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Gain-scheduled dynamic output feedback control for discrete-time LPV systems

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SUMMARY

This paper presents synthesis conditions for the design of gain-scheduled dynamic output feedback controllers for discrete-time linear parameter-varying systems. The state-space matrix representation of the plant and of the controller can have a homogeneous polynomial dependency of arbitrary degree on the scheduling parameter. As an immediate extension, conditions for the synthesis of a multiobjective $\mathscr{H}_\infty$ and $\mathscr{H}_2$ gain-scheduled dynamic feedback controller are also provided. The scheduling parameters vary inside a polytope and are assumed to be *a priori* unknown, but measured in real-time. If bounds on the rate of parameter variation are known, they can be taken into account, providing less conservative results. The geometric properties of the uncertainty domain are exploited to derive finite sets of linear matrix inequalities based on the existence of a homogeneous polynomially parameter-dependent Lyapunov function. An application of the control design to a realistic engineering problem illustrates the benefits of the proposed approach.

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1. INTRODUCTION

Dating back from the 1960s, classical gain-scheduling control techniques have been successfully applied in many engineering applications for the control of nonlinear systems. These techniques can be described as divide and conquer approaches, where the nonlinear control design is decomposed into a number of linear subproblems [1]. However, in the absence of a sound theoretical analysis, the classical gain-scheduled control designs come with no guarantees on robustness, performance or even nominal stability, as pointed out in the pioneering works [2–5]. As a result, these critical issues have been constantly reconsidered and reevaluated by the control community and a continuing effort is apparent to develop gain-scheduled control design techniques that guarantee stability and performance. Consequently, a distinction is made in the literature between the *classical ad hoc* gain-scheduling techniques and the so-called *modern* gain-scheduling approaches [1, 6].

In the classical techniques (see, for instance, [7–10]), the procedure to synthesize a gain-scheduled controller consists of the following steps. First, determine a family of linear time invariant

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(LTI) models by selecting different operating conditions of the system and then design an LTI controller for each LTI model. Next, based on the actual value of the varying parameters (measured or estimated online), schedule the local controllers using some interpolation method. The final step consists of checking the closed-loop stability and performance using extensive simulation. Although the system performance can be improved by increasing the number of local models (at the expense of increasing the computational burden), this approach may be unreliable, since the closed-loop stability and performance are only verified through simulations.

On the other hand, the modern design approaches start from a linear parameter-varying (LPV) representation of the system and derive direct synthesis conditions for a parameter-dependent controller. These techniques can be classified into Lyapunov-based approaches for parameter-dependent state-space systems and small-gain approaches for systems with a parameter-dependent linear fractional transformation (LFT) structure. One of the most critical issues in these modern methods is the choice of the Lyapunov function used to establish stability and performance. For linear systems, the most common approach is to choose a Lyapunov function V that is quadratic in the system state $x(t)$, that is, $V(t,x(t))=x(t)'P(t)x(t)$, where matrix $P(t)$ needs to be positive definite for all time instants. For LPV systems, the question then turns to choosing a parameterization for the dependency of matrix $P(t)$ on the scheduling parameters. Initially, many of the approaches used the concept of *quadratic stability* where the Lyapunov matrix is assumed independent of the scheduling parameter, since this choice results in numerically tractable optimization problems. However, quadratic stability generally leads to conservative results for practical applications since it assumes that the variation of the scheduling parameters is arbitrarily fast. Consequently, many works using parameter-dependent Lyapunov functions have been published to mitigate some of the conservatism associated with the quadratic stability-based approaches.

Using the small-gain theory, a finite number of convex conditions for $\mathscr{H}_\infty$ control synthesis was derived based on quadratic stability in [11, 12], while multiobjective $\mathscr{H}_2/\mathscr{H}_\infty$ synthesis conditions were presented in [13]. Besides using a parameter-independent Lyapunov function to assess the stability of the closed-loop system, another source of conservatism in these techniques is the use of a subclass of full-block scalings. To reduce this conservatism, the use of more general full-block scalings has been introduced in [14] while the use of parameter-dependent Lyapunov matrices was suggested in [15, 16].

Based on the Lyapunov theory (see, for example, [17]), conditions for the design of $\mathscr{H}_\infty$ gain-scheduled dynamic output feedback controllers are derived for discrete-time [18] and continuous-time [19] LPV state-space systems using the concept of quadratic stability and for continuous-time LPV systems with bounded parameter variation using affine and polytopic Lyapunov matrices [20, 21]. Observer-based dynamic output feedback synthesis techniques are derived using piecewise Lyapunov matrices in [22, 23]. More general, polynomial Lyapunov matrices have been used in [24] for continuous-time LPV systems with bounded variation to assess stability and in [25] to synthesize $\mathscr{H}_\infty$ state feedback controllers. Up to now, it seems that synthesis for dynamic output feedback based on polynomial Lyapunov matrices has not been discussed in the literature. Several other Lyapunov-based techniques have been proposed to derive a finite set of computable conditions to assess stability or to synthesize a controller for LPV systems. For example, in [26] a parameter-dependent weighted sum of Lyapunov matrices is suggested, together with a gridding technique to derive synthesis conditions for a gain-scheduled dynamic controller for continuous-time systems with a bound on the rate of parameter variation. Sum-of-Squares decomposition techniques have been exploited in [27] for both analysis and synthesis.

The aim of this paper is to provide synthesis conditions for the design of $\mathscr{H}_\infty$ and $\mathscr{H}_2$ gain-scheduled dynamic output feedback controllers for discrete-time homogeneous polynomially parameter-dependent linear systems with time-varying parameters belonging to a polytope. It is worth to emphasize that the contributions of this paper extend the recent results [28–30] and the state-of-the-art in the literature in the following directions. First, full-order dynamic output feedback controllers, instead of static gains, are addressed. Second, the plant system matrices can have a polynomially parameter-dependent representation of arbitrarily degree rather

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than the particular polytopic one. Likewise for the controller matrices. Third, polynomially parameter-dependent Lyapunov functions are used to guarantee closed-loop stability and performance. Fourth, a recently developed framework to incorporate known bounds on the rate of parameter variation inside a unit-simplex is exploited to reduce the conservatism of the proposed synthesis procedures. Fifth, as an extension of the proposed conditions for $\mathscr{H}_\infty$ and $\mathscr{H}_2$ dynamic output feedback control, a suboptimal multiobjective $\mathscr{H}_2/\mathscr{H}_\infty$ control design problem can be conveniently solved. Furthermore, an application of the proposed control design technique to a realistic vibroacoustic problem, based on experimentally obtained data, illustrates the benefits of the proposed approach and shows that the techniques can be used for realistic engineering problems.

The paper is organized as follows. First, some necessary notation is introduced in Section 2. Then, the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance of discrete-time LPV systems are recapitulated in Section 3 and the modeling of the uncertainty domain where the varying parameters and their rate of variation assume values is presented in Section 4. Afterwards, synthesis conditions are derived for $\mathscr{H}_\infty$, $\mathscr{H}_2$ and multiobjective gain-scheduled dynamic output feedback controllers in Section 5. These synthesis procedures are applied to a vibroacoustic problem with realistic numerical data in Section 6. Conclusions follow in Section 7.

2. NOTATION AND TERMINOLOGY

The set of real numbers is denoted by $\mathbb{R}$ and the set of natural (nonnegative integer) numbers by $\mathbb{N}$. Prime, $'$, is used to indicate the transpose. The space of square-summable sequences on $\mathbb{N}$ is given by

$$\ell_2^n \triangleq \left\{ f:\mathbb{N}\to\mathbb{R}^n \mid \sum_{k=0}^{\infty} f(k)' f(k) < \infty \right\}.$$

The corresponding 2-norm is defined as $\|x(k)\|_2^2 = \sum_{k=0}^{\infty} x(k)'x(k)$. The trace operator is denoted by $\mathrm{Tr}\{\cdot\}$, whereas the expectation operator is denoted by $\mathscr{E}\{\cdot\}$. Identity matrices (resp. zero matrices) are denoted as I (resp. 0) in case the size is clear from the context. The convex hull of a set X is denoted by $\mathrm{co}\{X\}$.

Definition 1 (*Unit-simplex*)
The unit-simplex Λ_N of dimension $N\in\mathbb{N}$, with $N\geqslant 2$, is given by

$$\Lambda_N = \left\{ \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N \mid \sum_{j=1}^{N} \xi_j = 1,\ \xi_j \geqslant 0,\ j = 1, \ldots, N \right\}.$$

Thus, the unit-simplex Λ_N contains all vectors consisting of N nonnegative real numbers that sum up to 1.

For $N\in\mathbb{N}$ and $g\in\mathbb{N}$, let $\mathscr{K}_N(g)$ be the set of N-tuples obtained from all possible combinations of N nonnegative integers k_j, $j=1,\ldots,N$, with sum $k_1+k_2+\cdots+k_N=g$, that is

$$\mathscr{K}_N(g) = \left\{ k = (k_1, k_2, \ldots, k_N) \in \mathbb{N}^N \mid \sum_{j=1}^{N} k_j = g \right\}.$$

Definition 2 (*Homogeneous polynomial*)
Given a unit-simplex Λ_N of dimension $N\in\mathbb{N}$, a polynomial $p(\alpha)$ with $\alpha\in\Lambda_N$ defined on $\mathbb{R}^N$ is called a homogeneous polynomial of degree $g\in\mathbb{N}$ if all its monomials have the same total degree g.

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For example, let $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\Lambda_3$, then the polynomial $p(\alpha)=2\alpha_1^4+\alpha_1^2\alpha_3^2-5\alpha_2^3\alpha_3+3\alpha_1\alpha_2\alpha_3^2$ is a homogeneous polynomial of degree $g=4$. In general, homogeneous polynomials of arbitrary degree g are parameterized as

$$A(\alpha)=\sum_{k\in\mathscr{K}_N(g)}\alpha^k A_k,$$

where α^k is a shorthand notation for $\alpha^k=\alpha_1^{k_1}\cdot\alpha_2^{k_2}\cdot\ldots\cdot\alpha_N^{k_N}=\prod_{j=1}^N\alpha_j^{k_j}$.

By definition, for N-tuples $k\in\mathbb{N}^N$ and $\tilde{k}\in\mathbb{N}^N$, $k\succcurlyeq\tilde{k}$ if $k_j\geqslant\tilde{k}_j$, for $j=1,\ldots,N$. Operations of summation $k+\tilde{k}$ and subtraction $k-\tilde{k}$ (whenever $k\succcurlyeq\tilde{k}$) are defined componentwise. Finally, for an N-tuple $k\in\mathbb{N}^N$, the coefficient $\pi(k)$ is defined as the product

$$\pi(k)\triangleq\prod_{j=1}^{N}k_j!$$

3. PERFORMANCE FOR DISCRETE-TIME LPV SYSTEMS

For discrete-time LPV systems, this section defines the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance and provides a characterization to compute guaranteed upper bounds for them. Note that this material has been presented with more details in [28].

The discrete-time LPV system H considered in this section is assumed to have a finite-dimensional state-space realization:

$$H:=\begin{cases}x(k+1)=A(\alpha(k))x(k)+B_w(\alpha(k))w(k), & x(0)=0,\\ z(k)=C_z(\alpha(k))x(k)+D_{zw}(\alpha(k))w(k), & \end{cases}\tag{1}$$

where $x(k)\in\mathbb{R}^{n_x}$ is the state, $w(k)\in\mathbb{R}^{n_w}$ the exogenous input and $z(k)\in\mathbb{R}^{n_z}$ the system output. The system matrices $A(\alpha(k))\in\mathbb{R}^{n_x\times n_x}$, $B_w(\alpha(k))\in\mathbb{R}^{n_x\times n_w}$, $C_z(\alpha(k))\in\mathbb{R}^{n_z\times n_x}$ and $D_{zw}(\alpha(k))\in\mathbb{R}^{n_z\times n_w}$ are real and bounded and have a homogeneous polynomial parameter-dependency of degree g on the scheduling parameter $\alpha(k)$ that assumes values in the unit-simplex Λ_N. To the best of the authors' knowledge, control design for this class of models has not yet been addressed in the literature. It is worth to emphasize there already exist LPV modeling techniques that provide models with polynomial dependency on the scheduling parameter (see, for instance, [31–34]). For this system H, its $\mathscr{H}_\infty$ performance is defined as follows.

Definition 3 (*$\mathscr{H}_\infty$ performance*)
Suppose that the system H is exponentially stable. Then, its $\mathscr{H}_\infty$ performance is defined as

$$\|H\|_\infty=\sup_{\|w(k)\|_2\neq 0}\frac{\|z(k)\|_2}{\|w(k)\|_2}\quad\text{with } w(k)\in\ell_2^{n_w}\quad\text{and}\quad z(k)\in\ell_2^{n_z}.$$

Based on the bounded real lemma, an upper bound for the $\mathscr{H}_\infty$ performance of system H can be computed using an extended LMI characterization, as shown in the following theorem.

Theorem 1 (*Extended $\mathscr{H}_\infty$ performance*)
Consider system H given by (1). If there exist a bounded matrix $G(\alpha)\in\mathbb{R}^{n_x\times n_x}$ and a bounded symmetric positive-definite matrix $P(\alpha)\in\mathbb{R}^{n_x\times n_x}$, for $\alpha\in\Lambda_N$, such that

$$\begin{bmatrix}P(\alpha(k+1)) & A(\alpha(k))G(\alpha(k)) & B_w(\alpha(k)) & 0\\ G(\alpha(k))'A(\alpha(k))' & G(\alpha(k))+G(\alpha(k))'-P(\alpha(k)) & 0 & G(\alpha(k))'C_z(\alpha(k))'\\ B_w(\alpha(k))' & 0 & \eta I & D_{zw}(\alpha(k))'\\ 0 & C_z(\alpha(k))G(\alpha(k)) & D_{zw}(\alpha(k)) & \eta I\end{bmatrix}>0,\tag{2}$$

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then the system H is exponentially stable and

$$\|H\|_{\infty} \leqslant \inf_{P(\alpha),\, G(\alpha),\, \eta} \eta.$$

The proof can be found in [35] and is a straightforward extension of the work presented in [36] for LTI systems. Matrix $P(\alpha)$ is called a Lyapunov matrix, whereas $G(\alpha)$ is called a slack variable.

Like in [28, 37–39], the $\mathscr{H}_2$ performance of discrete-time LPV systems is defined in this paper as follows.

Definition 4 ($\mathscr{H}_2$ *performance*)
Suppose that the system H is exponentially stable. Then, its $\mathscr{H}_2$ performance is defined as

$$\|H\|_2 = \limsup_{T\to\infty} \mathscr{E}\left\{\frac{1}{T}\sum_{k=0}^{T} z(k)' z(k)\right\}$$

when the system input $w(k)$ is a zero-mean white noise Gaussian process with identity covariance matrix.

The next theorem provides an extended LMI characterization for a guaranteed upper bound on the $\mathscr{H}_2$ performance of system H.

Theorem 2 (*Extended* $\mathscr{H}_2$ *performance*)
Consider system H given by (1). If there exist a bounded matrix $G(\alpha)\in\mathbb{R}^{n_x\times n_x}$ and bounded symmetric positive-definite matrices $P(\alpha)\in\mathbb{R}^{n_x\times n_x}$ and $W(\alpha)\in\mathbb{R}^{n_z\times n_z}$, for $\alpha\in\Lambda_N$, such that

$$\begin{bmatrix} P(\alpha(k+1)) & A(\alpha(k))G(\alpha(k)) & B_w(\alpha(k)) \\ G(\alpha(k))'A(\alpha(k))' & G(\alpha(k))+G(\alpha(k))'-P(\alpha(k)) & 0 \\ B_w(\alpha(k))' & 0 & I \end{bmatrix} > 0, \tag{3a}$$

$$\begin{bmatrix} W(\alpha(k))-D_{zw}(\alpha(k))D_{zw}(\alpha(k))' & C_z(\alpha(k))G(\alpha(k)) \\ G(\alpha(k))'C_z(\alpha(k))' & G(\alpha(k))+G(\alpha(k))'-P(\alpha(k)) \end{bmatrix} > 0, \tag{3b}$$

then the system H is exponentially stable and

$$\|H\|_2^2 \leqslant \inf_{P(\alpha),\, G(\alpha),\, W(\alpha)} \mathrm{Tr}\{W(\alpha)\}.$$

The proof can be found in [28].

4. MODELING OF THE UNCERTAINTY DOMAIN

This section briefly presents the modeling of the uncertainty domain where the varying parameters and their rate of variation assume values (for a more detailed presentation, see [28, 30]). In this modeling, the rate of variation of the parameters in one time instant, defined as

$$\Delta\alpha_j(k) = \alpha_j(k+1) - \alpha_j(k) \quad \text{for } j = 1, \ldots, N, \tag{4}$$

is assumed to be limited by an *a priori* known bound $b\in\mathbb{R}$ such that

$$-b \leqslant \Delta\alpha_j(k) \leqslant b \quad \text{for } j = 1, \ldots, N, \tag{5}$$

with $b\in[0,1]$. Since $\alpha(k)\in\Lambda_N$, it is clear from (4) that

$$\sum_{j=1}^{N} \Delta\alpha_j(k) = \sum_{j=1}^{N} \alpha_j(k+1) - \sum_{j=1}^{N} \alpha_j(k) = 0. \tag{6}$$

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As explained in [28, 30], the region where the vector[‡] $(\alpha, \Delta\alpha) \in \mathbb{R}^{2N}$ assumes values can be modeled by the polytope

$$\Gamma_b = \left\{ \delta \in \mathbb{R}^{2N} | \delta \in \text{co}\{g^1, \dots, g^M\},\ g^i = \begin{pmatrix} f^i \\ h^i \end{pmatrix},\ f^i \in \mathbb{R}^N,\ h^i \in \mathbb{R}^N, \right.$$
$$\left. \sum_{j=1}^{N} f_j^i = 1 \text{ with } f_j^i \geqslant 0,\ j = 1, \dots, N,\ \sum_{j=1}^{N} h_j^i = 0, \quad i = 1, \dots, M \right\},$$

defined as the convex combination of M vectors g^i. The vertices g^i of Γ_b can be constructed in a systematic way for a given b by searching for all possible solutions of the equalities $\sum_{j=1}^{N} \alpha_j = 1$ and $\sum_{j=1}^{N} \Delta\alpha_j = 0$ using the extreme points of the constraints given in (5), for $j = 1, \dots, N$. Once the vertices of the set Γ_b are defined, the convex characterization

$$(\alpha, \Delta\alpha) = \sum_{i=1}^{M} \begin{pmatrix} f^i \\ h^i \end{pmatrix} \gamma_i = \begin{bmatrix} \mathbf{F} \\ \mathbf{H} \end{bmatrix} \gamma \quad \text{with } \mathbf{F} = [f^1 \ \dots \ f^M],\ \mathbf{H} = [h^1 \ \dots \ h^M] \quad \text{and} \quad \gamma \in \Lambda_M \tag{7}$$

can be exploited in the derivations of the LMI conditions.

The next section derives synthesis conditions for gain-scheduled dynamic output feedback controllers based on the convex characterization (7).

5. FULL ORDER DYNAMIC OUTPUT FEEDBACK

This section considers the design of gain-scheduled dynamic output feedback controllers for the following discrete-time LPV system with a homogeneous polynomial parameter-dependency of degree g on the scheduling parameter α that takes values in the unit-simplex Λ_N

$$H := \begin{cases} x(k+1) = A(\alpha(k))x(k) + B_w(\alpha(k))w(k) + B_u(\alpha(k))u(k), \\ z(k) = C_z(\alpha(k))x(k) + D_{zw}(\alpha(k))w(k) + D_{zu}(\alpha(k))u(k), \\ y(k) = C_y(\alpha(k))x(k) + D_{yw}(\alpha(k))w(k), \end{cases} \tag{8}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state, $w(k) \in \mathbb{R}^{n_w}$ the exogenous disturbance input, $z(k) \in \mathbb{R}^{n_z}$ the performance output, $y(k) \in \mathbb{R}^{n_y}$ the measured output and $u(k) \in \mathbb{R}^{n_u}$ the control input. The aim is to provide a finite-dimensional set of LMIs for the synthesis of a strictly proper full-order dynamic output feedback controller

$$K := \begin{cases} x_c(k+1) = A_c(\alpha(k))x_c(k) + B_c(\alpha(k))y(k), \\ u(k) = C_c(\alpha(k))x_c(k), \end{cases} \tag{9}$$

with the controller state $x_c(k) \in \mathbb{R}^{n_x}$ and the system matrices $A_c(\alpha(k)) \in \mathbb{R}^{n_x \times n_x}$, $B_c(\alpha(k)) \in \mathbb{R}^{n_x \times n_y}$ and $C_c(\alpha(k)) \in \mathbb{R}^{n_u \times n_x}$, such that the closed-loop system

$$H_{cl} := \begin{cases} x_{cl}(k+1) = A_{cl}(\alpha(k))x_{cl}(k) + B_{cl}(\alpha(k))w(k), \\ z(k) = C_{cl}(\alpha(k))x_{cl}(k) + D_{cl}(\alpha(k))w(k), \end{cases} \tag{10}$$

‡In the modeling of the uncertainty domain, the vectors $\alpha \in \mathbb{R}^N$ and $\Delta\alpha \in \mathbb{R}^N$ represent *column* vectors, that is, $\alpha \in \mathbb{R}^{N \times 1}$ and $\Delta\alpha \in \mathbb{R}^{N \times 1}$. Likewise, the vector $(\alpha, \Delta\alpha)$ is a column vector $(\alpha, \Delta\alpha) \in \mathbb{R}^{2N \times 1}$. For reasons of compactness, this is not explicitly mentioned throughout the remainder of the paper.

with state $x_{cl}(k) = [x(k)' \;\; x_c(k)']' \in \mathbb{R}^{2n_x}$ and system matrices

$$A_{cl}(\alpha(k)) = \begin{bmatrix} A(\alpha(k)) & B_u(\alpha(k))C_c(\alpha(k)) \\ B_c(\alpha(k))C_y(\alpha(k)) & A_c(\alpha(k)) \end{bmatrix}, \quad B_{cl}(\alpha(k)) = \begin{bmatrix} B_w(\alpha(k)) \\ B_c(\alpha(k))D_{yw}(\alpha(k)) \end{bmatrix}, \tag{11}$$
$$C_{cl}(\alpha(k)) = [C_z(\alpha(k)) \;\; D_{zu}(\alpha(k))C_c(\alpha(k))], \quad D_{cl}(\alpha(k)) = [D_{zw}(\alpha(k))],$$

is exponentially stable for all possible trajectories of the parameter $\alpha(k) \in \Lambda_N$, with a guaranteed upper bound on the closed-loop $\mathscr{H}_\infty$ or $\mathscr{H}_2$ performance. The proposed LPV control synthesis procedures extend the techniques presented for LTI systems in [36], which in turn are inspired by Scherer *et al.* [40] and Masubuchi *et al.* [41]. In the following sections, the $\mathscr{H}_\infty$ control design is first discussed, after which the $\mathscr{H}_2$ design follows as a straightforward extension.

Note that the design of controllers with a feedthrough term from the measured output to the control input $D_c(\alpha(k))\,y(k)$ is also possible. However, in this work strictly proper controllers are chosen since this considerably simplifies the derivation of the synthesis procedures.

5.1. Gain-scheduled $\mathscr{H}_\infty$ dynamic output feedback control

The aim of this section is to derive a finite-dimensional set of LMI synthesis conditions for the design of strictly proper full-order gain-scheduled $\mathscr{H}_\infty$ dynamic output feedback controllers. Following the condition of Theorem 1, the $\mathscr{H}_\infty$ performance of the closed-loop system (10) is bounded by η if there exists a symmetric positive-definite matrix $\mathbf{P}(\alpha) \in \mathbb{R}^{2n_x \times 2n_x}$ and a matrix $\mathbf{G}(\alpha) \in \mathbb{R}^{2n_x \times 2n_x}$ such that the following matrix inequality holds:

$$\begin{bmatrix} \mathbf{P}(\alpha(k+1)) & A_{cl}(\alpha(k))\mathbf{G}(\alpha(k)) & B_{cl}(\alpha(k)) & 0 \\ \star & \mathbf{G}(\alpha(k)) + \mathbf{G}(\alpha(k))' - \mathbf{P}(\alpha(k)) & 0 & \mathbf{G}(\alpha(k))'C_{cl}(\alpha(k))' \\ \star & \star & \eta I & D_{cl}(\alpha(k))' \\ \star & \star & \star & \eta I \end{bmatrix} > 0$$
$$\text{for } k \geqslant 0. \tag{12}$$

It is clear that substituting for the closed-loop matrices (11) in the matrix inequality (12) yields a nonlinear matrix inequality, due to the multiplication of the unknown matrices of the controller (10) and the slack variable $\mathbf{G}(\alpha)$. Therefore, a suitable change of variables needs to be defined to transform the nonlinear matrix inequality (12) into an equivalent LMI. In the following, matrix $\mathbf{G}(\alpha)$ is chosen to be independent of the scheduling parameter α, that is, $\mathbf{G}(\alpha) = \mathbf{G}$. After the introduction of the nonlinear change of variables, some remarks about this choice are given.

5.1.1. Nonlinear change of variables. To start, define and partition matrices $K(\alpha)$, $\mathbf{P}(\alpha)$, $\mathbf{G}$ and $\mathbf{G}^{-1}$ as

$$K(\alpha) := \begin{bmatrix} A_c(\alpha) & B_c(\alpha) \\ C_c(\alpha) & 0 \end{bmatrix}, \quad \mathbf{P}(\alpha) := \begin{bmatrix} P(\alpha) & P_2(\alpha) \\ P_2(\alpha)' & P_3(\alpha) \end{bmatrix}, \quad \mathbf{G} := \begin{bmatrix} X & Z_1 \\ U & Z_2 \end{bmatrix}, \quad \mathbf{G}^{-1} := \begin{bmatrix} Y' & Z_3 \\ V' & Z_4 \end{bmatrix}.$$

From the definition of $\mathbf{G}$ and $\mathbf{G}^{-1}$, it is clear that the following relation must hold:

$$\mathbf{G}\mathbf{G}^{-1} = \begin{bmatrix} X & Z_1 \\ U & Z_2 \end{bmatrix} \begin{bmatrix} Y' & Z_3 \\ V' & Z_4 \end{bmatrix} = \begin{bmatrix} XY' + Z_1V' & XZ_3 + Z_1Z_4 \\ UY' + Z_2V' & UZ_3 + Z_2Z_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and, consequently, $XY' + Z_1V' = I$ and $UY' + Z_2V' = 0$. Now, introduce the parameter-independent transformation matrix

$$\mathbf{T} := \begin{bmatrix} I & Y' \\ 0 & V' \end{bmatrix}$$

and define the following nonlinear parameter-dependent change of variables:

$$\begin{bmatrix} Q(\alpha) & F(\alpha) \\ L(\alpha) & 0 \end{bmatrix} := \begin{bmatrix} V & YB_u(\alpha) \\ 0 & I \end{bmatrix} K(\alpha) \begin{bmatrix} U & 0 \\ C_y(\alpha)X & I \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix} A(\alpha)[X \quad 0], \tag{13a}$$

$$\begin{bmatrix} P(\alpha) & J(\alpha) \\ J(\alpha)' & H(\alpha) \end{bmatrix} := \mathbf{T}'\mathbf{P}(\alpha)\mathbf{T} \tag{13b}$$

$$S := YX + VU. \tag{13c}$$

The linearizing property of this transformation relies on the following identities:

$$\mathbf{T}' A_{\mathrm{cl}}(\alpha)\mathbf{G}\mathbf{T} = \begin{bmatrix} A(\alpha)X + B_u(\alpha)L(\alpha) & A(\alpha) \\ Q(\alpha) & YA(\alpha) + F(\alpha)C_y(\alpha) \end{bmatrix},$$

$$\mathbf{T}' B_{\mathrm{cl}}(\alpha) = \begin{bmatrix} B_w(\alpha) \\ YB_w(\alpha) + F(\alpha)D_{yw}(\alpha) \end{bmatrix},$$

$$C_{\mathrm{cl}}(\alpha)\mathbf{G}\mathbf{T} = [C_z(\alpha)X + D_{zu}(\alpha)L(\alpha) \;\; C_z(\alpha)], \quad D_{\mathrm{cl}}(\alpha) = D_{zw}(\alpha), \quad \mathbf{T}'\mathbf{G}\mathbf{T} = \begin{bmatrix} X & I \\ S & Y \end{bmatrix}.$$

Multiplying (12) with $T := \mathrm{diag}(\mathbf{T}, \mathbf{T}, I, I)$ on the right and with T' on the left yields the following linear matrix inequality in matrices X, $L(\alpha)$, Y, $F(\alpha)$, $Q(\alpha)$, S, $J(\alpha)$, and the symmetric positive-definite matrices $P(\alpha)$ and $H(\alpha)$

$$\Theta(\alpha(k)) = \begin{bmatrix} P(\alpha(k+1)) & J(\alpha(k+1)) & \Theta_{13}(\alpha(k)) & A(\alpha(k)) & B_w(\alpha(k)) & 0 \\ \star & H(\alpha(k+1)) & Q(\alpha(k)) & \Theta_{24}(\alpha(k)) & \Theta_{25}(\alpha(k)) & 0 \\ \star & \star & X + X' - P(\alpha(k)) & I + S' - J(\alpha(k)) & 0 & \Theta_{36}(\alpha(k)) \\ \star & \star & \star & Y + Y' - H(\alpha(k)) & 0 & C_z(\alpha(k))' \\ \star & \star & \star & \star & \eta I & D_{zw}(\alpha(k))' \\ \star & \star & \star & \star & \star & \eta I \end{bmatrix} > 0, \tag{14}$$

where

$$\Theta_{13}(\alpha(k)) = A(\alpha(k))X + B_u(\alpha(k))L(\alpha(k)),$$

$$\Theta_{24}(\alpha(k)) = YA(\alpha(k)) + F(\alpha(k))C_y(\alpha(k)),$$

$$\Theta_{25}(\alpha(k)) = YB_w(\alpha(k)) + F(\alpha(k))D_{yw}(\alpha(k)),$$

$$\Theta_{36}(\alpha(k)) = X'C_z(\alpha(k))' + L(\alpha(k))'D_{zu}(\alpha(k))'.$$

The congruence transformation T is full rank when $\mathbf{T}$ is full rank, which in turn is full rank if V is full rank. Based on an argument from [42], V can be assumed full rank without loss of generality. If the LMI (14) is satisfied, a gain-scheduled $\mathscr{H}_\infty$ controller can be recovered by inverting the nonlinear transformation (13a)

$$K(\alpha) = \begin{bmatrix} V^{-1} & -V^{-1}YB_u(\alpha) \\ 0 & I \end{bmatrix} \begin{bmatrix} Q(\alpha) - YA(\alpha)X & F(\alpha) \\ L(\alpha) & 0 \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ -C_y(\alpha)XU^{-1} & I \end{bmatrix},$$

yielding the following system matrices for the controller:

$$\begin{aligned} A_c(\alpha) &= V^{-1}(Q(\alpha) - YA(\alpha)X - YB_u(\alpha)L(\alpha) - F(\alpha)C_y(\alpha)X)U^{-1}, \\ B_c(\alpha) &= V^{-1}F(\alpha), \\ C_c(\alpha) &= L(\alpha)U^{-1}. \end{aligned} \tag{15}$$

It is clear that the controller matrices can be calculated whenever matrices U and V are nonsingular. In [36], it is shown that there always exist nonsingular matrices U and V for which (13c) holds, such that $VU = S - YX$. For invertible matrices U and V, (15) provides the matrices of a strictly proper full-order gain-scheduling $\mathscr{H}_\infty$ dynamic output feedback controller. Note that from each pair of feasible nonsingular matrices U and V, a new pair of nonsingular matrices $\widetilde{U} = \widetilde{T}^{-1}U$ and $\widetilde{V} = V\widetilde{T}$ can be constructed using any nonsingular matrix $\widetilde{T}$ since $\widetilde{V}\widetilde{U} = V\widetilde{T}\widetilde{T}^{-1}U = VU$. Applying this new pair $\widetilde{U}$ and $\widetilde{V}$ to construct the controller matrices (15) reveals that $\widetilde{T}$ constitutes a similarity transformation for the controller state x_c.

Remark 1

At this point, the choice of a constant slack variable $\mathbf{G}$ can be clarified. It is clear that a parameter-dependent $\mathbf{G}$ implies that matrices X, Y, U and V are parameter-dependent as well. This in turn, means that $\mathbf{T}$ and the congruence transformation T are parameter-dependent and therefore T needs to take the form

$$T(\alpha(k)) := \text{diag}(\mathbf{T}(\alpha(k+1)), \mathbf{T}(\alpha(k)), I, I),$$

where the first block is $\mathbf{T}(\alpha(k+1))$ to ensure that when (12) is multiplied with $T(\alpha(k))$ on the right and with $T(\alpha(k))'$ on the left, the nonlinear transformation (13b) can be used for both $\mathbf{P}(\alpha(k+1))$ in the first and $\mathbf{P}(\alpha(k))$ in the second diagonal block of (12). Applying this parameter-dependent congruence transformation $T(\alpha(k))$ yields (among others), the following expression:

$$\begin{aligned} &\mathbf{T}(\alpha(k+1))' A_{cl}(\alpha(k))\mathbf{G}(\alpha(k))\mathbf{T}(\alpha(k)) \\ &= \begin{bmatrix} A(\alpha(k))X(\alpha(k)) + B_u(\alpha(k))L(\alpha(k)) & A(\alpha(k)) \\ Q(\alpha(k)) & Y(\alpha(k+1))A(\alpha(k)) + F(\alpha(k))C_y(\alpha(k)) \end{bmatrix}, \end{aligned}$$

where the change of variables (13a) now becomes

$$\begin{aligned} Q(\alpha(k)) &= Y(\alpha(k+1))A(\alpha(k))X(\alpha(k)) + Y(\alpha(k+1))B_u(\alpha(k))C_c(\alpha(k))U(\alpha(k)) \\ &\quad + V(\alpha(k+1))B_c(\alpha(k))C_y(\alpha(k))X(\alpha(k)) + V(\alpha(k+1))A_c(\alpha(k))U(\alpha(k)), \\ F(\alpha(k)) &= V(\alpha(k+1))B_c(\alpha(k)), \\ L(\alpha(k)) &= C_c(\alpha(k))U(\alpha(k)). \end{aligned} \tag{16}$$

Note that this change of variables depends on the actual and the *future* value of the scheduling parameter. Since the construction of the controller matrices involves inverting this nonlinear transformation, the resulting controller not only depends on the actual value of the scheduling parameter $\alpha(k)$, but also on its future value $\alpha(k+1)$. Since in most applications, this value is not available in real-time, the gain-scheduled controller obtained by solving the matrix inequality (12) with a parameter-dependent slack variable $\mathbf{G}(\alpha)$ cannot be implemented in practice.

The fact that a constant slack variable $\mathbf{G}$ is used has the following side-effect. From (15), it is clear that parameter-independent matrices U, V, X and Y in combination with a homogeneous polynomial parameterization for $Q(\alpha)$, $F(\alpha)$ and $L(\alpha)$ yield controller matrices that have a homogeneous polynomial parameter-dependency as well. Furthermore, from (11) it is clear that a controller with homogeneous polynomially parameter-dependent controller matrices results in a closed-loop system that also has a homogeneous polynomial parameter-dependency. Inversion of (16), on the other hand, yields a more complex, rational dependency on the scheduling parameter $\alpha(k)$ and $\alpha(k+1)$.

5.1.2. Finite-dimensional set of sufficient LMI conditions. It is worth to emphasize that evaluating the LMI condition (14) for $\alpha(k)$ at all time instants $k\geqslant 0$ leads to an infinite-dimensional problem. However, using the following four steps, a finite-dimensional set of sufficient LMI conditions can be derived. The first step is to impose some particular structure on the Lyapunov matrix $P(\alpha)$. In this paper, $P(\alpha)$ is chosen to have a homogeneous polynomially parameter-dependent parameterization of a given degree p on the scheduling parameter $\alpha\in\Lambda_N$, that is,

$$P(\alpha)=\sum_{\ell\in\mathscr{K}_N(p)}\alpha^{\ell}P_{\ell},\quad \alpha\in\Lambda_N. \tag{17}$$

In the second step, the modeling of the uncertainty domain Γ_b of Section 4 is exploited. Using the linear relation (7), it is clear that at each time instant, there exists a $\gamma(k)\in\Lambda_M$, such that $\alpha(k)=\mathbf{F}\gamma(k)$. Consequently, using Theorem A.1 from Appendix A.2, a homogeneous polynomially parameter-dependent matrix $\widehat{P}(\gamma)=\sum_{t\in\mathscr{K}_M(p)}\gamma^t\widehat{P}_t$ can be constructed, such that $\widehat{P}(\gamma(k))\equiv P(\mathbf{F}\gamma(k))\equiv P(\alpha(k))$. Naturally, Theorem A.1 can also be used to construct homogeneous polynomial system matrices depending on γ, that are equivalent to the original system matrices of (8), for example $\widehat{A}(\gamma)=\sum_{n\in\mathscr{K}_M(g)}\gamma^n\widehat{A}_n$, such that $\widehat{A}(\gamma(k))\equiv A(\mathbf{F}\gamma(k))\equiv A(\alpha(k))$. Moreover, since $\alpha(k+1)=\alpha(k)+\Delta\alpha(k)$, it is clear from (7) that

$$\alpha(k+1)=\mathbf{F}\gamma(k)+\mathbf{H}\gamma(k)=(\mathbf{F}+\mathbf{H})\gamma(k)$$

and consequently a homogeneous polynomially parameter-dependent matrix, $\widetilde{P}(\gamma)=\sum_{t\in\mathscr{K}_M(p)}\gamma^t\widetilde{P}_t$ can be constructed, such that at each time instant $\widetilde{P}(\gamma(k))\equiv P((\mathbf{F}+\mathbf{H})\gamma(k))\equiv P(\alpha(k+1))$. The same procedure can be used to construct homogeneous polynomials $\widetilde{J}(\gamma(k))\equiv J((\mathbf{F}+\mathbf{H})\gamma(k))\equiv J(\alpha(k+1))$, $\widehat{Q}(\gamma(k))\equiv Q(\mathbf{F}\gamma(k))\equiv Q(\alpha(k))$, etc.

For the third step, one has that, as a result of the representation of the system matrices, the slack variable and the Lyapunov matrix at time instant k and $k+1$ can be written in terms of $\gamma(k)$. Thus, the LMI (14) can be rewritten with a dependency on $\gamma(k)$ as follows:

$$\Theta(\gamma(k))=\begin{bmatrix}\widetilde{P}(\gamma(k)) & \widetilde{J}(\gamma(k)) & \Theta_{13}(\gamma(k)) & \widehat{A}(\gamma(k)) & \widehat{B}_w(\gamma(k)) & 0\\ \star & \widetilde{H}(\gamma(k)) & \widehat{Q}(\gamma(k)) & \Theta_{24}(\gamma(k)) & \Theta_{25}(\gamma(k)) & 0\\ \star & \star & X+X'-\widehat{P}(\gamma(k)) & I+S'-\widehat{J}(\gamma(k)) & 0 & \Theta_{36}(\gamma(k))\\ \star & \star & \star & Y+Y'-\widehat{H}(\gamma(k)) & 0 & \widehat{C}_z(\gamma(k))'\\ \star & \star & \star & \star & \eta I & \widehat{D}_{zw}(\gamma(k))'\\ \star & \star & \star & \star & \star & \eta I\end{bmatrix}>0, \tag{18}$$

with

$$\Theta_{13}(\gamma(k))=\widehat{A}(\gamma(k))X+\widehat{B}_u(\gamma(k))\widehat{L}(\gamma(k)),$$

$$\Theta_{24}(\gamma(k))=Y\widehat{A}(\gamma(k))+\widehat{F}(\gamma(k))\widehat{C}_y(\gamma(k)),$$

$$\Theta_{25}(\gamma(k))=Y\widehat{B}_w(\gamma(k))+\widehat{F}(\gamma(k))\widehat{D}_{yw}(\gamma(k)),$$

$$\Theta_{36}(\gamma(k))=X'\widehat{C}_z(\gamma(k))'+\widehat{L}(\gamma(k))'\widehat{D}_{zu}(\gamma(k))'.$$

Notice that due to the change of variables, this LMI now depends on the time instant k only, as opposed to the LMI (14) that depends on both k and $k+1$. Therefore, a sufficient condition for the LMI (18) to hold for all time instants $k \geqslant 0$ is that the LMI

$$\Theta(\gamma)=\begin{bmatrix} \widetilde{P}(\gamma) & \widetilde{J}(\gamma) & \widehat{A}(\gamma)X+\widehat{B}_u(\gamma)\widehat{L}(\gamma) & \widehat{A}(\gamma) & \widehat{B}_w(\gamma) & 0 \\ \star & \widetilde{H}(\gamma) & \widehat{Q}(\gamma) & Y\widehat{A}(\gamma)+\widehat{F}(\gamma)\widehat{C}_y(\gamma) & Y\widehat{B}_w(\gamma)+\widehat{F}(\gamma)\widehat{D}_{yw}(\gamma) & 0 \\ \star & \star & X+X'-\widehat{P}(\gamma) & I+S'-\widehat{J}(\gamma) & 0 & X'\widehat{C}_z(\gamma)'+\widehat{L}(\gamma)'\widehat{D}_{zu}(\gamma)' \\ \star & \star & \star & Y+Y'-\widehat{H}(\gamma) & 0 & \widehat{C}_z(\gamma)' \\ \star & \star & \star & \star & \eta I & \widehat{D}_{zw}(\gamma)' \\ \star & \star & \star & \star & \star & \eta I \end{bmatrix} > 0, \tag{19}$$

holds for all $\gamma \in \Lambda_M$. It is important to stress that although the LMI is now *independent* of the time instant k, it is still infinite-dimensional since it needs to hold for all values of γ. However, in the fourth and final step, a finite-dimensional set of sufficient LMI conditions can be derived. This is the context of the following theorem.

Theorem 3 (*Gain-scheduled $\mathscr{H}_\infty$ dynamic output feedback control design*)
Consider system H, given by (8). Let $p \in \mathbb{N}$ be given. Let matrices $\mathbf{F}$ and $\mathbf{H}$, defined in (7), that characterize the uncertainty domain Γ_b be given. Then, the strictly proper full-order gain-scheduled dynamic output feedback controller (9) with controller matrices (15) stabilizes H with a guaranteed upper bound on the closed-loop $\mathscr{H}_\infty$ performance

$$\|H_{\rm cl}\|_\infty \leqslant \min_{P_\ell, H_\ell, J_\ell, F_\ell, L_\ell, Q_k, X, Y, S, \eta} \eta,$$

if there exist matrices $F_\ell \in \mathbb{R}^{n_x \times n_y}$, $J_\ell \in \mathbb{R}^{n_x \times n_x}$ and $L_\ell \in \mathbb{R}^{n_u \times n_x}$ and symmetric positive-definite matrices $H_\ell \in \mathbb{R}^{n_x \times n_x}$ and $P_\ell \in \mathbb{R}^{n_x \times n_x}$, for $\ell \in \mathscr{K}_N(p)$, and if there exist matrices $Q_k \in \mathbb{R}^{n_x \times n_x}$, for $k \in \mathscr{K}_N(g+p)$, and matrices $S \in \mathbb{R}^{n_x \times n_x}$, $X \in \mathbb{R}^{n_x \times n_x}$ and $Y \in \mathbb{R}^{n_x \times n_x}$ such that

$$\Theta_j=\begin{bmatrix} \sum\limits_{\substack{k\in\mathscr{K}_M(g)\\ j\geqslant k}} \frac{g!}{\pi(k)}\widetilde{P}_{j-k} & \sum\limits_{\substack{k\in\mathscr{K}_M(g)\\ j\geqslant k}} \frac{g!}{\pi(k)}\widetilde{J}_{j-k} & \Theta_{13,j} & \sum\limits_{\substack{k\in\mathscr{K}_M(p)\\ j\geqslant k}} \frac{p!}{\pi(k)}\widehat{A}_{j-k} & \sum\limits_{\substack{k\in\mathscr{K}_M(p)\\ j\geqslant k}} \frac{p!}{\pi(k)}\widehat{B}_{w,j-k} & 0 \\ \star & \sum\limits_{\substack{k\in\mathscr{K}_M(g)\\ j\geqslant k}} \frac{g!}{\pi(k)}\widetilde{H}_{j-k} & \widehat{Q}_j & \Theta_{24,j} & \Theta_{25,j} & 0 \\ \star & \star & \Theta_{33,j} & \Theta_{34,j} & 0 & \Theta_{36,j} \\ \star & \star & \star & \Theta_{44,j} & 0 & \sum\limits_{\substack{k\in\mathscr{K}_M(p)\\ j\geqslant k}} \frac{p!}{\pi(k)}\widehat{C}'_{z,j-k} \\ \star & \star & \star & \star & \frac{(g+p)!}{\pi(j)}\eta I & \sum\limits_{\substack{k\in\mathscr{K}_M(p)\\ j\geqslant k}} \frac{p!}{\pi(k)}\widehat{D}'_{zw,j-k} \\ \star & \star & \star & \star & \star & \frac{(g+p)!}{\pi(j)}\eta I \end{bmatrix} > 0, \tag{20}$$

for all $j \in \mathcal{K}_M(g+p)$, where

$$\Theta_{13,j} = \sum_{\substack{k\in\mathcal{K}_M(p)\\ j\succcurlyeq k}} \frac{p!}{\pi(k)}\widehat{A}_{j-k}X + \sum_{\substack{t\in\mathcal{K}_M(p)\\ j\succcurlyeq t}} \widehat{B}_{u,j-t}\widehat{L}_t,$$

$$\Theta_{24,j} = \sum_{\substack{k\in\mathcal{K}_M(p)\\ j\succcurlyeq k}} \frac{p!}{\pi(k)}Y\widehat{A}_{j-k} + \sum_{\substack{t\in\mathcal{K}_M(p)\\ j\succcurlyeq t}} \widehat{F}_t\widehat{C}_{y,j-t},$$

$$\Theta_{25,j} = \sum_{\substack{k\in\mathcal{K}_M(p)\\ j\succcurlyeq k}} \frac{p!}{\pi(k)}Y\widehat{B}_{w,j-k} + \sum_{\substack{t\in\mathcal{K}_M(p)\\ j\succcurlyeq t}} \widehat{F}_t\widehat{D}_{yw,j-t},$$

$$\Theta_{33,j} = \frac{(g+p)!}{\pi(j)}(X+X') - \sum_{\substack{k\in\mathcal{K}_M(g)\\ j\succcurlyeq k}} \frac{g!}{\pi(k)}\widehat{P}_{j-k}, \tag{21}$$

$$\Theta_{34,j} = \frac{(g+p)!}{\pi(j)}(I+S') - \sum_{\substack{k\in\mathcal{K}_M(g)\\ j\succcurlyeq k}} \frac{g!}{\pi(k)}\widehat{J}_{j-k},$$

$$\Theta_{36,j} = \sum_{\substack{k\in\mathcal{K}_M(p)\\ j\succcurlyeq k}} \frac{p!}{\pi(k)}X'\widehat{C}'_{z,j-k} + \sum_{\substack{t\in\mathcal{K}_M(p)\\ j\succcurlyeq t}} \widehat{L}'_t\widehat{D}'_{zu,j-t},$$

$$\Theta_{44,j} = \frac{(g+p)!}{\pi(j)}(Y+Y') - \sum_{\substack{k\in\mathcal{K}_M(g)\\ j\succcurlyeq k}} \frac{g!}{\pi(k)}\widehat{H}_{j-k}.$$

The matrix coefficients $\widehat{A}_n$, $\widehat{B}_{w,n}$, $\widehat{B}_{u,n}$, $\widehat{C}_{z,n}$, $\widehat{D}_{zw,n}$, $\widehat{D}_{zu,n}$, $\widehat{C}_{y,n}$ and $\widehat{D}_{yw,n}$, for $n \in \mathcal{K}_M(g)$, associated with the system matrices and the matrix coefficients $\widetilde{P}_t$, $\widehat{P}_t$, $\widetilde{H}_t$, $\widehat{H}_t$, $\widetilde{J}_t$, $\widehat{J}_t$, $\widehat{F}_t$ and $\widehat{L}_t$, for $t \in \mathcal{K}_M(p)$, and $\widehat{Q}_i$, for $i \in \mathcal{K}_M(g+p)$, associated with the optimization variables can be constructed using the linear relation (A2) of Theorem A.1 from Appendix A.2.

Proof

Take any $\gamma \in \Lambda_M$. Following the derivation in Appendix A, multiplying (20) with γ^j and summing for $j \in \mathcal{K}_M(g+p)$ yields (19). Since this LMI condition holds for any $\gamma \in \Lambda_M$, the inequality $\Theta(\gamma(k))>0$ holds for all time instants $k \geqslant 0$. Owing to the linear relation (7) involving $\alpha(k)$, $\alpha(k+1)$ and $\gamma(k)$, feasibility of $\Theta(\gamma(k))>0$ implies feasibility of (14), which, through the nonlinear transformation (13), proves that (12) holds for all time instants $k \geqslant 0$. Consequently, the closed-loop system (10) with system matrices (11) is exponentially stable with an upper bound η on its $\mathcal{H}_\infty$ performance. □

5.2. Gain-scheduled $\mathcal{H}_2$ dynamic output feedback control

Following a similar approach as above, LMI synthesis conditions can be derived for the design of strictly proper full-order gain-scheduled $\mathcal{H}_2$ dynamic output feedback controllers. First, the closed-loop matrices (11) are substituted in the conditions of Theorem 2 to ensure an upper bound on the $\mathcal{H}_2$ performance of the closed-loop system (10). Then, the change of variables (13) is used to linearize the obtained conditions. Finally, following the steps of Section 5.1.2, the modeling of

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the uncertainty domain Γ_b can be exploited to derive the parameter-dependent LMI conditions

$$\Psi(\gamma)=\begin{bmatrix} \widetilde{P}(\gamma) & \widetilde{J}(\gamma) & \widehat{A}(\gamma)X+\widehat{B}_u(\gamma)\widehat{L}(\gamma) & \widehat{A}(\gamma) & \widehat{B}_w(\gamma) \\ \star & \widetilde{H}(\gamma) & \widehat{Q}(\gamma) & Y\widehat{A}(\gamma)+\widehat{F}(\gamma)\widehat{C}_y(\gamma) & Y\widehat{B}_w(\gamma)+\widehat{F}(\gamma)\widehat{D}_{yw}(\gamma) \\ \star & \star & X+X'-\widehat{P}(\gamma) & I+S'-\widehat{J}(\gamma) & 0 \\ \star & \star & \star & Y+Y'-\widehat{H}(\gamma) & 0 \\ \star & \star & \star & \star & I \end{bmatrix}>0, \tag{22a}$$

$$\Phi(\gamma)=\begin{bmatrix} \widehat{W}(\gamma)-\widehat{D}_{zw}(\gamma)\widehat{D}_{zw}(\gamma)' & \widehat{C}_z(\gamma)X+\widehat{D}_{zu}(\gamma)\widehat{L}(\gamma) & \widehat{C}_z(\gamma) \\ \star & X+X'-\widetilde{P}(\gamma) & I+S'-\widehat{J}(\gamma) \\ \star & \star & Y+Y'-\widehat{H}(\gamma) \end{bmatrix}>0, \tag{22b}$$

which need to hold for all $\gamma\in\Lambda_M$. The next theorem provides a finite-dimensional set of sufficient LMI conditions for (22).

Theorem 4 (*Gain-scheduled $\mathscr{H}_2$ dynamic output feedback control design*)
Consider system H, given by (8). Let $p\in\mathbb{N}$ be given. Let matrices $\mathbf{F}$ and $\mathbf{H}$, defined in (7), that characterize the uncertainty domain Γ_b be given. Then, the strictly proper full-order gain-scheduled dynamic output feedback controller (9) with controller matrices (15) stabilizes H with a guaranteed upper bound on the closed-loop $\mathscr{H}_2$ performance

$$\|H_{\rm cl}\|_2^2\leqslant \min_{P_\ell,\,H_\ell,\,J_\ell,\,F_\ell,\,L_\ell,\,Q_k,\,X,\,Y,\,S,\,W_\ell,\,\overline{W}} \mathrm{Tr}\{\overline{W}\},$$

if there exist matrices $F_\ell\in\mathbb{R}^{n_x\times n_y}$, $J_\ell\in\mathbb{R}^{n_x\times n_x}$ and $L_\ell\in\mathbb{R}^{n_u\times n_x}$ and symmetric positive-definite matrices $H_\ell\in\mathbb{R}^{n_x\times n_x}$, $P_\ell\in\mathbb{R}^{n_x\times n_x}$ and $W_\ell\in\mathbb{R}^{n_z\times n_z}$, for $\ell\in\mathscr{K}_N(p)$, and if there exist matrices $Q_k\in\mathbb{R}^{n_x\times n_x}$, for $k\in\mathscr{K}_N(g+p)$, and matrices $S\in\mathbb{R}^{n_x\times n_x}$, $X\in\mathbb{R}^{n_x\times n_x}$ and $Y\in\mathbb{R}^{n_x\times n_x}$ and a symmetric positive-definite matrix $\overline{W}\in\mathbb{R}^{n_z\times n_z}$ such that

$$\Psi_j=\begin{bmatrix} \sum\limits_{\substack{k\in\mathscr{K}_M(g)\\ j\geqslant k}}\frac{g!}{\pi(k)}\widetilde{P}_{j-k} & \sum\limits_{\substack{k\in\mathscr{K}_M(g)\\ j\geqslant k}}\frac{g!}{\pi(k)}\widetilde{J}_{j-k} & \Theta_{13,j} & \sum\limits_{\substack{k\in\mathscr{K}_M(p)\\ j\geqslant k}}\frac{p!}{\pi(k)}\widehat{A}_{j-k} & \sum\limits_{\substack{k\in\mathscr{K}_M(p)\\ j\geqslant k}}\frac{p!}{\pi(k)}\widehat{B}_{w,j-k} \\ \star & \sum\limits_{\substack{k\in\mathscr{K}_M(g)\\ j\geqslant k}}\frac{g!}{\pi(k)}\widetilde{H}_{j-k} & \widehat{Q}_j & \Theta_{24,j} & \Theta_{25,j} \\ \star & \star & \Theta_{33,j} & \Theta_{34,j} & 0 \\ \star & \star & \star & \Theta_{44,j} & 0 \\ \star & \star & \star & \star & \frac{(g+p)!}{\pi(j)}I \end{bmatrix}>0, \tag{23a}$$

$$\Phi_i=\begin{bmatrix} \Phi_{11,i} & \Phi_{12,i} & \sum\limits_{\substack{k\in\mathscr{K}_M(g+p)\\ i\geqslant k}}\frac{(g+p)!}{\pi(k)}\widehat{C}_{z,i-k} \\ \star & \frac{(2g+p)!}{\pi(i)}(X+X')-\sum\limits_{\substack{k\in\mathscr{K}_M(2g)\\ i\geqslant k}}\frac{(2g)!}{\pi(k)}\widehat{P}_{i-k} & \frac{(2g+p)!}{\pi(i)}(I+S')-\sum\limits_{\substack{k\in\mathscr{K}_M(2g)\\ i\geqslant k}}\frac{(2g)!}{\pi(k)}\widehat{J}_{i-k} \\ \star & \star & \frac{(2g+p)!}{\pi(i)}(Y+Y')-\sum\limits_{\substack{k\in\mathscr{K}_M(2g)\\ i\geqslant k}}\frac{(2g)!}{\pi(k)}\widehat{H}_{i-k} \end{bmatrix}>0, \tag{23b}$$

$$\frac{p!}{\pi(\ell)}\overline{W}-W_\ell>0, \tag{23c}$$

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for all $j \in \mathcal{K}_M(g+p)$, $i \in \mathcal{K}_M(2g+p)$ and $\ell \in \mathcal{K}_N(p)$, respectively, where $\Theta_{13,j}$, $\Theta_{24,j}$, $\Theta_{25,j}$, $\Theta_{33,j}$, $\Theta_{34,j}$ and $\Theta_{44,j}$ are given in (21) and

$$\Phi_{11,i} = \sum_{\substack{k \in \mathcal{K}_M(2g) \\ i \succcurlyeq k}} \frac{(2g)!}{\pi(k)} \widehat{W}_{i-k} - \sum_{\substack{n \in \mathcal{K}_M(2g) \\ i \succcurlyeq n}} \frac{p!}{\pi(i-n)} \sum_{\substack{\ell \in \mathcal{K}_M(g) \\ n \succcurlyeq \ell}} \widehat{D}_{zw,\ell} \widehat{D}'_{zw,n-\ell},$$

$$\Phi_{12,i} = \sum_{\substack{k \in \mathcal{K}_M(g+p) \\ i \succcurlyeq k}} \frac{(g+p)!}{\pi(k)} \widehat{C}_{z,i-k} X + \sum_{\substack{j \in \mathcal{K}_M(g+p) \\ i \succcurlyeq j}} \frac{g!}{\pi(i-j)} \sum_{\substack{t \in \mathcal{K}_M(p) \\ j \succcurlyeq t}} \widehat{D}_{zu,j-t} \widehat{L}_t.$$

The coefficients $\widehat{A}_n$, $\widehat{B}_{w,n}$, $\widehat{B}_{u,n}$, $\widehat{C}_{z,n}$, $\widehat{D}_{zw,n}$, $\widehat{D}_{zu,n}$, $\widehat{C}_{y,n}$ and $\widehat{D}_{yw,n}$, for $n \in \mathcal{K}_M(g)$, associated with the system matrices and the matrix coefficients $\widetilde{P}_t$, $\widehat{P}_t$, $\widehat{W}_t$, $\widetilde{H}_t$, $\widehat{H}_t$, $\widetilde{J}_t$, $\widehat{J}_t$, $\widetilde{F}_t$ and $\widehat{L}_t$, for $t \in \mathcal{K}_M(p)$, and $\widehat{Q}_i$, for $i \in \mathcal{K}_M(g+p)$, associated with the optimization variables can be constructed using the linear relation (A2) of Theorem A.1 from Appendix A.2.

The proof can be constructed in an analogous way as the proof of Theorem 3, being thus omitted.

5.3. *Multiobjective gain-scheduled dynamic output feedback control*

The synthesis conditions for gain-scheduled $\mathcal{H}_\infty$ and $\mathcal{H}_2$ dynamic output feedback controllers of Sections 5.1 and 5.2 can be combined to impose different types of performance characterizations on different closed-loop input–output channels, thus yielding multiobjective gain-scheduled dynamic output feedback controllers [40]. To achieve this, selection matrices $S_{\mathbf{I},\mathrm{s}}$ and $S_{\mathbf{O},\mathrm{s}}$ can be used to, respectively, select the proper input and output channel for each performance specification, for $s = 1, \ldots, n_\mathrm{s}$, yielding the different open-loop channels

$$H_{\mathbf{IO},\mathrm{s}} := \begin{cases} x(k+1) = A(\alpha(k))x(k) + B_w(\alpha(k))S_{\mathbf{I},\mathrm{s}}w(k) + B_u(\alpha(k))u(k), \\ \quad z(k) = S_{\mathbf{O},\mathrm{s}}C_z(\alpha(k))x(k) + S_{\mathbf{O},\mathrm{s}}D_{zw}(\alpha(k))S_{\mathbf{I},\mathrm{s}}w(k) + S_{\mathbf{O},\mathrm{s}}D_{zu}(\alpha(k))u(k), \\ \quad y(k) = C_y(\alpha(k))x(k) + D_{yw}(\alpha(k))S_{\mathbf{I},\mathrm{s}}w(k). \end{cases}$$

Afterwards, the synthesis conditions of Theorems 3 and 4 can be applied to these different open-loop channels to obtain the multiobjective gain-scheduled dynamic output feedback controller. Obviously, the same matrices F_ℓ, L_ℓ, Q_k, S, X and Y, associated with the construction of the controller matrices (9), need to be used in the LMI conditions for the different performance specifications. On the other hand, to reduce the conservatism, different matrices P_ℓ, H_ℓ and J_ℓ can be defined for each different performance specification. This can be seen as an extension of the *G shaping paradigm*, presented for LTI systems in [36], to the design of full-order gain-scheduled dynamic output feedback controllers.

6. NUMERICAL RESULTS

This section presents the numerical results using the system data taken from [28], which in turn is based on the vibroacoustic setup used in [43]. The goal of the application is to attenuate the structural noise of a vibroacoustic system whose dynamics is highly sensitive to the ambient temperature. Since the temperature variation is slow, taking physical bounds on this variation into account during the control design can reduce the conservatism typically associated with control synthesis procedures based on quadratic stability that allow arbitrarily fast parameter variation.

Using the State-space Model Interpolation of Local Estimates technique presented in [33, 34], a homogeneous polynomially parameter-dependent 10th-order 2-input 1-output LPV model in the state-space form (8) is obtained. The two inputs, respectively, correspond to the disturbance input w and the control input u. The output y of the model is the sound pressure measured by a microphone.

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For all designs, the microphone signal y is used as both measured as well as performance output. For details on the setup, see [43]. For this system, the polynomial degree is chosen to be $g=1$ which implies that the resulting LPV model has in fact a polytopic parameter-dependency on α.

In the following, first $\mathscr{H}_\infty$ and $\mathscr{H}_2$ gain-scheduled dynamic output feedback controllers are designed using Theorems 3 and 4. Afterwards, multiobjective control design is considered. The resulting multiobjective controllers are verified through numerical simulation.

6.1. $\mathscr{H}_\infty$ and $\mathscr{H}_2$ dynamic output feedback design

The synthesis conditions of Theorem 3 and 4 are used to compute gain-scheduled dynamic output feedback controllers that guarantee an upper bound on the closed-loop $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performance from the disturbance w to the performance output y of the vibroacoustic setup (indicated as $\|T_{yw}\|_\infty$ and $\|T_{yw}\|_2$). To assess the impact of the bound b on the rate of parameter variation, controllers are designed for 101 equidistant values of b in the interval $[0, 1]$, that is, $b \in \{0, 0.01, 0.02, \ldots, 1\}$. The polynomial degree of the Lyapunov matrices is chosen to be $p=1$. The influence of the polynomial degree p on the achieved performance and on the computational time is analyzed in Section 6.3. For the $\mathscr{H}_\infty$ control design, $n_r=1241$ (number of LMI rows) and $n_v=922$ (number of scalar variables), whereas for the $\mathscr{H}_2$ control design $n_v=1243$ and $n_r=2082$. All problems are modeled in `Yalmip` [44] and solved using `SeDuMi` [45] within the `Matlab` environment.

The results of the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control designs are presented in Figure 1, which shows the obtained $\mathscr{H}_\infty$ upper bound η on $\|T_{yw}\|_\infty$ (Figure 1(a)) and the obtained $\mathscr{H}_2$ upper bound $\sqrt{\mathrm{Tr}\{\overline{W}\}}$ on $\|T_{yw}\|_2$ (Figure 1(b)) as a function of the bound b on the rate of parameter variation. Thin black solid lines indicate dynamic output feedback design. For comparison, the upper bound on the open-loop performance (thick dash-dotted), and the guaranteed closed-loop performance obtained using gain-scheduled state feedback (thick dashed), both computed using the LMI conditions of [28], are shown as well. It is clear that both the gain-scheduled state feedback and dynamic output feedback controllers dramatically outperform the open-loop system. Furthermore, it can be concluded for both the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control designs that the dynamic output feedback controllers achieve worse performance than the state feedback controllers. However, for the $\mathscr{H}_\infty$ control designs (see Figure 1(a)), the difference is small, as can be seen from the fact that the thick dashed line and thin black solid line almost coincide. For the $\mathscr{H}_2$ design, on the other hand, the difference is more pronounced. Keep in mind, though, that the dynamic controller is based on the measurement of the single output y only, whereas the state feedback controller has access to all 10 states. In practice, it is usually not possible to measure all states of the system, which makes the implementation of a state feedback controller infeasible.

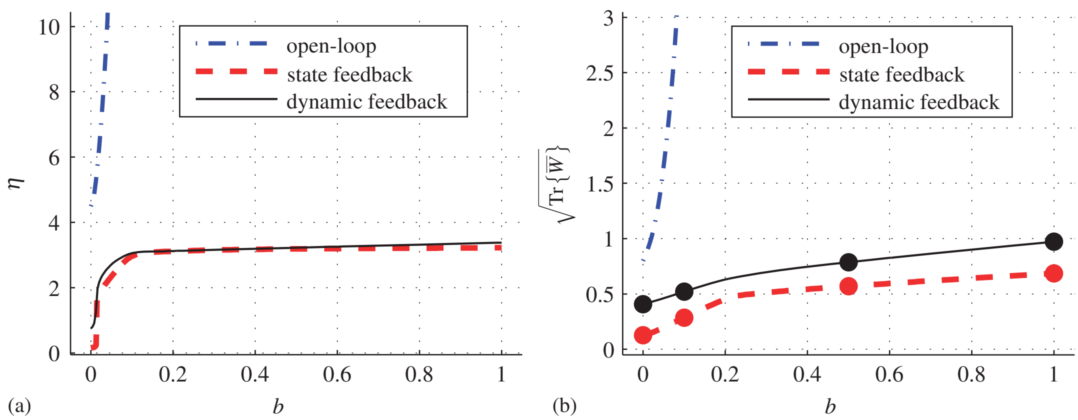

Figure 1. Comparison between the upper bound on the open-loop performance (dash-dotted) and the guaranteed upper bound on the closed-loop performance of the vibroacoustic setup obtained with state feedback (dashed) and dynamic (solid) output feedback using $p=1$, $d=0$: (a) $\mathscr{H}_\infty$ design and (b) $\mathscr{H}_2$ design.

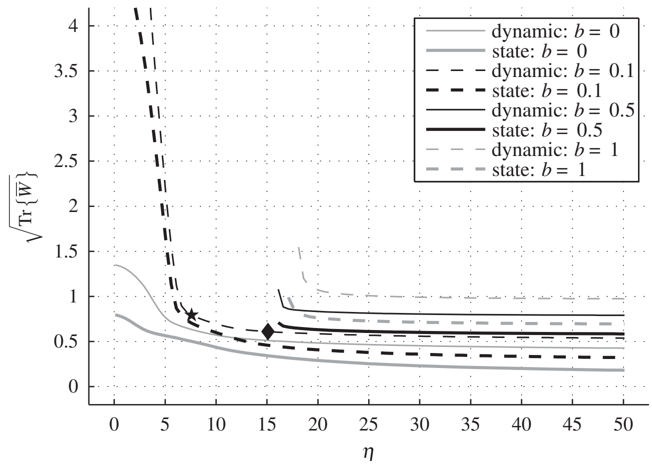

Figure 2. Trade-off between the imposed upper bound η on $\|T_{uw}\|_\infty$ and the guaranteed upper bound $\sqrt{\text{Tr}\{\overline{W}\}}$ on $\|T_{yw}\|_2$. Comparison between gain-scheduled state feedback (thick) and dynamic output feedback (thin).

6.2. *Multiobjective dynamic output feedback design*

For the multiobjective $\mathscr{H}_2/\mathscr{H}_\infty$ control design, the aim is to minimize an upper bound $\sqrt{\text{Tr}\{\overline{W}\}}$ on $\|T_{yw}\|_2$, while an upper bound η is imposed on the closed-loop $\mathscr{H}_\infty$ performance from the disturbance w to the control signal u (indicated as $\|T_{uw}\|_\infty$). This strategy is used to obtain realistic controllers that do not have excessively large control signals. Four bounds on the rate of parameter variation are considered: $b\in\{0, 0.1, 0.5, 1\}$ and the upper bound η on $\|T_{yw}\|_\infty$ takes 51 equidistant values in the interval $[0.1, 50.1]$. As above, the polynomial degree of the Lyapunov matrix is chosen as $p=1$.

Figure 2 shows the obtained trade-off between the prescribed $\mathscr{H}_\infty$ bound η and the obtained $\mathscr{H}_2$ bound $\sqrt{\text{Tr}\{\overline{W}\}}$. Thin lines indicate the dynamic output feedback control designs, whereas thick lines indicate state feedback control designs. The four cases for the bound b are indicated as follows: $b=0$ with gray solid lines, $b=0.1$ with black dashed lines, $b=0.5$ with black solid lines and $b=1$ with gray dashed lines. Several conclusions can be drawn. First, it is clear that, similar to the results presented in Figure 1, higher values of b result in a decrease in the guaranteed $\mathscr{H}_2$ closed-loop performance (manifested as an increase in $\sqrt{\text{Tr}\{\overline{W}\}}$). Second, tighter bounds on $\|T_{uw}\|_\infty$ yield a decrease in guaranteed closed-loop $\mathscr{H}_2$ performance as well. Third, like in Figure 1, it is clear that the state feedback controllers outperform the dynamic output feedback controllers. Fourth, it seems that for high values of η, all curves tend to a constant value. For each value of b, this value can be found in Figure 1(b) (indicated with circles), which shows the best guaranteed $\mathscr{H}_2$ performance that can be achieved since no bound on $\|T_{uw}\|_\infty$ is imposed.

6.3. *Influence of the polynomial degree p*

To check the influence of the polynomial degree p on the achieved performance and the numerical burden, the multiobjective control design for $b=0.1$ with a bound $\|T_{uw}\|_\infty\leqslant 7.6$ is repeated for $p\in\{1, 2, 3, 4\}$. For $p=0$, the set of LMIs is infeasible and no dynamic output feedback controller can be designed based on quadratic stability of the closed-loop system. Table I shows the results. It is clear that the closed-loop performance improves significantly for higher values of p, as can be seen from column two, which shows a decrease in the obtained upper bound $\sqrt{\text{Tr}\{\overline{W}\}}$, and column three, which shows the relative difference ε between the performance obtained for higher p and the performance obtained for $p=1$. This, however, comes at the price of an important increase in numerical complexity as can be concluded from the increased number of variables n_v, number of LMI rows n_r and solver time (indicated by `SeDuMi`). For $p=1$, the synthesis conditions

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Table I. Numerical complexity associated with the multiobjective control design for $b=0.1$, $\|T_{uw}\|_\infty \leqslant 7.6$ and $p\in\{1,2,3,4\}$. No feasible solution has been found for $p=0$.

p	$\sqrt{\text{Tr}\{\overline{W}\}}$	ε (%)	n_{V}	n_{r}	Calculation time (s)
1	0.7926	/	1784	3004	226.28
2	0.7510	5.2527	2425	7421	799.66
3	0.7293	7.9982	3066	15919	2313.17
4	0.7207	9.0753	3707	30829	6086.82

already involve a high numerical burden. However, it should be noticed that this computational time is purely off-line. Now, the only online computational time involved is the time required to compute the system matrices for the dynamic controller using formula (12). In this equation, the matrix operations that do not depend on α can be performed off-line and the operations on the parameter-dependent matrices basically amount to the computation of polynomial functions and of ordinary matrix products, which are fast to perform. Therefore, the proposed gain-scheduled controller can be implemented in a real-time application.

7. CONCLUSIONS

New LMI conditions are presented for the synthesis of strictly proper full-order gain-scheduled multiobjective $\mathscr{H}_2/\mathscr{H}_\infty$ dynamic output feedback controllers for discrete-time homogeneous polynomially parameter-dependent linear parameter-varying systems based on polynomially parameter-dependent Lyapunov functions. The synthesis procedures explicitly take an *a priori* known bound on the rate of parameter variation into account, thus reducing the conservatism generally associated with methods that allow arbitrarily fast parameter variation.

Numerical results presented for a vibroacoustic application show the potential of the proposed control design. As the rate of temperature variation is obviously bounded in practice, the application of synthesis procedures that consider bounds on rate of variation is particularly interesting in this control problem. For a specified bound on the rate of parameter variation, a specific multiobjective controller can be designed such that a variety of closed-loop performance specifications are met, which allows to compute trade-off curves between various conflicting design criteria.

APPENDIX A

A.1. Finite-dimensional set of LMIS

The aim of this appendix is to derive finite-dimensional sets of sufficient LMI conditions that guarantee the parameter-dependent LMIs (19) and (22). To this end, it is necessary to introduce two calculus results for homogeneous polynomials.

A.1.1. Preliminaries. First, the product of two homogeneous polynomials $R(\alpha)$ and $Q(\alpha)$ of degree f and g, respectively, can be computed as follows:

$$R(\alpha)Q(\alpha)=\left(\sum_{i\in\mathscr{K}_N(f)}\alpha^i R_i\right)\left(\sum_{j\in\mathscr{K}_N(g)}\alpha^j Q_j\right)=\sum_{i\in\mathscr{K}_N(f)}\sum_{j\in\mathscr{K}_N(g)}\alpha^{i+j}R_iQ_j.$$

Defining $k=i+j$ yields $i=k-j$, whenever $k\succcurlyeq j$, such that it is possible to rewrite the product $R(\alpha)Q(\alpha)$ as the following homogeneous polynomial of degree $f+g$:

$$R(\alpha)Q(\alpha)=\sum_{k\in\mathscr{K}_N(f+g)}\alpha^k\sum_{\substack{j\in\mathscr{K}_N(g)\\ k\succcurlyeq j}}R_{k-j}Q_j. \tag{A1}$$

Second, throughout this appendix, homogenization is used as a generic tool.

Definition A.1 (*Homogenization*)
For $\alpha\in\Lambda_N$ and a given monomial $m(\alpha)$ of degree d, the homogenization of degree g of $m(\alpha)$ is obtained by multiplying $m(\alpha)$ with

$$1=\left(\sum_{j=1}^{N}\alpha_j\right)^g,$$

which, using an extension of the well-known binomial expansion, is equal to the homogeneous polynomial

$$1=\left(\sum_{j=1}^{N}\alpha_j\right)^g=\sum_{\ell\in\mathscr{K}_N(g)}\frac{g!}{\pi(\ell)}\alpha^\ell.$$

A.1.2. Derivation of Theorem 3. The aim of this section is to derive a finite-dimensional set of LMI conditions that guarantees that the parameter-dependent LMI (19) holds. The left-hand side $\Theta(\gamma)$ of this LMI is a polynomial matrix of degree $g+p$. Therefore, all blocks of $\Theta(\gamma)$ need to be written as homogeneous polynomially parameter-dependent matrices of degree $g+p$. In this appendix, this derivation is shown for each block of $\Theta(\gamma)$ using the two calculus results introduced above.

Derivation for blocks $(1,1)$, $(1,2)$ *and* $(2,2)$. Block $(1,1)$ of $\Theta(\gamma)$, $\widetilde{P}(\gamma)$, is of degree p and therefore a homogenization of degree g is necessary. This yields

$$\left(\sum_{i=1}^{M}\gamma_i\right)^g\widetilde{P}(\gamma)=\left(\sum_{k\in\mathscr{K}_M(g)}\frac{g!}{\pi(k)}\gamma^k\right)\left(\sum_{t\in\mathscr{K}_M(p)}\gamma^t\widetilde{P}_t\right)=\sum_{j\in\mathscr{K}_M(g+p)}\gamma^j\sum_{\substack{k\in\mathscr{K}_M(g)\\ j\succcurlyeq k}}\frac{g!}{\pi(k)}\widetilde{P}_{j-k}.$$

The derivation for blocks $(1,2)$ and $(2,2)$ is analogous.

Derivation for blocks $(3,3)$, $(3,4)$ *and* $(4,4)$. Block $(3,3)$ of $\Theta(\gamma)$, $X+X'-\widehat{P}(\gamma)$, consists of two parts. The first part, $X+X'$, is constant and therefore a homogenization of degree $g+p$ is necessary. This yields

$$\left(\sum_{i=1}^{M}\gamma_i\right)^{g+p}(X+X')=\sum_{j\in\mathscr{K}_M(g+p)}\gamma^j\frac{(g+p)!}{\pi(j)}(X+X').$$

The second block, $\widehat{P}(\gamma)$, is of degree p and thus, a homogenization of degree g is necessary. Following the same steps as for block $(1,1)$, this gives

$$\left(\sum_{i=1}^{M}\gamma_i\right)^g\widehat{P}(\gamma)=\sum_{j\in\mathscr{K}_M(g+p)}\gamma^j\sum_{\substack{k\in\mathscr{K}_M(g)\\ j\succcurlyeq k}}\frac{g!}{\pi(k)}\widehat{P}_{j-k}.$$

Combining both results yields

$$X+X'-\widehat{P}(\gamma)=\sum_{j\in\mathscr{K}_M(g+p)}\gamma^j\left(\frac{(g+p)!}{\pi(j)}(X+X')-\sum_{\substack{k\in\mathscr{K}_M(g)\\ j\succcurlyeq k}}\frac{g!}{\pi(k)}\widehat{P}_{j-k}\right).$$

The derivation for blocks $(3,4)$ and $(4,4)$ is analogous.

Derivation for blocks $(1,4)$, $(1,5)$, $(4,6)$ *and* $(5,6)$. Block $(1,4)$ of $\Theta(\gamma)$, $\widehat{A}(\gamma)$, is of degree g and therefore a homogenization of degree p is necessary. This yields

$$\left(\sum_{i=1}^{M}\gamma_i\right)^p\widehat{A}(\gamma)=\left(\sum_{k\in\mathscr{K}_M(p)}\frac{p!}{\pi(k)}\gamma^k\right)\left(\sum_{\ell\in\mathscr{K}_M(g)}\gamma^\ell\widehat{A}_\ell\right)=\sum_{j\in\mathscr{K}_M(g+p)}\gamma^j\sum_{\substack{k\in\mathscr{K}_M(p)\\ j\succcurlyeq k}}\frac{p!}{\pi(k)}\widehat{A}_{j-k}.$$

The derivation for blocks $(1,5)$, $(4,6)$ and $(5,6)$ is analogous.

Derivation for blocks $(1,3)$, $(2,4)$, $(2,5)$ *and* $(3,6)$. Block $(1,3)$ of $\Theta(\gamma)$, $\widehat{A}(\gamma)X+\widehat{B}_u(\gamma)\widehat{L}(\gamma)$, consists of two parts. The homogenization of the first part $\widehat{A}(\gamma)X$ can be constructed by multiplying the homogenization of block $(1,4)$ with X on the right. The second part $\widehat{B}_u(\gamma)\widehat{L}(\gamma)$ is of degree $g+p$ and therefore no homogenization is necessary:

$$\widehat{B}_u(\gamma)\widehat{L}(\gamma)=\sum_{j\in\mathscr{K}_M(g+p)}\gamma^j\sum_{\substack{t\in\mathscr{K}_M(p)\\ j\succcurlyeq t}}\widehat{B}_{u,j-t}\widehat{L}_t.$$

Combining both results yields

$$\widehat{A}(\gamma)X+\widehat{B}_u(\gamma)\widehat{L}(\gamma)=\sum_{j\in\mathscr{K}_M(g+p)}\gamma^j\left(\sum_{\substack{k\in\mathscr{K}_M(p)\\ j\succcurlyeq k}}\frac{p!}{\pi(k)}\widehat{A}_{j-k}X+\sum_{\substack{t\in\mathscr{K}_M(p)\\ j\succcurlyeq t}}\widehat{B}_{u,j-t}\widehat{L}_t\right).$$

The derivation for blocks $(2,4)$, $(2,5)$ and $(3,6)$ is analogous.

Derivation for blocks $(5,5)$ *and* $(6,6)$. Blocks $(5,5)$ and $(6,6)$ of $\Theta(\gamma)$, given by ηI, are constant and therefore a homogenization of degree $g+p$ is necessary. This yields

$$\left(\sum_{i=1}^{M}\gamma_i\right)^{g+p}\eta I=\sum_{j\in\mathscr{K}_M(g+p)}\gamma^j\frac{(g+p)!}{\pi(j)}\eta I.$$

A.1.3. Derivation of Theorem 4. The aim of this section is to derive a finite-dimensional set of LMI conditions that guarantees that the parameter-dependent LMIs (22a) and (22b) hold. Since the left-hand side $\Psi(\gamma)$ of (22a) is a diagonal block of the left-hand side $\Theta(\gamma)$ of (19), the first set of LMIs (23a) can be readily obtained as a diagonal block from the LMIs (20). The left-hand side $\Phi(\gamma)$ of LMI (22b) is a polynomial matrix of degree $2g+p$. Therefore, all blocks of $\Phi(\gamma)$ need to be written as homogeneous polynomially parameter-dependent matrices of degree $2g+p$. This derivation is discussed next.

Derivation for block $(1,1)$. Block $(1,1)$, given by $\widehat{W}(\gamma)-\widehat{D}_{zw}(\gamma)\widehat{D}_{zw}(\gamma)'$, consists of two terms. The term $\widehat{W}(\gamma)$ is of degree p, whereas the term $\widehat{D}_{zw}(\gamma)\widehat{D}_{zw}(\gamma)'$ is of degree $2g$. Therefore, a homogenization is necessary to make both terms of degree $2g+p$. For the first term, this yields

$$\left(\sum_{i=1}^{M}\gamma_i\right)^{2g}\widehat{W}(\gamma)=\left(\sum_{k\in\mathscr{K}_M(2g)}\frac{(2g)!}{\pi(k)}\gamma^k\right)\left(\sum_{\ell\in\mathscr{K}_M(p)}\gamma^\ell\widehat{W}_\ell\right)=\sum_{i\in\mathscr{K}_M(2g+p)}\gamma^i\sum_{\substack{k\in\mathscr{K}_M(2g)\\ i\succcurlyeq k}}\frac{(2g)!}{\pi(k)}\widehat{W}_{i-k}.$$

Likewise, for the second term, this yields

$$\begin{aligned}\left(\sum_{i=1}^{M}\gamma_i\right)^{p}\widehat{D}_{zw}(\gamma)\widehat{D}_{zw}(\gamma)'&=\left(\sum_{k\in\mathscr{K}_M(p)}\frac{p!}{\pi(k)}\gamma^k\right)\left(\sum_{\ell\in\mathscr{K}_M(g)}\gamma^\ell\widehat{D}_{zw,\ell}\right)\left(\sum_{m\in\mathscr{K}_M(g)}\gamma^m\widehat{D}_{zw,m}\right)'\\&=\sum_{i\in\mathscr{K}_M(2g+p)}\gamma^i\sum_{\substack{n\in\mathscr{K}_M(2g)\\ i\succcurlyeq n}}\frac{p!}{\pi(i-n)}\sum_{\substack{\ell\in\mathscr{K}_M(g)\\ n\succcurlyeq\ell}}\widehat{D}_{zw,\ell}\widehat{D}'_{zw,n-\ell}.\end{aligned}$$

Combining both results gives

$$\begin{aligned}\widehat{W}(\gamma)-\widehat{D}_{zw}(\gamma)\widehat{D}_{zw}(\gamma)'=&\sum_{i\in\mathscr{K}_M(2g+p)}\gamma^i\\&\times\left(\sum_{\substack{k\in\mathscr{K}_M(2g)\\ i\succcurlyeq k}}\frac{(2g)!}{\pi(k)}\widehat{W}_{i-k}-\sum_{\substack{n\in\mathscr{K}_M(2g)\\ i\succcurlyeq n}}\frac{p!}{\pi(i-n)}\sum_{\substack{\ell\in\mathscr{K}_M(g)\\ n\succcurlyeq\ell}}\widehat{D}_{zw,\ell}\widehat{D}'_{zw,n-\ell}\right).\end{aligned}$$

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Derivation for block $(1,3)$. Block $(1,3)$, given by $\widehat{C}_z(\gamma)$, is of degree g and therefore a homogenization of degree $g+p$ is necessary. This yields

$$\left(\sum_{i=1}^{M}\gamma_i\right)^{g+p}\widehat{C}_z(\gamma)=\left(\sum_{k\in\mathscr{K}_M(g+p)}\frac{(g+p)!}{\pi(k)}\gamma^k\right)\left(\sum_{\ell\in\mathscr{K}_M(g)}\gamma^\ell\widehat{C}_{z,\ell}\right)$$

$$=\sum_{i\in\mathscr{K}_M(2g+p)}\gamma^i\sum_{\substack{k\in\mathscr{K}_M(g+p)\\ i\succcurlyeq k}}\frac{(g+p)!}{\pi(k)}\widehat{C}_{z,i-k}.$$

Derivation for block $(1,2)$. Block $(1,2)$, given by $\widehat{C}_z(\gamma)X+\widehat{D}_{zu}(\gamma)\widehat{L}(\gamma)$, consists of two parts. The homogenization of the first part can be constructed by multiplying the homogenization of block $(1,3)$ with X on the right. The second part is of degree $g+p$ and therefore a homogenization of degree g is necessary. This yields

$$\left(\sum_{i=1}^{M}\gamma_i\right)^{g}\widehat{D}_{zu}(\gamma)\widehat{L}(\gamma)=\left(\sum_{k\in\mathscr{K}_M(g)}\frac{g!}{\pi(k)}\gamma^k\right)\left(\sum_{\ell\in\mathscr{K}_M(g)}\gamma^\ell\widehat{D}_{zu,\ell}\right)\left(\sum_{t\in\mathscr{K}_M(p)}\gamma^t\widehat{L}_t\right)$$

$$=\sum_{i\in\mathscr{K}_M(2g+p)}\gamma^i\sum_{\substack{j\in\mathscr{K}_M(g+p)\\ i\succcurlyeq j}}\frac{g!}{\pi(i-j)}\sum_{\substack{t\in\mathscr{K}_M(p)\\ j\succcurlyeq t}}\widehat{D}_{zu,j-t}\widehat{L}_t.$$

Combining both results gives

$$\widehat{C}_z(\gamma)X+\widehat{D}_{zu}(\gamma)\widehat{L}(\gamma)=\sum_{i\in\mathscr{K}_M(2g+p)}\gamma^i$$

$$\times\left(\sum_{\substack{k\in\mathscr{K}_M(g+p)\\ i\succcurlyeq k}}\frac{(g+p)!}{\pi(k)}\widehat{C}_{z,i-k}X+\sum_{\substack{j\in\mathscr{K}_M(g+p)\\ i\succcurlyeq j}}\frac{g!}{\pi(i-j)}\sum_{\substack{t\in\mathscr{K}_M(p)\\ j\succcurlyeq t}}\widehat{D}_{zu,j-t}\widehat{L}_t\right).$$

Derivation for blocks $(2,2)$, $(2,3)$ *and* $(3,3)$. Block $(2,2)$, given by $X+X'-\widehat{P}(\gamma)$, consists of two parts. The first part is constant and therefore a homogenization of degree $2g+p$ is necessary. This yields

$$\left(\sum_{i=1}^{M}\gamma_i\right)^{2g+p}(X+X')=\sum_{i\in\mathscr{K}_M(2g+p)}\gamma^i\frac{(2g+p)!}{\pi(i)}(X+X').$$

The second block is of degree p and therefore a homogenization of degree $2g$ is necessary. This gives

$$\left(\sum_{i=1}^{M}\gamma_i\right)^{2g}\widehat{P}(\gamma)=\sum_{i\in\mathscr{K}_M(2g+p)}\gamma^i\sum_{\substack{k\in\mathscr{K}_M(2g)\\ i\succcurlyeq k}}\frac{(2g)!}{\pi(k)}\widehat{P}_{i-k}.$$

Combining both results yields

$$X+X'-\widehat{P}(\gamma)=\sum_{i\in\mathscr{K}_M(2g+p)}\gamma^i\left(\frac{(2g+p)!}{\pi(i)}(X+X')-\sum_{\substack{k\in\mathscr{K}_M(2g)\\ i\succcurlyeq k}}\frac{(2g)!}{\pi(k)}\widehat{P}_{i-k}\right).$$

The derivation for blocks $(2,3)$ and $(3,3)$ is analogous.

Derivation for LMI (23c). The set of LMIs given by (23c) is a sufficient condition for $\overline{W}>W(\alpha)$. Since $W(\alpha)$ is a homogeneous polynomial of degree p, a homogenization of the constant left-hand side $\overline{W}$ of degree p is necessary. This yields

$$\left(\sum_{i=1}^{N}\alpha_i\right)^{p}\overline{W}=\sum_{\ell\in\mathscr{K}_N(p)}\alpha^\ell\frac{p!}{\pi(\ell)}>W(\alpha)=\sum_{\ell\in\mathscr{K}_N(p)}\alpha^\ell W_\ell.$$

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A.2. Change of variables

In this appendix, a change of variables is presented for homogeneous polynomials. It is shown that if there exists a linear relation $\alpha = \mathbf{F}\gamma$ between two variables $\alpha \in \Lambda_N$ and $\gamma \in \Lambda_M$, then for any homogeneous polynomial of degree g in α, a homogeneous polynomial of degree g in γ can be constructed such that both homogeneous polynomials are identical under the restriction $\alpha = \mathbf{F}\gamma$.

Theorem A.1 (*Change of variables for homogeneous polynomials*)
Given a homogeneous polynomial $A(\alpha)$ of degree g in the variable $\alpha \in \Lambda_N$,

$$A(\alpha) = \sum_{\ell \in \mathscr{K}_N(g)} \alpha^\ell A_\ell.$$

Suppose that there exists the following linear relation $\alpha = \mathbf{F}\gamma$, with $\mathbf{F} \in \mathbb{R}^{N\times M}$, between $\alpha \in \Lambda_N$ and $\gamma \in \Lambda_M$. Then, there exists a homogeneous polynomial

$$\widehat{A}(\gamma) = \sum_{t \in \mathscr{K}_M(g)} \gamma^t \widehat{A}_t,$$

of degree g, such that $A(\alpha) \equiv A(F\gamma) \equiv \widehat{A}(\gamma)$. Moreover, the coefficients $\widehat{A}_t$ of $\widehat{A}(\gamma)$ can be constructed from the coefficients A_ℓ of $A(\alpha)$, using the following linear combination:

$$\widehat{A}_t = \sum_{\ell \in \mathscr{K}_N(g)} \sum_{\substack{k \in \mathscr{K}_{\mathbf{M}_N}(\ell) \\ \sum_{j=1}^N k_j = t}} \frac{\pi(\ell)}{\pi(k)} \left(\prod_{j=1}^N \prod_{i=1}^M \mathbf{F}_{(j,i)}^{k_{j,i}} \right) A_\ell, \tag{A2}$$

where the notation

$$\sum_{\substack{k \in \mathscr{K}_{\mathbf{M}_N}(\ell) \\ \sum_{j=1}^N k_j = t}}$$

implies that in this summation over $k \in \mathscr{K}_{\mathbf{M}_N}(\ell)$, only those terms should be considered for which $t = \sum_{j=1}^N k_j$.

Proof
The proof follows by construction. First, denote the columns and the rows of matrix $\mathbf{F} \in \mathbb{R}^{N\times M}$ as $\mathbf{F}_{(:,i)} \in \mathbb{R}^{N\times 1}$, for $i = 1, \ldots, M$, and $\mathbf{F}_{(j,:)} \in \mathbb{R}^{1\times M}$, for $j = 1, \ldots, N$, respectively. Thus, it is clear that

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \mathbf{F}\gamma = [\mathbf{F}_{(:,1)} \ \ldots \ \mathbf{F}_{(:,M)}] \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_M \end{pmatrix} = \begin{bmatrix} \mathbf{F}_{(1,:)} \\ \vdots \\ \mathbf{F}_{(N,:)} \end{bmatrix} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_M \end{pmatrix} \quad \text{and} \quad \alpha_j = \sum_{i=1}^M \mathbf{F}_{(j,i)} \gamma_i$$

for $j = 1, \ldots, N$.

Using this linear relation between $\alpha \in \Lambda_N$ and $\gamma \in \Lambda_M$, it can be seen that

$$A(\alpha) = \sum_{\ell \in \mathscr{K}_N(g)} \alpha^\ell A_\ell = \sum_{\ell \in \mathscr{K}_N(g)} \left(\prod_{j=1}^N \alpha_j^{\ell_j} \right) A_\ell = \sum_{\ell \in \mathscr{K}_N(g)} \left(\prod_{j=1}^N \left(\sum_{i=1}^M \mathbf{F}_{(j,i)} \gamma_i \right)^{\ell_j} \right) A_\ell,$$

and, using an extension of the well-known binomial expansion, it follows that

$$A(\alpha) = \sum_{\ell \in \mathcal{K}_N(g)} \left(\prod_{j=1}^{N} \left(\sum_{k_j \in \mathcal{K}_M(\ell_j)} \frac{\ell_j!}{\pi(k_j)} \mathbf{F}_{(j,:)}^{k_j} \gamma^{k_j} \right) \right) A_\ell$$

$$= \sum_{\ell \in \mathcal{K}_N(g)} \sum_{k_1 \in \mathcal{K}_M(\ell_1)} \sum_{k_2 \in \mathcal{K}_M(\ell_2)} \cdots \sum_{k_N \in \mathcal{K}_M(\ell_N)} \frac{\ell_1!}{\pi(k_1)} \mathbf{F}_{(1,:)}^{k_1} \gamma^{k_1} \frac{\ell_2!}{\pi(k_2)} \mathbf{F}_{(2,:)}^{k_2} \gamma^{k_2} \cdots \frac{\ell_N!}{\pi(k_N)} \mathbf{F}_{(N,:)}^{k_N} \gamma^{k_N} A_\ell$$

$$= \sum_{\ell \in \mathcal{K}_N(g)} \sum_{k_1 \in \mathcal{K}_M(\ell_1)} \sum_{k_2 \in \mathcal{K}_M(\ell_2)} \cdots \sum_{k_N \in \mathcal{K}_M(\ell_N)} \frac{\pi(\ell)}{\prod_{j=1}^{N} \pi(k_j)} \left(\prod_{j=1}^{N} \prod_{i=1}^{M} \mathbf{F}_{(j,i)}^{k_{j,i}} \right) \gamma^{\sum_{j=1}^{N} k_j} A_\ell.$$

Now, defining the vector $\mathbf{M}_N = (M, M, \ldots, M) \in \mathbb{N}^N$ and denoting the Cartesian product of the sets $\mathcal{K}_M(\ell_j)$, for $j = 1, \ldots, N$ as

$$\mathcal{K}_{\mathbf{M}_N}(\ell) = \mathcal{K}_M(\ell_1) \times \mathcal{K}_M(\ell_2) \times \cdots \times \mathcal{K}_M(\ell_N),$$

the $\mathbf{M}_N$-tuple k can be defined as $k = (k_1, k_2, \ldots, k_N) \in \mathcal{K}_{\mathbf{M}_N}(\ell)$. Then, $\prod_{j=1}^{N} \pi(k_j)$ can be written in shorthand as $\prod_{j=1}^{N} \pi(k_j) = \pi(k)$ and it follows that $A(\alpha)$ can be written as

$$A(\alpha) = \sum_{\ell \in \mathcal{K}_N(g)} \sum_{k \in \mathcal{K}_{\mathbf{M}_N}(\ell)} \frac{\pi(\ell)}{\pi(k)} \left(\prod_{j=1}^{N} \prod_{i=1}^{M} \mathbf{F}_{(j,i)}^{k_{j,i}} \right) \gamma^{\sum_{j=1}^{N} k_j} A_\ell.$$

Defining

$$\sum_{j=1}^{N} k_j = t \quad \text{such that } t \in \mathcal{K}_M\left(\sum_{j=1}^{N} \ell_j \right) = \mathcal{K}_M(g)$$

yields

$$A(\alpha) = \sum_{t \in \mathcal{K}_M(g)} \gamma^t \sum_{\ell \in \mathcal{K}_N(g)} \sum_{\substack{k \in \mathcal{K}_{\mathbf{M}_N}(\ell) \\ \sum_{j=1}^{N} k_j = t}} \frac{\pi(\ell)}{\pi(k)} \left(\prod_{j=1}^{N} \prod_{i=1}^{M} \mathbf{F}_{(j,i)}^{k_{j,i}} \right) A_\ell = \sum_{t \in \mathcal{K}_M(g)} \gamma^t \widehat{A}_t = \widehat{A}(\gamma),$$

with $\widehat{A}_t$ given by (A2). □

Note that since the coefficients A_ℓ of the homogeneous polynomial can be both scalars and matrices, the theorem holds for homogeneous polynomials as well as homogeneous polynomially parameter-dependent matrices.

Example A.1

Consider, as an example, a homogeneous polynomially parameter-dependent matrix of degree $g = 1$, that is, matrix $A(\alpha)$ is a polytopic matrix, given by

$$A(\alpha) = \sum_{\ell=1}^{N} \alpha_\ell A_\ell.$$

In this case, matrix $\widehat{A}(\gamma)$ has a polytopic structure as well and it can be shown that the generic change of variables of Theorem A.1 becomes

$$\widehat{A}(\gamma) = \sum_{t=1}^{M} \gamma_t \sum_{\ell=1}^{N} \mathbf{F}_{(\ell,t)} A_\ell.$$

This special case for the change of variables has been presented in [28].

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