mathbb{Z}^d} \|f \cdot T_{\alpha k} \chi_{Q_\alpha}\|_\infty = \sum_{k \in \mathbb{Z}^d} \|f\|_{\infty, \alpha k + Q_\alpha}$$

is an equivalent norm for W for each $\alpha > 0$ (cf. [13, Prop. 4.1.7]).

The Köthe dual of W is the amalgam space

$$W^\sim = W^\sim(\mathbb{R}^d) = (L^1, \ell^\infty)$$
$$= \{f \text{ measurable} : fh \in L^1(\mathbb{R}^d) \text{ for all } h \in W(\mathbb{R}^d)\},$$

with norm

$$\|f\|_{W^\sim} = \sup_{\|h\|_W = 1} |\langle f, h \rangle| = \sup_{k \in \mathbb{Z}^d} \int_{[0,1]^d} |f(x+k)| \, dx = \sup_{k \in \mathbb{Z}^d} \|f \cdot T_k \chi_{[0,1]^d}\|_1.$$

By [1, Thm. 1.2.9], $W^\sim$ is a closed, norm-fundamental subspace of the dual space W^* of W.

Both W and $W^\sim$ are invariant under the translation operators T_x, and the corresponding operator norms are uniformly bounded in x. In fact, translations are isometries for W and $W^\sim$ under the equivalent norms $\int_{\mathbb{R}^d} \|f \cdot T_x \chi_{[0,1]^d}\|_\infty \, dx$ and $\sup_x \|f \cdot T_x \chi_{[0,1]^d}\|_1$, respectively.

Note that with $g \in W$, the STFT $S_g f$ is well defined as a function on $\mathbb{R}^{2d}$ for every $f \in W^\sim$. Since

$$\|f\|_{W^\sim}^p = \sup_{k \in \mathbb{Z}^d} \|f \cdot T_k \chi_{[0,1]^d}\|_1^p$$
$$\le \sup_{k \in \mathbb{Z}^d} \|f \cdot T_k \chi_{[0,1]^d}\|_p^p \le \sum_{k \in \mathbb{Z}^d} \|f \cdot T_k \chi_{[0,1]^d}\|_p^p = \|f\|_p^p,$$

we see that L^p is embedded in $W^\sim$ for $1 \le p \le \infty$. In our interpretation $W^\sim$ is the correct *function* space on which the operators of Gabor theory can be defined in a meaningful way within the context of L^p-theory.

2.2. *Gabor frames and the modulation spaces.* Given a window $g \in L^2(\mathbb{R}^d)$ and given $\alpha, \beta > 0$, we say that the collection

$$\mathcal{G}(g, \alpha, \beta) = \{M_{\beta n} T_{\alpha k} g : k, n \in \mathbb{Z}^d\}$$

is a *Gabor frame* for $L^2(\mathbb{R}^d)$ if there exist constants $A, B > 0$ (called *frame bounds*) such that

$$\forall f \in L^2(\mathbb{R}^d), \quad A\|f\|_2^2 \le \sum_{k,n \in \mathbb{Z}^d} |\langle f, M_{\beta n} T_{\alpha k} g \rangle|^2 \le B\|f\|_2^2.$$

In this case, there exists a *dual window* $\gamma \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(\gamma, \alpha, \beta)$ is also a Gabor frame for $L^2(\mathbb{R}^d)$ and such that

$$(2) \qquad f = \sum_{k,n \in \mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} \gamma \rangle M_{\beta n} T_{\alpha k} g$$

$$= \sum_{k,n \in \mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g \rangle M_{\beta n} T_{\alpha k} \gamma$$

for all $f \in L^2(\mathbb{R}^d)$. The series in (2) converge unconditionally in L^2, and by the frame definition, the ℓ^2-norm of the sequence $(\langle f, M_{\beta n} T_{\alpha k} g \rangle)$ of Gabor coefficients is an equivalent norm for L^2. For detailed discussion of frames, we refer to [3], [13].

Under stronger assumptions on g, the expansions in (2) are valid not only in L^2 but in the entire class of function spaces known as the modulation spaces. For detailed discussion of these spaces we refer to [11], [6], [12]. The appropriate window class is the Feichtinger algebra

$$(3) \qquad M^1(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : S_f f \in L^1(\mathbb{R}^{2d})\},$$

which is a subspace of the Wiener algebra W.

Suppose $g \in M^1$. Then the modulation space $M^{p,q}$ is

$$(4) \qquad M^{p,q}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : S_g f \in L^{p,q}(\mathbb{R}^{2d})\}$$

with norm

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |S_g f(x, y)|^p \, dx \right)^{q/p} dy \right)^{1/q}.$$

$M^{p,q}$ is independent of the choice of $g \in M^1$ in the sense of equivalent norms. M^1 defined by (3) coincides with $M^{1,1}$ defined by (4).

Suppose $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for L^2 with window $g \in M^1$. In this case the dual window γ will lie in M^1 as well [6] (the analogous statement requiring only $g \in W$ fails in general). Further, the expansions (2) are valid in each space $M^{p,q}$ with unconditional convergence of the series in the norm

of $M^{p,q}$ if $1 \leq p, q < \infty$ and unconditional weak* convergence otherwise. Moreover, the $\ell^{p,q}$ norm of the sequence of Gabor coefficients of f is an equivalent norm for $M^{p,q}$, i.e.,

$$\|f\|_{M^{p,q}} \asymp \|(\langle f, M_{\beta n} T_{\alpha k} g \rangle)\|_{\ell^{p,q}} = \Big(\sum_{n \in \mathbb{Z}^d} \Big(\sum_{k \in \mathbb{Z}^d} |\langle f, M_{\beta n} T_{\alpha k} g \rangle|^p \Big)^{q/p} \Big)^{1/q}.$$

The remarks above can be extended to the case of weighted modulation spaces. The space $M^{2,2}$ coincides with L^2 (and more generally, the Sobolev space H^s coincides with a weighted modulation space), but L^p does not coincide with any modulation space when $p \neq 2$.

2.3. *Discussion of results*: *Gabor analysis on L^p and $W^{\sim}$*. Our goal is to show that the Gabor expansions in (2) are valid in some sense on L^p even though L^p is not a modulation space when $p \neq 2$. Furthermore, we will define an appropriate sequence space s^p such that membership of f in L^p is characterized by membership of the corresponding sequence of Gabor coefficients in s^p. The precise definition of s^p is given in Section 3.2 below via an application of Littlewood–Paley theory. With that definition, our main result can be stated as follows.

THEOREM 1. *Let $g, \gamma \in W(\mathbb{R}^d)$ be such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$. Let $1 < p < \infty$. Then for $f \in W^{\sim}$, the following statements are equivalent*:

(a) $f \in L^p(\mathbb{R}^d)$,
(b) $C_g f = (\langle f, M_{\beta n} T_{\alpha k} g \rangle)_{k,n \in \mathbb{Z}^d} \in s^p$,
(c) $C_\gamma f = (\langle f, M_{\beta n} T_{\alpha k} \gamma \rangle)_{k,n \in \mathbb{Z}^d} \in s^p$.

Moreover, in case any one of these holds, we have the following norm equivalences:

$$(5) \qquad \|f\|_p \asymp \|(\langle f, M_{\beta n} T_{\alpha k} g \rangle)\|_{s^p} \asymp \|(\langle f, M_{\beta n} T_{\alpha k} \gamma \rangle)\|_{s^p}.$$

REMARK 1. When the definition of s^p is considered, it is clear that if $p = 1$ or $p = \infty$ then the right-hand side of (5) fails to characterize L^1 or L^∞. We remarked in a preprint version of this paper that it is likely that $\|c\|_{s^1}$ characterizes some type of Hardy-like space on $\mathbb{R}^d$, and indeed such a result has recently been obtained by Gilbert and Lakey [9].

The proof of Theorem 1 is achieved by a careful examination of the following analysis and synthesis operators. Many questions about Gabor frames can be answered by studying the boundedness properties of these operators on appropriate function spaces, and indeed this will be our approach to proving the convergence of Gabor frame expansions in L^p.

DEFINITION 1. For a window function g and given $\alpha, \beta > 0$, the *synthesis operator* or *reconstruction mapping* is the operator R_g which maps sequences

$c = (c_{kn})_{k,n \in \mathbb{Z}^d}$ to functions on $\mathbb{R}^d$ according to the formal definition

$$(6) \qquad R_g c = \sum_{k,n \in \mathbb{Z}^d} c_{kn} \, M_{\beta n} T_{\alpha k} g.$$

The formal adjoint of R_g is the *analysis operator* or *coefficient mapping* C_g, which maps a function f to its corresponding sequence of Gabor coefficients according to the definition

$$(7) \qquad C_g f(k, n) = \langle f, M_{\beta n} T_{\alpha k} g \rangle, \qquad k, n \in \mathbb{Z}^d.$$

With the above terminology, the Gabor expansions in (2) amount to a factorization of the identity on L^2 as

$$(8) \qquad I = R_g C_\gamma = R_\gamma C_g.$$

With $g \in M^1$, the series (6) defining $R_g c$ converges unconditionally in $M^{p,q}$ when $c \in \ell^{p,q}$ (weak* unconditionally if $p = \infty$ or $q = \infty$), and the synthesis operator is a bounded mapping $R_g : \ell^{p,q} \to M^{p,q}$ with adjoint $C_g : M^{p,q} \to \ell^{p,q}$. Moreover, the factorizations in (8) are valid on $M^{p,q}$ (see [6], [12]).

In considering L^p-spaces, several additional difficulties arise. First, because we are now outside the modulation space setting, the series in (6) cannot converge unconditionally, and part of the difficulty is simply to assign meaning to the formal series (6). We will see that it is most appropriate to define $R_g c$ in terms of an iterated series rather than the double sum appearing in (6). Further, we will define a sequence space s^p on which R_g is defined and which it maps boundedly into L^p.

Second, in addition to deriving the boundedness properties of R_g, C_g, we must show that the factorization of the identity (8) is valid on L^p. In order to obtain this result for the broadest feasible window class (the Wiener algebra W), it is necessary to understand the mapping properties of R_g, C_g on as large a function space as possible. This leads us to consider C_g as a mapping on $W^\sim$. We determine the appropriate sequence space w such that $C_g : W^\sim \to w$ and $R_g : w \to W^\sim$ and show that the factorization of the identity (8) holds on the large space $W^\sim$. We obtain this by demonstrating that the Walnut representation of $R_\gamma C_g$ is valid on $W^\sim$ in a weak sense.

While we view the above-mentioned results on $W^\sim$ as being of independent interest, we note that they could be omitted if we were content to restrict the class of windows. At the same time, the validity of Theorem 1 would extend to larger spaces than $W^\sim$. For instance, if we require $g, \gamma \in M^1$, then R_g is defined on ℓ^∞ and the series (6) defining $R_g c$ converges unconditionally in the weak*-topology on $M^\infty = (M^1)^*$ (cf. [12]). Further, the identity $R_\gamma C_g = I$ holds on M^∞. Consequently, *if $g, \gamma \in M^1$, then the equivalent conditions of Theorem 1 hold whenever $f \in M^\infty$.* While M^∞ is a strictly larger Banach space than $W^\sim$, it includes distributions as

well as functions. In a similar way, *if we restrict g, γ still further to lie in the Schwartz space S, then Theorem 1 holds for f ∈ S′.*

However, restricting the window class beyond W is in some sense unnatural for analysis on L^p. In particular, one natural window in this case (and indeed, the inspiration for our results) is $\chi_{[0,1]^d}$, which does not lie in M^1 or S. For Gabor analysis on L^p, the natural window class seems to be the space W.

3. Results

3.1. *An estimate.* In this section we prove an estimate that will play a key role in our analysis. We first require the following elementary fact.

LEMMA 1. *Let $\alpha, \beta > 0$ be given. Let K be the maximum number of $(1/\beta)\mathbb{Z}^d$-translates of $Q_{1/\beta}$ required to cover any $\alpha\mathbb{Z}^d$-translate of Q_α, i.e.,*

$$(9) \qquad K = \max_{k \in \mathbb{Z}^d} \# \left\{ l \in \mathbb{Z}^d : \left| \left(\frac{l}{\beta} + Q_{1/\beta} \right) \cap (\alpha k + Q_\alpha) \right| > 0 \right\}.$$

Then given $1 \le p < \infty$ we have, for any $1/\beta$-periodic function $m \in L^p(Q_{1/\beta})$ and any $k \in \mathbb{Z}^d$,

$$\|m\|_{p,\alpha k + Q_\alpha}^p \le K \|m\|_{p,Q_{1/\beta}}^p.$$

The estimate we require is as follows. We will use the letter C to denote a constant whose value may differ from occurrence to occurrence.

PROPOSITION 1. *Fix $1 \le p < \infty$ and $\alpha, \beta > 0$. Let $g \in W(\mathbb{R}^d)$. Then there exists a constant $C = C(p, \alpha, \beta) > 0$ such that if $\{m_k\}_{k \in \mathbb{Z}^d}$ is any sequence of $1/\beta$-periodic functions in $L^p(Q_{1/\beta})$ with $\sum_k \|m_k\|_{p,Q_{1/\beta}}^p < \infty$, then $\sum_k m_k \cdot T_{\alpha k} g$ converges unconditionally in $L^p(\mathbb{R}^d)$ and satisfies*

$$(10) \qquad \left\| \sum_{k \in \mathbb{Z}^d} m_k \cdot T_{\alpha k} g \right\|_p \le C \|g\|_W \left(\sum_{k \in \mathbb{Z}^d} \|m_k\|_{p,Q_{1/\beta}}^p \right)^{1/p}.$$

Proof. If F is a finite subset of $\mathbb{Z}^d$, then

$$(11) \quad \left\| \sum_{k \in F} m_k \cdot T_{\alpha k} g \right\|_p = \left\| \sum_{k \in F} \sum_{l \in \mathbb{Z}^d} m_k \cdot T_{\alpha k} g \cdot T_{\alpha(k+l)} \chi_{Q_\alpha} \right\|_p$$

$$\le \sum_{l \in \mathbb{Z}^d} \|g \cdot T_{\alpha l} \chi_{Q_\alpha}\|_\infty \left\| \sum_{k \in F} |m_k| \cdot T_{\alpha(k+l)} \chi_{Q_\alpha} \right\|_p.$$

We need to estimate the local pieces in (11). Since the translates of χ_{Q_α} are disjoint, we have

$$(12) \quad \Big\| \sum_{k \in F} |m_k| \cdot T_{\alpha(k+l)} \chi_{Q_\alpha} \Big\|_p^p = \int_{\mathbb{R}^d} \Big(\sum_{k \in F} |m_k(x)| \, \chi_{\alpha(k+l)+Q_\alpha}(x) \Big)^p dx$$

$$= \sum_{k \in F} \int_{\alpha(k+l)+Q_\alpha} |m_k(x)|^p \, dx$$

$$\leq K \sum_{k \in F} \|m_k\|_{p, Q_{1/\beta}}^p,$$

where K is defined by (9). Since (1) is an equivalent norm for W, there is a constant B depending only on α such that $\|g\|_{W,\alpha} \leq B\|g\|_W$. Combining this with (11) and (12), we obtain

$$\Big\| \sum_{k \in F} m_k \cdot T_{\alpha k} g \Big\|_p \leq \sum_{l \in \mathbb{Z}^d} \|g \cdot T_{\alpha l} \chi_{Q_\alpha}\|_\infty \Big(K \sum_{k \in F} \|m_k\|_{p, Q_{1/\beta}}^p \Big)^{1/p}$$

$$\leq BK^{1/p} \|g\|_W \Big(\sum_{k \in F} \|m_k\|_{p, Q_{1/\beta}}^p \Big)^{1/p},$$

with B and K depending only on α and β. Thus the series $\sum m_k \cdot T_{\alpha k} g$ is unconditionally Cauchy in L^p-norm, hence converges unconditionally, and furthermore (10) holds by passing to the limit. ■

REMARK 2. A similar argument can be applied when $p < 1$ if g is in the amalgam space (L^∞, ℓ^p) defined by the norm $(\sum_{k \in \mathbb{Z}^d} \|f \cdot T_{k\alpha} \chi_{Q_\alpha}\|_\infty^p)^{1/p}$.

3.2. *Gabor meets Littlewood–Paley.* The intuitive meaning of Gabor sums is easy to understand and provides the main idea for our subsequent characterization of $L^p(\mathbb{R}^d)$ in terms of the Gabor coefficients $\langle f, M_{\beta n} T_{\alpha k} g \rangle$. For motivation, assume that g has essentially compact support in a neighborhood of 0. Summing over the frequencies βn in the Gabor expansion (2) first, we obtain formally for each $k \in \mathbb{Z}^d$ a trigonometric series

$$m_k(x) = \sum_{n \in \mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} \gamma \rangle e^{2\pi i \beta n x}$$

with period $1/\beta$. We can then recast (2) formally as

$$(13) \qquad\qquad f = \sum_{k \in \mathbb{Z}^d} m_k \cdot T_{\alpha k} g,$$

and m_k can be interpreted as the expansion into a Fourier series of that piece of f that is supported in a neighborhood of αk. It is now plausible that the entire L^p-norm of f can be controlled by summing these local L^p-norms. The key ingredients in making this idea precise are Proposition 1 and the classical theorem of Littlewood and Paley.

Let $1 < p < \infty$. Then a $1/\beta$-periodic function m_k lies in $L^p(Q_{1/\beta})$ if and only if it can be written as a Fourier series

$$m_k(x) = \sum_{n \in \mathbb{Z}^d} c_{kn} e^{2\pi i \beta n x}$$

in the sense that the square partial sums

$$s_N m_k = \sum_{|n| \leq N} c_{kn} e^{2\pi i \beta n x}, \qquad |n| = \max_j |n_j|,$$

converge to m_k in L^p-norm (cf. [14], [19]). The convergence is conditional; other sequences of partial sums need not converge. Littlewood–Paley theory allows us to write the L^p-norm of m_k in terms of the coefficients c_{kn} in the Fourier series of m_k. Define

$$\Lambda_0 = \{(n_1, \ldots, n_d) \in \mathbb{Z}^d : |n_i| \leq 1\},$$

and let Λ_j be the corona

$$\Lambda_j = \{(n_1, \ldots, n_d) \in \mathbb{Z}^d : |n_i| \leq 2^j\} \setminus \{(n_1, \ldots, n_d) \in \mathbb{Z}^d : |n_i| \leq 2^{j-1}\}.$$

Then for $1 < p < \infty$ we have the following norm equivalence (cf. [17, Section 3.4.4] or [4, Chapter 7]):

$$(14) \qquad \|m_k\|_{p, Q_{1/\beta}} \asymp \left(\int_{Q_{1/\beta}} \left(\sum_{j=0}^{\infty} \left| \sum_{n \in \Lambda_j} c_{kn} e^{2\pi i \beta n x} \right|^2 \right)^{p/2} dx \right)^{1/p}.$$

This equivalence fails if $p = 1$ or $p = \infty$.

Motivated by (13) and (14), we introduce a sequence space s^p defined by the following norm:

$$\|c\|_{s^p} := \left(\sum_{k \in \mathbb{Z}^d} \int_{Q_{1/\beta}} \left(\sum_{j=0}^{\infty} \left| \sum_{n \in \Lambda_j} c_{kn} e^{2\pi i \beta n x} \right|^2 \right)^{p/2} dx \right)^{1/p}.$$

In particular, if $c \in s^p$, then $m_k(x) = \sum_n c_{kn} e^{2\pi i \beta n x}$ converges for each k (as a limit of square partial sums) and is in $L^p(Q_{1/\beta})$. By (14), we therefore have

$$\|c\|_{s^p} \asymp \left(\sum_{k \in \mathbb{Z}^d} \|m_k\|_{p, Q_{1/\beta}}^p \right)^{1/p},$$

so s^p is isomorphic to the Banach space $\ell^p(L^p(Q_{1/\beta}))$.

A discussion similar to the above leads us to a suitable definition of the synthesis operator acting on s^p.

DEFINITION 2. Given $c \in s^p$, we define $R_g c$ by the iterated series

$$(15) \qquad R_g c = \sum_{k \in \mathbb{Z}^d} m_k \cdot T_{\alpha k} g, \quad \text{where} \quad m_k(x) = \sum_{n \in \mathbb{Z}^d} c_{kn} e^{2\pi i \beta n x}.$$

In most of the standard situations occurring in Gabor theory the double sum defining $R_g c$ in (6) converges unconditionally. However, for the L^p-theory it is essential to consider iterated sums as in (15) and conditional convergence of these series. In particular, Proposition 1 implies immediately that $R_g c$ is well defined as the iterated sum in (15) and is an element of L^p, and that $R_g : s^p \to L^p$ is bounded. We formalize this as the following result.

COROLLARY 1. *If $g \in W(\mathbb{R}^d)$ and $1 < p < \infty$, then the synthesis operator R_g defined by (15) is a bounded mapping from s^p into $L^p(\mathbb{R}^d)$. Specifically,*

$$\|R_g c\|_p \leq C \|g\|_W \|c\|_{s^p},$$

where C depends only on α, β, and p.

This corollary leaves open the question of whether the double sum in (6) converges for $c \in s^p$. Addressing this issue requires using the convergence properties of Fourier series, which we will do in Section 3.6.

By the duality theorem for Banach space valued ℓ^p-spaces [2], we have $\ell^p(L^p(Q_{1/\beta}))^* = \ell^{p'}((L^{p'}(Q_{1/\beta}))$, where $1/p' + 1/p = 1$. Consequently, the dual space of s^p is $(s^p)^* = s^{p'}$. By considering finite sequences (which are dense in s^p), it is not difficult to show that C_g defined by (7) is indeed the adjoint of $R_g : s^p \to L^p$ defined by (15). Applying duality to Corollary 1, we therefore obtain the following estimate for the analysis operator.

COROLLARY 2. *If $g \in W(\mathbb{R}^d)$ and $1 < p < \infty$, then the analysis operator C_g defined by (7) is a bounded mapping from $L^p(\mathbb{R}^d)$ into s^p. Specifically,*

$$\|C_g f\|_{s^p} = \|(\langle f, M_{\beta n} T_{\alpha k} g \rangle)\|_{s^p} \leq C \|g\|_W \|f\|_p,$$

where C depends only on α, β, and p.

The combination of Corollaries 1 and 2 implies that $R_\gamma C_g$ maps L^p into itself. Hence, if f is, say, a function in $W^\sim$ whose sequence of Gabor coefficients $C_g f$ lies in s^p, then $\widetilde{f} = R_\gamma C_g f \in L^p$. However, we cannot yet prove Theorem 1. In order to conclude from this that $f \in L^p$, we need to know that the factorization of the identity $R_\gamma C_g = I$ is valid in some sense on the space $W^\sim$. This is accomplished in the following section by extending the Walnut representation of $R_\gamma C_g$ to $W^\sim$.

3.3. *Gabor analysis on $W^\sim$.* We first require the following lemma due to Walnut [18, Lemma 2.1]. The form of this lemma as given below is proved in [12, Lemma 6.3.1].

LEMMA 2. *Given $g, \gamma \in W$, define for each $n \in \mathbb{Z}^d$ the correlation function*

$$G_n(x) = \sum_{k \in \mathbb{Z}^d} \overline{g}(x - n/\beta - \alpha k) \gamma(x - \alpha k) = \sum_{k \in \mathbb{Z}^d} T_{\alpha k + n/\beta} \overline{g}(x) T_{\alpha k} \gamma(x).$$

Then

$$\sum_n \|G_n\|_\infty \le (1/\alpha + 1)^d (2\beta + 2)^d \|g\|_W \|\gamma\|_W.$$

Given a sequence of functions $f_k \in W^\sim$, we will say that the series $\sum_k f_k$ converges unconditionally in the $\sigma(W^\sim, W)$-topology if $\sum_k \langle f_k, h \rangle$ converges unconditionally for each $h \in W$. By [1, Cor. 1.5.3], $W^\sim$ is a closed subspace of W^* and is $\sigma(W^\sim, W)$-complete.

We can now prove the following estimate for $W^\sim$ analogous to the estimate proved in Proposition 1 for L^p.

PROPOSITION 2. *Fix $\alpha, \beta > 0$ and $g \in W(\mathbb{R}^d)$. Then there exists a constant $C = C(\alpha, \beta) > 0$ such that if $\{m_k\}_{k \in \mathbb{Z}^d}$ is any sequence of $1/\beta$-periodic functions in $L^1(Q_{1/\beta})$ with $\sup \|m_k\|_{1, Q_{1/\beta}} < \infty$, then $\sum_k m_k \cdot T_{\alpha k} g$ converges unconditionally in the $\sigma(W^\sim, W)$-topology and satisfies*

$$\Big\| \sum_{k \in \mathbb{Z}^d} m_k \cdot T_{\alpha k} g \Big\|_{W^\sim} \le C \|g\|_W (\sup_{k \in \mathbb{Z}^d} \|m_k\|_{1, Q_{1/\beta}}).$$

Proof. Let $h \in W$ and define $G_k(x) = \sum_n T_{\alpha k + n/\beta} g \cdot T_{n/\beta} \overline{h}$. Then using the periodicity of m_k and applying Lemma 2 (with the roles of α and $1/\beta$ interchanged), we obtain

$$\sum_{k \in \mathbb{Z}^d} |\langle m_k \cdot T_{\alpha k} g, h \rangle| = \sum_{k \in \mathbb{Z}^d} \Big| \int_{\mathbb{R}^d} m_k(x) T_{\alpha k} g(x) \overline{h}(x) \, dx \Big|$$

$$\le \sum_{k \in \mathbb{Z}^d} \int_{Q_{1/\beta}} |m_k(x)| \Big| \sum_{n \in \mathbb{Z}^d} T_{\alpha k + n/\beta} g(x) T_{n/\beta} \overline{h}(x) \Big| \, dx$$

$$\le \sum_{k \in \mathbb{Z}^d} \|m_k\|_{1, Q_{1/\beta}} \|G_k\|_\infty$$

$$\le C (\sup_{k \in \mathbb{Z}^d} \|m_k\|_{1, Q_{1/\beta}}) \|g\|_W \|h\|_W,$$

with C depending only on α and β. The partial sums of $\sum_k m_k \cdot T_{\alpha k} g$ are therefore Cauchy in the $\sigma(W^\sim, W)$-topology, hence converge since $W^\sim$ is $\sigma(W^\sim, W)$-complete. ∎

While Fourier series do not converge in general in $L^1(Q_{1/\beta})$, it is still true that Fourier coefficients are unique. The preceding proposition therefore suggests that the synthesis operator may be defined on the following sequence space.

DEFINITION 3. We denote by w the space of all sequences $c = (c_{kn})$ for which there exist functions $m_k \in L^1(Q_{1/\beta})$ such that the Fourier coefficients of m_k are c_{kn}, i.e.,

$$\widehat{m}_k(n) = \beta^d \int_{Q_{1/\beta}} m_k(x) e^{-2\pi i \beta n x} \, dx = c_{kn},$$

and such that

$$\|c\|_w = \sup_{k \in \mathbb{Z}^d} \|m_k\|_{1,Q_{1/\beta}} < \infty.$$

Clearly, $s^p \subseteq w$ for $1 < p < \infty$.

DEFINITION 4. We extend the definition of the synthesis operator to w by defining $R_g c$ for $c \in w$ by

$$(16) \quad R_g c = \sum_{k \in \mathbb{Z}^d} m_k \cdot T_{\alpha k} g, \quad \text{where } m_k \in L^1(Q_{1/\beta}) \text{ satisfies } \widehat{m}_k(n) = c_{kn},$$

By Proposition 2, the series (16) converges unconditionally in the $\sigma(W^\sim, W)$-topology, and we have the following result.

COROLLARY 3. *If* $g \in W(\mathbb{R}^d)$, *then the synthesis operator* R_g *defined by* (16) *is a bounded mapping from* w *into* $W^\sim(\mathbb{R}^d)$. *Specifically,*

$$\|R_g c\|_{W^\sim} \leq C \|g\|_W \|c\|_w,$$

where C *depends only on* α *and* β.

REMARK 3. If $c \in s^p$ then

$$m_k(x) = \sum_n c_{kn} \, e^{2\pi i \beta n x} \in L^p(Q_{1/\beta}) \subset L^1(Q_{1/\beta}).$$

Consequently, for such c the definitions of $R_g c$ given by (15) and (16) coincide.

To prove the boundedness of the analysis operator on $W^\sim$, it is more convenient to argue directly than by duality; this also provides an explicit expression for the functions m_k.

PROPOSITION 3. *If* $g \in W(\mathbb{R}^d)$, *then the analysis operator* C_g *defined by* (7) *is a bounded mapping from* $W^\sim(\mathbb{R}^d)$ *into* w. *Specifically, there exists* $C = C(\alpha, \beta)$ *such that*

$$\|C_g f\|_w = \|(\langle f, M_{\beta n} T_{\alpha k} g \rangle)_{k,n \in \mathbb{Z}^d}\|_w \leq C \|g\|_W \|f\|_{W^\sim}.$$

Moreover, the functions $m_k \in L^1(Q_{1/\beta})$ *which satisfy*

$$\widehat{m}_k(n) = C_g f(k, n)$$

are given by

$$(17) \qquad m_k(x) = \beta^{-d} \sum_{n \in \mathbb{Z}^d} (f \cdot T_{\alpha k} \overline{g})(x - n/\beta).$$

Proof. Fix $f \in W^\sim$. Then since $g \in W$ we have $f \cdot T_{\alpha k} \overline{g} \in L^1(\mathbb{R}^d)$ for every $k \in \mathbb{Z}^d$, and therefore the functions m_k given by (17) are well defined

elements of $L^1(Q_{1/\beta})$. Further,

$$
\begin{aligned}
\widehat{m}_k(n) &= \beta^d \int\limits_{Q_{1/\beta}} m_k(x)e^{-2\pi i\beta nx}\,dx \\
&= \int\limits_{Q_{1/\beta}} \sum_{l\in\mathbb{Z}^d}(f\cdot T_{\alpha k}\overline{g})(x-l/\beta)\,e^{-2\pi i\beta n(x-l/\beta)}\,dx \\
&= \int\limits_{\mathbb{R}^d}(f\cdot T_{\alpha k}\overline{g})(x)\,e^{-2\pi i\beta nx}\,dx = \langle f, M_{\beta n}T_{\alpha k}g\rangle \\
&= C_g f(k,n),
\end{aligned}
$$

and

$$
\begin{aligned}
\|m_k\|_{1,Q_{1/\beta}} &= \beta^{-d}\int\limits_{Q_{1/\beta}}\Big|\sum_{l\in\mathbb{Z}^d}(f\cdot T_{\alpha k}\overline{g})(x-l/\beta)\Big|\,dx \\
&\le \beta^{-d}\int\limits_{\mathbb{R}^d}|(f\cdot T_{\alpha k}\overline{g})(x)|\,dx \le \beta^{-d}\|f\|_{W^\sim}\|T_{\alpha k}g\|_W \\
&\le C\|f\|_{W^\sim}\|g\|_W.
\end{aligned}
$$

Therefore $C_g f \in w$ and $\|C_g f\|_w = \sup_k \|m_k\|_{L^1(Q_{1/\beta})} \le C\|f\|_{W^\sim}\|g\|_W.$ ∎

The Walnut representation of the Gabor frame operator was introduced in [18]. We next show that the Walnut representation of the Gabor frame operator is valid on $W^\sim$.

PROPOSITION 4 (Walnut's representation). *Let $g,\gamma \in W(\mathbb{R}^d)$ and let G_n be the associated sequence of correlation functions as defined in Lemma 2. Then for $f \in W^\sim(\mathbb{R}^d)$ we have*

$$
(18) \qquad R_\gamma C_g f = \beta^{-d}\sum_{n\in\mathbb{Z}^d} G_n \cdot T_{n/\beta}f,
$$

where the right-hand series converges absolutely in $W^\sim$-norm and unconditionally in the $\sigma(W^\sim, W)$-topology.

Proof. By Lemma 2 we have $\sum\|G_n\|_\infty < \infty$. Hence

$$
\sum_{n\in\mathbb{Z}^d}\|G_n\cdot T_{n/\beta}f\|_{W^\sim} \le \sum_{n\in\mathbb{Z}^d}\|G_n\|_\infty\|T_{n/\beta}f\|_{W^\sim} \le C\|f\|_{W^\sim}\sum_{n\in\mathbb{Z}^d}\|G_n\|_\infty,
$$

so the right-hand side of (18) converges absolutely.

Now let m_k be defined by (17). Then $R_\gamma C_g f = \sum_k m_k\cdot T_{\alpha k}\gamma$, where this series converges in the $\sigma(W^\sim, W)$-topology. Therefore, for $h \in W$,

$$
\begin{aligned}
\langle R_\gamma C_g f, h\rangle &= \sum_{k\in\mathbb{Z}^d}\int\limits_{\mathbb{R}^d} m_k(x)T_{\alpha k}\gamma(x)\overline{h}(x)\,dx \\
&= \beta^{-d}\sum_{k\in\mathbb{Z}^d}\int\limits_{\mathbb{R}^d}\sum_{n\in\mathbb{Z}^d} T_{n/\beta}f(x)T_{\alpha k+n/\beta}\overline{g}(x)T_{\alpha k}\gamma(x)\overline{h}(x)\,dx
\end{aligned}
$$

$$= \beta^{-d} \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} T_{n/\beta} f(x) T_{\alpha k + n/\beta} \overline{g}(x) T_{\alpha k} \gamma(x) \overline{h}(x) \, dx$$

$$= \beta^{-d} \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} T_{n/\beta} f(x) G_n(x) \overline{h}(x) \, dx = \beta^{-d} \sum_{n \in \mathbb{Z}^d} \langle G_n \cdot T_{n/\beta} f, h \rangle,$$

from which (18) follows. The interchanges of integration and summation are justified by Fubini's Theorem as follows. Let

$$\widetilde{G}_n(x) = \sum_{k \in \mathbb{Z}^d} |g(x - n/\beta - \alpha k) \gamma(x - \alpha k)|$$

denote the correlation functions corresponding to $|g|$, $|\gamma|$. Since these functions lie in W, by Lemma 2 we have $\sum \|\widetilde{G}_n\|_\infty < \infty$. Therefore,

$$\sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} |T_{n/\beta} f(x) T_{\alpha k + n/\beta} g(x) T_{\alpha k} \gamma(x) h(x)| \, dx$$

$$= \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |T_{n/\beta} f(x) h(x) \widetilde{G}_n(x)| \, dx$$

$$\leq \sum_{n \in \mathbb{Z}^d} \|\widetilde{G}_n\|_\infty \int_{\mathbb{R}^d} |T_{n/\beta} f(x) h(x)| \, dx$$

$$\leq C \|f\|_{W^\sim} \|h\|_W \sum_{n \in \mathbb{Z}^d} \|\widetilde{G}_n\|_\infty < \infty. \quad \blacksquare$$

COROLLARY 4. *Let $g, \gamma \in W(\mathbb{R}^d)$ be such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$. Then $R_\gamma C_g = R_g C_\gamma = I$ holds as an identity on $W^\sim$.*

Proof. Restricted to $\ell^2 = s^2$, the definitions of the synthesis operator given by (15) and (16) coincide. The frame hypothesis implies that the identity $R_\gamma C_g = I$ holds on L^2. This then implies by [12, Thm. 7.3.1] that

$$\beta^{-d} G_0 = 1 \text{ a.e.} \quad \text{and} \quad G_n = 0 \text{ a.e. for } n \neq 0.$$

Consequently, for $f \in W^\sim$ we obtain $R_\gamma C_g f = \beta^{-d} \sum_n G_n \cdot T_{n/\beta} f = f$, as desired. $\blacksquare$

3.4. *Proof of Theorem 1.* Now we can prove Theorem 1, characterizing L^p in terms of Gabor coefficients.

Proof of Theorem 1. (a)$\Rightarrow$(b). Assume first that $f \in L^2 \cap L^p$. Then since $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$, we have $f = R_\gamma C_g f$. With C denoting the larger of the constants appearing in Corollaries 1 and 2, we therefore have

$$\|f\|_p = \|R_\gamma C_g f\|_p \leq C \|\gamma\|_W \|C_g f\|_{s^p} \leq C^2 \|g\|_W \|\gamma\|_W \|f\|_p.$$

Since $L^2 \cap L^p$ is dense in L^p, this inequality then extends to all of L^p, yielding the first norm equivalence in (5).

(b)$\Rightarrow$(a). Assume that $f \in W^\sim$ is such that $C_g f \in s^p$. Then by Corollary 1, the function $\widetilde{f} = R_\gamma(C_g f)$ is in L^p. On the other hand, by Corollary 4 we have $R_\gamma C_g f = f$. Thus $f = \widetilde{f} \in L^p$.

(a)$\Leftrightarrow$(c). This is similar to (a)$\Leftrightarrow$(b), using the factorization $I = R_g C_\gamma$. $\blacksquare$

3.5. *Characterization via the square function.* We briefly present in this section some other approaches to characterizing L^p.

EXAMPLE 1. Consider the case $\alpha = 1/\beta$. Given a sequence $c = (c_{kn}) \in s^p$, define $m_k(x) = \sum_n c_{kn} e^{2\pi i \beta n x}$. Then, since the cubes $\alpha k + Q_\alpha$ have overlaps of measure zero, we have the following norm equivalences for the space s^p:

$$\|c\|_{s^p} \asymp \left(\sum_{k \in \mathbb{Z}^d} \|m_k\|_{p,Q_\alpha}^p \right)^{1/p}$$

$$= \left(\sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |m_k(x)|^p T_{\alpha k} \chi_{Q_\alpha}(x) \, dx \right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^d} \left(\sum_{k \in \mathbb{Z}^d} |m_k(x) T_{\alpha k} \chi_{Q_\alpha}(x)|^2 \right)^{p/2} dx \right)^{1/p}$$

$$= \left\| \left(\sum_{k \in \mathbb{Z}^d} \left| \sum_{n \in \mathbb{Z}^d} c_{kn} M_{\beta n} T_{\alpha k} \chi_{Q_\alpha} \right|^2 \right)^{1/2} \right\|_p.$$

This resembles very much the characterization of L^p in terms of the square function of the coefficients in the Haar basis (or in any wavelet basis; cf. [15, Ch. 6.2, Thm. 2]), and is more in the spirit of the approach of [10]. In this regard, it is interesting to note that L^p can be easily characterized by using a continuous analog of the square function for the short-time Fourier transform, as in the following proposition.

PROPOSITION 5. *Let $0 < p \leq \infty$ and $1 \leq r \leq \infty$, and fix $g \in L^{2r}(\mathbb{R}^d)$ and $h \in L^{2r'}(\mathbb{R}^d)$ such that $g\overline{h} \not\equiv 0$. Then for $f \in L^p(\mathbb{R}^d)$,*

$$(19) \qquad \|f\|_p = \|\overline{g}h\|_2^{-1} \left\| \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} S_g f(x,y) M_y T_x h \, dy \right|^2 dx \right)^{1/2} \right\|_p.$$

Proof. Assume first that f, g, and h all lie in the Schwartz class. In this case $S_g f(x,y) = (f \cdot T_x \overline{g})^\wedge(y) \in L^1(\mathbb{R}^d)$. Hence the inversion formula for the Fourier transform is valid pointwise in the following calculation:

$$\int_{\mathbb{R}^d} S_g f(x,y) M_y T_x h(t) \, dy = \int_{\mathbb{R}^d} (f \cdot T_x \overline{g})^\wedge(y) \, e^{2\pi i y t} h(t-x) \, dy$$

$$= f(t) \overline{g}(t-x) h(t-x).$$

Therefore, the integral over the "square function" can be spelled out as

follows:

$$\left\| \left(\underset{\mathbb{R}^d}{\int} \Big| \underset{\mathbb{R}^d}{\int} S_g f(x,y) M_y T_x h \, dy \Big|^2 dx \right)^{1/2} \right\|_p^p$$

$$= \underset{\mathbb{R}^d}{\int} \Big| \underset{\mathbb{R}^d}{\int} |f(t)\overline{g}(t-x)h(t-x)|^2 \, dx \Big|^{p/2} dt = \|f\|_p^p \, \|\overline{g}h\|_2^p.$$

By Hölder's inequality,

$$\int |g(x)|^2 |h(x)|^2 \, dx \leq \|g\|_{2r}^2 \, \|h\|_{2r'}^2 < \infty.$$

Therefore identity (19) extends to all $f \in L^p(\mathbb{R}^d)$, $g \in L^{2r}(\mathbb{R}^d)$, and $h \in L^{2r'}(\mathbb{R}^d)$ by density and continuity. ■

3.6. *Convergence of Gabor expansions in L^p.* The partial sums of Fourier series converge in L^p-norm (a "classical" result) and almost everywhere (the deepest result in harmonic analysis). We can apply this knowledge to obtain similar statements for the partial sums of Gabor expansions.

Given a sequence $c = (c_{kn}) \in s^p$, let us write

$$S_{K,N} c = \sum_{|k| \leq K} \sum_{|n| \leq N} c_{kn} M_{\beta n} T_{\alpha k} g,$$

with the understanding that

$$S_{K,\infty} c = \sum_{|k| \leq K} \sum_{n \in \mathbb{Z}^d} c_{kn} M_{\beta n} T_{\alpha k} g = \sum_{|k| \leq K} m_k \cdot T_{\alpha k} g$$

and

$$S_{\infty,\infty} c = R_g c.$$

Note that in the context of Fourier series in higher dimensions it is important to interpret the norm $| \cdot |$ as the maximum norm on $\mathbb{R}^d$, i.e., $|n| = \max_{j=1,\dots,d} |n_j|$.

The following statement, that Gabor expansions converge in L^p, is perhaps surprising in light of the known facts that Gabor expansions converge unconditionally in the modulation spaces but cannot converge unconditionally in L^p. On the other hand, the convergence indicated is clearly "conditional" in the sense that the order of summation is critical.

PROPOSITION 6. *Let $1 < p < \infty$ be fixed. Assume that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$, and suppose that $g, \gamma \in W(\mathbb{R}^d)$.*

(a) *If $c = (c_{kn}) \in s^p$, then the partial sums $S_{K,N}c$ converge to $R_g c$ in L^p-norm.*

(b) *If $f \in L^p(\mathbb{R}^d)$, then the partial sums*

$$S_{K,N}(\langle f, M_{\beta n}T_{\alpha k}\gamma \rangle) = \sum_{|k| \leq K} \sum_{|n| \leq N} \langle f, M_{\beta n}T_{\alpha k}\gamma \rangle M_{\beta n}T_{\alpha k}g,$$

$$S_{K,N}(\langle f, M_{\beta n}T_{\alpha k}g \rangle) = \sum_{|k| \leq K} \sum_{|n| \leq N} \langle f, M_{\beta n}T_{\alpha k}g \rangle M_{\beta n}T_{\alpha k}\gamma,$$

of the Gabor expansions given in (2) converge in L^p-norm to f.

Proof. (a) Assume that $c \in s^p$, and write

$$s_N m_k = \sum_{|n| \leq N} c_{kn} e^{2\pi i \beta n x}$$

for the partial sums of the Fourier series of m_k. We know (see e.g. [14], [19]) that $\{e^{2\pi i \beta n x}\}_{n \in \mathbb{Z}^d}$ forms a basis for $L^p(Q_{1/\beta})$. Consequently,

$$\lim_{N \to \infty} \|s_N m_k - m_k\|_{p, Q_{1/\beta}} = 0$$

and

$$\sup_N \|s_N m_k\|_{p, Q_{1/\beta}} \leq C_1 \|m_k\|_{p, Q_{1/\beta}}$$

for some constant $C_1 > 0$. Now, given $\varepsilon > 0$, we can find K_0 such that

$$\forall K \geq K_0, \quad \Big(\sum_{|k| \geq K} \|m_k\|_{p, Q_{1/\beta}}^p \Big)^{1/p} < \varepsilon.$$

Then we can find N_0 such that

$$\forall N \geq N_0, \quad \sup_{|k| \leq K_0} \|s_N m_k - m_k\|_{p, Q_{1/\beta}} < \frac{\varepsilon}{(2K_0 + 1)^{d/p}}.$$

Write the remainder term $R_g c - S_{K,N}c$ as

$$S_{\infty,\infty}c - S_{K,N}c = (S_{\infty,\infty}c - S_{K_0,\infty}c) + (S_{K_0,\infty}c - S_{K_0,N}c) + (S_{K_0,N}c - S_{K,N}c).$$

Let $C_2 = C\|g\|_W$, where $C = C(p, \alpha, \beta)$ is the constant introduced in Proposition 1. Then for any $K \geq K_0$ and $N \geq N_0$, we estimate

$$\|R_g c - S_{K,N}c\|_p \leq \Big\| \sum_{|k| > K_0} m_k \cdot T_{\alpha k}g \Big\|_p + \Big\| \sum_{|k| \leq K_0} (m_k - s_N m_k) \cdot T_{\alpha k}g \Big\|_p$$

$$+ \Big\| \sum_{K_0 < |k| \leq K} s_N m_k \cdot T_{\alpha k}g \Big\|_p$$

$$\leq C_2 \Big(\sum_{|k| > K_0} \|m_k\|_{p, Q_{1/\beta}}^p \Big)^{1/p}$$

$$+ C_2 \Big(\sum_{|k| \leq K_0} \|m_k - s_N m_k\|_{p, Q_{1/\beta}}^p \Big)^{1/p} + C_2 \Big(\sum_{K_0 < |k| \leq K} \|s_N m_k\|_{p, Q_{1/\beta}}^p \Big)^{1/p}$$

$$\leq C_2 \varepsilon + C_2 \varepsilon + C_1 C_2 \varepsilon,$$

where we have applied Proposition 1 several times.

(b) If $f \in L^p(\mathbb{R}^d)$ then $(\langle f, M_{\beta n} T_{\alpha k} \gamma \rangle) \in s^p$ and $(\langle f, M_{\beta n} T_{\alpha k} g \rangle) \in s^p$ by Corollary 2, so the result follows from part (a). ∎

EXAMPLE 2. We give one example of a conditionally convergent Gabor expansion in $L^p(\mathbb{R}^d)$. We know that while $\{e^{2\pi i \beta n x}\}$ does form a basis for $L^p(Q_{1/\beta})$ for all $1 < p < \infty$, it is a conditional basis when $p \neq 2$ (cf. [14]). Fix $p \neq 2$ and let $m \in L^p(Q_{1/\beta})$ be any particular function whose Fourier series converges conditionally. That is, for this m we have $s_N m \to m$ in $L^p(Q_{1/\beta})$, but there exist other sequences of partial sums that do not converge. Therefore, if we set $\alpha = 1/\beta$, $g = \chi_{Q_{1/\beta}}$, and $f = m \cdot \chi_{Q_{1/\beta}}$, then the partial sums $S_{K,N} = s_N m \cdot \chi_{Q_{1/\beta}}$ in the Gabor expansion for f will converge to f in $L^p(\mathbb{R}^d)$, but there will exist other sequences of partial sums that will not converge. Thus, the Gabor expansion of this f converges conditionally in $L^p(\mathbb{R}^d)$.

3.7. *Almost everywhere convergence.* Finally, we consider the pointwise convergence of Gabor expansions. For simplicity, we will restrict to the case where the window g is compactly supported.

PROPOSITION 7. *Let $1 < p < \infty$ be fixed. Assume that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$, and suppose that g is compactly supported and $\widehat{g} \in L^1(\mathbb{R}^d)$. Then for any $f \in L^p(\mathbb{R}^d)$, the partial sums $S_{K,N}(x) = \sum_{|k| \leq K} \sum_{|n| \leq N} \langle f, M_{\beta n} T_{\alpha k} \gamma \rangle M_{\beta n} T_{\alpha k} g(x)$ converge to $f(x)$ for a.e. x.*

Proof. The assumption on g guarantees that both g and γ are in W (see [6]).

Let $c_{kn} = \langle f, M_{\beta n} T_{\alpha k} \gamma \rangle$. Then $(c_{kn}) \in s^p$ by Theorem 1, and therefore $m_k \in L^p(Q_{1/\beta})$ for each k. By the multidimensional Carleson–Hunt theorem [5], there exists a set $Z \subseteq \mathbb{R}^d$ of measure zero such that if $x \notin Z$ then $s_N m_k(x) \to m_k(x)$ for all k. Fix $x \notin Z$, and let K_0 be large enough that $T_{\alpha k} g(x) = 0$ for all $|k| \geq K_0$. Then

$$\lim_{K, N \to \infty} S_{K,N}(x) = \sum_{|k| \leq K_0} \lim_{N \to \infty} \sum_{|n| \leq N} s_N m_k(x) \, T_{\alpha k} g(x)$$

$$= \sum_{|k| \leq K_0} m_k(x) \, T_{\alpha k} g(x) = f(x).$$

Thus the Gabor expansion converges almost everywhere. ∎

References

[1]   C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[2]   J. Bergh and J. Löfström, *Interpolation Spaces*: *An Introduction*, Springer, Berlin, 1976.

[3]   I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia 1992.

[4]   R. E. Edwards and G. I. Gaudry, *Littlewood–Paley and Multiplier Theory*, Springer, Berlin, 1977.

[5]   C. Fefferman, *On the convergence of multiple Fourier series*, Bull. Amer. Math. Soc. 77 (1971), 744–745.

[6]   H. G. Feichtinger and K. Gröchenig, *Gabor frames and time-frequency analysis of distributions*, J. Funct. Anal. 146 (1997), 464–495.

[7]   H. G. Feichtinger, K. Gröchenig and D. Walnut, *Wilson bases and modulation spaces*, Math. Nachr. 155 (1992), 7–17.

[8]   G. B. Folland, *Harmonic Analysis on Phase Space*, Ann. of Math. Stud. 122, Princeton Univ. Press, Princeton, NJ, 1989.

[9]   J. E. Gilbert and J. D. Lakey, *On the characterization of the local Hardy space by Gabor frames*, preprint, 2000.

[10]   L. Grafakos and C. Lennard, *Characterization of $L^p(\mathbb{R}^n)$ using the Gabor frame*, J. Fourier Anal. Appl. 7 (2001), 101–126.

[11]   K. Gröchenig, *Describing functions*: *atomic decompositions versus frames*, Monatsh. Math. 112 (1991), 1–41.

[12]   —, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[13]   C. E. Heil and D. F. Walnut, *Continuous and discrete wavelet transforms*, SIAM Rev. 31 (1989), 628–666.

[14]   Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover, New York, 1976.

[15]   Y. Meyer, *Wavelets and Operators*, Cambridge Univ. Press, Cambridge, 1992.

[16]   J. Ramanathan and T. Steger, *Incompleteness of sparse coherent states*, Appl. Comput. Harmon. Anal. 1 (1994), 209–215.

[17]   H.-J. Schmeisser and H. Triebel, *Topics in Fourier Analysis and Function Spaces*, Geest & Portig, Leipzig, 1987.

[18]   D. Walnut, *Continuity properties of the Gabor frame operator*, J. Math. Anal. Appl. 165 (1992), 479–504.

[19]   A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1959.

Department of Mathematics
The University of Connecticut
Storrs, Connecticut 06269-3009, U.S.A.
E-mail: groch@math.uconn.edu

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332-0160, U.S.A.
E-mail: heil@math.gatech.edu