# Gallai's Path Decomposition for 2-degenerate Graphs 

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#### Abstract

Gallai's path decomposition conjecture states that if $G$ is a connected graph on $n$ vertices, then the edges of $G$ can be decomposed into at most $\left\lceil\frac{n}{2}\right\rceil$ paths. A graph is said to be an odd semi-clique if it can be obtained from a clique on $2 k+1$ vertices by deleting at most $k-1$ edges. Bonamy and Perrett asked if the edges of every connected graph $G$ on $n$ vertices can be decomposed into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths unless $G$ is an odd semi-clique. A graph $G$ is said to be 2-degenerate if every subgraph of $G$ has a vertex of degree at most 2 . In this paper, we prove that the edges of any connected 2-degenerate graph $G$ on $n$ vertices can be decomposed into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths unless $G$ is a triangle.


Keywords: Path decomposition, Gallai's path decomposition, 2-degenerate graphs, Outer-planar graphs, Seriesparallel graphs

## 1 Introduction

All graphs in this paper are simple, undirected and finite. A path decomposition of a graph is a partition of the edge set of the graph into paths. Gallai made the following conjecture on path decomposition:
Conjecture 1 (Gallai's path decomposition conjecture). If $G$ is a connected graph on $n$ vertices, then the edges of $G$ can be decomposed into at most $\left\lceil\frac{n}{2}\right\rceil$ paths.
Lovasz (1968) proved that the edges of a connected graph $G$ on $n$ vertices can be decomposed into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths and cycles. An even subgraph $I$ of $G$ is defined as a subgraph induced by all the vertices having an even degree in $G$. Lovásд (1968) proved the conjecture for graphs whose even subgraph consists of a single vertex. Pyber (1996) later proved that the conjecture holds if the even subgraph of a graph is a forest. Fan (2005) proved that the conjecture holds if each block of the even subgraph is a triangle-free graph of maximum degree at most three. Botler and Sambinelli (2021) generalized the result of Fan (2005).
Bonamy and Perrett (2019) proved Gallai's conjecture for all graphs with maximum degree at most five. A graph is said to be an odd semi-clique if it can be obtained from a clique on $2 k+1$ vertices by deleting at most $k-1$ edges. Bonamy and Perrett (2019) asked if the edges of every connected graph $G$ on $n$ vertices can be decomposed into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths unless $G$ is an odd semi-clique. Botler et al. (2020) proved that the edges of any connected graph $G$ with treewidth at most 3 can be decomposed into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths unless $G$ is $K_{3}$ or $K_{5}-e$. Botler et al. (2019) proved that the edges of triangle-free planar
graphs can be decomposed into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths. More recently, Chu et al. 2021 proved that the edges of any connected graph $G$ with maximum degree 6 in which the vertices of degree 6 form an independent set can be decomposed into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths, unless $G$ is $K_{3}, K_{5}$ or $K_{5}-e$.

Our result: A graph $G$ is said to be $k$-degenerate if every subgraph of $G$ has a vertex of degree at most $k$. In this paper, we study the path decomposition in 2-degenerate graphs. The class of 2-degenerate graphs properly include outer-planar graphs, series-parallel graphs and planar graphs of girth at least 5 as subclasses. We prove that:
Theorem 1. Let $G$ be a connected 2-degenerate graph on $n$ vertices. Then the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths unless $G$ is a triangle.

Note that any triangle requires two paths. We can even extend the result to 2-degenerate graphs which are not connected, but no component being a triangle. We have the following corollary:
Corollary 1. Let $G$ be a 2-degenerate graph on $n$ vertices such that none of the components of $G$ is a triangle. Then the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths.

## 2 Preliminaries

The degree of a vertex $v$ in a graph $G=(V, E)$ is denoted by $d_{G}(v)$. A vertex of degree 0 is said to be isolated. The set of neighbours of a vertex $v$ in $G$ is denoted by $N_{G}(v)$. A walk is a non-empty alternating sequence $v_{0}, e_{0}, v_{1}, e_{1}, \ldots, e_{k-1}, v_{k}$ of vertices and edges in a graph, such that any edge in the sequence has the vertex immediately before and after it as its endpoints in the graph. A path is a walk in which all vertices and edges are distinct. The number of edges in a path $P$ is called its length and is denoted as $|P|$. The notation $v_{i} P v_{j}$ denotes the subpath of the path $P$ starting at the vertex $v_{i}$ and ending at the vertex $v_{j}$. For a path $P$ and an edge $x y \notin P$ and $x y$ incident on one of the endpoints of $P, P \cup x y$ denotes the extended path obtained by adding $x y$ to $P$. A closed walk is a walk that starts and ends at the same vertex. A cycle is a closed walk where all the edges are distinct, and the only repeated vertices are the first and the last vertices. The number of edges in a cycle $C$ is called its length and is denoted as $|C|$. For an edge $x y \in E$, where $x, y \in V, G-x y$ will denote the graph obtained by deletion of the edge $x y$. Similarly, if $F$ denotes any subset of edges of the edge set $E$, then $G-F$ denotes the graph obtained after the removal of all the edges of $F$ from $G$. For a vertex $x \in V, G-x$ will denote the graph obtained by deletion of the vertex $x$ and all the edges incident on $x$. Similarly, the graph $G-\left\{x_{0}, x_{1}, \ldots, x_{i}\right\}$ denotes the graph obtained if multiple vertices $x_{0}, x_{1}, \ldots, x_{i}$ and the edges incident on them are removed from $G$. An odd component is a component of a graph with an odd number of vertices. Similarly, an even component is a component of a graph with an even number of vertices. We say that a graph $G$ on $n$ vertices has a valid decomposition if there exists a decomposition of the edges of $G$ into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths. For any missing definitions, please refer Diestel (2005).
Definition 1 (Vertex removal order). Let $x_{1}, x_{2}, \ldots, x_{n}$ be an ordering of vertices of $G$. We denote by $H_{i}$, the graph obtained by removing vertices $x_{1}, \ldots, x_{i-1}$ from $G$. If $G$ is a 2-degenerate graph on $n$ vertices, then there exists an ordering $x_{1}, x_{2}, \ldots, x_{n}$ of vertices such that $d_{H_{i}}\left(x_{i}\right) \leq 2$, for every $i, 1 \leq i \leq n$

## 3 Proof of Theorem 1

Let $G$ be a minimum counterexample to the theorem statement with respect to the number of vertices. It is easy to see that $n>3$ since any connected graph with at most three vertices is either a path or a triangle,
and the theorem holds in those cases.
Observation 1. Let $H$ be a subgraph of $G$ with $n-i$ non-isolated vertices, for $i>0$, and suppose that none of the components in $H$ is a triangle. Since $G$ is a minimum counterexample, the edges of $H$ can be decomposed into at most $\left\lfloor\frac{n-i}{2}\right\rfloor$ paths.
Lemma 1. Let $P$ be a path in the graph $G$. Let $T_{1}, T_{2}, \ldots, T_{j}$, for $j \geq 0$, be the components of $G-E(P)$ that are triangles. Then the edges of $P$ and the components $T_{1}, T_{2}, \ldots, T_{j}$ can be decomposed into $j+1$ paths.

Proof: We prove the statement using induction on the number of triangle components. Clearly, if $j=0$, the statement holds. Suppose the statement holds for $j-1$ triangles. Now we prove that the statement holds if $G-E(P)$ has $j$ triangle components $T_{1}, T_{2}, \ldots, T_{j}$. Let the endpoints of the path $P$ be vertices $a$ and $b$. While travelling along $P$, starting at $a$, let $x$ be the vertex in $P$ where a triangle first intersects $P$. Without loss of generality, let $T_{j}$ be this triangle. Let $x, y$ and $z$ be the vertices of $T_{j}$. The triangle $T_{j}$ may intersect $P$ at one, two or three vertices. This gives us the following three cases. In all these cases, we decompose the edges of $P \cup T_{j}$ into two paths $Q$ and $R$, each of which will be used later.
Case 1. $P$ intersects $T_{j}$ at three vertices.
Let the vertices of $T_{j}$ intersect the path $P$ in the order $x, y, z$. See Figure 11. We can define two paths $Q$ and $R$, such that $Q=a P x \cup x y \cup y z \cup z w$, where $w$ is a neighbour of $z$ and $z w \in z P y$ and $R=w P y \cup y P x \cup x z \cup z P b$.


Fig. 1: Three vertices of triangle $T_{j}$ intersect $P$. (a) The path $P$ is shown with a dashed line, while the edges of $T_{j}$ are shown in solid lines. (b) Decomposition of the edges of $T_{j}$ and $P$ into two paths $Q$ (thin line) and $R$ (bold line).

Case 2. P intersects $T_{j}$ at two vertices.
Let the vertices of $T_{j}$ intersect the path $P$ in the order $x, y$. See Figure 2. We can define two paths $Q$ and $R$, such that $Q=a P x \cup x y \cup y w$, where $w$ is a neighbour of $y$ and $y w \in x P y$ and $R=$ $w P x \cup x z \cup z y \cup y P b$.


Fig. 2: Two vertices of triangle $T_{j}$ intersect $P$. (a) The path $P$ is shown with a dashed line, while the edges of $T_{j}$ are shown in solid lines. (b) Decomposition of the edges of $T_{j}$ and $P$ into two paths $Q$ (thin line) and $R$ (bold line).

Case 3. $P$ intersects $T_{j}$ at one vertex.
Let the vertex of $T_{j}$ that intersects the path $P$ be $x$. See Figure 3. We can define two paths $Q$ and $R$ such that $Q=a P x \cup x y \cup y z$ and $R=z x \cup x P b$.


Fig. 3: One vertex of triangle $T_{j}$ intersects $P$. (a) The path $P$ is shown with a dashed line, while the edges of $T_{j}$ are shown in solid lines. (b) Decomposition of the edges of $T_{j}$ and $P$ into two paths $Q$ (thin line) and $R$ (bold line).

Let $G^{\prime}=G-E(Q)$. It is easy to see that $G^{\prime}$ does not have any triangle components using the following argument: since $T_{j}$ is the first triangle to intersect $P$, the removal of $a P x$ does not produce any other triangle components. The removal of the rest of the edges in the path $Q$ does not create any new triangle components either, because in Case 1 and Case 22, the vertices $x, y, z$ and $w$ are still connected to the path $R$ and in Case 3, $x$ and $z$ are still connected to the path $R$, while $y$ becomes isolated.

Path $Q$ does not contain edges from triangles $T_{1}, T_{2}, \ldots, T_{j-1}$. Let $G^{\prime \prime}=G^{\prime}-E(R)=G-E(Q \cup R)$. Note that $G^{\prime \prime}=G-E\left(P \cup T_{j}\right)$. Therefore, the removal of the path $R$ from $G^{\prime}$ will create $j-1$ triangle
components $T_{1}, T_{2}, \ldots, T_{j-1}$. By inductive hypothesis, the edges of $R$ and the edges of $T_{1}, T_{2}, \ldots, T_{j-1}$ can be decomposed into $j$ paths. These $j$ paths, along with the path $Q$, give us the required $j+1$ paths.

Claim 1. $G$ does not contain two vertices with degree at most 2 .
Proof: Let $u$ and $v$ be the closest pair of vertices in $G$ with degrees at most 2. Our proof relies on removing the edges of a path or a cycle that isolates the vertices $u$ and $v$. In each case of the proof that follows, we will attempt to get a valid decomposition of the edges of $G$, using a valid decomposition of the edges of a subgraph of $G$ having at most $n-2$ non-isolated vertices.

Let $P^{\prime}$ be a shortest path connecting vertices $u$ and $v$. Suppose $\left|P^{\prime}\right|>2$. If $d_{G}(u)=1$ and $d_{G}(v)=1$, then let $P=P^{\prime}$. Otherwise, if either or both of $u$ and $v$ have degree 2, then extend the path $P^{\prime}$ to include all the edges incident on $u$ and $v$. Let $P$ be this extended path. Removal of the edges of $P$ from $G$ removes all the edges incident on $u$ and $v$, making them isolated.

On the other hand, suppose $\left|P^{\prime}\right| \leq 2$. If $d_{G}(u)=1$ and $d_{G}(v)=1$, let $P=P^{\prime}$. If either or both of $u$ and $v$ have degree 2, then extend the path $P^{\prime}$ to include all the edges incident on $u$ and $v$ to get a walk $L$. Since $\left|P^{\prime}\right| \leq 2$, we can infer that $|L| \leq 4$ and no edge is repeated. Thus the walk $L$ is a path or a cycle. If the walk $L$ is a path, let $P=L$. Otherwise, it is a cycle of length either 3 or 4 . Note that $u$ and $v$ are isolated vertices in $G-E(L)$.

Now consider the case when $\left|P^{\prime}\right|>2$ or $\left|P^{\prime}\right| \leq 2$ and $L$ is a path. The path $P$ is such that $G-E(P)$ has at least two isolated vertices $u$ and $v$. Let $j \geq 0$ be the number of triangle components of $G-E(P)$. Let $H$ denote the graph obtained from $G-E(P)$, after the $j$ triangle components are removed. $H$ has at most $n-2-3 j$ non-isolated vertices. Since the graph $G$ is a minimum counterexample, by Observation 1 , the edges in $H$ can be decomposed into at most $\left\lfloor\frac{n-2-3 j}{2}\right\rfloor$ paths. By Lemma 1 , the edges of the $j$ triangle components together with $P$ require $j+1$ paths. This implies that the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n-2-3 j}{2}\right\rfloor+j+1 \leq\left\lfloor\frac{n}{2}\right\rfloor$ paths.

Now we are left with the case $\left|P^{\prime}\right| \leq 2$ and $L$ is a cycle, which we denote as $C$.
Subclaim 1. If the edges of $G-E(C)$ can be decomposed into $k$ paths, then the edges of $G$ can be decomposed into at most $k+1$ paths.

Proof: As mentioned before, the length of $C$ is either 3 or 4 . We will consider these cases separately. Let $W$ be a path in the path decomposition of $G-E(C)$ that intersects $C$ in $G$. Let vertices $a$ and $b$ be the two endpoints of this path $W$. We will demonstrate that the edges of $W$ and $C$ together can be decomposed into at most 2 paths in $G$. If the length of cycle $C$ is 4 , let $x$ and $y$ be the common neighbours of $u$ and $v$. The path $W$ can intersect $C$ in $G$ at either one or both of $x$ and $y$. If the path $W$ intersects $C$ at both $x$ and $y$ (in that order), we can define two paths $W_{1}$ and $W_{2}$, such that $W_{1}=a W x \cup x v \cup v y \cup y u$ and $W_{2}=u x \cup x W y \cup y W b$. If the path $W$ intersects $C$ at only one vertex (let this be $x$ ), we can define two paths $W_{1}$ and $W_{2}$ such that $W_{1}=a W x \cup x v \cup v y \cup y u$ and $W_{2}=u x \cup x W b$.

If the cycle $C$ is of length $3, W$ can intersect $C$ at exactly one vertex (let this be $x$ ). We can define two paths $W_{1}=a W x \cup x v \cup v u$ and $W_{2}=u x \cup x W b$. Therefore, the edges of $W$ and $C$ together can be
decomposed into at most two paths in $G$. In other words, if the edges of $G-E(C)$ can be decomposed into $k$ paths, then the edges of $G$ can be decomposed into at most $k+1$ paths.

Subclaim 2. Let $T$ be a triangle component of $G-E(C)$. Then, the edges of $T$ and $C$ together can be decomposed into at most two paths.

Proof: As mentioned earlier, $C$ is either of length 3 or 4 . The triangle $T$ and the cycle $C$ can have either one or two vertices in common. If $T$ and $C$ have one vertex in common, then there exist two adjacent vertices of degree 2 in $T$ and hence, in $G$. By definition, $u$ and $v$ are the closest pair of vertices in $G$, whose degrees are at most 2 . Therefore, $u$ and $v$ also have to be adjacent. So, $C$ can only be of length 3 here. Hence, if at most one vertex in $T$ intersects $C$, the resulting graph is a graph with two triangles intersecting at a common vertex, say $x$. It is easy to see that the edges in this graph can be decomposed into at most two paths. If $T$ and $C$ have two common vertices, the length of $C$ has to be 4, because $u$ and $v$ cannot be part of $T$. Let the common vertices in $C$ and $T$ be $x$ and $y$. Let $a$ be the third vertex in $T$. It is easy to see that the edges of $C$ and $T$ together require at most two paths : namely, $P_{1}=v y \cup y a \cup a x \cup x u$ and $P_{2}=u y \cup y x \cup x v$.

If $G-E(C)$ has no triangle component, then by Subclaim 1 and Observation 11, the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n}{2}\right\rfloor$ paths. Suppose $T$ is a triangle component of $G-E(C)$. Clearly, at most one triangle component is present in $G-E(C)$ when $|C|=3$ and that triangle component will contain the vertex $x$. Because $u$ and $v$ are the closest pair of vertices with degree at most 2 in $G$, at most one triangle component is present in $G-E(C)$ when $|C|=4$ and that triangle component will contain both the vertices $x$ and $y$. By Subclaim 2 , the edges of $C$ and $T$ can be decomposed into at most 2 paths. This implies that the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n-2-3}{2}\right\rfloor+2 \leq\left\lfloor\frac{n}{2}\right\rfloor$ paths.

Observation 2. Let $H$ be a graph on $n$ vertices that consists of two odd components $I$ and $J$ with $n_{I}$ and $n_{J}$ vertices respectively, with $n_{I}+n_{J}=n$, and suppose none of these components is a triangle. Suppose the edges of $I$ and $J$ admit a path decomposition with at most $\left\lfloor\frac{n_{I}}{2}\right\rfloor$ and $\left\lfloor\frac{n_{J}}{2}\right\rfloor$ paths respectively. Then the edges of $H$ can be decomposed into at most $\left\lfloor\frac{n_{I}+n_{J}}{2}\right\rfloor-1$ paths, which is one path less than the bound given by the conjecture for a graph with $n_{I}+n_{J}$ vertices.

(a)

(b)

Fig. 4: (a) Claim 2; Vertex $v$ cannot be a pendant vertex. (b) Claim 4: Vertex $x$ cannot be a cut vertex. Any vertex which is drawn as a circle has all its edges depicted in the figure. Rectangular vertices may have edges not depicted in the figure.

By Claim 1. $G$ contains exactly one vertex of degree at most 2 . Let this vertex be $v$. Let $N_{G}(v)=\{x\}$ if $d_{G}(v)=1$ and $N_{G}(v)=\{x, y\}$ if $d_{G}(v)=2$.

Now if we consider the 2-degenerate ordering of vertices of $G$ as given in Definition 11, the vertex $v$ will be the first vertex in the ordering. By claim 11, all other vertices have degree at least 3 in $G$. Note that $G-v$ is again 2-degenerate and hence contains a vertex of degree at most 2 . We can infer that the vertex with degree at most 2 in $G-v$ has to be either $x$ or $y$. Without loss of generality, let $x$ be that vertex. Thus $d_{G}(x)=3$. Let $N_{G}(x)=\{v, w, z\}$.
Claim 2. $G$ does not contain a pendant vertex.
Proof: Suppose $d_{G}(v)=1$. Since $G-\{v, x\}$ is also a 2-degenerate graph, it is easy to see that at least one of $w$ or $z$ in $G$ must have degree 3. Without loss of generality, let it be $w$ (See Fig 4a).
Case 1. $x$ is not a cut vertex in $G-v$.
Let $G^{\prime}=G-\{v x, x w\}$. If removing edges $v x$ and $x w$ produced a triangle component, then $G$ contains at least two vertices with degree at most 2 and violates Claim 11. Since $x$ is not a cut vertex in $G-v$, there exists a path from $w$ to $z$, which does not go through any edges incident on $x$. Let $P^{\prime}$ be a shortest path between $w$ and $z$ that does not contain $x$. Let $P=P^{\prime} \cup x z$. Let $G^{\prime \prime}=G^{\prime}-E(P)$. Let $j \geq 0$ be the number of triangle components in $G^{\prime \prime}$. Let $H$ denote the graph obtained from $G^{\prime \prime}$, after the $j$ triangle components are removed. In $H, x$ and $v$ are isolated. By Observation 11, the edges in $H$ can be decomposed into at most $\left\lfloor\frac{n-2-3 j}{2}\right\rfloor$ paths. The vertex $w$ has degree 1 in $H$. The path ending at $w$ in $H$ can be extended to $v$, by adding edges $x w$ and $v x$. By Lemma1, the $j$ triangle components together with $P$ require $j+1$ paths. Then, the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n-2-3 j}{2}\right\rfloor+j+1 \leq\left\lfloor\frac{n}{2}\right\rfloor$ paths.
Case 2. $x$ is a cut vertex in $G-v$.
Since $x$ is a cut vertex in $G-v$, removal of $x$ will create two components, say $I$ and $J$. Let $J$ be the component containing vertex $w$ and $I$ be the component containing vertex $z$. Since the components $I$ and $J$ are 2-degenerate, they must have a vertex with degree at most 2. By Claim 11, we can see that such a vertex can not have degree $\leq 2$ in $G$. Therefore, we can conclude that $d_{G}(z)=d_{G}(w)=3$.

Suppose both $J$ and $I$ have an odd number of vertices. By Observation 2 , the edges of components $J$ and $I$ together can be decomposed into at most $\left\lfloor\frac{n-2}{2}\right\rfloor-1$ paths (note that $G-\{v, x\}$ can not have any triangle components. Otherwise, $G$ contains at least two vertices with degree at most 2 and violates Claim 11). So we can use up to two paths to decompose the remaining edge set of $\{v x, x w, x z\}$, without going over the allowed limit of $\left\lfloor\frac{n}{2}\right\rfloor$ paths for $G$. Now if both $I$ and $J$ do not have an odd number of vertices, then at least one of the components has an even number of vertices. Since $d_{G}(z)=d_{G}(w)=3$, without loss of generality, let the component $J$ containing vertex $w$ have an even number of vertices. Here the proof splits into two cases again.
Case 2.1. $w$ is a cut vertex in $J$.
Since $w$ is a cut vertex, the removal of $w$ in $J$ creates two components $J_{1}$ and $J_{2}$. Since $J$ has an even number of vertices, one of the components in $J$ (after removal of $w$ ) has an odd number of vertices and the other component has an even number of vertices. Let $J_{1}$ be the odd component and $J_{2}$ be the even component. Let $a$ be the neighbour of $w$ in $J_{1}, b$ be its neighbour in $J_{2}$. The graph induced by $J_{2}$ and
$w$ is an odd component. Therefore, by Observation 2 , the edges of component $J$ (excluding the edge $a w)$ require $\left(\left\lfloor\frac{n_{J}}{2}\right\rfloor-1\right)$ paths, where $n_{J}$ is the number of vertices in $J$. Note that the edges inside the component $I$, regardless of whether it has odd or even number of vertices, can be decomposed into at most $\left\lfloor\frac{n_{I}}{2}\right\rfloor$ paths by Observation 1, where $n_{I}$ is the number of vertices in $I$. We need one path for edges $a w, x w$, and $x z$. The edge $v x$ requires another path in the path decomposition of $G$. So, the edges of $G$ can be decomposed into at $\operatorname{most}\left(\left\lfloor\frac{n_{J}}{2}\right\rfloor-1\right)+1+\left\lfloor\frac{n_{I}}{2}\right\rfloor+1 \leq\left\lfloor\frac{n}{2}\right\rfloor$ paths.
Case 2.2. $w$ is not a cut vertex in $J$.
Since the graph $J$ is also 2-degenerate, and because of Claim 1, at least one of the vertices adjacent to $w$ must be a degree 3 vertex. Let it be vertex $a$. Let $G^{\prime}=G-\{v x, x w, w a\}$. Now we show that $G^{\prime}$ has no triangle component. Let $Q$ denote the path $v x \cup x w \cup w a$. Note that $G^{\prime}=G-E(Q)$. Let us assume that a triangle component $T$ is present in $G^{\prime}$. Clearly, $T$ must intersect with $Q$ at more than two vertices. Otherwise, $T$ contains at least one vertex with degree 2 in $G$, which violates Claim 11. Therefore all the vertices of $T$ must intersect $Q$ and they must be non-adjacent in $Q$. But that is not possible for the path $Q$ containing only three edges. Therefore, no triangle component exists in $G^{\prime}$.

Since $w$ is not a cut vertex, there exists a path between its neighbours $a$ and $b$ that does not go through any of the edges incident on $w$. Let $P^{\prime}$ be a shortest such path. Let $P=P^{\prime} \cup b w$. Let $G^{\prime \prime}=G^{\prime}-E(P)$. Let $k \geq 0$ be the number of triangle components of $G^{\prime \prime}$. Let $H$ denote the graph obtained from $G^{\prime \prime}$, after the $k$ triangle components are removed. In $H, w$ and $v$ are isolated. By Observation 1, edges in $H$ can be decomposed into at most $\left\lfloor\frac{n-2-3 k}{2}\right\rfloor$ paths. By Lemma 1, the $k$ triangle components together with $P$ require $k+1$ paths. The vertex $a$ has degree 1 in $H$. The path ending at $a$ can be extended till $v$ without needing any additional paths (note that extending the path to $v$ does not create a cycle because $v$ and $w$ are isolated vertices in $H$ and the vertex $x$ is a cut vertex by our assumption). This takes care of edges $a w, w x, x v$. Then, the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n-2-3 k}{2}\right\rfloor+k+1 \leq\left\lfloor\frac{n}{2}\right\rfloor$ paths.

From Claim 1 and Claim 22, we can conclude that $v$ is the only vertex in $G$ with degree at most 2, and that $d_{G}(v)=2$.

## Claim 3. G does not contain a degree 2 cut vertex.

Proof: Suppose $v$ is a cut vertex in $G$, whose removal separates the graph into components $X$ and $Y$ with $n_{X}$ and $n_{Y}$ vertices respectively. Let $x$ and $y$ be the neighbours of $v$ in components $X$ and $Y$ respectively. Consider the graph $G-v$. Let component $X^{\prime}=X+v x$, with $n_{X}+1$ vertices. If components $X^{\prime}$ and $Y$ are not triangles, then by Observation 11 , the edges of $X^{\prime}$ and $Y$ can be decomposed into $\left\lfloor\frac{n_{X}+1}{2}\right\rfloor$ paths and $\left\lfloor\frac{n_{Y}}{2}\right\rfloor$ paths respectively. The path in $X^{\prime}$ that ends at $v$ can be extended by adding back the edge $v y$. This does not create any new paths. Clearly, component $X^{\prime}$ is not a triangle. By Claim 11, component $Y$ cannot be a triangle either. Hence, the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n_{X}+1}{2}\right\rfloor+\left\lfloor\frac{n_{Y}}{2}\right\rfloor \leq$ $\left\lfloor\frac{n}{2}\right\rfloor$ paths, which is a contradiction. This implies that $v$ cannot be a cut vertex in $G$.

Claim 4. The vertex $x$ cannot be a cut vertex in $G$.
Proof: Suppose $x$ is a cut vertex in $G$. Since $v$ can not be a cut vertex, there exists a path between its neighbours $x$ and $y$, that does not use any of the edges incident on $v$ (See Fig 4b). Let $z$ be the neighbour of $x$ along this path (note that $z$ can be $y$ itself, but that will not affect our proof). Let $G^{\prime}=G-\{x z, v x\}$.

Then, $G^{\prime}$ has two components by our assumption that $x$ is a cut vertex. Clearly, neither components are triangle component. Let $n_{A}$ and $n_{B}$ denote the number of vertices in the two components. Since $G$ is a minimum counterexample, by Observation 11, we can decompose the edges of the components into at most $\left\lfloor\frac{n_{A}}{2}\right\rfloor$ and $\left\lfloor\frac{n_{B}}{2}\right\rfloor$ paths respectively. Note that vertices $v, y$ and $z$ are in the same component, while vertex $x$ is in a different component. Since vertex $v$ and vertex $x$ each have degree 1 in their respective components, we can add the removed edges $v x$ and $x z$ without creating any new paths. This can be done by adding the edge $v x$ to the path ending at $v$, and the edge $x z$ to the path ending at $x$, in their respective components. So, the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n_{A}}{2}\right\rfloor+\left\lfloor\frac{n_{B}}{2}\right\rfloor \leq\left\lfloor\frac{n}{2}\right\rfloor$ paths, which is a contradiction.

By Claim $\dagger, x$ is not a cut vertex. Recall that $N_{G}(x)=\{v, z, w\}$. Based on the degrees of neighbours of $x$ we have the following cases:


Fig. 5: Figure depicts scenarios if $x$ is not a cut vertex. Case 1: $x$ has a degree 3 neighbour $z$. Case 2.1: $x$ has no degree 3 neighbour and $d_{G}(y)=3$. Case 2.2: $x$ has no degree 3 neighbour and $d_{G}(y)=4$. Any vertex which is drawn as a circle has all its edges depicted in the figure. Rectangular vertices may have edges not depicted in the figure.

## Case 1. $x$ has a degree 3 neighbour.

Without loss of generality, let that neighbour be $z$ (See Fig 5a). Let $G^{\prime}=G-\{x z\}$. Since $x$ is not a cut vertex in $G$, there exists a path from $z$ to $v$, that does not go through $x$. Let $P^{\prime}$ be a shortest such path. Let $P=P^{\prime} \cup z t \cup x v$, where $t$ is the other neighbour of $z$ that does not lie on $P^{\prime}$. Let $G^{\prime \prime}=G^{\prime}-E(P)$. Let $j \geq 0$ be the number of triangle components of $G^{\prime \prime}$. Note that removal of the path $P$ isolates vertices $z$ and $v$. Let $H$ denote the graph obtained from $G^{\prime \prime}$, after the $j$ triangle components are removed. The edges in $H$ can be decomposed into at most $\left\lfloor\frac{n-2-3 j}{2}\right\rfloor$ paths. The vertex $x$ has degree 1 in $H$. The edge $x z$ can be added back by extending the path in $H$ that ends at $x$. The $j$ triangle components together with $P$ require $j+1$ paths, by Lemma11. Then, the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n-2-3 j}{2}\right\rfloor+j+1$ $\leq\left\lfloor\frac{n}{2}\right\rfloor$ paths. Thus $d_{G}(z) \geq 4$ and $d_{G}(w) \geq 4$.

Case 2. $x$ has no degree 3 neighbour.
Based on the degree of $y$, we have following subcases:

Case 2.1. $d_{G}(y)=3$.
Note that the case of $y$ having a degree 3 neighbour is similar to Case 11, with $y$ playing the role of $x$. Vertices $v, x$ and $y$ are the first three vertices in the vertex removal order given by Definition 1. Let some vertex $a$ be the next vertex in the order after $v, x, y$. Vertex $v$ is the only vertex with degree 2 in $G$. Since $G-\{v, x, y\}$ is also 2-degenerate, vertex $a$ has to be a neighbour of $x$ or $y$. But $x$ and $y$ have no neighbour with degree 3 . Therefore, $d_{G}(a) \geq 4$. Since vertex $a$ has degree 2 in $G-\{v, x, y\}$, $a$ must be a neighbour of both $x$ and $y$ and $d_{G}(a)=4$. Without loss of generality, let $z$ be $a$.(See Fig 5b).

Recall that $w$ is the other neighbour of $x$. Let $G^{\prime}=G-\{v x, v y\}$. Let $P=y z \cup z x \cup x w$. Let $G^{\prime \prime}=G^{\prime}-E(P)$. Let $j \geq 0$ be the number of triangle components of $G^{\prime \prime}$. The vertices $x$ and $v$ are isolated now. Let $H$ denote the graph obtained from $G^{\prime \prime}$, after the $j$ triangle components are removed. The edges in $H$ can be decomposed into at most $\left\lfloor\frac{n-2-3 j}{2}\right\rfloor$ paths. $d_{H}(y)=1$. So, the path ending at $y$ can be extended to include edges $v y$ and $v x$. The $j$ triangle components and $P$ require at most $j+1$ paths. Then, the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n-2-3 j}{2}\right\rfloor+j+1 \leq\left\lfloor\frac{n}{2}\right\rfloor$ paths. Therefore, $d_{G}(y) \geq 4$.
Case 2.2. $d_{G}(y) \geq 4$.
Vertices $v$ and $x$ are the first two vertices in the vertex removal order given by Definition 11. Since $G-\{v, x\}$ is also 2-degenerate, the next vertex in the order must also be a neighbour of $x$ or $v$. Recall that $d_{G}(y) \geq 4$ and $x$ has no neighbour with degree 3 or less. Since $G-\{v, x\}$ is 2-degenerate and must have a vertex with degree at most $2, d_{G}(y)=4$ and $y$ has to be a neighbour of $x$ (See Fig 5c).

Let $G^{\prime}=G-\{x y, x v\}$. If removing edges $x y$ and $x v$ created a triangle component in $G^{\prime}$, then $G$ contains at least two vertices with degree at most 2 and that violates Claim 11. Since $x$ is not a cut vertex, there exists a path from $z$ to $v$ that does not go through any of the incident edges on $x$. Let $P^{\prime}$ be a shortest such path. Let $P=P^{\prime} \cup z x$. Let $G^{\prime \prime}=G^{\prime}-E(P)$. Let $j \geq 0$ be the number of triangle components of $G^{\prime \prime}$. Let $H$ denote the graph obtained from $G^{\prime \prime}$, after the $j$ triangle components are removed. The vertices $x$ and $v$ are isolated in $H$. The vertex $y$ in $H$ is of degree 1 . So, the edges $x y$ and $x v$ can be added to the path ending at $y$ in $H$. Then, the edges of $G$ can be decomposed into at most $\left\lfloor\frac{n-2-3 j}{2}\right\rfloor+j+1 \leq\left\lfloor\frac{n}{2}\right\rfloor$ paths.

Therefore, we infer that $G$ is not a counterexample. This completes our proof for Theorem 1 .

## 4 Conclusion

We have proved Gallai's path decomposition conjecture for 2-degenerate graphs. Botler et al, (2020) proved the conjecture for graphs with treewidth at most 3 and Botler et al. (2019) proved the conjecture for triangle-free planar graphs. Note that these graphs are proper subclasses of 3-degenerate graphs. Hence proving the conjecture for 3-degenerate graphs would generalize the results in Botler et al. (2020) and Botler et al. (2019).

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