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GALERKIN'S PROCEDURE FOR NONLINEAR PERIODIC SYSTEMS

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SOMMAIRE

L'auteur se propose de déterminer si un système périodique non-linéaire du type vectoriel

$$\frac{dx}{dt} = X(x, t)$$

peut toujours être traité avec succès par la méthode approchée de Galerkin, c'est-à-dire en admettant que la solution d'ordre m est représentée par un polynome trigonométrique à m termes de la forme $a_n \cos nt + b_n \sin nt$, dont les coefficients sont déterminés de manière à satisfaire une certaine équation moyennée. Les conditions de validité de cette étude sont moins larges que celles envisagées par Cesari ; en revanche, les résultats sont susceptibles d'une application pratique immédiate. On suppose que X et ses dérivées par rapport à x sont continûment différentiables en x et en t, et que la solution périodique considérée est "isolée", c'est-à-dire qu'il n'existe pas d'autres solutions périodiques voisines.

On démontre que, dans ces conditions, l'existence d'une solution périodique isolée entraîne celle d'une approximation de Galerkin d'ordre aussi élevé qu'on le désire, et qu'un certain opérateur linéaire lié à la matrice Jacobienne de X reste alors borné ; et que si cet opérateur est borné, l'existence d'une approximation de Galerkin d'ordre aussi élevé qu'on veut entraîne celle d'une solution périodique isolée.

Des exemples numériques sont donnés.

§0. INTRODUCTION.

Given a real nonlinear periodic system

$$\frac{dx}{dt} = X(x, t), \tag{0.1}$$

where x and $X(x, t)$ are the vectors of the same dimension and $X(x, t)$ is

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periodic in t with period 2π. To seek a periodic solution of (0.1), first, we take a trigonometric polynomial

$$x_m(t) = a_o + \sqrt{2} \sum_{n=1}^{m} (a_n \cos nt + b_n \sin nt) \tag{0.2}$$

with undetermined coefficients $(a_o, a_1, b_1, \ldots, a_m, b_m)$ and, next, we determine these coefficients so that $x_m(t)$ may satisfy the equation

$$\begin{aligned} \frac{dx_m(t)}{dt} &= \frac{1}{2\pi} \int_0^{2\pi} X[x_m(s), s]\, ds \\ &+ \frac{1}{\pi} \sum_{n=1}^{m} \Big\{ \cos nt \int_0^{2\pi} X[x_m(s), s] \cos ns\, ds \\ &+ \sin nt \int_0^{2\pi} X[x_m(s), s] \sin ns\, ds \Big\}. \end{aligned} \tag{0.3}$$

Then it is guessed that the trigonometric polynomial $x_m(t)$ determined in the above way will be a periodic approximate solution of the given system (0.1) provided m is large. This procedure is nothing but Galerkin's procedure applied to the system (0.1). In the present paper, we shall call the trigonometric polynomial (0.2) satisfying (0.3) the *Galerkin approximation of the order* m. The question in the present paper is whether Galerkin's procedure applied to a nonlinear periodic system (0.1) is always successful or not.

This problem was studied by Cesari [1]. He studied the problem under very mild conditions and he gave the very general conditions that Galerkin's procedure may be successful in seeking a periodic solution of a nonlinear system. But his approach is based on a theorem on invariance of degree of topological mapping and the conditions of that theorem are not specialized into the form connected directly with the given system. So, when his results are applied to practical problems, some more techniques are needed and this does not seem to be an easy work usually.

Such being the case, setting some more conditions upon the given system and restricting a periodic solution somewhat, we tried to get the results more convenient to practical application. We assumed $X(x, t)$ and its derivatives with respect to x are both continuously differentiable with respect to x and t and we restricted a periodic solution to such one that the multipliers of the equation of first variation with respect to the periodic solution are all different from one. In the present paper, we shall call such a periodic solution an *isolated periodic solution*, because there is no other periodic solution in the neighborhood of such a periodic solution. The condition of smoothness of $X(x, t)$ and the restriction to an isolated periodic solution will not be severe limitations from a standpoint of practical application.

The equation (0.3) can be written as follows :

$$
\begin{cases}
F_0^{(m)}(\alpha) \overset{\text{def}}{=} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} X\,[x_m(s),\ s]\ ds = 0, \\[2ex]
F_n^{(m)}(\alpha) \overset{\text{def}}{=} \dfrac{1}{\sqrt{2}\pi} \displaystyle\int_0^{2\pi} X\,[x_m(s),\ s] \cos ns\ ds - nb_n = 0, \\[2ex]
G_n^{(m)}(\alpha) \overset{\text{def}}{=} \dfrac{1}{\sqrt{2}\pi} \displaystyle\int_0^{2\pi} X\,[x_m(s),\ s] \sin ns\ ds + na_n = 0
\end{cases}
\tag{0.4}
$$

$$(n = 1,\ 2,\ \ldots,\ m),$$

where
$$\alpha = (a_0,\ a_1,\ b_1,\ \ldots,\ a_m,\ b_m)$$

and
$$x_m(t) = a_0 + \sqrt{2} \sum_{n=1}^{m} (a_n \cos nt + b_n \sin nt). \tag{0.5}$$

Since the coefficients of Galerkin approximations are determined by (0.4), in the present paper, we shall call the equation (0.4) the *determining equation of Galerkin approximations*.

Our results are :

Under the conditions of smoothness of $X(x, t)$,

1/ the existence of an isolated periodic solution lying inside the region of definition of $X(x, t)$ always implies the existence of a Galerkin approximation of any high order and the boundedness of a certain linear operator connected with the Jacobi-matrix of $X(x, t)$ with respect to x ;

2/ the existence of a Galerkin approximation of the sufficiently high order always implies the existence of an isolated exact periodic solution provided the boundedness condition of 1/ is satisfied.

From the first result, the uniform convergence of the Galerkin approximation is readily proved. From the second result, there are given a requisite and an error estimate for a Galerkin approximation which can affirm the existence of an exact periodic solution.

The determining equation of Galerkin approximations is nonlinear, so, at first sight, it seems very difficult to solve such a nonlinear system of equations. But the examples of the last paragraph show it will be not so difficult in practical problems to solve the nonlinear determining equation by a computer if we use Newton's method. The reason is in the rapid convergence of the Fourier series. Moreover, in numerical solution of the determining equation, we do not need the explicit form of the determining equation if we use some techniques of Fourier analysis.

However, if the explicit form of the determining equation is wanted, it can be generated also by a computer provided the right-hand side of (0.1) is a polynomial of x.

The programs for computation in the present paper have been all written by A. Reiter of the Mathematics Research Center and the computation has been carried out by CDC 1604 of the University of Wisconsin. Here the author wishes to express his hearty thanks to Mr. A. Reiter for his earnest and constant co-operation.

§1. PRELIMINARIES.

In the present paper, we use Euclidean norms for vectors and matrices and denote them by the symbol $\| \ldots \|$. For continuous periodic vector-functions, we use two kinds of norms. Namely, let $f(t)$ be a continuous periodic vector-function with period 2π. Then, in the present paper, we use two kinds of norms $\|f\|$ and $\mathbf{|} f \mathbf{|}$, which are defined as follows :

$$\|f\| = \left[\frac{1}{2\pi} \int_0^{2\pi} \|f(t)\|^2 \, dt \right]^{1/2},$$

$$\mathbf{|} f \mathbf{|} = \max_t \|f(t)\| .$$

Here $\|f(t)\|$ is a norm of the vector $f(t)$.

The approach of the present paper is based on the following three propositions.

Proposition 1. *Given a linear periodic system*

$$\frac{dx}{dt} = A(t)x + \varphi(t), \tag{1.1}$$

where $A(t)$ is a continuous periodic matrix with period 2π and $\varphi(t)$ is a continuous periodic vector with the same period. If the multipliers of the corresponding homogeneous system

$$\frac{dy}{dt} = A(t)y \tag{1.2}$$

are all different from one, then (1.1) *has one and only one periodic solution with period 2π, which is given by*

$$x(t) = \int_0^{2\pi} H(t, s)\varphi(s)\, ds \tag{1.3}$$

where $H(t, s)$ is a piece-wise continuous periodic matrix such that

$$H(t, s) = \begin{cases} \Phi(t)\, [E - \Phi(2\pi)]^{-1}\, \Phi^{-1}(s) \\ \qquad\qquad 0 \leq s \leq t \leq 2\pi, \\ \Phi(t)\, [E - \Phi(2\pi)]^{-1}\, \Phi(2\pi)\, \Phi^{-1}(s) \\ \qquad\qquad \textit{for } 0 \leq t \leq s \leq 2\pi \end{cases} \tag{1.4}$$

and $$H(t, s) = H(t + 2m\pi, s + 2n\pi) \quad (m, n : \textit{integers}). \tag{1.5}$$

Here E is a unit matrix and $\Phi(t)$ is a fundamental matrix of (1.2) *such that $\Phi(0) = E$.*

The formula (1.3) defines a linear mapping H in a space of continuous periodic functions. Consequently, the norms of this linear mapping are defined corresponding to the norms of continuous periodic functions. We shall denote them by $\|H\|$ and $\left|\!\left| H \right|\!\right|$. Then, by means of Schwartz' inequality, it is readily seen that

$$\|H\| \leq \left[\int_0^{2\pi} \int_0^{2\pi} \sum_{k,l} H_{kl}^2(t, s)\, ds\, dt \right]^{1/2}, \tag{1.6}$$

$$\left|\!\left| H \right|\!\right| \leq \left[2\pi \cdot \max_t \int_0^{2\pi} \sum_{k,l} H_{kl}^2(t, s)\, ds \right]^{1/2}, \tag{1.7}$$

where $H_{kl}(t, s)$ are the elements of the matrix $H(t, s)$.

In what follows, we shall call the linear mapping H defined by (1.3) the *H-mapping corresponding to a given matrix $A(t)$.*

Proposition 2. *Given a real system of equations*

$$F(\alpha) = 0, \tag{1.8}$$

where α and $F(\alpha)$ are the vectors of the same dimension and $F(\alpha)$ is a continuously differentiable function of α defined in some region Ω of α.

Assume that (1.8) *has an approximate solution $\alpha = \hat{\alpha}$ for which the determinant of a Jacobi-matrix $J(\alpha)$ of $F(\alpha)$ with respect to α does not vanish and that there are a positive constant δ and a non-negative constant $\varkappa < 1$ such that*

(i) $\Omega_\delta = \{\alpha \mid \|\alpha - \hat{\alpha}\| \leqq \delta\} \subset \Omega$,

(ii) $\|J(\alpha) - J(\hat{\alpha})\| \leqq \varkappa/M'$ *for any* $\alpha \in \Omega_\delta$,

(iii) $\dfrac{M'r}{1 - \varkappa} \leqq \delta$,

where r and M' are the numbers such that

$$\|F(\hat{\alpha})\| \leqq r \quad \text{and} \quad \|J^{-1}(\hat{\alpha})\| \leqq M'.$$

Then the system (1.8) *has one and only one solution $\alpha = \bar{\alpha}$ in Ω_δ and*

$$\|\bar{\alpha} - \hat{\alpha}\| \leqq \frac{M'r}{1 - \varkappa}.$$

This proposition can be proved by Newton's iterative process :

$$\alpha_{n+1} = \alpha_n - J^{-1}(\hat{\alpha})\, F(\alpha_n) \quad (n = 0, 1, 2, \ldots),$$

where $\alpha_0 = \hat{\alpha}$.

Proposition 3. *Given a real system of differential equations*

$$\frac{dx}{dt} = X(x, t). \tag{1.9}$$

Here x *and* $X(x,t)$ *are the vectors of the same dimension and* $X(x,t)$ *is periodic in* t *with period* 2π *and is continuously differentiable with respect to* x *for* $x \in D$ *and* $t \in L$ *where* D *is a given region of* x *and* L *is a real line.*

Assume (1.9) *has a periodic approximate solution* $x = \bar{x}(t)$ *lying in* D *and there are a continuous periodic matrix* $A(t)$, *a positive constant* δ *and a non-negative constant* $\varkappa < 1$ *such that*

(i) *the multipliers of the linear homogeneous system*

$$\frac{dy}{dt} = A(t)\, y$$

are all different from one,

(ii) $D_\delta = \{x(t) \mid \|x(t) - \bar{x}(t)\| \leqq \delta\} \subset D$,

(iii) $\|\Psi[x(t), t] - A(t)\| \leqq \varkappa / M_1$ *for any* $x(t) \in D_\delta$,

(iv) $\dfrac{M_1 r}{1 - \varkappa} \leqq \delta$.

Here

$\Psi(x, t)$ *is a Jacobi-matrix of* $X(x,t)$ *with respect to* x; M_1 *is a positive constant such that*

$$\|H\| \leqq M_1 ,$$

where H *is a H-mapping corresponding to* $A(t)$; r *is a positive constant such that*

$$\left\|\frac{d\bar{x}(t)}{dt} - X[\bar{x}(t), t]\right\| \leqq r.$$

Then the given system (1.9) *has one and only one periodic solution* $x = \hat{x}(t)$ *in* D_δ *and this is an isolated periodic solution. Further, for* $x = \hat{x}(t)$, *it holds that*

$$\|\hat{x}(t) - \bar{x}(t)\| \leqq \frac{M_1 r}{1 - \varkappa}. \qquad (1.10)$$

This proposition can be proved by means of the iterative process :

$$x_{n+1}(t) = \int_0^{2\pi} H(t, s)\, [X(x_n(s), s) - A(s) x_n(s)]\, ds$$

$$(n = 0, 1, 2, \ldots),$$

where $x_0(t) = \bar{x}(t)$ and $H(t, s)$ is a matrix of the H-mapping corresponding to $A(t)$.

§2. THE EXISTENCE OF A GALERKIN APPROXIMATION

2.1 A truncated trigonometric polynomial of a periodic solution. Let $f(t)$ be a continuous periodic vector-function with period 2π and let its Fourier series be

$$f(t) \sim c_0 + \sqrt{2} \sum_{n=1}^{\infty} (c_n \cos nt + d_n \sin nt),$$

where $c_0, c_1, d_1, c_2, d_2, \ldots$ are vectors. Then the trigonometric polynomial

$$f_m(t) = c_0 + \sqrt{2} \sum_{n=1}^{m} (c_n \cos nt + d_n \sin nt)$$

is a truncated trigonometric polynomial of a given periodic function $f(t)$. In the sequel, we shall denote such truncation of a periodic function by P_m and write a truncated polynomial $f_m(t)$ of a periodic function $f(t)$ as follows :

$$f_m(t) = P_m f(t).$$

If we put $\gamma = (c_0, c_1, d_1, \ldots, c_m, d_m)$, then it is readily seen that

$$\|f_m\| = \|\gamma\|.$$

This property will be used often in the sequel.

For a continuously differentiable periodic function, there holds

Lemma 2.1. *Let $f(t)$ be a continuously differentiable periodic vector-function with period 2π. Then,*

$$\|f - P_m f\| \leqq \sigma(m) \, \|\dot{f}\| \leqq \sigma(m) \, \|\dot{f}\|,$$

$$\|f - P_m f\| \leqq \sigma_1(m) \, \|\dot{f}\|,$$

where $\cdot = d/dt$ *and*

$$\begin{cases} \sigma(m) = \sqrt{2} \left[\dfrac{1}{(m+1)^2} + \dfrac{1}{(m+2)^2} + \cdots \right]^{1/2}, \\ \sigma_1(m) = \dfrac{1}{m+1}. \end{cases}$$

For $\sigma(m)$, it holds that

$$\frac{\sqrt{2}}{m+1} < \sigma(m) < \frac{\sqrt{2}}{\sqrt{m}}.$$

This can be proved easily by means of Schwartz' inequality and Parseval's equality.

If we apply Lemma 2.1 to a periodic solution of a differential equation, then we have the following lemma concerning its truncated trigonometric polynomials.

Lemma 2.2. *Given a real periodic system*

$$\frac{dx}{dt} = X(x, t), \tag{2.1}$$

where x *and* $X(x,t)$ *are the vectors of the same dimension and* $X(x,t)$ *is periodic in* t *with period* 2π. *We assume that* $X(x,t)$ *and its derivatives with respect to* x *are continuously differentiable with respect to* x *and* t *in the region* $D \times L$, *where* D *is a closed bounded region of* x *and* L *is a real line.*

Let K, K_1 *and* K_2 *be the non-negative constants such that*

$$\begin{cases} K = \max_{D \times L} \|X(x, t)\|, \qquad K_1 = \max_{D \times L} \|\Psi(x, t)\|, \\ K_2 = \max_{D \times L} \left\| \frac{\partial X(x, t)}{\partial t} \right\|, \end{cases} \tag{2.2}$$

where $\Psi(x,t)$ *is a Jacobi-matrix of* $X(x,t)$ *with respect to* x.

Then, if there is a periodic solution $x = \hat{x}(t)$ *of* (2.1) *lying in* D, *it holds that*

(i) $|\hat{x} - \hat{x}_m| \leqq \sigma(m)K$,

(ii) $\|\hat{x} - \hat{x}_m\| \leqq \sigma_1(m)K$,

(iii) $|\dot{\hat{x}} - \dot{\hat{x}}_m| \leqq \sigma(m)(KK_1 + K_2)$,

where $x_m(t) = P_m \hat{x}(t)$.

From this lemma, readily follows the following corollary.

Corollary. *If* $x = \hat{x}(t)$ *is an isolated periodic solution of* (2.1) *lying inside* D, *then there exists a positive integer* m_0 *such that, for any* $m \geqq m_0$,

(i) $\hat{x}_m(t) \in D$;

(ii) *the multipliers of the linear homogeneous system*

$$\frac{dy}{dt} = \Psi[\hat{x}_m(t), t]\, y$$

are all different from one and the H-mapping H_m *corresponding to* $\Psi[\hat{x}_m(t), t]$ *is equi-bounded, namely, there exists a non-negative constant* M *such that*

$$\|H_m\|,\ |H_m| \leqq M ;$$

(iii) $\frac{d}{dt}\Psi[\hat{x}_m(t), t]$ *is equi-bounded, namely, there exists a non-negative constant* K_3 *such that*

$$\left\| \frac{d}{dt} \Psi[\hat{x}_m(t), t] \right\| \leqq K_3.$$

2.2 The Jacobi-matrix of the determining equation of Galerkin approximations. Let $J_m(\alpha)$ be the Jacobi-matrix of the left member of the determining equation (0.4) of Galerkin approximations. To find the basic properties of $J_m(\alpha)$, let us consider a linear system

$$J_m(\alpha)\,\xi + \gamma = 0, \tag{2.3}$$

where $\xi = (u_0, u_1, v_1, \ldots, u_m, v_m)$ and $\gamma = (c_0, c_1, d_1, \ldots, c_m, d_m)$. If we put

$$\begin{cases} y(t) = u_0 + \sqrt{2}\sum_{n=1}^{m} (u_n \cos nt + v_n \sin nt), \\ \varphi(t) = c_0 + \sqrt{2}\sum_{n=1}^{m} (c_n \cos nt + d_n \sin nt), \end{cases}$$

then, by the definition of $J_m(\alpha)$, corresponding to (2.3), we have a differential system

$$\frac{dy(t)}{dt} = P_m \Psi[x_m(t), t]\, y(t) + \varphi(t), \tag{2.4}$$

where $x_m(t)$ is of the form (0.5).

Now we can prove

Lemma 2.3. *We assume the conditions of* Lemma 2.2 *and further we assume the system* (2.1) *has an isolated periodic solution* $x = \hat{x}(t)$ *lying inside* D. *Taking* m_0 *sufficiently large, we consider the differential system*

$$\frac{dy}{dt} = P_m \Psi[\hat{x}_m(t), t]\, y + \varphi(t) \tag{2.5}$$

for $m \geq m_0$, *where* $\hat{x}_m(t) = P_m \hat{x}(t)$ *and* $\varphi(t)$ *is an arbitrary continuous periodic function with period* 2π.

Then, for any periodic solution $y = y(t)$ *of* (2.5), *if it exists, it holds that*

$$\|y\| \leq \frac{M[1 + K_1\sigma_1(m)]}{1 - M(K_3 + K_1^2)\,\sigma_1(m)}\,\|\varphi\|.$$

The equation (2.5) is rewritten as follows :

$$\frac{dy}{dt} = \Psi[\hat{x}_m(t), t]\, y + \varphi(t) + \eta,$$

where

$$\eta = -(I - P_m)\,\Psi[\hat{x}_m(t), t]\, y.$$

Here I is an identical operator. Estimating η by means of Lemma 2.1, we can prove the lemma using Corollary of Lemma 2.2.

Let

$$\hat{x}(t) = \hat{a}_o + \sqrt{2}\sum_{n=1}^{\infty}(\hat{a}_n \cos nt + \hat{b}_n \sin nt) \tag{2.6}$$

be the Fourier series of an isolated periodic solution $x = \hat{x}(t)$ of (2.1) lying inside D. Then, from the above lemma, readily follow the following corollaries.

Corollary 1. *There exists a positive integer m_o such that*

$$\det J_m(\hat{\alpha}) \neq 0$$

for any $m \geqq m_o$, where

$$\hat{\alpha} = (\hat{a}_o, \hat{a}_1, \hat{b}_1, \ldots, \hat{a}_m, \hat{b}_m). \tag{2.7}$$

Corollary 2. *There is a positive integer m_o such that, for any $m \geqq m_o$, $J_m^{-1}(\hat{\alpha})$ exists and*

$$\|J_m^{-1}(\hat{\alpha})\| \leqq \frac{M\,[1 + K_1\sigma_1(m)]}{1 - M(K_3 + K_1^2)\,\sigma_1(m)}.$$

For the difference $J_m(\alpha') - J_m(\alpha'')$, we can prove

Lemma 2.4. *We assume the conditions of Lemma 2.2. Let K_4 be a positive constant such that*

$$K_4 = \left[\max_{D \times L} \sum_{k,l,p} \left(\frac{\partial_t^2 X_k(x,t)}{\partial x_l \partial x_p}\right)^2\right]^{1/2},$$

where $X_k(x,t)$ and x_l are respectively the components of the vectors $X(x,t)$ and x.

Then, if both of

$$x'(t) = a_o' + \sqrt{2}\sum_{n=1}^{\infty}(a_n' \cos nt + b_n' \sin nt)$$

and

$$x''(t) = a_o'' + \sqrt{2}\sum_{n=1}^{\infty}(a_n'' \cos nt + b_n'' \sin nt)$$

belong to D together with $\theta x'(t) + (1 - \theta)\, x''(t)$ $(0 \leqq \theta \leqq 1)$, then

$$\|J_m(\alpha') - J_m(\alpha'')\| \leqq K_4\ \|x' - x''\| \leqq K_4\ \sqrt{2m+1}\ \|\alpha' - \alpha''\|,$$

where $\alpha' = (a_o', a_1', b_1', \ldots, a_m', b_m')$ and $\alpha'' = (a_o'', a_1'', b_1'', \ldots, a_m'', b_m'')$.

This can be proved making use of the correspondence between (2.3)' and (2.4).

2.3 The existence of a Galerkin approximation. The existence of a Galerkin approximation is affirmed from that of an isolated periodic solution by the following theorem.

Theorem 1. *Given a real nonlinear periodic system*

$$\frac{dx}{dt} = X(x, t) \tag{2.8}$$

where x and X(x, t) are the real vectors of the same dimension and X(x, t) is periodic in t with period 2π. We assume that X(x, t) and its derivatives with respect to x are continuously differentiable with respect to x and t in the region D × L, where D is a closed bounded region of x and L is a real line.

Then, if there is an isolated periodic solution $x = \hat{x}(t)$ of (2.8) lying inside D, there exists a Galerkin approximation $x = \bar{x}_m(t)$ of any order $m \geqq m_0$ lying in D provided m_0 is taken sufficiently large.

Proof. Take a small positive number δ_0 so that

$$U = \{x(t) \mid \|x(t) - \hat{x}(t)\| \leqq \delta_0\} \subset D.$$

Then, by Lemma 2.2, $P_m\hat{x}(t) = \hat{x}_m(t) \in U \subset D$ for any $m \geqq m_0$ provided m_0 is sufficiently large. For such m, we have

$$\frac{d\hat{x}_m(t)}{dt} = P_m X[\hat{x}_m(t), t] + R_m(t), \tag{2.9}$$

where

$$R_m(t) = P_m\{X[\hat{x}(t), t] - X[\hat{x}_m(t), t]\}.$$

By Lemma 2.2, it is readily seen that

$$\|R_m\| \leqq KK_1\sigma_1(m). \tag{2.10}$$

For brevity, let us write the determining equation (0.4) in the vector form as follows :

$$F^{(m)}(\alpha) = 0. \tag{2.11}$$

Then (2.9) and (2.10) implies

$$F^{(m)}(\hat{\alpha}) = \rho^{(m)}$$

and

$$\|\rho^{(m)}\| \leqq KK_1\sigma_1(m), \tag{2.12}$$

where $\hat{\alpha}$ is a vector defined in (2.7). This says $\alpha = \hat{\alpha}$ is an approximate solution of (2.11). Therefore let us now apply Proposition 2 to the equation (2.11).

By Lemma 2.2, it is easily seen that $F^{(m)}(\alpha)$ is determinate in the region

$$\Omega_m = \left\{ \alpha \mid \|\alpha - \hat{\alpha}\| \leqq \frac{\delta_o - K\,\sigma(m)}{\sqrt{2m+1}} \right\}. \tag{2.13}$$

If m_0 is sufficiently large, then, for any $m \geqq m_0$, by Corollaries 1 and 2 of Lemma 2.3, $J_m^{-1}(\hat{\alpha})$ exists and

$$\|J_m^{-1}(\hat{\alpha})\| \leqq M' = \frac{M\,[1 + K_1\,\sigma_1(m_o)]}{1 - M(K_3 + K_1^2)\,\sigma_1(m_o)}. \tag{2.14}$$

Let $\varkappa$ be an arbitrary number such that $0 < \varkappa < 1$. Then, if we take sufficiently large positive integer $m_1 \geqq m_0$, then, for any $m \geqq m_1$, we can take δ_m such that

$$\frac{M'KK_1}{1-\varkappa}\,\sigma_1(m) \leqq \delta_m \leqq \frac{1}{\sqrt{2m+1}} \min\left(\frac{\varkappa}{K_4 M'},\ \delta_o - K\,\sigma(m_o)\right). \tag{2.15}$$

Then, by (2.12) ~ (2.15) and Lemma 2.4, we can easily prove that the conditions of Proposition 2 are all fulfilled for $\delta = \delta_m$. Thus we see that the determining equation (2.11) has one and only one solution $\alpha = \bar{\alpha}$ in Ω_{δ_m}. This proves the theorem. Q. E. D.

2.4 <u>Errors and some properties of Galerkin approximations.</u> An error estimate of a Galerkin approximation is given by the following theorem.

<u>Theorem 2.</u> *We assume the conditions of* Theorem 1. *Let* $x = \hat{x}(t)$ *be an isolated periodic solution of* (2.8) *lying inside D and* $x = \bar{x}_m(t)$ *be its Galerkin approximation affirmed in* Theorem 1. *If we take* m_0 *sufficiently large, then, for any positive integer* $m \geqq m_0$, *we have*

$$\|\bar{x}_m - \hat{x}\| \leqq \frac{M'KK_1}{1-\varkappa}\cdot\frac{\sqrt{2m+1}}{m+1} + K\,\sigma(m), \tag{2.16}$$

$$\|\dot{\bar{x}} - \dot{\hat{x}}\| \leqq (K_2 + 2KK_1)\,\sigma(m) + \frac{M'KK_1^2}{1-\varkappa}\cdot\frac{\sqrt{2m+1}}{m+1}, \tag{2.17}$$

where

$\varkappa$ is an arbitrary fixed number such that $0 < \varkappa < 1$;

K, K_1 and K_2 are the constants defined in Lemma 2.2 ;

$\sigma(m)$ is a number defined in Lemma 2.1 ;

M' is a constant defined in (2.14).

Proof. The inequality (2.16) can be easily proved by means of Proposition 2 and Lemma 2.2.

Now, by the definition, for the Galerkin approximation $x = \bar{x}_m(t)$, it holds that

$$\frac{d\bar{x}_m(t)}{dt} = P_m X[\bar{x}_m(t),\ t]. \tag{2.18}$$

This can be rewritten as follows :

$$\frac{d\bar{x}_m(t)}{dt} = X[\bar{x}_m(t), t] + \eta_m(t), \tag{2.19}$$

where

$$\eta_m(t) = -(I - P_m) X[\bar{x}_m(t), t].$$

By Lemma 2.1, it is readily seen that

$$\|\eta_m\| \leq \sigma(m)(KK_1 + K_2). \tag{2.20}$$

Then we can get (2.17) readily from the equality

$$\frac{d\bar{x}_m(t)}{dt} - \frac{d\hat{x}(t)}{dt} = \{X[\bar{x}_m(t), t] - X[\hat{x}(t), t]\} + \eta_m(t). \qquad \text{Q.E.D.}$$

From this theorem, readily follows the following corollary.

Corollary. *If there is an isolated periodic solution $x = \hat{x}(t)$ of* (2.8) *lying inside D, then its Galerkin approximation $x = \bar{x}_m(t)$ affirmed in* Theorem 1 *converges uniformly to the original solution $x = \hat{x}(t)$ together with its first order derivative as $m \longrightarrow \infty$.*

By Theorem 2, if we take m_0 sufficiently large, then, for any $m \geqq m_0$, the conclusions of Corollary of Lemma 2.2 are all valid for $\bar{x}_m(t)$ as well as for $\hat{x}_m(t)$. Thus, by means of Corollaries 1 and 2 of Lemma 2.3, we have

Theorem 3. *We assume the conditions of* Theorem 1 *and suppose* (2.8) *has an isolated periodic solution lying inside D. Let*

$$\bar{x}_m(t) = \bar{a}_0 + \sqrt{2}\sum_{n=1}^{m}(\bar{a}_n \cos nt + \bar{b}_n \sin nt)$$

be its Galerkin approximation affirmed in Theorem 1 *and suppose m_0 is sufficiently large.*

Then, for any $m \geqq m_0$,

$$\det J_m(\bar{\alpha}) \neq 0,$$

and there exists a constant M' such that

$$\|J_m^{-1}(\bar{\alpha})\| \leq M',$$

where $\bar{\alpha} = (\bar{a}_0, \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_m, \bar{b}_m)$. Further, the multipliers of the linear homogeneous system

$$\frac{dy}{dt} = \Psi[\bar{x}_m(t), t]\, y$$

are all different from one and the H-mapping H_m corresponding to $\Psi[x_m(t), t]$

is equi-bounded, namely, there is a non-negative constant M_1 such that

$$\|H_m\|,\ \|H_m\| \leqq M_1.$$

§3. THE EXISTENCE OF AN ISOLATED PERIODIC SOLUTION.

According to Theorem 3, let us assume the equi-boundedness of the H-mapping H_m. Then we can get the following theorem.

Theorem 4. *We assume the conditions of* Theorem 1.

Let m_0 be a certain positive integer and Δ be a region such that its $\varepsilon(>0)$-neighrhood is contained in D.

Then, if there is a Galerkin approximation $x = \bar{x}_m(t)$ of any order $m \geqq m_0$ lying in Δ such that, for the H-mapping H_m corresponding to $\Psi[\bar{x}_m(t), t]$, $\|H_m\|$ is equi-bounded, then there exists one and only one exact isolated periodic solution $x = \hat{x}(t)$ of (2.8) *in the neighborhood of $x = \bar{x}_m(t)$ and, between $\bar{x}_m(t)$ and $\hat{x}(t)$, the following inequality holds :*

$$\|\hat{x} - \bar{x}_m\| \leqq \frac{M_1}{1-\varkappa}(KK_1 + K_2)\,\sigma(m), \tag{3.1}$$

where

$\varkappa$ *is an arbitrary fixed number such that* $0 < \varkappa < 1$;

K, K_1 *and* K_2 *are the constants defined in* Lemma 2.2 ;

$\sigma(m)$ *is a number defined in* Lemma 2.1 ;

M_1 *is a non-negative constant such that*

$$\|H_m\| \leqq M_1.$$

Proof. For $x = \bar{x}_m(t)$, (2.18) holds and we have (2.19) and (2.20).

Let $\varkappa$ be an arbitrary fixed number such that $0 < \varkappa < 1$. Then, if we take sufficiently large positive integer $m_1 \geqq m_0$, then we can take δ such that

$$\frac{M_1}{1-\varkappa}(KK_1 + K_2)\frac{\sqrt{2}}{\sqrt{m_1}} \leqq \delta \leqq \min\left(\varepsilon,\ \frac{\varkappa}{M_1K_4}\right), \tag{3.2}$$

where K_4 is a constant defined in Lemma 2.4. Then, by means of (2.20) and (3.2), we can easily prove that, for any $m \geqq m_1$, the conditions of Proposition 3 are all fulfilled for $\bar{x}(t) = \bar{x}_m(t)$ and $A(t) = \Psi[\bar{x}_m(t), t]$. Thus we see that there exists one and only one exact isolated periodic solution $x = \hat{x}(t)$ of (2.8) in D_3. The inequality (3.1) readily follows from (2.10). Q. E. D.

Remark. In Theorem 4, we have assumed that the multipliers of the linear homogeneous system

$$\frac{dy}{dt} = \Psi[\bar{x}_m(t), t]$$

are all different from one. However this can be proved if we assume the existence and the equi-boundedness of $J_*^{-1}(\bar{\alpha})$ where $\bar{\alpha}$ is a vector corresponding to the Fourier coefficients of $\bar{x}_*(t)$. But the equi-boundedness of the H-mapping corresponding to $\Psi[\bar{x}_*(t), t]$ which is also assumed in Theorem 4, seems not to follow from sole assumption of the existence and the equi-boundedness of $J_*^{-1}(\bar{\alpha})$.

§4. NUMERICAL EXAMPLES

4.1 Example 1.

$$\ddot{x} + 1.5^2 x + (x - 1.5 \sin t)^3 = 2 \sin t. \tag{4.1}$$

If $x = x(t)$ is a solution of (4.1), then $-x(-t)$ and $-x(t+\pi)$ are also the solutions. Therefore, if the periodic solution of (4.1) is unique, then the Fourier series of such a periodic solution must be of the form

$$x(t) = a_1 \sin t + a_3 \sin 3t + \dots .$$

Now, let us seek the 3rd order Galerkin approximation of the above form.

The equation (4.1) can be written in the form of first order system as follows :

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -1.5^2 x - (x - 1.5 \sin t)^3 + 2 \sin t. \end{cases} \tag{4.2}$$

Let the desired 3rd order Galerkin approximation be

$$\begin{cases} x = x(t) = a_1 \sin t + a_3 \sin 3t, \\ y = y(t) = \dot{x}(t) = a_1 \cos t + 3a_3 \cos 3t, \end{cases}$$

then, after solving the determining equation, *we get the following Galerkin approximation :*

$$x = \bar{x}(t) = 1.59941 \sin t - 0.00004 \sin 3t. \tag{4.3}$$

Let H be the H-mapping corresponding to

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1.5^2 & 0 \end{pmatrix},$$

then, by (1.7), we find

$$\|H\| \leqq \frac{13\sqrt{2}}{12} \pi = 4.81312 \dots < 4.8132.$$

Let $\Psi(x, y, t)$ be the Jacobi-matrix of the right member of (4.2), then it is readily seen that

$$\|\Psi(x, y, t) - A(t)\| \leqq 3(\delta + 0.09945)^2$$

for $x = x(t)$ such that $|x(t) - \bar{x}(t)| \leqq \delta$.

Since

$$|\ddot{\bar{x}}(t) + 1.5^2\, \bar{x}(t) + (\bar{x}(t) - 1.5 \sin t)^3 - 2 \sin t| < 0.000025,$$

according to Proposition 3, let us seek δ and $\varkappa (< 1)$ such that

$$\begin{cases} 3(\delta + 0.09945)^2 \leqq \dfrac{\varkappa}{4.8132}, \\ \dfrac{4.8132 \times 0.000025}{1 - \varkappa} \leqq \delta. \end{cases} \tag{4.4}$$

If we suppose $\delta \leqq 0.0002$, then (4.4) can be replaced by the stronger inequalities

$$\frac{0.00012033}{1 - \varkappa} \leqq \delta \leqq \frac{\varkappa}{1.439} - 0.09945.$$

From this, we readily see that (4.4) is valid for $\varkappa = 0.144$ and δ such that $0.000141 \leqq \delta \leqq 0.0002$. Thus, by Proposition 3, we see that *the given equation* (4.1) *has an exact isolated periodic solution* $x = \hat{x}(t)$ *and that*

$$|\bar{x}(t) - \hat{x}(t)| \leqq 0.000141$$

for the Galerkin approximation (4.3).

4.2 Example 2.

$$\ddot{x} + x^3 = \sin t. \tag{4.5}$$

This equation is taken up in Cesari's paper [1].

First, we compute the Galerkin approximation of the form

$x(t) = b_1 \sin t$ and we find $b_1 = 1.4923$.

Next, starting from

$$b_1 = 1.4923 \quad a_0 = a_1 = a_2 = b_2 = \cdots = a_{15} = b_{15} = 0,$$

we solve the determining equation by means of Newton's method and *we get the desired Galerkin approximation as follows :*

$$\begin{aligned} \bar{x}(t) = {} & 1.4311\,89037 \sin t - 0.1269\,15530 \sin 3t \\ & + 0.0097\,54734 \sin 5t - 0.0007\,63601 \sin 7t \\ & + 0.0000\,59845 \sin 9t - 0.0000\,04691 \sin 11t \\ & + 0.0000\,00368 \sin 13t - 0.0000\,00029 \sin 15t. \end{aligned} \tag{4.6}$$

For this Galerkin approximation, we see that

$$|\ddot{\bar{x}} + \bar{x}^3 - \sin t| \leq r = 713 \times 10^{-9}. \tag{4.7}$$

To find the value of M_1 (Cf. Theorem 4), first, we compute the fundamental matrix of the equation of first variation by means of the Runge-Kutta method with mesh-size $2^{-6}\pi$. Next, we compute the matrix $H(t,s)$ by means of (1.4) and, finally, we compute the value of M_1 by means of (1.7). We have computed the integral $\int_0^{2\pi} \sum_{k,l} H_{kl}^2(t,s)ds$ by Simpson's rule. The obtained value of M_1 is

$$M_1 = 11.4107. \tag{4.8}$$

From (4.7) and (4.8), we see by Proposition 3 that *the given equation* (4.5) *has an exact isolated periodic solution* $x = \hat{x}(t)$ *and that*

$$|\bar{x}(t) - \hat{x}(t)| \leq 8.231 \times 10^{-6}$$

for the Galerkin approximation (4.6).

4.3 Example 3.

$$\ddot{x} - \lambda(1 - x^2)\dot{x} + x - \lambda \sin t = 0. \tag{4.9}$$

This is a *van der Pol equation with a harmonic forcing term.*
For $\lambda = 0.1$, we get the following Galerkin approximation :

$$\begin{aligned} x = \bar{x}(t) = &- 2.3787\ 85902 \cos t &&- 0.1423\ 30099 \sin t \\ &- 0.0046\ 46924 \cos 3t &&+ 0.0418\ 67539 \sin 3t \\ &+ 0.0012\ 23706 \cos 5t &&+ 0.0002\ 15278 \sin 5t \\ &+ 0.0000\ 09756 \cos 7t &&- 0.0000\ 39873 \sin 7t \\ &- 0.0000\ 01358 \cos 9t &&- 0.0000\ 00430 \sin 9t \\ &- 0.0000\ 00019 \cos 11t &&+ 0.0000\ 00047 \sin 11t \\ &+ 0.0000\ 00002 \cos 13t &&+ 0.0000\ 00001 \sin 13t \\ &+ 0.0000\ 00000 \cos 15t &&- 0.0000\ 00000 \sin 15t. \end{aligned} \tag{4.10}$$

Checking the conditions of Proposition 3, we see that, *for* $\lambda = 0.1$, *the given equation* (4.9) *has one and only one isolated periodic solution* $x = \hat{x}(t)$ *in the neighborhood of the Galerkin approximation* (4.10) *and that*

$$|\bar{x}(t) - \hat{x}(t)| \leq 2.50 \times 10^{-7}$$

for the Galerkin approximation (4.10).

Remark 1. In the course of computation of M_1 we have computed the multipliers of the equation of first variation. They are 0.87449 and 0.35980. This says *the periodic solution whose existence is certified by the above computation will be stable.*

Remark 2. It is easily found by the method of averaging that, *for sufficiently small* $\lambda > 0$, *the equation* (4.9) *has a unique stable periodic solution* whose first approximation is

$$x = a_0 \cos t - \frac{\lambda}{8}(a_0 - 1) \sin 3t$$

where $a_0 \approx -2.3830$ is a unique real root of the equation

$$a_0^3 - 4a_0 + 4 = 0.$$

For $\lambda = 0.1$, the above *first approximation becomes*

$$x = -2.3830 \cos t + 0.04229 \sin 3t. \tag{4.11}$$

Comparing the above results with ours, we see :

1/ *the conclusion on the existence of a stable periodic solution is right for* $\lambda = 0.1$ (1) ;

2/ *the first approximation* (4.11) *has not sufficient accuracy.*

REFERENCE

[1] Cesari, L., *Functional analysis and periodic solutions of nonlinear differential equations*, Contributions to Differential Equations, 1(1963), 149-187.

DISCUSSION

M. FORBAT : M. Urabe n'a-t-il pas de remarque à faire au sujet du cas où le système algébrique fournissant les coefficients de l'approximation de Galerkin admettrait plusieurs solutions ? Je signale à cette occasion que l'excellent ouvrage du Professeur Kauderer (Nichtlineare Mechanik, Springer, Berlin 1959) décrit de nombreuses méthodes approchées, souvent précieuses.

M. URABE : L'équation déterminante n'a naturellement pas toujours une solution unique ; mais elle fournit une seule solution pour chaque solution périodique isolée. Si l'équation proposée possède *n* solutions périodiques isolées, les équations déterminantes ont *n* solutions qui donnent les approximations de Galerkin satisfaisant aux conditions de la proposition 2. Lorsqu'on résoud numériquement les équations déterminantes, on obtient telle ou telle solution suivant les valeurs de départ.

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(1) Note that the conclusion derived by the method of averaging has no certification for its validity for a given value of λ unless the conclusion is checked for the given value of λ. Our method is, of course, one of the methods available for checking the conclusion for the given value of λ.

Je pense que la méthode de Newton est très commode pour résoudre le système non linéaire pour de nombreuses équations

M. SETHNA : Pour l'équation de Duffing

$$\ddot{x} + x + x^3 = \cos \gamma t,$$

on trouve que si $\frac{1}{3} - \varepsilon < \gamma < \frac{1}{3} + \varepsilon$, $0 < \varepsilon \ll 1$, l'application de la méthode en première approximation en partant de

$$x = a_1^{(1)} \cos \gamma t$$

donne une amplitude $a_1^{(1)}$ qui diffère considérablement de celle qu'on obtient par la deuxième approximation

$$x = a_1^{(2)} \cos \gamma t + a_3^{(2)} \cos 3 \gamma t$$

Est-ce que votre procédé permet de prévoir un tel comportement ?

M. URABE : Il est bien évident que dans les cas critiques il faut recourir à des techniques spéciales ; mais parfois la méthode des perturbations permet de trouver les valeurs de départ.

M. CESARI : L'exemple de l'équation $\ddot{x} + x^3 = \sin t$, traité tant par Cesari que par Urabe, peut éclairer la question. Dès la première approximation, trois solutions périodiques apparaissent comme possibles ; on n'a analysé que celle qui est harmonique, et pour elle on a démontré l'existence d'une solution périodique exacte voisine. On avait eu le sentiment que pour arriver à des conclusions analogues en ce qui concerne les deux autres solutions, il eût fallu recourir à des approximations de Galerkin d'ordres plus élevés. La façon de faire de M. Urabe, qui fait intervenir des approximations de Galerkin d'ordres supérieurs obtenues à l'aide de calculateurs analogiques et de la méthode de Newton, avec comme conséquences la possibilité de discuter l'existence et d'évaluer les bornes de l'erreur, me semble très adéquate et donne d'excellents résultats.

M. ROSENBERG : M. Forbat a mentionné le livre de Kauderer. Il est vrai qu'on y trouve de nombreux exemples de procédés numériques, mais on n'y démontre jamais l'existence de solutions périodiques avant d'admettre qu'il y en a.

Additional References

For details of the proof, computation and application, see:

1. M. Urabe, Galerkin's procedure for nonlinear periodic systems, Arch. Rational Mech. Anal., 20(1965), 120-152,
2. M. Urabe and A. Reiter, Numerical computation of nonlinear forced oscillations by Galerkin's procedure, J. Math. Anal. Appl., 14(1966), 107-140,
3. M. Urabe, Periodic solutions of differential systems, Galerkin's procedure and the method of averaging, J. Differential Equations, 2(1966), 265-280.