Gallai's Conjecture For Graphs of Girth at Least Four

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Abstract

In 1966, Gallai conjectured that for any simple, connected graph G having n vertices, there is a path-decomposition of G having at most $\lceil \frac{n}{2} \rceil$ paths. In this paper, we show that for any simple graph G having girth $g \geq 4$, there is a path-decomposition of G having at most $\frac{p(G)}{2} + \left\lfloor \left(\frac{g+1}{2g} \right) q(G) \right\rfloor$ paths, where $p(G)$ is the number of vertices of odd degree in G and $q(G)$ is the number of non-isolated vertices of even degree in G.

Keywords: Graph, girth, path-decomposition.

1 Introduction

A **decomposition** of a graph G is a collection of subgraphs $\mathcal{P}$ such that each edge of G belongs to exactly one subgraph of $\mathcal{P}$. A **path-decomposition** is a decomposition consisting of paths. A **path-cycle-decomposition** is a decomposition consisting of paths and cycles. The following is a well-known old conjecture due to Gallai:

1.1 Conjecture (Gallai, 1966)
For any simple, connected graph G having n vertices, there is a path-decomposition having at most $\lceil \frac{n}{2} \rceil$ paths.

We shall first give a brief overview of some of the previous work on this conjecture. In [5], Lovász showed the following important result:

1.2 Theorem (Lovász)
For any simple, connected graph G having n vertices, there is a path-cycle-decomposition $\mathcal{P}$ such that $|\mathcal{P}| \leq \lfloor \frac{n}{2} \rfloor$.

As a consequence of the proof of the above theorem, Gallai's conjecture holds for all graphs with only odd vertices.

1.3 Theorem (Lovász)
For any simple, connected graph G having n vertices where all vertices are odd, there is a path-decomposition with $\frac{n}{2}$ paths.

For a graph G, the **E-subgraph** of G is the subgraph induced by the even vertices of G. We shall denote such a subgraph by G_{ev}. In [6], Pyber strengthened Theorem 1.3 by showing the following:

1.4 Theorem (Pyber)
For any simple graph G having n vertices, if G_{ev} is a forest, then G has a path-decomposition with at most $\lfloor \frac{n}{2} \rfloor$ paths.

Fan [4] subsequently improved upon Pyber's Theorem.

1.5 Theorem (Fan)
For any simple graph G having n vertices, if every block of G_{ev} is a triangle-free graph where every vertex has degree at most three, then G has a path-decomposition with at most $\lceil \frac{n}{2} \rceil$ paths.

It is easy to see that if the assumption of connectedness is omitted from Gallai's conjecture, then it is false. While a bound of $\frac{n}{2}$ is not possible for disconnected graphs, Donald [3] showed that $\frac{3}{4}n$ is possible. This bound was later improved to $\frac{2}{3}n$ by Dean and Koudier [2]. In their paper, they actually proved a stronger result which we will describe. For a graph G, let $p(G)$ denote the number of odd vertices and let $q(G)$ denote the number of non-isolated even vertices. Dean and Koudier proved the following:

1.6 Theorem (Dean, Koudier)
For any simple graph having n vertices, there is a path-decomposition with at most $\frac{p(G)}{2} + \lfloor \frac{2}{3}q(G) \rfloor$ paths.

The bound of $\frac{2}{3}n$ can be seen to be best possible by taking a graph consisting of vertex-disjoint triangles. In this paper, we show that the bound in the above paper can be greatly improved when the girth is larger. We shall prove the following theorem:

1.7 Theorem
For every simple graph G having girth $g \geq 4$, there is a path-decomposition having at most $\frac{p(G)}{2} + \left\lfloor \left(\frac{g+1}{2g}\right) q(G) \right\rfloor$ paths.

We have attempted to make this paper as self-contained as possible. While the material in Sections 2 and 3 is largely known, our treatment involving directed graphs is different and contains various concepts and notation which will be used later. In Section 3, we shall provide various lemmas, two of which can be found in [4]. Notably, we give a short proof of Lemma 3.1 based on directed graphs. In Section 4, we begin our assault on the main theorem by proving a lemma which will be a key component in the proof of Theorem 1.7. The proof of the main theorem is given in Sections 5 and 6. It hinges on the case $g = 4$ and this case is dealt with in the last section.

For two subgraphs H and K of a graph we let $H \cup K$ denote the subgraph with vertex set $V(H) \cup V(K)$ and edge set $E(H) \cup E(K)$. For a graph G and a vertex $u \in V(G)$, we shall let $E_G(u)$ denote the set of edges incident with u. For a subset $A \subseteq E_G(u)$, we let $N_{G,A}(u)$ denote the set of vertices v for which $uv \in A$. The graphs considered in this paper are simple. As such, we shall often denote a path by its sequence of vertices, eg. $v_1 v_2 \cdots v_k$. For a path P and vertices u and v lying on P let $P[uv]$ denote the portion of the path lying between (and including) u and v. In this paper, we shall frequently make use of the classic "lollipop" construction for a path. Suppose $P = v_0 v_1 \cdots v_k$ is a path where $k \geq 2$ and $e = v_i v_k$ is an edge where $1 \leq i < k-1$. We refer to $P \cup \{e\}$ as a **P-lollipop**. Now $P' = P[v_0 v_i] \cup P[v_{i+1} v_k] \cup \{e\}$ is seen to be a new path starting at v_0 and terminating at v_{i+1}. We shall refer to P' as the path obtained from the lollipop $P \cup \{e\}$. In the event that $e = v_0 v_k$, there are two possible paths obtained from the lollipop.

2 The Lovász Construction

In this section, we shall describe the path and cycle constructions used in the proof of Theorem 1.2. Here we use a new approach using directed graphs which is inspired in part by the work of Cai [1].

Let $\overrightarrow{G}$ be a directed graph. For every vertex $v \in V(\overrightarrow{G})$, let $d^+_{\overrightarrow{G}}(v)$ and $d^-_{\overrightarrow{G}}(v)$ denote the out-degree and in-degree at v, respectively. For every subset $S \subseteq V(\overrightarrow{G})$, let $d^+_{\overrightarrow{G}}(S)$ and $d^-_{\overrightarrow{G}}(S)$ denote the number of out-directed arcs leaving S and the number of in-directed arcs entering S, respectively. For every vertex v let $\delta_{\overrightarrow{G}}(v) = d^+_{\overrightarrow{G}}(v) - d^-_{\overrightarrow{G}}(v)$. For every subset $S \subseteq V(\overrightarrow{G})$, let $\delta_{\overrightarrow{G}}(S) = \delta^+_{\overrightarrow{G}}(S) - \delta^-_{\overrightarrow{G}}(S)$.

We have the following elementary lemma whose proof is left to the reader.

2.1 Lemma

(i) For every subset $S \subseteq V(\overrightarrow{G})$, we have $\delta_{\overrightarrow{G}}(S) = \sum_{v\in S} \delta_{\overrightarrow{G}}(v)$.

(ii) $\delta_{\overrightarrow{G}}(V(\overrightarrow{G})) = 0$.

(iii) For every $S \subseteq V(\overrightarrow{G})$ where $\delta_{\overrightarrow{G}}(S) > 0$, there is a directed path from a vertex $x \in S$ where $\delta_{\overrightarrow{G}}(x) > 0$ to a vertex $y \in V(\overrightarrow{G})\backslash S$ where $\delta_{\overrightarrow{G}}(y) < 0$.

Let G be a simple graph and let $\mathcal{P}$ be a path-cycle-decomposition. For every vertex $v \in V(G)$, we let $\mathcal{P}(v)$ denote the number of paths of $\mathcal{P}$ which terminate at v. Let x be a vertex of G. We shall define a directed graph $\overrightarrow{G}_{\mathcal{P},x}$ associated with $\mathcal{P}$ and x in the the following manner. We define the vertex set of $\overrightarrow{G}_{\mathcal{P},x}$ to be $N_G(x) \cup \{x\}$. We shall define the arcs of $\overrightarrow{G}_{\mathcal{P},x}$ using the paths of $\mathcal{P}$. For every path $P \in \mathcal{P}$ and every terminal vertex y of P belonging to $N_G(x)$ we shall define an arc as follows: if x is not on the path P or $xy \in E(P)$, then let yx be an arc of $\overrightarrow{G}_{\mathcal{P},x}$. Otherwise, if x is on the path P, let $C = xyy_1y_2\cdots y_kx$ be the unique (undirected) cycle of $P\cup\{xy\}$ which contains the edge xy. We define yy_k to be an arc of $\overrightarrow{G}_{\mathcal{P},x}$. By the way in which we have defined the arcs of $\overrightarrow{G}_{\mathcal{P},x}$ we see that $d^-_{G_{\mathcal{P},x}}(v) \le 1$ and $d^+_{G_{\mathcal{P},x}}(v) = \mathcal{P}(v)$ for all $v \in N_G(x)$. Furthermore, $d^+_{G_{\mathcal{P},x}}(x) = 0$.

Let $A \subseteq E(G)$ be a set of edges incident with x and let $H = G\backslash A$. Suppose that H has a path-cycle-decomposition $\mathcal{P}$. Let $xy \in A$ and suppose there exists a directed path $\overrightarrow{P}$ in $\overrightarrow{G}_{\mathcal{P},x}$ from y to x. Let $\overrightarrow{P} = y_0y_1\cdots y_kx$ where $y_0 = y$. We observe that $y_i \in N_H(x)$ for $i = 1,\ldots,k$. Thus $d^-_{\overrightarrow{G}_{\mathcal{P},x}}(y_i) = 1,\ i = 1,\ldots,k$. We call $\overrightarrow{P}$ a $(\mathbf{y},\mathbf{x})-$**path** of $\overrightarrow{G}_{\mathcal{P},x}$. For every arc $y_{i-1}y_i, \imath = 1,2,\ldots,k$ in $\overrightarrow{P}$, there is a corresponding path $Q_{i-1} \in \mathcal{P}$ where y_{i-1} is a terminal vertex of Q_{i-1} and y_ix is an edge of Q_{i-1}. Furthermore, the arc y_kx in $\overrightarrow{P}$ corresponds to a path $Q_k \in \mathcal{P}$ where y_k is a terminal vertex of Q_k and x is not a vertex of Q_k. We call the $k+1$-tuple $(Q_0,Q_1,\ldots,Q_k)$ a $(\mathbf{y},\mathbf{x})$**-path-fan** of $\mathcal{P}$. We also refer to $(Q_0,Q_1,\ldots,Q_k)$ as the path-fan corresponding to $\overrightarrow{P}$ (conversely, $\overrightarrow{P}$ is the directed path corresponding to $(Q_0,Q_1,\ldots,Q_k)$). For such a path-fan, we shall define new paths $Q'_i,\ i = 0,1,\ldots,k$ as follows:

$$Q'_i = \begin{cases} (Q_i\backslash\{y_{i+1}x\}) \cup \{y_ix\}, & \text{for } i = 0,\ldots,k-1; \\ Q_i \cup y_ix, & \text{for } i = k. \end{cases}$$

For $i = 0, \ldots, k-1$, the path Q'_i is seen to be the path obtained from the Q_i-lollipop $Q_i \cup \{xy_i\}$. We call $(Q'_0, Q'_1, \ldots, Q'_k)$ the **extended path-fan** of $(Q_0, Q_1, \ldots, Q_k)$. One can easily check that

$$\mathcal{P}^* = (\mathcal{P} \backslash \{Q_0, Q_1, \ldots, Q_k\}) \cup \{Q'_0, \ldots, Q'_k\}$$

is a path-cycle-decomposition of $H \cup \{xy\}$. Moreover, $|\mathcal{P}^*| = |\mathcal{P}|$, $\mathcal{P}^*(x) = \mathcal{P}(x) + 1$, $\mathcal{P}^*(y) = \mathcal{P}(y) - 1$, and $\mathcal{P}^*(v) = \mathcal{P}(v)$ for all $v \in V(G) \backslash \{x, y\}$. We shall refer to $\mathcal{P}^*$ as a **xy-extension** of $\mathcal{P}$ at x. In shorthand, we shall simply write $\mathcal{P} \underset{xy}{\overset{x}{\to}} \mathcal{P}^*$. When such a xy-extension exists, we shall say that the edge xy is **$\mathcal{P}$-addible at x**. Equivalently, xy is $\mathcal{P}$-addible at x if there exists a (y, x)-path in $\overrightarrow{G}_{\mathcal{P},x}$.

Suppose that xy and xz are both $\mathcal{P}$-addible edges at x. Then there are (y, x)- and (z, x)- paths in $\overrightarrow{G}_{\mathcal{P},x}$, say

$$\overrightarrow{P}_y = y_0 y_1 \cdots y_s x \quad \text{and} \quad \overrightarrow{P}_z = z_0 z_1 \cdots z_t x$$

respectively. The vertices $y_1, y_2, \ldots, y_s$ and $z_1, z_2, \ldots, z_t$ all have in-degree one in $\overrightarrow{G}_{\mathcal{P},x}$. Because of this, $\overrightarrow{P}_y$ and $\overrightarrow{P}_z$ share only one common vertex, namely x. Let $(Q_0, \ldots, Q_s)$ and $(R_0, \ldots, R_t)$ be the $(y, x)-$ and (z, x)-path-fans corresponding to $\overrightarrow{P}_y$ and $\overrightarrow{P}_z$, respectively. Since $\overrightarrow{P}_y$ and $\overrightarrow{P}_z$ share only the vertex x, the paths $Q_0, Q_1, \ldots, Q_{s-1}, R_0, R_1, \ldots, R_{t-1}$ are distinct. However, when z_t is also a terminal vertex of Q_s, it is possible that $Q_s = R_t$. See Figure 1. Suppose this is the case. Let C be the cycle $C = Q_s \cup \{xy_s, xz_t\}$. Let $(Q'_0, \ldots, Q'_s)$ and $(R'_0, \ldots, R'_t)$ be the extended path-fans for $(Q_0, \ldots, Q_s)$ and $(R_0, \ldots, R_t)$ respectively. We shall define a path-cycle-decomposition $\mathcal{P}^*$ of $H \cup \{xy, xz\}$ as follows: let

$$\mathcal{P}^* = (\mathcal{P} \backslash \{Q_0, Q_1, \ldots, Q_s, R_0, R_1, \ldots, R_t\}) \cup \{Q'_0, Q'_1, \ldots, Q'_{s-1} R'_0, R'_1, \ldots, R'_{t-1}, C\}.$$

We observe that $\mathcal{P}^*$ contains one less path than $\mathcal{P}$ but one more cycle (namely, C). The set $\mathcal{P}^*$ is seen to be a path-cycle-decomposition of $H \cup \{xy, xz\}$ where $|\mathcal{P}^*| = |\mathcal{P}|$. We shall refer to the set $\{xy_s, xz_t\}$ as a **$\mathcal{P}$-pair** of edges at x. In shorthand, we shall denote the above transformation from $\mathcal{P}$ to $\mathcal{P}^*$ as $\mathcal{P} \underset{\{xy,xz\}}{\overset{x}{\rightrightarrows}} \mathcal{P}^*$. Here $\rightrightarrows$ is used to emphasize that the transformation results in a path-cycle-decomposition containing a cycle which contains xy and xz.

Let $B \subseteq A$ and suppose that each edge of B is $\mathcal{P}$-addible at x. We say that an edge $xy \in B$ is **$\mathcal{P}$-solitary** at x with respect to B if there is no $\mathcal{P}$-pair of edges in B which contains xy. Suppose that all edges of B

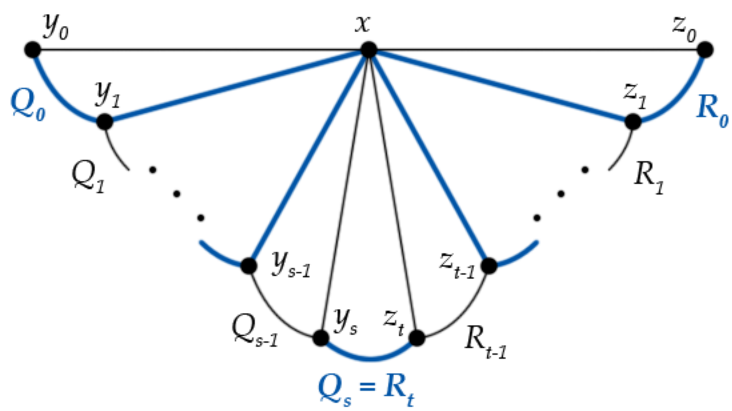

Figure 1: Two path-fans where $Q_s = R_t$

are $\mathcal{P}$-solitary. Let $B = \{xy_1, xy_2, \dots, xy_t\}$. Then there is a sequence of path-cycle-decompositions $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_t$ where

$$\mathcal{P} \underset{xy_1}{\overset{x}{\rightarrow}} \mathcal{P}_1 \underset{xy_2}{\overset{x}{\rightarrow}} \mathcal{P}_2 \cdots \mathcal{P}_{t-1} \underset{xy_t}{\overset{x}{\rightarrow}} \mathcal{P}_t.$$

The collection $\mathcal{P}_t$ is seen to be a path-cycle-decomposition for $H \cup B$ having the properties that $\mathcal{P}_t(x) = \mathcal{P}(x) + |B|$, $\mathcal{P}_t(y_i) = \mathcal{P}(y_i) - 1$, $i = 1, \dots, t$, and $\mathcal{P}_t(v) = \mathcal{P}(v)$ for all $v \in V(G) \backslash \{x, y_1, \dots, y_t\}$. In shorthand, we shall simply write $\mathcal{P} \underset{B}{\overset{x}{\rightarrow}} \mathcal{P}_t$. We shall say that the set B is $\mathcal{P}$-addible at x.

More generally, when not all the edges of B are necessarily solitary, B can be partitioned as $B = B_0 \cup B_1 \cup \cdots B_k$ where B_0 is the set of all $\mathcal{P}$-solitary edges of B, and $B_1, B_2, \dots, B_k$ are the sets of $\mathcal{P}$-pairs contained in B. Then there exists a sequence of path-cycle-decompositions $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k, \mathcal{P}_{k+1}$ such that

$$\mathcal{P} \underset{B_0}{\overset{x}{\rightarrow}} \mathcal{P}_1 \underset{B_1}{\overset{x}{\rightrightarrows}} \mathcal{P}_2 \cdots \mathcal{P}_k \underset{B_k}{\overset{x}{\rightrightarrows}} \mathcal{P}_{k+1}.$$

$\mathcal{P}_{k+1}$ is seen to be a path-cycle-decomposition of $H \cup B$ where $\mathcal{P}_{k+1}$ has k more cycles than $\mathcal{P}$ (containing the pairs $B_1, B_2, \dots, B_k$) and k fewer paths. We see that $|\mathcal{P}^{k+1}| = |\mathcal{P}|$ and each of the k new cycles created contain x. In shorthand, we shall denote such a transformation by $\mathcal{P} \underset{B}{\overset{x}{\Rrightarrow}} \mathcal{P}^{k+1}$.

3 Some Lemmas about Path-Decompositions

In this section, we shall provide some lemmas, the first two of which are due to Fan [4]. For the purpose of being self-contained we shall provide proofs of these lemmas. Notably, we shall provide a different, short proof of the first lemma which is based on the directed graph $\overrightarrow{G}_{\mathcal{P},x}$ introduced in Section 2.

3.1 Lemma
Let x be a vertex in G and let $A = \{xy_1, xy_2, \ldots, xy_k\}$ be a set of edges which are incident with x. Let $\mathcal{P}$ be a path-decomposition of $H = G \backslash A$. If $\sum_{i=1}^{k} \mathcal{P}(y_i) > |\{v \in N_G(x) \mid \mathcal{P}(v) = 0\}|$, then there exists $y \in \{y_1, \ldots, y_k\}$ such that xy is $\mathcal{P}$-addible at x.

Proof. Suppose $\sum_{i=1}^{k} \mathcal{P}(y_i) > |\{v \in N_G(x) \mid \mathcal{P}(v) = 0\}|$. Let $Y = \{y_1, \ldots, y_k\}$ and $Z = \{v \in N_G(x) \mid \mathcal{P}(v) = 0\}$. One sees that

$$\begin{aligned} \delta_{\overrightarrow{G}_{\mathcal{P},x}}(y) &= \mathcal{P}(y), \ \forall y \in Y \backslash Z, \\ \delta_{\overrightarrow{G}_{\mathcal{P},x}}(z) &\geq -1, \ \forall z \in Z, \\ \delta_{\overrightarrow{G}_{\mathcal{P},x}}(v) &\geq 0, \ \forall v \in N_G(x) \backslash (Y \cup Z), \\ \text{and } \delta_{\overrightarrow{G}_{\mathcal{P},x}}(x) &\leq 0. \end{aligned}$$

Let $S = Y \cup Z$. From the above, we see that $\delta_{\overrightarrow{G}_{\mathcal{P},x}}(Y) = \sum_{i=1}^{k} \mathcal{P}(y_i)$ and $\delta_{\overrightarrow{G}_{\mathcal{P},x}}(Z) \geq -|Z|$. Thus by our assumptions, we see that $\delta_{\overrightarrow{G}_{\mathcal{P},x}}(S) = \delta_{\overrightarrow{G}_{\mathcal{P},x}}(Y) + \delta_{\overrightarrow{G}_{\mathcal{P},x}}(Z) > 0$. It is also seen that $\delta_{\overrightarrow{G}_{\mathcal{P},x}}(x) < 0$. By Lemma 2.1 (iii), it follows that there is a directed path in $\overrightarrow{G}_{\mathcal{P},x}$ from a vertex $y \in Y$ where $\delta_{\overrightarrow{G}_{\mathcal{P},x}}(y) > 0$ to the vertex x. It now follows that the edge xy is $\mathcal{P}$-addible at x. $\square$

3.2 Lemma
Let x be a vertex in a graph G, $A \subseteq E_G(x)$ and let $H = G \backslash A$. Let $r = |\{v \in N_G(x) \mid \mathcal{P}(v) = 0\}|$. Suppose that $\mathcal{P}$ is a path-decomposition of H where $\mathcal{P}(y) \geq 1$ for all $y \in N_{G,A}(x)$. Then there exists a $\mathcal{P}$-addible set $B \subseteq A$ and a path-decomposition $\mathcal{P}^*$ of $H \cup B$ such that $|B| \geq \left\lceil \frac{|A|-r}{2} \right\rceil$ and $\mathcal{P} \xrightarrow[B]{x} \mathcal{P}^*$.

Moreover, if $r = 0$, then for all edges $xy \in A$, we may choose B such that $xy \in B$.

Proof. By induction on A. Suppose $|A| = 1$ and $A = \{xy\}$. If $r \geq 1$, then there is nothing to prove. So we may assume that $r = 0$. Since $\mathcal{P}(y) \geq 1$, Lemma 3.1 implies that xy is $\mathcal{P}$-addible at x. The lemma is now seen to hold by taking $B = A$.

Suppose that the lemma holds when $|A| = k$ (where $k \geq 1$). Suppose $|A| = k+1$. If $r \geq k+1$, then there is nothing to prove. So we may assume that $|A| > r$. Then

$$\sum_{xy \in A} \mathcal{P}(y) \geq |A| > r = |\{v \in N_G(x) \mid \mathcal{P}(v) = 0\}|.$$

By Lemma 3.1, there exists $xy \in A$ such that xy is $\mathcal{P}$-addible at x. Note that when $r = 0$, every edge of A is $\mathcal{P}$-addible and hence xy can be any edge of A. Let $\mathcal{P} \underset{xy}{\overset{x}{\rightarrow}} \mathcal{P}'$ and let $A' = A \backslash \{xy\}$ and $H' = H \cup \{xy\}$. Let $r' = |\{v \in N_G(x) \mid \mathcal{P}'(v) = 0\}|$. Given that $\mathcal{P}'(y) = \mathcal{P}(y) - 1$, $\mathcal{P}'(x) = \mathcal{P}(x) + 1$, and $\mathcal{P}'(v) = \mathcal{P}(v)$ for all $v \in N_G(x) \backslash \{x, y\}$, it is seen that $r' \leq r+1$ and $\mathcal{P}'(y) \geq 1$ for all $xy \in A'$. Since $|A'| = k$, we can apply the inductive assumption to $H', A', \mathcal{P}'$ and r'. Thus there exists a $\mathcal{P}'$-addible set $B' \subseteq A'$ at x such that $|B'| \geq \left\lceil \frac{|A'| - r'}{2} \right\rceil$. Let $B = B' \cup \{xy\}$. Then B is seen to be $\mathcal{P}$-addible at x. Furthermore

$$|B| = |B'| + 1 \geq \left\lceil \frac{|A'| - r'}{2} \right\rceil + 1 \geq \left\lceil \frac{|A| - r - 2}{2} \right\rceil + 1 = \left\lceil \frac{|A| - r}{2} \right\rceil.$$

The proof now follows by induction.

In the case where $r = 0$, the above proof shows that for any edge $e \in A$, we may choose the set B so that it contains e. $\square$

Let G be a simple graph and let $F = G_{ev}$. Let $x \in V(F)$ and let $H = G \backslash E_F(x)$. Assume that $\mathcal{P}$ is a path-decomposition for H. We shall need the following two technical lemmas.

3.3 Lemma
Suppose that x is an odd vertex of F and $d_F(v) \geq 4$ for at most one vertex $v \in N_F(x)$. Then G has a path-decomposition $\mathcal{P}^*$ where $|\mathcal{P}^*| = |\mathcal{P}|$, $\mathcal{P}^*(x) = \mathcal{P}(x) + 1$, and $\mathcal{P}^*(v) = \mathcal{P}(v)$ for all $v \in V(G) \backslash (N_G(x) \cup \{x\})$.

Proof. Let $E_F(x) = \{xy_1, xy_2, \ldots, xy_{2k+1}\}$. We observe that all the vertices in $N_G(x) \cup \{x\}$ are odd in H and as such $\mathcal{P}(v) \geq 1$ for all $v \in N_G(x) \cup \{x\}$. Now Lemma 3.1 implies that each edge $xy_i,\ i = 1, \ldots, 2k+1$ is $\mathcal{P}$-addible at x. Thus if $d_F(x) = 1$ (or $k = 0$), then the desired path-decomposition $\mathcal{P}^*$ of G is obtained via $\mathcal{P} \underset{xy_1}{\overset{x}{\rightarrow}} \mathcal{P}^*$. Thus we may assume that $d_F(x) \geq 3$ and

$k \geq 1$. Given that $d_F(v) \geq 4$ for at most one vertex $v \in N_F(x)$, we may assume that $d_F(y_i) \leq 3,\ i = 2, \dots, 2k+1$.

By Lemma 3.2, there exists $B \subseteq A$ such that $xy_1 \in B,\ |B| \geq k+1$, and B is $\mathcal{P}$-addible at x. We may assume that $B = \{xy_1, xy_2, \dots, xy_l\}$ where $l \geq k+1$. Let $H_0 = H \cup B$, and for $i = 1, 2, \dots, 2k+1-l$ let $H_i = H_{i-1} \cup \{xy_{l+i}\}$. Let $\mathcal{P}_0$ be a path-decomposition of H_0 where $\mathcal{P} \underset{B}{\overset{x}{\rightarrow}} \mathcal{P}_0$. Then $\mathcal{P}_0(x) = \mathcal{P}(x) + |B| \geq 3,\ \mathcal{P}_0(y_i) = \mathcal{P}(y_i) - 1,\ i = 1, \dots, l$, and $\mathcal{P}_0(v) = \mathcal{P}(v)$ for all $v \in V(G) \backslash \{x, y_1, \dots, y_l\}$. Since $d_F(y_{l+1}) \leq 3$, we have $|N_F(y_{l+1}) \backslash \{x\}| \leq 2$. Thus

$$|\{v \in N_G(y_{l+1}) \bigm| \mathcal{P}_0(v) = 0\}| \leq 2 < \mathcal{P}_0(x).$$

Thus by Lemma 3.1, the edge xy_{l+1} is $\mathcal{P}_0$-addible at y_{l+1} and there is a path-decomposition $\mathcal{P}_1$ of H_1 where $\mathcal{P}_0 \underset{xy_{l+1}}{\overset{y_{l+1}}{\rightarrow}} \mathcal{P}_1$ and $\mathcal{P}_1(x) = \mathcal{P}_0(x) - 1$. Suppose now that for some $1 \leq j < 2k+1-l$ we have the sequence of path-decompositions $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_j$ where

$$\mathcal{P} \underset{B}{\overset{x}{\rightarrow}} \mathcal{P}_0 \underset{xy_{l+1}}{\overset{y_{l+1}}{\rightarrow}} \mathcal{P}_1 \rightarrow \cdots \rightarrow \mathcal{P}_{j-1} \underset{xy_{l+j}}{\overset{y_{l+j}}{\rightarrow}} \mathcal{P}^j.$$

Then

$$\mathcal{P}_j(x) = \mathcal{P}(x) + |B| - j = \mathcal{P}(x) + l - j \geq 3.$$

Again, since $d_F(y_{l+j+1}) \leq 3$ we have that $|\{v \in N_G(y_{l+j+1}) \bigm| \mathcal{P}_{j+1}(v) = 0\}| \leq 2 < \mathcal{P}_j(x)$. Thus by Lemma 3.1, the edge xy_{l+j+1} is $\mathcal{P}_j$-addible at y_{l+j+1} and consequently there is path-decomposition $\mathcal{P}_{j+1}$ of H_{j+1} where $\mathcal{P}_j \underset{xy_{l+j+1}}{\overset{y_{l+j+1}}{\rightarrow}} \mathcal{P}_{j+1}$. Proceeding by induction, it follows that there is a sequence

$$\mathcal{P} \underset{B}{\overset{x}{\rightarrow}} \mathcal{P}_0 \underset{xy_{l+1}}{\overset{y_{l+1}}{\rightarrow}} \mathcal{P}_1 \rightarrow \cdots \rightarrow \mathcal{P}_{2k-l} \underset{xy_{2k+1}}{\overset{y_{2k+1}}{\rightarrow}} \mathcal{P}^{2k+1-l}$$

where $\mathcal{P}^* = \mathcal{P}^{2k+1-l}$ is a path-decomposition for G. We note that $|\mathcal{P}^*| = |\mathcal{P}|$ and $\mathcal{P}^*(x) = \mathcal{P}(x) + l - (2k+1-l) \geq \mathcal{P}(x) + 1 \geq 2$. □

3.4 Lemma
If $d_F(x) = 3$, then either

(i) There exists a path-decomposition $\mathcal{P}^*$ of G where $\mathcal{P}^*(x) \geq 2$ and $\mathcal{P}^*(v) = \mathcal{P}(v)$ for all $v \in V(G) \backslash (N_F(x) \cup \{x\})$

or

(ii) There exists $z \in N_F(x)$ such that $d_F(z) \geq 4$.

Proof. Suppose that $d_F(x) = 3$. We observe that the vertices of $N_G(x) \cup \{x\}$ are odd in H and consequently $\mathcal{P}(v) \geq 1$ for all $v \in N_G(x) \cup \{x\}$. It follows by Lemma 3.2 that there exists a $\mathcal{P}$-addible set $B \subset E_F(x)$ where $|B| = 2$, and there is a path-decomposition $\mathcal{P}_0$ of $H_0 = H \cup B$ such that $\mathcal{P} \underset{B}{\overset{x}{\to}} \mathcal{P}_0$. Then $\mathcal{P}_0(x) = \mathcal{P}(x) + 2 \geq 3$ and $\mathcal{P}_0(v) = \mathcal{P}(v)$ for all $v \in N_G(v) \backslash (N_{G,B}(x) \cup \{x\})$. Let $\{z\} = N_F(x) \backslash N_{F,B}$. Suppose that $|\{v \in N_G(z) \mid \mathcal{P}_0(v) = 0\}| \leq 2$. Since $\mathcal{P}_0(x) \geq 3$, Lemma 3.1 implies that the edge xz is $\mathcal{P}_0$-addible at z. Let $\mathcal{P}_1$ be a path-decomposition for $H_0 \cup \{xz\} = G$ where $\mathcal{P}_0 \underset{xz}{\overset{z}{\to}} \mathcal{P}_1$. Then $\mathcal{P}^* = \mathcal{P}_1$, has the properties as described in (i). On the other hand, if $|\{v \in N_G(z) \mid \mathcal{P}_0(v) = 0\}| \geq 3$, then seeing as $\{v \in V(G) \mid \mathcal{P}(v) = 0\} \subseteq V(F) \backslash \{x\}$, it follows that $d_F(z) \geq 4$. Thus (ii) holds in this case. $\square$

4 Path-Cycle-Decompositions into Path-Decompositions

A major step towards proving the main theorem is showing that certain path-cycle-decompositions can be transformed into path-decompositions having almost the same cardinality. The lemma presented in this section serves just this purpose.

4.1 Lemma
Let G be a simple graph having girth g where $g \geq 4$. Let P be a path and let $\mathcal{C}$ be an edge-disjoint collection of cycles having the property that $E(C) \cap E(P) = \emptyset$ and $V(C) \cap V(P) \neq \emptyset$ for all $C \in \mathcal{C}$. Let H be the subgraph where $H = P \cup \bigcup_{C \in \mathcal{C}} C$. If $|V(P)| + |\mathcal{C}| \leq 2g$, then there is a path decomposition of H having at most $|\mathcal{C}| + 1$ paths.

Proof. By induction on $|\mathcal{C}|$. If $|\mathcal{C}| = 0$ (that is, $\mathcal{C} = \emptyset$) then $\{P\}$ is the desired path decomposition of H. Assume that the lemma holds for all P and $\mathcal{C}$ where $|\mathcal{C}| \leq k$ for some $k \geq 0$. Suppose $|\mathcal{C}| = k+1$ and let $v_1, v_2, \ldots, v_p$ be the consecutive vertices of P. We observe that $p \leq 2g - |\mathcal{C}| \leq 2g - 1$.

We shall first assume that $v_1, v_p \in \bigcup_{C \in \mathcal{C}} V(C)$. The bulk of the proof is devoted to this case. Let $C, D \in \mathcal{C}$ where $v_1 \in V(C)$ and $v_p \in V(D)$. Note that it is possible that $C = D$. Suppose there exists a vertex $v \in V(C) \backslash V(P)$ where $vv_1 \in E(C)$. Then $P' = P \cup vv_1$ is seen to be a path having $p' = p + 1$ vertices. Let P_0 be the path $C \backslash \{vv_1\}$. Let $\mathcal{C}' = \mathcal{C} \backslash \{C\}$ and let $H' = H \backslash E(P_0)$. Now P' and $\mathcal{C}'$ are such that $p' + |\mathcal{C}'| = p + |\mathcal{C}| \leq 2g$. Since $|\mathcal{C}'| = |\mathcal{C}| - 1 = k$, there exists a path decomposition $\mathcal{P}'$ of H' having at most $|\mathcal{C}'| + 1$ paths. Now $\mathcal{P} = \mathcal{P}' \cup \{P_0\}$ is seen to be a path decomposition of H where $|\mathcal{P}| = |\mathcal{P}'| + 1 \leq |\mathcal{C}'| + 2 = |\mathcal{C}| + 1$. In this case, the lemma holds for $\mathcal{C}$. Thus we may assume that no such vertex v exists for v_1. Letting x_1, x_2 be the vertices such that $x_1v_p,\ x_2v_p \in E(C)$, we may assume that $x_1, x_2 \in V(P)$. Similarly, if y_1, y_2 are vertices such that $x_1v_1,\ x_2v_1 \in E(D)$, we may assume that $y_1, y_2 \in V(P)$.

Since G has girth g, it follows that $x_1, x_2 \in \{v_g, v_{g+1}, \dots, v_p\}$ and $y_1, y_2 \in \{v_1, \dots, v_g\}$. Given that $p \leq 2g-1$, there are only three possible pairs for $\{x_1, x_2\}$, namely $\{v_g, v_{p-1}\}$, $\{v_g, v_p\}$, and $\{v_{g+1}, v_p\}$. Likewise, the three possible pairs for $\{y_1, y_2\}$ are $\{v_1, v_g\}$, $\{v_2, v_g\}$, and $\{v_1, v_{g-1}\}$. Notice that when $\{x_1, x_2\} = \{v_g, v_p\}$ we have that $p \geq 2g-2$ and $1 \leq |\mathcal{C}| \leq 2$. When $\{x_1, x_2\} = \{v_g, v_{p-1}\}$ or $\{v_{g+1}, v_p\}$, then $p = 2g-1$ and $|\mathcal{C}| = 1$. We shall consider three cases:

Case A: $\{x_1, x_2\} = \{v_g, v_p\}$.

Proof In this case, if $\{y_1, y_2\} = \{v_1, v_g\}$, then $v_1v_gv_pv_1$ is seen to be a triangle of G, contrary to our assumption that $g \geq 4$. Thus $\{y_1, y_2\} = \{v_1, v_{g-1}\}$ and $C = D$. Since $v_1v_gv_{g-1}v_pv_1$ is seen to be a cycle, it follows that $g = 4$. Suppose $v_2 \notin V(C)$. Let $P' = (P \backslash \{v_1v_2\}) \cup \{v_1v_p\}$ and let $P_0 = (C \backslash \{v_1v_p\}) \cup \{v_1v_2\}$. Then P_0 and P' are seen to be paths. Let $\mathcal{C}' = \mathcal{C} \backslash \{C\}$ and let $H' = H \backslash E(P_0)$. Applying the inductive assumption with P' and $\mathcal{C}'$ in place of P and $\mathcal{C}$, there exists a path-decomposition $\mathcal{P}'$ for H' having at most $|\mathcal{C}'| + 1 = |\mathcal{C}|$ paths. Now $\mathcal{P} = \mathcal{P}' \cup \{P_0\}$ is seen to be a path-decomposition of H having at most $|\mathcal{C}| + 1$ paths. From the above, we may assume that $v_2 \in V(C)$ and (by symmetry) $v_{p-1} \in V(C)$.

Let Q be the path $C \backslash \{v_1v_p\}$. Let v be the neighbour of v_{p-1} on Q such that v lies on the path $Q[v_{p-1}v_p]$. Suppose $v \notin V(P)$. Let P' be the path $(P \backslash \{v_{p-1}v_p\}) \cup v_{p-1}v \cup \{v_1v_p\}$ and let P_0 be the path obtained from the Q-lollipop $Q \cup \{v_{p-1}v_p\}$ (noting that $v_{p-1}v \notin E(P_0)$). Let $H' = H \backslash E(P_0)$. Applying the inductive assumption (as before), we can find a path-decomposition $\mathcal{P}'$ for H' such that $\mathcal{P} = \mathcal{P}' \cup \{P_0\}$ is a path-decomposition of G having at most $|\mathcal{C}| + 1$ paths. Thus we may assume that $v \in V(P)$. Given that $g = 4$ and $p \leq 2g - 1 = 7$ there is only one choice for v, namely $v = v_2$. Given that v lies on the path $Q[v_{p-1}v_p]$, when we travel from v_1 to v_p along Q, we visit the vertices v_4, v_{p-1}, v_2, v_3 in this order. Let v' be the vertex adjacent to v_2 on the path $Q[v_2v_3]$. Clearly $v' \notin V(P)$ since G is triangle-free. Let $P' = v_1v_pv_3 \cup P[v_3v_{p-1}] \cup v_{p-1}v_2v'$ and let $P_0 = Q[v'v_3] \cup P[v_3v_1] \cup Q[v_1v_{p-1}] \cup v_{p-1}v_p$. See Figure 2 . One sees that P' and P_0 are paths of G which form a path-decomposition of $P \cup C$. Let $H' = H \backslash E(P_0)$ and let $\mathcal{C}' = \mathcal{C} \backslash \{C\}$. Now P' and $\mathcal{C}'$ can play the roles of

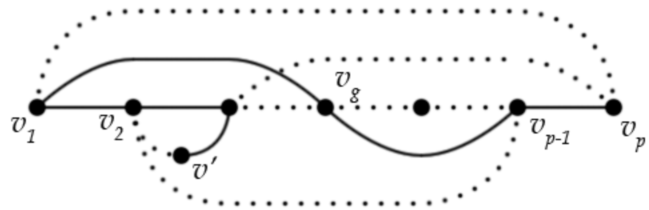

Figure 2: The paths P' (dotted line) and P_0 (solid line)

P and $\mathcal{C}$ in H'. Applying the inductive assumption, there exists a path-decomposition $\mathcal{P}'$ of H' containing at most $|\mathcal{C}'| + 1 = |\mathcal{C}|$ paths. Now $\mathcal{P} = \mathcal{P}' \cup \{P_0\}$ is seen to be a path-decomposition of H having at most $|\mathcal{C}| + 1$ paths. This completes the proof of Case A. $\square$

Case B: $\{x_1, x_2\} = \{v_{g+1}, v_p\}$.

Proof As remarked before, $p = 2g - 1$ and $|\mathcal{C}| = 1$ in this case. Thus $C = D$. If $\{y_1, y_2\} = \{v_1, v_g\}$, then by symmetry, we could apply the proof of Case A (with y_1, y_2 playing the role of x_1, x_2). Thus we may assume that $\{y_1, y_2\} = \{v_1, v_{g-1}\}$. We see that $v_1 v_{g+1} v_g v_{g-1} v_p v_1$ is a 5-cycle, and thus it follows that $g = 4$ or 5 and $p = 7$ or 9. When $v_2 \notin V(C)$ or $v_{p-1} \notin V(P)$, then one can apply the same arguments as in Case A. Thus we may assume that $v_2, v_{p-1} \in V(P)$. Let Q be the path $C \backslash \{v_1 v_p\}$ and let v be the neighbour of v_{p-1} on Q such that v lies on the path $Q[v_{p-1} v_p]$. If $v \notin V(P)$, then we can apply the same arguments as in Case A. Thus we may assume $v \in V(P)$, or more accurately, $v \in \{v_2, v_3, \ldots, v_{g-2}, v_g\}$. Since any choice for v in this set is seen to create a 3- or 4-cycle, we conclude that $g = 4$ and $p = 7$. There is now only one choice for v, namely $v = v_2$. We now construct similar paths P' and P_0 as we did in the proof of Case A, and use our inductive assumption to find a path-decomposition of H having at most $|\mathcal{C}| + 1$ paths. $\square$

Case C: $\{x_1, x_2\} = \{v_g, v_{p-1}\}$.

Proof From Cases A and B, we may assume by symmetry that $\{y_1, y_2\} = \{v_2, v_g\}$. Then $v_1 v_2 v_p v_{p-1} v_1$ is seen to be a 4-cycle and hence $g = 4$, $p = 7$, and $\mathcal{C} = \{C\}$. We need only show that H has a path-decomposition with two paths. Let $Q = C\backslash\{v_1, v_p\}$. Then Q is seen to be path from v_2 to $v_{p-1} = v_6$. The path Q may or may not contain either of the vertices v_3 or v_5. There are five possibilities given below, where we list the occurrence (or non-occurrence) of v_3 or v_5 in the order which they appear when we move along Q from v_2 to v_6. For example, the sequence v_2, v_5, v_6 means that Q passes through v_5 en route from v_2 to v_6, but it does not pass through v_3. The possible sequences are:

i) v_2, v_6 ii) v_2, v_3, v_6 iii) v_2, v_5, v_6 iv) v_2, v_3, v_5, v_6 v) v_2, v_5, v_3, v_6.

For each of the possibilities i)-v), we shall choose a vertex v on the path Q as follows:

If i) occurs, let v be a vertex in $V(Q)\backslash\{v_2, v_6\}$.
If ii) occurs, let v be a vertex in $V(Q[v_2 v_3])\backslash\{v_2, v_3\}$.
If iii) occurs, let v be a vertex in $V(Q[v_5, v_6])\backslash\{v_5, v_6\}$.
If iv) or v) occurs, let v be a vertex in $V(Q[v_3, v_5])\backslash\{v_3, v_5\}$.

See Figures 3 and 4. For the each of the cases i) - v), we shall define two paths P_0 and P_1. For cases i) and ii), let $P_0 = Q[vv_6] \cup v_6v_5v_4v_7v_2v_1$ and $P_1 = Q[vv_2] \cup v_2v_3v_4v_1v_6v_7$. For case iii), let $P_0 = Q[vv_2] \cup v_2v_3v_4v_1v_6v_7$ and $P_1 = Q[vv_6] \cup v_6v_5v_4v_7v_2v_1$. For case iv), let $P_0 = Q[vv_3] \cup v_3v_2v_7v_6v_5v_4v_1$ and $P_1 = Q[vv_6] \cup v_6v_1v_2 \cup Q[v_2v_3] \cup v_3v_4v_7$. For case v, let $P_0 = Q[vv_3] \cup v_3v_2v_7v_6v_5v_4v_1$ and $P_1 = Q[vv_2] \cup v_2v_1v_6 \cup Q[v_6v_3] \cup v_3v_4v_7$. It can be readily checked that $\{P_0, P_1\}$ is a path-decomposition of H in each case. This completes the proof of Case C. $\square$

In the case where $v_1, v_p \in \bigcup_{C\in\mathcal{C}} V(C)$, the proof of follows from Cases A, B, and C above. Suppose now that one of v_1 or v_p does not belong to $\bigcup_{C\in\mathcal{C}} C$, and without loss of generality, we may assume this is true for v_p. Let $j = \max\{i \mid v_i \in \bigcup_{C\in\mathcal{C}} V(C)\}$. Let $C \in \mathcal{C}$ be a cycle containing v_j. Let z_1 and z_2 be the neighbours of v_j on C where $v_jz_1, v_jz_2 \in E(C)$. If $z_1 \notin V(P)$, then $P' = P[v_1v_j] \cup v_jz_1$ and $P_0 = (C\backslash\{v_jz_1\}) \cup P[v_jv_p]$ are seen to be paths. Letting $H' = H\backslash E(P_0)$, $\mathcal{C}' = \mathcal{C}\backslash\{C\}$, and applying the inductive assumption to H', we see that there is a path-decomposition $\mathcal{P}'$ having at most $|\mathcal{C}'| + 1 = |\mathcal{C}|$ paths. Now $\mathcal{P} = \mathcal{P}' \cup \{P_0\}$ is seen to be a path-decomposition for H having at most $|\mathcal{C}| + 1$ paths. Thus we may assume that $z_1 \in V(P)$ and similarly, $z_w \in V(P)$. Given that G has girth g, we are lead to the conclusion that $\{z_1, z_2\} = \{v_1, v_{g-1}\}$, $p = 2g-1$, $j = 2g-2$, and $\mathcal{C} = \{C\}$. Thus $v_1 \in V(C)$ and $v_1v_{2g-2} \in E(C)$. Let x_1 be the neighbour of v_1

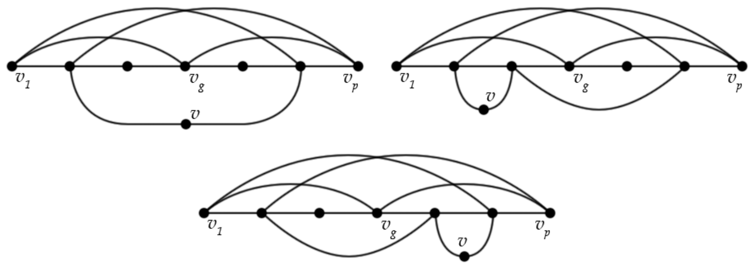

Figure 3: Cases (i),(ii), and (iii)

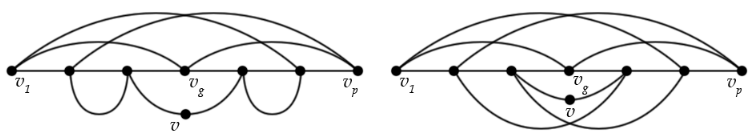

Figure 4: Cases (iv) and (v)

along C where $x_1 \neq v_{2g-2}$. If $x_1 \notin V(P)$, then we can use previous arguments to find the desired path-decomposition for H. Thus we may assume that $x_1 \in V(P)$. Considering that G has girth g, there is only one possibility for x_1, namely $x_1 = v_g$. The situation we have now is very similar to the one treated in Case C. By adapting the arguments there, one can find the desired path-decomposition for H in this case. We leave the details to the reader. This completes the proof of the lemma.

$\square$

5 Proof of the Main Theorem: Part I

In this section, we shall begin our proof of the main theorem. For a graph G having girth g, we let $\xi(G) = \frac{p(G)}{2} + \left\lfloor \left(\frac{g+1}{2g}\right) q(G) \right\rfloor$. Suppose that the theorem is false and let G be a counterexample having fewest possible edges. Suppose that G has girth g where $g \geq 4$. Then G has no path-decomposition with $\xi(G)$ or fewer paths. We shall establish a number of properties for G and show that $g = 4$. For convenience, we shall let $F = G_{ev}$. For each vertex $v \in V(F)$, we define ϵ_v where $\epsilon_v = 0$ or $\epsilon_v = 1$, depending on whether v is even or odd in F. For all $v \in V(G)$, let $d'_G(v) = d_G(v) + \epsilon_v$.

(5.1) $d_F(x) \neq 1$ *for all* $x \in V(F)$.

Proof Suppose to the contrary that there exists $x \in V(G)$ where $d_F(x) = 1$. Let y be the unique neighbour of x in F and let $H = G\backslash xy$. Then x and y are odd vertices in H, and moreover, each neighbour of x in G is odd. Let $\mathcal{P}$ be a path-decomposition of H having fewest paths. By assumption, $|\mathcal{P}| \leq \xi(H)$. Since each neighbour of x in G is odd, we have that $\mathcal{P}(v) \geq 1$ for all $v \in N_G(x)$. Now by Lemma 3.1, the edge xy is $\mathcal{P}$-addible at x and hence G has a path-decomposition with $|\mathcal{P}| \leq \xi(H) \leq \xi(G)$ paths. This gives a contradiction. $\square$

(5.2) *For all odd vertices x in F there exist vertices $y_1, y_2 \in N_F(x)$ such that $d_F(y_i) \geq 4,\ i = 1, 2$.*

Proof Suppose to the contrary that there exists an odd vertex x of F where $d_F(v) \geq 4$ for at most one vertex $v \in N_F(x)$. Let $H = G\backslash E_F(x)$. By assumption, H has a path-decomposition $\mathcal{P}$ with at most $\xi(H)$ paths. By Lemma 3.3, G has a path-decomposition $\mathcal{P}^*$ where $|\mathcal{P}^*| = |\mathcal{P}|$. Since $|\mathcal{P}| \leq \xi(H) \leq \xi(G)$, this gives a contradiction. $\square$

(5.3) *For all even, non-isolated vertices x in F there exist vertices $y_1, y_2 \in N_F(x)$ such that $d_F(y_i) \geq 3,\ i = 1, 2$.*

Proof Suppose to the contrary that there exists an even, non-isolated vertex x of F for which $d_F(v) \geq 3$ for at most vertex $v \in N_F(x)$. Since x is non-isolated, there exists $y \in N_F(x)$ such that $d_F(y) \leq 2$. By (**5.1**), we have that $d_F(y) \geq 2$, and hence it follows that $d_F(y) = 2$. Let z be the unique neighbour in $N_F(y)\backslash\{x\}$. Let $A = E_F(x)\backslash\{xy\}$. Let $G' = G\backslash\{yz\}$ and $H' = G'\backslash A$. Let $\mathcal{P}$ be a path-decomposition of H' having at most $\xi(H')$ paths. Let F' be the E-subgraph of G'. We observe that y and z are odd vertices of G' and hence $d_{F'}(x) = d_F(x) - 1$ is odd. Furthermore, $d_{F'}(v) \geq 3$ for at most one vertex $v \in N_{F'}(x)$. Applying Lemma 3.3, there is a path-decomposition $\mathcal{P}^*$ for G' where $|\mathcal{P}^*| = |\mathcal{P}| \leq \xi(H')$ and $\mathcal{P}^*(x) \geq \mathcal{P}(x) + 1$. Noting that $\mathcal{P}(x) \geq 1$ since x is odd in H', it follows that $\mathcal{P}^*(x) \geq 2$. Furthermore, $\mathcal{P}(z) \geq 1$ since z is odd in H'. Thus $\mathcal{P}^*(z) = \mathcal{P}(z) \geq 1$. Now Lemma 3.1 implies that the edge yz is $\mathcal{P}^*$-addible at y. Thus G has a path-decomposition with $|\mathcal{P}^*| = |\mathcal{P}| \leq \xi(H')$ paths. However, it is seen that $\xi(H') \leq \xi(G)$, and with this we reach a contradiction. $\square$

From (**5.1**), (**5.2**), and (**5.3**) we obtain the following:

(5.4) *For all non-isolated vertices x in F there exist vertices $y_1, y_2 \in N_F(x)$ such that $d_F(y_i) \geq 3,\ i = 1, 2$.*

(5.5) *F has maximum degree at most $2g-2$.*

Proof. Suppose to the contrary that there is a vertex $x \in V(F)$ where $d_F(x) \geq 2g-1$. Let $H = G \backslash E_F(x)$. Then H has a path-decomposition $\mathcal{P}$ having at most $\xi(H)$ paths, and we see that $\mathcal{P}(v) \geq 1$ for all $v \in N_G(x)$ (since every vertex $v \in N_G(x)$ is odd in H). Thus each edge of $E_F(x)$ is $\mathcal{P}$-addible at x. Using the Lovàsz construction, there is a path-cycle-decomposition $\mathcal{P}^*$ of G where $\mathcal{P} \underset{E_F(x)}{\overset{x}{\Rrightarrow}} \mathcal{P}^*$. We observe that $\mathcal{P}^*$ has at most $\lfloor \frac{d_F(x)}{2} \rfloor$ cycles. We have that $p(H) = p(G) + d'_F(v)$ and $q(H) = q(G) - d'_F(v)$. Thus

$$\begin{aligned}
\xi(H) &= \frac{p(G)}{2} + d'_F(v) + \left\lfloor \frac{g+1}{2g}(q(G) - d'_F(v)) \right\rfloor \\
&\leq \xi(G) - \left\lfloor \left(\frac{g+1}{2g} - \frac{1}{2g} \right) d'_F(v) \right\rfloor \\
&\leq \xi(G) - \left\lfloor \frac{1}{2g} d'_F(v) \right\rfloor .
\end{aligned}$$

Let a and b be nonnegative integers such that $d'_F(v) = a(2g) + b$ where $0 \leq b < 2g$. Since $d_F(v) \geq 2g-1$, we see that $d'_F(v) \geq 2g$ and hence $a \geq 1$. From the above, we see that $\xi(H) \leq \xi(G) - a$. By Lemma 4.1, we can replace any edge-disjoint set of cycles $\mathcal{C}$ containing v by $|\mathcal{C}| + 1$ paths, provided $|\mathcal{C}| \leq 2g-1$. Since there are at most $\left\lfloor \frac{d_F(x)}{2} \right\rfloor \leq a \cdot g + \frac{b}{2}$ cycles in $\mathcal{P}^*$ (all of which contain v), we can partition the set of cycles of $\mathcal{P}^*$ into at most a sets having at most $2g$ cycles in each. Thus we can replace the cycles in each of these sets by paths so as to obtain a path-decomposition of G containing at most $|\mathcal{P}^*| + a$ paths. From the above, we see that $|\mathcal{P}^*| + a \leq \xi(H) + a \leq \xi(G)$. This yields a contradiction. $\square$

For every path P in F, we shall define a functions $\mu_P, \eta : V(P) \to \mathbf{Z}^+$ where $\mu_P(v) = |N_F(v) \backslash V(P)| + \epsilon_v$ and $\eta_P(v) = \left\lfloor \frac{|N_F(v) \backslash V(P)|}{2} \right\rfloor$ $\forall v \in V(P)$. We also define $\mu(P) = \sum_{v \in V(P)} \mu_P(v)$ and $\eta(P) = \sum_{v \in V(P)} \eta_P(v)$. It is a straightforward exercise to show that $\mu(P)$ is always even. When the vertices of P share no common neighbours in $V(G) \backslash V(P)$, $\mu(P)$ represents the number of new odd vertices created when one deletes all the edges of F incident with P. We say that P is an **F-path** if

(i) $d_F(v) \geq 3$ for all $v \in V(P)$.

(ii) $\mu(P) \geq 2g$.

Using (**5.4**), it is easy to see that for any vertex v where $d_F(v) \geq 3$, there is an F-path starting at v. We say that an F-path is **minimal** if it does not properly contain another F-path as a subgraph. By (**5.5**), it follows that any F-path must have at least two vertices. Furthermore, it is not difficult to show that for any F-path P we have $\mu(P) \geq 2(|V(P)|+1)$. Thus it follows that for any minimal F-path P, $|V(P)| \leq g-1$.

(5.6) *For any minimal F-path P we have that $\eta(P) + |V(P)| \leq 2g$.*

Proof For all $v \in V(P)$ we have $\mu_P(v) \geq 2\eta_P(v)$. Thus $\mu(P) \geq 2\eta(P)$. Let $v_1, v_2, \ldots, v_p$ be the consecutive vertices of P. By the minimality of P, it follows that $\sum_{i=1}^{p-1} \mu_P(v_i) < 2g$. Thus $\mu_P(v_1) < 2g - \sum_{i=2}^{p-1} \mu_P(v_i)$ and hence $\mu_P(v_1) \leq 2(g-1) - \sum_{i=2}^{p-1} \mu_P(v_i)$, seeing as $\sum_{i=2}^{p-1} \mu_P(v_i)$ is even. Likewise, we have that $\sum_{i=2}^{p} \mu_P(v_i) < 2g$ and hence $\mu_P(v_p) \leq 2(g-1) - \sum_{i=2}^{p-1} \mu_P(v_i)$. Thus $\mu_P(v_1) + \mu_P(v_p) \leq 4(g-1) - 2\sum_{i=2}^{p-1} \mu_P(v_i)$ and hence $\mu(P) \leq 4(g-1) - \sum_{i=2}^{p-1} \mu_P(v_i)$. Since $d_F(v_i) \geq 3$ for all vertices v_i, it follows that $\mu_P(v_i) \geq 2$. Thus $\sum_{i=2}^{p-1} \mu_P(v_i) \geq 2(p-2)$. Consequently, we see that $\mu(P) \leq 4(g-1) - 2(p-2)$. Therefore, $\eta(P) \leq \frac{\mu(P)}{2} \leq 2g - p$. From this it follows that $\eta(P) + p \leq 2g$. □

(5.7) *Let $P = v_1, v_2, \ldots, v_p$ be a minimal F-path. Then we have the following:*

(i) *v_1 and v_p share common neighbours in $V(F) \backslash V(P)$.*

(ii) *$p = g - 1$.*

(iii) *$d_F(v) \in \{3, 4\}$ for all $v \in V(P)$.*

Proof (i) Suppose to the contrary that v_1 and v_p share no common neighbours in $V(F) \backslash V(P)$. Then no two vertices of P share a common neighbour in $V(F) \backslash V(P)$. Let $H = G \backslash E(P)$. For $i = 1, \ldots, p$ let $B_i = E_F(v_i) \backslash E(P)$. Let $B = \bigcup_{i=1}^{p} B_i$ and let $H_0 = H \backslash B$. We first observe that $p(H_0) = p(G) + \mu(P)$ and $q(H_0) = q(G) - \mu(P)$. For $i = 1, \ldots, p$ let $H_i = H \cup \bigcup_{j=1}^{i} B_j$. Let $\mathcal{P}_0$ be a path-decomposition of H_0 having at most $\xi(H_0)$ paths. Given that $\mu(P) \geq 2g$, we have that

$$\begin{aligned} \xi(H_0) &= \frac{p(G)}{2} + \frac{\mu(P)}{2} + \left\lfloor \frac{g+1}{2g}(q(G) - \mu(P)) \right\rfloor \\ &\leq \xi(G) - \left\lfloor \frac{\mu(P)}{2g} \right\rfloor \\ &\leq \xi(G) - 1. \end{aligned}$$

Thus $\xi(H_0) \leq \xi(G) - 1$. We observe that for $i = 1, \dots, p$ and $j = i, \dots p$, the vertices of $N_H(v_j)$ are odd vertices of H_{i-1}. In particular, the vertices of $N_H(v_1)$ are odd in H_0. Thus $\mathcal{P}(v) \geq 1$ for all $v \in N_H(v_1)$ and hence there exists a path-cycle-decomposition $\mathcal{P}_1$ of H_1 where $\mathcal{P}_0 \overset{v_1}{\underset{B_1}{\Rrightarrow}} \mathcal{P}_1$. Then $\mathcal{P}_1$ has at most $\left\lfloor \frac{|B_1|}{2} \right\rfloor = \eta_P(v_1)$ cycles, all of which contain v_1. Similarly, since all the vertices of $N_H(v_2)$ are odd in H_1, there is a path-cycle-decomposition $\mathcal{P}_2$ of H_2 where $\mathcal{P}_1 \overset{v_2}{\underset{B_2}{\Rrightarrow}} \mathcal{P}_2$, and $\mathcal{P}_2$ has at most $\left\lfloor \frac{|B_2|}{2} \right\rfloor = \eta_P(v_2)$ cycles, all of which contain v_2. Continuing in the same fashion, we obtain a sequence of path-cycle-decompositions

$$\mathcal{P}_0 \overset{v_1}{\underset{B_1}{\Rrightarrow}} \mathcal{P}_1 \overset{v_2}{\underset{B_2}{\Rrightarrow}} \mathcal{P}_2 \Rrightarrow \cdots \Rrightarrow \mathcal{P}_{p-1} \overset{v_p}{\underset{B_p}{\Rrightarrow}} \mathcal{P}_p.$$

Now $\mathcal{P} = \mathcal{P}_p$ is seen to be a decomposition of H where $|\mathcal{P}| = |\mathcal{P}_0|$ and $\mathcal{P}$ contains at most $\sum_{i=1}^{p} \eta_P(v_i) = \eta(P)$ cycles, where each cycle contains at least one vertex of P. Let $\mathcal{C}$ be the set of cycles in $\mathcal{P}$. Since $|\mathcal{C}| \leq \eta(P)$ and $\eta(P) + p \leq 2g$ (by (**5.6**)), we see that $|\mathcal{C}| + p \leq 2g$. Now Lemma 4.1 implies that there is a path-decomposition $\mathcal{P}'$ for the subgraph induced by $P \cup \bigcup_{C \in \mathcal{C}} C$ where $|\mathcal{P}'| \leq |\mathcal{C}| + 1$. Let $\mathcal{P}^* = (\mathcal{P} \backslash \mathcal{C}) \cup \mathcal{P}'$. Then $\mathcal{P}^*$ is seen to be a path-decomposition of G where $|\mathcal{P}^*| = |\mathcal{P}| - |\mathcal{C}| + |\mathcal{P}'| \leq |\mathcal{P}| + 1$. Given that $|\mathcal{P}| = |\mathcal{P}_0| \leq \xi(H_0)$, it follows that $|\mathcal{P}^*| \leq \xi(H_0) + 1 \leq \xi(G)$. This gives a contradiction. Thus v_1 and v_p share common neighbours in $V(F) \backslash V(P)$. □

Proof (ii) Since G has girth g, (i) implies that $p \geq g - 1$. From our prior remarks, we also know that $p \leq g - 1$. Thus it follows that $p = g - 1$. □

Proof (iii) Suppose to the contrary that $d_F(v_i) \geq 5$ for some $i \in \{1, \dots, g-1\}$. If $i < g - 1$, then the path $P' = P[v_1 v_{g-2}]$ is seen to satisfy $\mu(P') \geq 2g$. On the other hand, if $i = g - 1$, then $P' = P[v_2 v_{g-1}]$ is seen to satisfy $\mu(P') \geq 2g$. Thus P properly contains an F-path, contradicting the minimality of P. Therefore $d_F(v_i) \in \{3, 4\},\ i = 1, \dots, g-1$. □

As a direct consequence of (**5.7**) (ii), we have the following:

(5.8) *Any F-path with $g - 1$ vertices is minimal.*

(5.9) $d_F(v) \leq 4$ *for all* $v \in V(G)$.

Proof Suppose x is a vertex of G where $d_F(x) \geq 3$. By (**5.4**), we can construct a path F-path P starting at x where $d_F(v) \geq 3$ for all $v \in V(P)$ and $|V(P)| = g-1$. By (**5.8**), F is minimal. It now follows by (**5.7**) (iii) that $d_F(x) \leq 4$. $\square$

We shall now show that G has girth $g = 4$.

(5.10) $g = 4$.

Proof By (**5.9**), we have $d_F(v) \leq 4$ for all $v \in V(G)$. Now using (**5.2**) and (**5.4**), one can show that there exists a vertex $v_1 \in V(G)$ where $d_F(v_1) = 4$. Using (**5.4**), we can construct an F-path Let $P = v_1 v_2 \cdots v_{g-1}$ having $g-1$ vertices. Such a path is minimal by (**5.8**). By (**5.7**) (ii), v_1 and v_p have common neighbours in $V(F) \backslash V(P)$. Note that if v_1 and v_{g-1} have more than one common neighbour in $V(F) \backslash V(P)$, then G has a 4-cycle. Thus we may assume that v_1 and v_{g-1} have exactly one common neighbour in $V(F) \backslash V(P)$, which we denote by v_g. We see that v_1 and v_{g-1} are the only two vertices of P having a common neighbour in $V(F) \backslash V(P)$. By (**5.9**), we have that $d_F(v_i) \leq 4$ for all $v \in V(G)$.

Let $H = G \backslash (E(P) \cup \{v_{g-1}v_g\})$. For $i = 1, 2, \ldots, g-1$ let $B_i = E_F(v_i) \backslash (E(P) \cup \{v_1 v_g, v_{g-1}v_g\}$. Let P' be the path $P \cup v_{g-1}v_g$ and let $B = \bigcup_{i=1}^{g-1} B_i$. Let $H_0 = H \backslash B$ and for $i = 1, \ldots, g-1$ let $H_i = H_{i-1} \cup B_i$. Given that $3 \leq d_F(v_i) \leq 4$, $i = 1, \ldots, g-1$, it is straightforward to show that $p(H_0) = p(G) + 2g$ (and $q(H_0) = q(G) - 2g)$ and $\xi(H_0) = \xi(G) - 1$. Let $\mathcal{P}_0$ be a path-decomposition for H_0 having at most $\xi(H_0)$ paths. We may now proceed in a manner similar to the proof of (**5.7**). There exists a path-cycle-decomposition $\mathcal{P}_1$ for H_1 where $\mathcal{P}_0 \underset{B_1}{\overset{v_1}{\Rrightarrow}} \mathcal{P}_1$ and $\mathcal{P}_1$ has at most $\left\lfloor \frac{|B_1|}{2} \right\rfloor \leq 1$ cycles, all of which contain v_1. Similarly, there exists a path-cycle-decomposition $\mathcal{P}_2$ for H_2 where $\mathcal{P}_1 \underset{B_2}{\overset{v_2}{\Rrightarrow}} \mathcal{P}_2$ and $\mathcal{P}_2$ has at most $\left\lfloor \frac{|B_2|}{2} \right\rfloor \leq 1$ cycles, all of which contain v_2. Continuing in the same fashion, we obtain a sequence of decompositions

$$\mathcal{P}_0 \underset{B_1}{\overset{v_1}{\Rrightarrow}} \mathcal{P}_1 \underset{B_2}{\overset{v_2}{\Rrightarrow}} \mathcal{P}_2 \Rrightarrow \cdots \mathcal{P}_{g-2} \underset{B_{g-1}}{\overset{v_{g-1}}{\Rrightarrow}} \mathcal{P}_{g-1}.$$

Now $\mathcal{P} = \mathcal{P}_{g-1}$ is seen to be path-cycle-decomposition of H where $|\mathcal{P}| = |\mathcal{P}_0|$ and $\mathcal{P}$ contains at most $g-1$ cycles, each of which contains at least one vertex of P'. Let $\mathcal{C}$ be the set of such cycles. Since $|\mathcal{C}| + |V(P')| \leq g-1+g = 2g-1 < 2g$, Lemma 4.1 implies that there is a path-decomposition $\mathcal{P}'$ for the subgraph $P' \cup \bigcup_{C \in \mathcal{C}} C$ where $|\mathcal{P}'| \leq |\mathcal{C}| + 1$. Let $\mathcal{P}^* = (\mathcal{P} \backslash \mathcal{C}) \cup \mathcal{P}'$. Then $\mathcal{P}^*$ is seen to be a path-decomposition of G where $|\mathcal{P}^*| = |\mathcal{P}| - |\mathcal{C}| + |\mathcal{P}'| \leq |\mathcal{P}| + 1$. Given that $|\mathcal{P}| = |\mathcal{P}_0| \leq \xi(H_0) \leq \xi(G) - 1$, it follows that $|\mathcal{P}^*| \leq \xi(G)$. This gives a contradiction. □

6 Proof of the Main Theorem: Part II

In this section, we shall complete the second part of the proof of the main theorem. From (**5.10**) in the previous section, we know that $g = 4$. We also note that by (**5.9**), $d_F(v) \leq 4$ for all $v \in V(G)$. We shall establish several more properties of G which will culminate in showing that any non-trivial component of F is isomorphic to the complete bipartite graph $K_{3,4}$.

(6.1) *No two vertices of degree four in F are adjacent.*

Proof Suppose to the contrary that x and y are vertices of degree four in F which are adjacent. Let $e = xy$ and let $X = E_F(x)\backslash\{e\}$ and $Y = E_F(y)\backslash\{e\}$. Let $N_F(x) = \{y, x_1, x_2, x_3\}$ and $N_F(y) = \{x, y_1, y_2, y_3\}$. Let $H_0 = G\backslash(X \cup Y)$. One sees that the vertices in $N_G(x) \cup N_G(y)$ are odd in H_0. Let $\mathcal{P}_0$ be a path-decomposition of H_0 where $|\mathcal{P}_0| \leq \xi(H_0)$. Then $\xi(H_0) = \frac{p(G)}{2} + 4 + \left\lfloor \frac{5}{8}(q(G) - 8) \right\rfloor = \xi(G) - 1$. We also observe that $\mathcal{P}_0(v) \geq 1$ for all $v \in N_G(x) \cup N_G(y)$. Let $H_1 = H_0 \cup X$ and let F_1 be the even-subgraph of H_1. The vertices of $\{x, x_1, x_2, x_3\}$ are seen to even in H_1 and $d_{F_1}(x) = 3$. We shall consider two cases.

Suppose there exists a path-decomposition $\mathcal{P}_1$ of H_1 where $\mathcal{P}_1(x) \geq 2$ and $\mathcal{P}_1(v) = \mathcal{P}_0(v) \geq 1$ for all $v \in V(G)\backslash(\{x, x_1, x_2, x_3\})$. Then $\mathcal{P}_1(v) = \mathcal{P}_0(v) \geq 1$ for all $v \in N_G(y)$. Applying Lemma 3.2, there exists a subset $Y_0 \subseteq Y$ such that $|Y_0| = 2$ and Y_0 is $\mathcal{P}_1$-addible at y. We may assume that $Y_0 = \{yy_1, yy_2\}$. Then there is a path-decomposition $\mathcal{P}^*$ of $G\backslash\{yy_3\}$ where $\mathcal{P}_1 \xrightarrow[Y_0]{y} \mathcal{P}^*$ and $|\mathcal{P}^*| = |\mathcal{P}_0| \leq \xi(H_0) = \xi(G) - 1$. Let $P' = yy_3$ be the path induced by the edge yy_3. Then $\mathcal{P}^* \cup \{P'\}$ is a path-decomposition of G having at most $\xi(G)$ paths. This gives a contradiction.

Suppose no such path-decomposition $\mathcal{P}_1$ for H_1 as described above exists. Then Lemma 3.4 implies that there exists $z \in N_{F_1}(x)$ for which $d_{F_1}(x) \geq 4$. This implies that $|N_F(z)\backslash(N_F(x) \cup N_F(y))| \geq 3$ and $d_F(z) = 4$. From this, it also follows that $N_F(y) \cap N_F(z) = \{x\}$. Let $P = zxy$. Then P is a path such that $\mu(P) \geq 8$ and consequently P is an F-path. Moreover, (**5.8**) implies that P is minimal. However, the vertices y and z have no common neighbour in $V(F)\backslash V(P)$. This contradicts (**5.7**) (i). □

(6.2) *Let $x \in V(F)$ where $d_F(x) = 4$. Let $A \subset E(F)$ and let $H_0 = G\backslash A$. Let $\mathcal{P}_0$ be a path-decomposition of H_0 where $\mathcal{P}_0(v) \geq 1$ for all $v \in N_G(x) \cup \{x\}$. Suppose $X \subseteq A$ where $|X| = 3$ and $X \subset E_F(x)$. Then there is a path-decomposition $\mathcal{P}^*$ of $H_0 \cup X$ such that $\mathcal{P}^*(x) \geq 2$ and $\mathcal{P}^*(v) = \mathcal{P}_0(v)$ for all $v \in V(G)\backslash(N_{G,X}(x) \cup \{x\})$.*

Proof Similar to the proof of Lemma 3.4, one can show that if no such path-decomposition $\mathcal{P}^*$ exists for $H_0 \cup X$, then there is a vertex $z \in N_F(x)$ such that $d_F(x) \geq 4$ (and hence $d_F(z) = 4$). However, x and z would be adjacent vertices in F which have degree four. This contradicts (**6.1**). Thus such a path-decomposition $\mathcal{P}^*$ must exist. $\square$

(6.3) *Suppose $x, y \in V(F)$ where $d_F(x) = d_F(y) = 4$. If $N_F(x) \cap N_F(y) \neq \emptyset$, then $N_F(x) = N_F(y)$.*

Proof By (**6.1**), x and y are non-adjacent. Suppose first that $|N_F(x) \cap N_F(y)| = 1$ and $N_F(x) \cap N_F(y) = \{z\}$. Let X be the set of edges incident with x in $F\backslash z$ and let Y be the set of edges incident with y in F. Let $H_0 = G\backslash(X \cup Y)$ and let $\mathcal{P}_0$ be a path-decomposition for H_0 where $\mathcal{P}_0$ has at most $\xi(H_0)$ paths. Then $\xi(H_0) = \xi(G) - 1$ and $\mathcal{P}_0(v) \geq 1$ for all $v \in N_G(x) \cup N_G(y)$. By (**6.2**), there exists a path-decomposition $\mathcal{P}_1$ of $H_1 = H_0 \cup X$ such that $\mathcal{P}_1(x) \geq 2$ and $\mathcal{P}_1(v) = \mathcal{P}_0(v) \geq 1$ for all $v \in N_G(y)$. By Lemma 3.2, there exists $Y_1 \subseteq Y$ such that $|Y_1| = 2$ and Y_1 is $\mathcal{P}_1$-addible at y. Let $\mathcal{P}_2$ be a path-decomposition of $H_1 \cup Y_1$ where $\mathcal{P}_1 \xrightarrow[Y_1]{y} \mathcal{P}_2$. Let P' be the path induced by the edges of $Y\backslash Y_1$. Then $\mathcal{P}^* = \mathcal{P}_2 \cup \{P'\}$ is seen to be a path-decomposition of G having $|\mathcal{P}_1| + 1 = |\mathcal{P}_0| + 1 \leq \xi(H_0) + 1 = \xi(G)$ paths. This gives a contradiction.

Suppose $|N_F(x) \cap N_F(y)| = 2$ and let $N_F(x) \cap N_F(y) = \{w, z\}$. Let X (respectively, Y) be the set of edges incident with x (respectively, y) in $F\backslash w$ (respectively, $F\backslash z$). Let $H_0 = G\backslash(X \cup Y)$ and let $\mathcal{P}_0$ be a path-decomposition of H_0 where $\mathcal{P}_0$ has a most $\xi(H_0)$ paths. Then $\xi(H_0) = \xi(G) - 1$ and $\mathcal{P}_0(v) \geq 1$ for all $v \in N_G(x) \cup N_G(y)$. By (**6.2**), there is a path-decomposition $\mathcal{P}_1$ of $H_1 = H_0 \cup X$ such that $|\mathcal{P}_1| = |\mathcal{P}_0|$, $|\mathcal{P}_1(x)| \geq 2$, and $\mathcal{P}_1(v) \geq 1$ for all $v \in N_G(y)\backslash\{z\}$ (and $\mathcal{P}_1(z) \geq 0$). Thus Lemma 3.1 implies that there is an edge $e \in Y$ for which e is $\mathcal{P}_1$-addible at y. Let $\mathcal{P}_2$ be a path-decomposition of $H_1 \cup \{e\}$ where $\mathcal{P}_1 \xrightarrow[e]{y} \mathcal{P}_2$. Let P' be the path induced by the edges of $Y\backslash\{e\}$. Then $\mathcal{P}^* = \mathcal{P}_2 \cup \{P'\}$ is a path-decomposition of G having $|\mathcal{P}^*| = |\mathcal{P}_2| + 1 = |\mathcal{P}_1| + 1 = |\mathcal{P}_0| + 1 \leq \xi(H_0) + 1 = \xi(G)$ paths. This gives a contradiction.

Suppose $|N_F(x) \cap N_F(y)| = 3$ and let $N_F(x) \cap N_F(y) = \{u, w, z\}$. By (**5.4**), x has a least two neighbours of degree three in F. Because of this, we may assume that $d_F(u) = 3$. Let X be the set of edges incident with x in $F \backslash u$. Let e be the edge incident with u in $F \backslash \{x, y\}$ and let f be the edge incident with y in $F \backslash \{u, w, z\}$. Let $H_0 = G \backslash (X \cup \{e, f\})$ and let $\mathcal{P}_0$ be a path-decomposition of H_0 where $\mathcal{P}_0$ has at most $\xi(H_0)$ paths. We see that $\xi(H_0) = \xi(G) - 1$. We observe that all the vertices in $N_G(x) \cup N_G(y) \cup N_G(u)$ are all odd in H_0. Thus $\mathcal{P}_0(v) \geq 1$ for all vertices v in this set. By (**6.2**), there is a path-decomposition $\mathcal{P}_1$ of $H_1 = H_0 \cup X$ where $|\mathcal{P}_1| = |\mathcal{P}_0|$, $\mathcal{P}_1(x) \geq 2$ and $\mathcal{P}_1(v) = \mathcal{P}_0(v) \geq 1$ for all $v \in N_G(u) \backslash \{x\}$. Thus by Lemma 3.1, e is $\mathcal{P}_1$-addible at u. Let $\mathcal{P}_2$ be a path-decomposition of $H_1 \cup \{e\}$ where $\mathcal{P}_1 \underset{e}{\overset{u}{\to}} \mathcal{P}_2$. Let P' be the path induced by the edge f. Then $\mathcal{P}^* = \mathcal{P}_2 \cup \{P'\}$ is seen to be a path-decomposition of G where having at most $\xi(H_0) + 1 = \xi(G)$ paths. This gives a contradiction.

From the above, we conclude that $|N_F(x) \cap N_F(y)| = 4$ and hence $N_F(x) = N_F(y)$. □

(6.4) *Each non-trivial component of F is isomorphic to $K_{3,4}$.*

Proof Let K be a non-trivial component of F. It is an easy exercise to show that K has two vertices of degree four in F which share a common neighbour. Let x and y be two such vertices. By (**6.3**), we have $N_F(x) = N_F(y)$. Let $N_F(x) = \{u, v, w, z\}$. By (**5.4**), at least two vertices in $\{u, v, w, z\}$ have degree at least three in F. We may assume that $d_F(u) \geq 3$ and $d_F(v) \geq 3$. Now (**6.1**) implies that $d_F(u) = d_F(v) = 3$.

Suppose first that $N_F(u) \neq N_F(v)$. Let e (respectively, f) be the edge in $F \backslash \{x, y\}$ incident with u (respectively, v). Let $g = xw$ and $h = yz$. Let $H_0 = G \backslash \{e, f, g, h\}$. Then $\xi(H_0) = \xi(G) - 1$. Let $\mathcal{P}_0$ be a path-decomposition of H_0 having at most $\xi(H_0)$ paths. All vertices in $\bigcup_{v' \in \{u,v,w,z\}} N_G(v')$ are seen to be odd in H_0. By Lemma 3.1, both e and f are $\mathcal{P}_0$-addible at u and v respectively. Let $\mathcal{P}_1$ be a path-decomposition of $H_1 = H_0 \cup \{e\}$ where $\mathcal{P}_0 \underset{e}{\overset{u}{\to}} \mathcal{P}_1$. Let $\mathcal{P}_2$ be a path-decomposition of $H_2 = H_1 \cup \{f\}$ where $\mathcal{P}_1 \underset{f}{\overset{v}{\to}} \mathcal{P}_2$. We observe that $\mathcal{P}_2(u) \geq 2$, $\mathcal{P}_2(v) \geq 2$, and $\mathcal{P}_2(v') \geq 1$ for all $v' \in N_G(x)$. Thus Lemma 3.1 implies that g is $\mathcal{P}_2$-addible at x. Let $\mathcal{P}_3$ be a path-decomposition of $G \backslash \{h\} = H_2 \cup \{g\}$ where $\mathcal{P}_2 \underset{g}{\overset{x}{\to}} \mathcal{P}_3$. Let P' be the path induced by the edge h. Now $\mathcal{P}^* = \mathcal{P}_3 \cup \{P'\}$ is seen to be a path-decomposition of G where $|\mathcal{P}^*| \leq \xi(H_2) + 1 \leq \xi(H_0) + 1 = \xi(G)$. This gives a contradiction. Thus $N_F(u) = N_F(v)$. The above argument also shows that for any two vertices $u', v' \in \{u, v, w, z\}$ where $d_F(u') =$

$d_F(v') = 3$, $N_F(u') = N_F(v')$. Let $N_F(u) = \{x, y, t\}$. Suppose that u and v are the only two vertices in $\{u, v, w, z\}$ having degree three in F. Let $H_0 = G \backslash \{xw, xz, yv, ut\}$. Let $\mathcal{P}_0$ be a path-decomposition of H_0 having at most $\xi(H_0)$ paths. Then y, u, w, z, t, v are all odd vertices in H_0 and consequently $\mathcal{P}_0(v') \geq 1$ for all $v' \in \{v, y, w, z, u, t\}$. Furthermore, $\mathcal{P}_0(v') \geq 1$ for all $v' \in N_G(y)$. By Lemma 3.1, the edge yv is $\mathcal{P}_0$-addible at y. Let $\mathcal{P}_1$ be a path-decomposition of $H_1 = H_0 \cup \{yv\}$ where $\mathcal{P}_0 \underset{yv}{\overset{y}{\to}} \mathcal{P}_1$. Then $\mathcal{P}_1(v) \geq 0$, $\mathcal{P}_1(y) \geq 2$, and $\mathcal{P}_1(v') = 0$ for at most one vertex $v' \in N_G(x)$. Given that $\mathcal{P}_1(w) \geq 1$ and $\mathcal{P}_1(z) \geq 1$, Lemma 3.2 implies that at least one of xw or xz is $\mathcal{P}_1$-addible at x. Without loss of generality, we may assume that xw is $\mathcal{P}_1$-addible. Let $\mathcal{P}_2$ be a path-decomposition of $H_2 = H_1 \cup \{xw\}$ where $\mathcal{P}_1 \underset{xw}{\overset{x}{\to}} \mathcal{P}_2$. Then $\mathcal{P}_2(x) \geq 1$, $\mathcal{P}_2(w) \geq 0$, and $\mathcal{P}_2(v') \geq 1$ for all $v' \in N_G(u)$. Now Lemma 3.1 implies that ut is $\mathcal{P}_2$-addible at u. Let $\mathcal{P}_3$ be a path-decomposition of $H_3 = H_2 \cup \{ut\}$ where $\mathcal{P}_2 \underset{ut}{\overset{u}{\to}} \mathcal{P}_3$. Then $\mathcal{P}_3(u) \geq 2$, $\mathcal{P}_3(t) \geq 0$, and $\mathcal{P}_3(v') \geq 1$ for all $v' \in N_G(z)$. By Lemma 3.1, xz is $\mathcal{P}_3$-addible at z. Let $\mathcal{P}_4$ be a path-decomposition of $G = H_3 \cup \{xz\}$ where $\mathcal{P}_3 \underset{xz}{\overset{z}{\to}} \mathcal{P}_4$. Since $|\mathcal{P}_4| = |\mathcal{P}_0| \leq \xi(H_0) \leq \xi(G)$, this gives a contradiction.

From the above, at least one of the vertices w or z has degree three in F. Without loss of generality, we may assume that $d_F(w) = 3$. By our previous observation, it follows that $N_F(u) = N_F(v) = N_F(w) = \{x, y, t\}$. If $d_F(t) = 3$, then (**5.2**) implies that at least two of the vertices of $\{u, v, w\}$ have degree four in F. However, since x and y have degree four in F, this would contradict (**6.1**). Thus $d_F(t) = 4$. Now (**6.3**) implies that $N_F(t) = N_F(x) = N_F(y)$. We now see that $K \simeq K_{3,4}$. $\square$

If F only has non-trivial components, then it follows by Theorem 1.3 that G has a path-decomposition with at most $\frac{p(G)}{2}$ paths. Thus F has at least one non-trivial component, say F_1. By (**6.4**), we have that $F_1 \simeq K_{3,4}$. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ be a bipartition of the vertices of F_1, where $x_i y_j \in E(F_1)$ for all $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$. Suppose first that $F = F_1$. Let $H_0 = G \backslash \{x_1y_1, x_1y_2, x_1y_3, x_2y_4\}$ and let $\mathcal{P}_0$ be a path-decomposition for H_0 having at most $\xi(H_0)$ paths. We observe that $p(H_0) = p(G) + 6$ and $q(H_0) = q(G) - 6 = 1$. Thus we see that $\xi(H_0) = \xi(G) - 1$. Given that $d_F(y_i) = 3,\ i = 1, 2, 3, 4$, it follows from (**6.2**) that $H_1 = H_0 \cup \{x_1y_1, x_1y_2, x_1y_3\}$ has a path-decomposition $\mathcal{P}_1$ where $|\mathcal{P}_1| = |\mathcal{P}_0|$. Let P' be the path induced by the edge x_2y_4. Then $\mathcal{P}^* = \mathcal{P}_1 \cup \{P'\}$ is seen to be a path-decomposition of G having $|\mathcal{P}_1| + 1 = |\mathcal{P}_0| + 1 \leq \xi(H_0) + 1 = \xi(G)$ paths. This gives a contradiction.

From the above, we see that $F \neq F_1$ and hence there must be other components of F other than F_1. Let P be a shortest path from F_1 to another component of F which we will call F_2. If P terminates in X, then we may assume that its terminal vertex is x_3. Otherwise, if P terminates in Y, we may assume its terminal vertex is y_4. Suppose first that P terminates at x_3. Let $H_0 = G\backslash(\{x_1y_1, x_1y_2, x_1y_3, x_2y_4\} \cup E(P))$. Observe that all the vertices of P are odd in H_0. Thus we see $p(H_0) = p(G) + 8$, $q(H_0) = q(G) - 8$ and $\xi(H_0) = \xi(G) - 1$. By (**6.2**), there exists a path-decomposition $\mathcal{P}_1$ of $H_1 = H_0 \cup \{x_1y_1, x_1y_2, x_1y_3\}$ where $|\mathcal{P}_1| = |\mathcal{P}_0|$, $\mathcal{P}_1(x_1) \geq 2$, and $\mathcal{P}_1(v) = \mathcal{P}_0(v) \geq 1$ for all $v \in N_G(y_4)$. By Lemma 3.1, the edge x_2y_4 is $\mathcal{P}_1$-addible at y_4. Let $\mathcal{P}_2$ be a path-decomposition of $H_2 = H_1 \cup \{x_2y_4\}$ where $\mathcal{P}_1 \underset{x_2y_4}{\overset{y_4}{\rightarrow}} \mathcal{P}_2$. Now $\mathcal{P}^* = \mathcal{P}_2 \cup \{P\}$ is seen to be a path-decomposition of G having at most $\xi(H_0) + 1 = \xi(G)$ paths. This gives a contradiction.

Suppose instead that P terminates at y_4. Let

$$H_0 = G\backslash(\{x_1y_1, x_1y_2, x_1y_3, x_2y_4, x_3y_4\} \cup E(P))$$

and let $\mathcal{P}_0$ be a path-decomposition of H_0 having at most $\xi(H_0)$ paths. Again, we see that $\xi(H_0) = \xi(G) - 1$. By (**6.2**), there is a path-decomposition $\mathcal{P}_1$ of $H_1 = H_0 \cup \{x_1y_1, x_1y_2, x_1y_3\}$ where $|\mathcal{P}_1| = |\mathcal{P}_0|$, $\mathcal{P}_1(x_1) \geq 2$, and $\mathcal{P}_1(v) \geq 1$ for all $v \in N_G(y_4)$. Thus x_2y_4 is $\mathcal{P}_1$-addible at y_4. Let $\mathcal{P}_2$ be a path-decomposition of $H_2 = H_1 \cup \{x_2y_4\}$ where $\mathcal{P}_1 \underset{x_2y_4}{\overset{y_4}{\rightarrow}} \mathcal{P}_2$. Let $P' = P \cup x_3y_4$. Then $\mathcal{P}^* = \mathcal{P}_2 \cup \{P'\}$ is seen to be a path-decomposition of G having at most $\xi(H_0) + 1 = \xi(G)$ paths. This gives a final contradiction and the proof is complete.

References

[1] Cai, L., *Path decompositions of multigraphs*, J. Graph Theory **19** (1995), 297-307.

[2] Dean, N., Kouider, M., *Gallai's conjecture for disconnected graphs*, Discrete Math., **213** (2000) 43-54.

[3] Donald, A., *An upper bound for the path number of a graph*, Journal of Graph Theory **4** (1980) 189-201.

[4] Fan, G., *Path decompositions and Gallai's conjecture*, Journal of Comb. Theory, **93** (2003) 117-125.

[5] Lovász, L., *On covering of graphs*, in Theory of Graphs, Proc. Coll. Tihany, 1966.

[6] Pyber, L., *Covering the edges of a connected graphs by paths*, Journal of Comb. Theory, **66** (1996) 152-159.