International Journal of Algebra, Vol. 3, 2009, no. 20, 993 - 999

Galois Extensions with a Galois Commutator Subring

George Szeto

Department of Mathematics
Bradley University
Peoria, Illinois 61625, USA
szeto@bradley.edu

Lianyong Xue

Department of Mathematics
Bradley University
Peoria, Illinois 61625, USA
lxue@bradley.edu

Abstract

Let B be a Galois extension of B^G with Galois group G, Δ the commutator subring of B^G in B, and $G|_\Delta$ the restriction of G to Δ. Equivalent conditions are given for a Galois extension Δ of Δ^G with Galois group $G|_\Delta$. It is shown that the following statements are equivalent: (1) Δ is a Galois extension of Δ^G with Galois group induced by and isomorphic with G/N where $N = \{g \in G \,|\, g(x) = x \text{ for all } x \in \Delta\}$. (2) $B^G\Delta$ is a Galois extension of B^G with Galois group induced by and isomorphic with G/N and Δ is a finitely generated and projective module over Δ^G. (3) B is a composition of two Galois extensions: $B \supset B^G\Delta$ with Galois group N and $B^G\Delta \supset B^G$ with Galois group induced by and isomorphic with G/N such that Δ is a finitely generated and projective module over Δ^G. Consequently, more results can be derived for several well known classes of Galois extensions such as DeMeyer-Kanzaki Galois extensions, Azumaya Galois extensions, and Hirata separable Galois extensions.

Mathematics Subject Classification: 16S35, 16W20

Keywords: separable extensions, Azumaya algebras, Galois extensions, DeMeyer-Kanzaki Galois extensions, Azumaya Galois extensions

1 Introduction

Let T be a ring extension of S and $V_T(S)$ the commutator subring of S in T. Properties of $V_T(S)$ play an important role for central simple algebras, Azumaya algebras, Hirata separable extensions, Galois extensions for rings, and Hopf Galois extensions ([1]–[2], [4]–[9]). Let B be a Galois extension of B^G with Galois group G. Then $V_B(B^G) = \oplus \sum_{g\in G} J_g$ where $J_g = \{b \in B \,|\, bx = g(x)b$ for all $x \in B\}$ ([4], Proposition 1). In [1], it was shown that if B is a Galois extension of an Azumaya algebra B^G over C^G where C is the center of B, then $V_B(B^G)$ is a Galois algebra over C^G with Galois group induced by and isomorphic with G ([1], Theorem 2). Also, $V_B(B^G)$ is investigated for a Hirata separable Galois extension B with Galois group G ([8]). The purpose of the present paper is to characterize a Galois extension B of B^G with Galois group G such that $V_B(B^G)$ is a Galois extension with Galois group induced by G. We shall show the following equivalent conditions: Let B be a Galois extension of B^G with Galois group G and $\Delta = V_B(B^G)$. (1) Δ is a Galois extension of Δ^G with Galois group isomorphic with G/N where $N = \{g \in G \,|\, g(x) = x$ for all $x \in \Delta\}$. (2) $B^G\Delta$ is a Galois extension of B^G with Galois group isomorphic with G/N and Δ is a finitely generated and projective module over Δ^G. (3) B is a composition of two Galois extensions: (i) $B \supset B^G\Delta$ with Galois group N and (ii) $B^G\Delta \supset B^G$ with Galois group isomorphic with G/N such that $J_{\overline{g}}^{(\Delta)}$ is a finitely generated projective module over Δ^G for each $\overline{g} \in G/N$ where $J_{\overline{g}}^{(\Delta)} = \{b \in \Delta \,|\, bx = g(x)b$ for all $x \in \Delta\}$. Consequently, more results can be derived for several well known classes of Galois extensions such as DeMeyer-Kanzaki Galois extensions, Azumaya Galois extensions, and Hirata separable Galois extensions.

2 Definitions and Notations

Throughout this paper, B will represent a ring with 1, C the center of B, G a finite automorphism group of B, B^G the set of elements in B fixed under each element in G, and $B * G$ the skew group ring of G over B, that is, $B * G$ is a free left B-module in which the multiplication is given by $gb = g(b)g$ for $b \in B$ and $g \in G$.

Let A be a subring of a ring B with the same identity 1. Following the definitions and notations as given in [9], we denote $V_B(A)$ the commutator (also called centralizer) subring of A in B. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., k$ for some integer $k\}$ such that $\sum a_i b_i = 1$, and $\sum ba_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. We call B a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m$

for some integer $m\}$ such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. Such a set $\{a_i, b_i\}$ is called a G-Galois system for B. A Galois extension B of B^G is called a Galois algebra over B^G if B^G is contained in C, and a central Galois algebra if B is a Galois extension of C. We called B a center Galois extension with Galois group G if C is a Galois algebra over C^G with Galois group $G|_C \cong G$, and a commutator Galois extension of B^G with Galois group G if $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$. A Galois extension B of B^G with Galois group G is called an Azumaya Galois extension if B^G is an Azumaya C^G-algebra. A Galois extension B of B^G with Galois group G is called a DeMeyer-Kanzaki Galois extension if B is an Azumaya algebra over C which is a Galois algebra over C^G with Galois group $G|_C \cong G$. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule, and B is called a Hirata separable Galois extension of B^G if it is a Galois and a Hirata separable extension of B^G.

3 Characterizations

In this section, let B be a Galois extension of B^G with Galois group G and $\Delta = V_B(B^G)$. We shall characterize B with a Galois commutator Δ with Galois group induced by G. We begin with some basic facts.

Lemma 3.1 *Let T be a ring and G an automorphism group of T. Then (1) $V_T(T^G)$ is a G-invariant subring of T and (2) $(V_T(T^G))^G$ is contained in the center of $V_T(T^G)$.*

Proof. (1) For any $g \in G$, $a \in V_T(T^G)$, and $x \in T^G$, we have that $g(a)x = g(ax) = g(xa) = xg(a)$, so $g(a) \in V_T(T^G)$.

(2) holds because $(V_T(T^G))^G = T^G \cap (V_T(T^G))$ which is contained in the center of $V_T(T^G)$.

Let $N = \{g \in G \mid g(a) = a$ for all $a \in V_T(T^G)\}$. Then part (1) in Lemma 3.1 implies that N is a normal subgroup of G, and $V_T(T^G)$ is an algebra over $(V_T(T^G))^G$ by part (2). We shall employ a well known fact for a Galois extension.

Lemma 3.2 *Let B be a Galois extension of B^G with Galois group G and A a G-invariant subring of B under the action of G. If A is a Galois extension of B^G with Galois group induced by and isomorphic with G, then $A = B$.*

Now we show the main theorem in this section.

Theorem 3.3 *Let B be a Galois extension of B^G with Galois group G, $\Delta = V_B(B^G)$, and $D = \Delta^G$. Then the following statements are equivalent: (1) Δ is a Galois algebra over D with Galois group induced by and isomorphic with G/N where $N = \{g \in G \,|\, g(x) = x$ for all $x \in \Delta\}$. (2) $B^G\Delta$ is a Galois extension of B^G with Galois group induced by and isomorphic with G/N and Δ is a finitely generated and projective module over D. (3) B is a composition of two Galois extensions: $B \supset B^G\Delta$ with Galois group N and $B^G\Delta \supset B^G$ with Galois group induced by and isomorphic with G/N such that $J_{\overline{g}}^{(\Delta)}$ is a finitely generated projective module over D for each $\overline{g} \in G/N$ where $J_{\overline{g}}^{(\Delta)} = \{b \in \Delta \,|\, bx = g(x)b$ for all $x \in \Delta\}$.*

Proof. (1) $\Longrightarrow$ (2) Since Δ is a Galois algebra over D where $D = \Delta^G$ which in contained in the center of Δ by Lemma 3.1, Δ is a finitely generated and projective module over D. Let $\{a_i, b_i \in \Delta \,|\, i = 1, 2, ..., m\}$ be a Galois system for Δ. Then $B^G\Delta$ is a Galois extension of $(B^G\Delta)^G$ $(= B^G)$ with Galois group induced by and isomorphic with G/N for Δ because $B^G\Delta$ can take $\{a_i, b_i \in B^G\Delta \,|\, i = 1, 2, ..., m\}$ as a Galois system.

(2) $\Longrightarrow$ (1) By hypothesis, $B^G\Delta$ is a Galois extension of B^G with Galois group induced by and isomorphic with G/N, so, by Theorem 1 in [3], the skew group ring

$$(B^G\Delta) * (G/N) \cong \mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta).$$

Denoting G/N by $\overline{G}$, we have that

$$\alpha : (B^G\Delta) * \overline{G} \cong \mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta)$$

by $(\alpha(\sum_{\overline{g}\in\overline{G}} a_{\overline{g}}\overline{g}))(x) = \sum_{\overline{g}\in\overline{G}} a_{\overline{g}}\overline{g}(x)$ for each $x \in B^G\Delta$. Then

$$\Delta * \overline{G} = V_{B^G\Delta*\overline{G}}(B^G) \cong V_{\mathrm{Hom}_{B^G(B^G\Delta, B^G\Delta)}}(\alpha(B^G)).$$

Next we claim that $V_{\mathrm{Hom}_{B^G(B^G\Delta, B^G\Delta)}}(\alpha(B^G)) = \mathrm{Hom}_D(\Delta, \Delta)$ where $D = \Delta^{\overline{G}} = \Delta^G$. In fact, let $f \in \mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta)$ such that $f \cdot \alpha(r) = \alpha(r) \cdot f$ for each $r \in B^G$. Then for each $t \in \Delta$, $f(t)r = f(tr) = f(rt) = f(\alpha(r)(t)) = (f \cdot \alpha(r))(t)) = (\alpha(r) \cdot f)(t)) = rf(t)$. This implies that $f(t) \in \Delta$. Thus $f : \Delta \longrightarrow \Delta$; and so $V_{\mathrm{Hom}_{B^G(B^G\Delta, B^G\Delta)}}(\alpha(B^G)) \subset \mathrm{Hom}_D(\Delta, \Delta)$ (for $D = B^G \cap \Delta$ by Lemma 3.1). Conversely, let $f \in \mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta)$ such that $f \in \mathrm{Hom}_D(\Delta, \Delta)$. We claim that $f \cdot \alpha(r) = \alpha(r) \cdot f$ for each $r \in B^G$. In fact, for each $s \in B^G$ and each $a \in \Delta$, $(f \cdot \alpha(r))(sa) = f(rsa) = f(ars) = f(a)(rs) = rf(a)s = rf(as) = rf(sa) = (\alpha(r) \cdot f)(sa)$ (for $f(a) \in \Delta$). Thus $f \cdot \alpha(r) = \alpha(r) \cdot f$ for each $r \in B^G$. But then $f \in V_{\mathrm{Hom}_{B^G(B^G\Delta, B^G\Delta)}}(\alpha(B^G))$. This proves that

$$V_{\mathrm{Hom}_{B^G(B^G\Delta, B^G\Delta)}}(\alpha(B^G)) = \mathrm{Hom}_D(\Delta, \Delta).$$

Therefore, $\alpha : \Delta * \overline{G} \cong \mathrm{Hom}_D(\Delta, \Delta)$. Moreover, by hypothesis, Δ is a finitely generated and projective module over D, so Δ is a Galois algebra over D with Galois group isomorphic with $\overline{G}$ ([3], Theorem 1).

(2) $\Longrightarrow$ (3) Since $B^G\Delta$ is a Galois extension of B^G with Galois group induced by and isomorphic with $\overline{G}$ $(= G/N)$, B^N containing $B^G\Delta$ is also a Galois extension of B^G $(= B^{\overline{G}})$ with Galois group isomorphic with $\overline{G}$; and so $B^N = B^G\Delta$ by Lemma 3.2. But then $B \supset B^G\Delta$ is a Galois extension with Galois group N and $B^G\Delta \supset B^G$ is a Galois extension with Galois group induced by and isomorphic with $\overline{G}$ $(= G/N)$ such that Δ is a finitely generated and projective module over D. Noting that $V_{B^G\Delta}(B^G) = \Delta = \oplus \sum_{\overline{g} \in \overline{G}} J_{\overline{g}}^{(\Delta)}$ ([4], Proposition 1 and Theorem 1). we have that $J_{\overline{g}}^{(\Delta)}$ is a finitely generated projective module over D for each $\overline{g} \in G/N$.

(3) $\Longrightarrow$ (2) is clear.

By Theorem 3.3, we shall derive some consequences for several well known classes of Galois extensions. We recall that B is a center Galois extension with Galois group G if B is a Galois extension with Galois group G such that its center C is a Galois algebra over C^G with Galois group $G|_C \cong G$, and B is a commutator Galois extension of B^G with Galois group G if $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$.

Corollary 3.4 *Let B be a Galois extension of B^G with Galois group G. If $B = B^G C$ such that C is finitely generated and projective over C^G, then B a center Galois extension with Galois group G.*

Corollary 3.5 *Let B be a Galois extension of B^G with Galois group G. If $B = B^G\Delta$ such that Δ is finitely generated and projective over Δ^G, then B a commutator Galois extension with Galois group G.*

Remark 3.6 *Since a DeMeyer-Kanzaki Galois extension is also a center Galois extension and an Azumaya Galois extension is a commutator Galois extension ([1], Theorem 2), Corollary 3.4 and Corollary 3.5 hold for the classes of DeMeyer-Kanzaki Galois extensions and Azumaya Galois extensions.*

Corollary 3.7 *Let B be a Hirata separable Galois extension of B^G with Galois group G. If $B = B^G\Delta$, then Δ is a Galois algebra with Galois group induced by and isomorphic with G/N.*

Proof. Since B is a Hirata separable Galois extension of B^G with Galois group G, J_g is a finitely generated and projective rank one module over C^G for each $g \in G$ ([8], Theorem 2). Hence $\Delta = \oplus \sum_{g\in G} J_g$ is a finitely generated and projective module over C^G. Thus Δ is a Galois algebra over D with Galois group induced by and isomorphic with G/N by Theorem 3.3.

4 The Galois commutator

In section 3, we characterize a Galois extension B with a Galois commutator subring Δ. In this section, we shall give an equivalent condition for Δ as a composition of a central Galois algebra and a commutative Galois algebra. Thus we derive an expression for B as a composition of three Galois extensions. We keep the notations of N, $\overline{G}$, and $J_{\overline{g}}^{(\Delta)}$ as given in section 3.

Lemma 4.1 *Let B be a Galois extension of B^G with Galois group G. Then $\oplus \sum_{h\in N} J_{gh} \subset J_{\overline{g}}^{(\Delta)}$ for each $g \in G$ and $\Delta = \sum_{\overline{g}\in\overline{G}} J_{\overline{g}}^{(\Delta)}$.*

Proof. Since B is a Galois extension of B^G with Galois group G, $\Delta = V_B(B^G) = \oplus \sum_{g\in G} J_g = \oplus \sum_{\overline{g}\in\overline{G}} \sum_{h\in N} J_{gh}$ ([4], Proposition 1). For any $a \in J_{gh}$ and $x \in \Delta$, we have that $ax = (gh)(x)a = g(x)a = \overline{g}(x)a$, so $a \in J_{\overline{g}}^{(\Delta)}$; and so $J_{gh} \subset J_{\overline{g}}^{(\Delta)}$. Thus $\Delta = \oplus \sum_{\overline{g}\in\overline{G}} \sum_{h\in N} J_{gh} = \sum_{\overline{g}\in\overline{G}} J_{\overline{g}}^{(\Delta)}$.

Theorem 4.2 *Let B be a Galois extension of B^G with Galois group G such that Δ is a Galois algebra over D with Galois group induced by and isomorphic with G/N, Z the center of Δ, and $K = \{g \in G \,|\, g(a) = a$ for all $a \in Z\}$. Then Δ is a central Galois algebra over Z with Galois group induced by and isomorphic with K/N and Z is a commutative Galois algebra over Z^G with Galois group induced by and isomorphic with $(G/N)/(K/N)$ if and only if $J_{gh} = \{0\}$ for each $g \notin K$ and $h \in N$.*

Proof. Since Δ is a Galois algebra over D with Galois group induced by and isomorphic with G/N, $\Delta = \oplus \sum_{\overline{g}\in\overline{G}} J_{\overline{g}}^{(\Delta)}$. Moreover, Δ is a central Galois algebra over Z with Galois group induced by and isomorphic with K/N if and only if $J_{\overline{g}}^{(\Delta)} = \{0\}$ for each $\overline{g} \notin K/N$, and in this case, Z is a commutative Galois algebra over Z^G with Galois group induced by and isomorphic with $(G/N)/(K/N)$ ([4], Proposition 3). But then $J_{\overline{g}}^{(\Delta)} = \oplus \sum_{h\in N} J_{gh}$ for each $\overline{g} \in \overline{G}$ by Lemma 4.1. Thus $J_{\overline{g}}^{(\Delta)} = \{0\}$ for each $\overline{g} \notin K/N$ if and only if $J_{gh} = \{0\}$ for each $g \notin K$ and $h \in N$.

Corollary 4.3 *Let B be a Galois extension of B^G with Galois group G such that Δ is a Galois algebra over D with Galois group induced by and isomorphic with G/N, Z the center of Δ, and $K = \{g \in G \,|\, g(a) = a \text{ for all } a \in Z\}$. If $J_{gh} = \{0\}$ for each $g \notin K$ and $h \in N$, then B is a composition of three Galois extensions: (1) $B \supset B^G\Delta$ with Galois group N, (2) $B^G\Delta \supset B^GZ$ with Galois group induced by and isomorphic with K/N, and (3) $B^GZ \supset B^G$ with Galois group induced by and isomorphic with $\overline{G}/\overline{K}$ where $\overline{G} = G/N$ and $\overline{K} = K/N$.*

ACKNOWLEDGEMENTS. This paper was written under the support of a Caterpillar Fellowship at Bradley University. The authors would like to thank Caterpillar Inc. for the support.

References

[1] R. Alfaro and G. Szeto, Skew Group Rings which are Azumaya, *Comm. in Algebra,* **23**(6) (1995), 2255-2261.

[2] R. Alfaro and G. Szeto, On Galois Extensions of an Azumaya Algebra, *Comm. in Algebra,* **25**(6) (1997), 1873-1882.

[3] F.R. DeMeyer, Some Notes on the General Galois Theory of Rings, *Osaka J. Math.,* **2** (1965), 117 - 127.

[4] T. Kanzaki, On Galois Algebra Over A Commutative Ring, *Osaka J. Math.,* **2** (1965), 309-317.

[5] H. F. Kreimer and M. Takouchi, Hopf Algebras and Galois Extensions of an Algebra, *Indiana Univ. Math J.,* **30** (1981), 675-692.

[6] P. Nuss, Galois-Azumaya Extensions and the Brauer Group of a Commutative Ring, *Bull. Belg. Math. Soc. Vol.*, **13** (2006), 247-270.

[7] M. Ouyang, Azumaya Extensions and Galois Correspondence, *Algebra Colloquium,* **7**(1) (2000), 43-57.

[8] K. Sugano, On a Special Type of Galois Extensions, *Hokkaido J. Math.,* **9** (1980) 123-128.

[9] G. Szeto and L. Xue, On Galois Extensions of a Separable Algebra, *International Mathematical Forum*, **3**(14) (2008), 677-683.

Received: March, 2009