MANAGEMENT SCIENCE
Vol. 34, No. 6, June 1988
Printed in U.S.A.

GAME-THEORETIC OPTIMAL PORTFOLIOS*

ROBERT BELL AND THOMAS M. COVER
The Rand Corporation
Departments of Statistics and Electrical Engineering,
Stanford University, Stanford, California 94305

We show, for a wide variety of payoff functions, that the expected log optimal portfolio is also game theoretically optimal in a single play or in multiple plays of the stock market. Thus there is no essential conflict between good short-term and long-run performance. Both are achieved by maximizing the conditional expected log return.
(PORTFOLIO; GAME THEORY; LOG INVESTMENT)

1. Introduction

Suppose an investor is faced with a collection of m stocks $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ drawn according to some known joint distribution function $F(\mathbf{x})$. We shall assume the stock values X_i are nonnegative. The random variable X_i is the value of a one unit investment in the ith stock. A portfolio is a vector $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbf{B} = \{\mathbf{b} \in \mathbf{R}^m: b_i \geq 0, \sum b_i = 1\}$, with the interpretation that b_i is the proportion of wealth allocated to stock i. The random capital S resulting from investment portfolio $\mathbf{b}$ is given by $S = \sum b_i X_i = \mathbf{b}^t\mathbf{X}$.

We examine the two-person zero-sum game with payoff $E\phi(S_1/S_2)$, where ϕ is any nondecreasing function, and $S_1 = \mathbf{b}_1^t\mathbf{X}$, $S_2 = \mathbf{b}_2^t\mathbf{X}$ are the random capitals resulting from portfolio strategies $\mathbf{b}_1$ and $\mathbf{b}_2$ against a market vector $\mathbf{X} \geq 0$ drawn according to some known distribution $F(\mathbf{x})$. Let $\tilde{S}_1$, $\tilde{S}_2$ denote random capitals obtained by fair randomization of S_1 and S_2. How does one outperform another investor according to the criterion $E\phi(\tilde{S}_1/\tilde{S}_2)$?

It will be shown that a certain portfolio $\mathbf{b}^*$ is the heart of the solution of all such games. More specifically, the game with payoff $E\phi(\tilde{S}_1/\tilde{S}_2)$ is solved for either player by first employing fair randomization to the initial capital, where the randomization depends only on the function ϕ, and then distributing the resultant random capital according to the portfolio $\mathbf{b}^*$. In this sense $\mathbf{b}^*$ is competitively optimal for all games $\phi(\tilde{S}_1/\tilde{S}_2)$. The game theoretic optimal portfolio $\mathbf{b}^*$ is characterized as that portfolio maximizing $E \ln \mathbf{b}^t\mathbf{X}$. Thus it has optimal asymptotic properties as well.

Specifically, we consider a two-person zero-sum game with payoff function $E\phi(S_1/S_2)$. Players 1 and 2 each start with one unit of capital. A strategy for player i consists of a choice of a "fair" distribution function $G_i(w)$, $G_i(0^-) = 0$, $\int w dG_i(w) \leq 1$, and a choice of portfolio $\mathbf{b}_i \in \mathbf{B}$. Player i then exchanges his unit of capital for the fair random variable (r.v.) $W_i \sim G_i(w)$, and distributes the result W_i of this gamble across the stocks according to portfolio $\mathbf{b}_i$. We assume that W_1, W_2, and $\mathbf{X}$ are independent r.v.'s. The payoff to player 1 for the game is defined to be

$$E\phi(W_1\mathbf{b}_1^t\mathbf{X}/W_2\mathbf{b}_2^t\mathbf{X}) = \int \phi(w_1\mathbf{b}_1^t\mathbf{x}/w_2\mathbf{b}_2^t\mathbf{x})dG_1(w_1)dG_2(w_2)dF(\mathbf{x}). \tag{1.1}$$

We call this the stock *market* ϕ-*game*. If

$$\inf_{\mathbf{b}_2, G_2} \sup_{\mathbf{b}_1, G_1} E\phi(W_1\mathbf{b}_1^t\mathbf{X}/W_2\mathbf{b}_2^t\mathbf{X}) = \sup_{\mathbf{b}_1, G_1} \inf_{\mathbf{b}_2, G_2} E\phi(W_1\mathbf{b}_1^t\mathbf{X}/W_2\mathbf{b}_2^t\mathbf{X}) = \mathbf{v}, \tag{1.2}$$

then $\mathbf{v}$ is the value of the stock market ϕ-game.

* Accepted by William T. Ziemba; received July 1987.

0025-1909/88/3406/0724$01.25

Let $\tilde{S}_1 = W_1\mathbf{b}_1'\mathbf{X}$ and $\tilde{S}_2 = W_2\mathbf{b}_2'\mathbf{X}$. It can be seen that all of the following payoff functions can be written in the form $E\phi(\tilde{S}_1/\tilde{S}_2)$: (a) $P(\tilde{S}_1 > \tilde{S}_2)$, (b) $P(\tilde{S}_1 > t\tilde{S}_2)$, (c) $E\tilde{S}_1/\tilde{S}_2$, (d) $Ee^{\tilde{S}_1/\tilde{S}_2}$, (e) $E \ln (\tilde{S}_1/\tilde{S}_2)$, (f) $E(\tilde{S}_1/(\tilde{S}_1 + \tilde{S}_2))$, (g) $E \min \{\tilde{S}_1/\tilde{S}_2, a\}$, and (h) the expected number of factors (rounded off to the nearest integer) by which $\tilde{S}_1$ exceeds $\tilde{S}_2$. The payoff function $P(\tilde{S}_1 > \tilde{S}_2)$, previously considered and solved in Bell and Cover (1980), is obtained when we let ϕ be the indicator function of $[1, \infty)$.

We first consider in §2 the two-person zero-sum game with payoff function $E\phi(W_1/W_2)$, where W_1 and W_2 are independent fair random variables (i.e., $W_i \geq 0$, $EW_i \leq 1$). We denote this the *primitive ϕ-game* because it does not involve a portfolio selection. §3 establishes the equivalence of $ES/S^* \leq 1$ for all $S \in \mathbf{S}$, and $E \ln S/S^* \leq 0$ for all $S \in \mathbf{S}$, if $\mathbf{S}$ is a convex family of random variables. This equivalence is the key to the proofs in §§4, 5, and 6.

§4 establishes that the minimax strategies for the portfolio ϕ-game (ϕ nondecreasing) are $W_1^*\mathbf{b}^{*\prime}\mathbf{X}$ and $W_2^*\mathbf{b}^{*\prime}\mathbf{X}$, where W_1^* and W_2^* are the minimax strategies for the primitive ϕ-game and $\mathbf{b}^*$ maximizes $E \ln \mathbf{b}'\mathbf{X}$. It follows that the solution of the portfolio ϕ-game factors into two parts:

(1) a purely game theoretic randomization of the initial capital, depending only on ϕ, and

(2) an allocation of the resulting capital according to $\mathbf{b}^*$.

These results hold up for multistage market games, as shown in §§5 and 6. Consequently, for ϕ nondecreasing, the log optimal portfolio $\mathbf{b}^*$ is the optimal allocation of resources for any two-person zero-sum game with payoff function $E\phi(\tilde{S}_1/\tilde{S}_2)$. Development of the optimal asymptotic properties of $\mathbf{b}^*$ can be found in Finkelstein and Whitley (1981), Thorp (1969), Breiman (1961), Algoet and Cover (1988) and a critical discussion of such portfolios can be found in Samuelson (1967, 1969). An algorithm for calculation $\mathbf{b}^*$ is described in Cover (1984). A development of robust portfolios is given in Cover and Gluss (1986).

2. Pure Optimal Strategies for the Primitive ϕ-Game

Consider the primitive ϕ-game with payoff function $E\phi(W_1/W_2)$, and strategies $W_1, W_2 \in \mathbf{W}$, the set all nonnegative random variables with mean ≤ 1. We shall call $\mathbf{W}$ the set of fair r.v.'s. As yet, there is no stock market or portfolio selection in the problem. We first wish to determine conditions on ϕ such that no randomization is needed to achieve the value of the game.

In the primitive ϕ-game, players 1 and 2 choose independent random variables $W_1 \sim G_1$, $W_2 \sim G_2$, where G_1, G_2 belong to the set $\mathbf{G}$ of distribution functions with expected value ≤ 1 and support set $[0, \infty)$. The payoff to player 1 is

$$E\phi(W_1/W_2) = \iint \phi(w_1/w_2)dG_1(w_1)dG_2(w_2). \tag{2.1}$$

Distributions G_1^* and G_2^* are optimal strategies if they satisfy the saddlepoint conditions

$$\iint \phi dG_1 dG_2^* \leq \iint \phi dG_1^* dG_2^* \leq \iint \phi dG_1^* dG_2, \tag{2.2}$$

for all $G_1, G_2 \in \mathbf{G}$. The value $\mathbf{v}_\phi$ of the game is given by $\mathbf{v}_\phi = \int \phi dG_1^* dG_2^*$.

THEOREM 1. *The primitive ϕ-game has pure optimal strategies $W_1^* = W_2^* \equiv 1$ if and only if $\phi'(1) \geq 0$ exists and*

$$((t-1)/t)\phi'(1) \leq \phi(t) - \phi(1) \leq (t-1)\phi'(1), \tag{2.3}$$

for all $t > 0$. In this case the value of the game is $\mathbf{v}_\phi = \phi(1)$.

REMARK. The family of functions $\phi_\alpha(t) = t^\alpha$, $t > 0$, satisfies the conditions of Theorem 1 if $0 \le \alpha \le 1$. The value of such a game is 1. We conclude, for example, that if two gamblers were to walk into a fair casino with the agreement that player 1 should receive $E(W_1/W_2) - 1$ from player 2, then neither should gamble.

The following proof is not essential for what follows.

PROOF. Without loss of generality assume that $\phi(1) = 0$. Let $\tilde{\phi}(t) = -\phi(1/t)$. Since $E\tilde{\phi}(W_2) \le 0$ iff $E\phi(1/W_2) \ge 0$, we note that the optimal strategies satisfy $W_1^* = W_2^* \equiv 1$ iff $E\phi(W_1) \le 0$ for all $W_1 \in \mathbf{W}$ and $E\tilde{\phi}(W_2) \le 0$ for all $W_2 \in \mathbf{W}$.

For $0 < \delta < 1$ and $\eta > 0$, let

$$W_1 = \begin{cases} 1-\delta, & \text{with probability } \eta/(\delta+\eta), \\ 1+\eta, & \text{with probability } \delta/(\delta+\eta), \end{cases} \quad \text{and} \tag{2.4}$$

$$W_2 = \begin{cases} 1-\eta/(1+\eta), & \text{with probability } \delta(1+\eta)/[\delta(1+\eta)+\eta(1-\delta)], \\ 1+\delta/(1-\delta), & \text{with probability } \eta(1-\delta)/[\delta(1+\eta)+\eta(1-\delta)]. \end{cases}$$

Note that $EW_1 = EW_2 = 1$. Now $E\phi(W_1) \le 0$ implies

$$\eta\phi(1-\delta) + \delta\phi(1+\eta) \le 0. \tag{2.5}$$

Similarly, $E\tilde{\phi}(W_2) \le 0$ implies

$$\eta(1-\delta)\tilde{\phi}[1+\delta/(1-\delta)] + \delta(1+\eta)\tilde{\phi}[1-\eta/(1+\eta)] \le 0,$$

which implies

$$\eta(1-\delta)\phi(1-\delta) + \delta(1+\eta)\phi(1+\eta) \ge 0. \tag{2.7}$$

These inequalities can be rewritten as

$$\frac{-\phi(1-\delta)}{\delta}\,\frac{1-\delta}{1+\eta} \le \frac{\phi(1+\eta)}{\eta} \le \frac{-\phi(1-\delta)}{\delta}. \tag{2.8}$$

Letting $\delta \downarrow 0$ for fixed η in the first inequality implies that ϕ is left continuous at 1. A complementary analysis of the second inequality implies that ϕ is right continuous and thus continuous at 1.

Taking limits of (2.8) as $\eta \downarrow 0$ yields

$$-\phi(1-\delta)/\delta + \phi(1-\delta) \le \varliminf_{\eta\to 0} \phi(1+\eta)/\eta \le \varlimsup_{\eta\downarrow 0} \phi(1+\eta)/\eta \le -\phi(1-\delta)/\delta. \tag{2.9}$$

Since ϕ is continuous and equals 0 at 1, the lim inf and lim sup are arbitrarily close together implying that $\lim_{\eta\downarrow 0} \phi(1+\eta)/\eta$ exists. Since this limit exists, (2.9) implies that $\lim_{\delta\downarrow 0} -\phi(1-\delta)/\delta$ also exists and that the two limits are equal. Taking limits of (2.9) as $\delta \downarrow 0$ and letting $t = 1+\eta$ now gives the desired result. Finally, relaxing the condition $\phi(1) = 0$, we apply the same analysis to the game $\phi(t) - \phi(1)$, to obtain (2.3).

3. Convex Families

The log optimal portfolio $\mathbf{b}^*$ has the property $E \ln (\mathbf{b}^{*t}\mathbf{X}/\mathbf{b}^t\mathbf{X}) \ge 0$ for all $\mathbf{b} \in \mathbf{B} = \{\mathbf{b}: \sum b_i = 1, b_i \ge 0\}$. The crucial fact needed for the game theoretic results in the next section is the existence of a portfolio $\mathbf{b}^{**}$ such that $E(\mathbf{b}^t\mathbf{X}/\mathbf{b}^{**t}\mathbf{X}) \le 1$, for all $\mathbf{b} \in \mathbf{B}$. That the portfolios $\mathbf{b}^*$ and $\mathbf{b}^{**}$ are the same is a consequence of the following more general result on convex families of random variables.

DEFINITION. $\mathbf{S}$ is said to be a *convex family* of random variables if the members of $\mathbf{S}$ are defined on the same probability space and if $S_1, S_2 \in \mathbf{S}$ implies $\lambda S_1 + (1-\lambda)S_2 \in \mathbf{S}$, for all $0 \le \lambda \le 1$.

EXAMPLE 1. Random returns generated by portfolios. Let $\mathbf{X} \sim F(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^\infty$. Then the set of random variables

$$\mathbf{S} = \{S_b: S_b = \sum_{i=1}^{\infty} b_i \mathbf{X}_i,\ b_i \geq 0,\ \sum b_i = 1\},$$

is a convex family.

EXAMPLE 2. Random returns generated by constrained portfolios. If the set of allowed portfolios $\mathbf{b}$ is a convex set $\mathbf{B}_0$, then $\mathbf{S} = \{S_b = \mathbf{b}^t\mathbf{X}: \mathbf{b} \in \mathbf{B}_0\}$ is a convex family.

EXAMPLE 3. Random returns generated by portfolios based on the past. Consider n sequential plays against a market $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n \sim F(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$, $\mathbf{x}_i \in \mathbf{R}_+^m$, $i = 1, 2, \ldots, n$. Let $\mathbf{b}_1, \mathbf{b}_2(\mathbf{x}_1), \ldots, \mathbf{b}_n(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1})$ be a collection of functions $\mathbf{b}_i: (\mathbf{R}^m)^{(i-1)} \to \mathbf{B}$, and let $\mathbf{S} = \{\prod_{i=1}^n \mathbf{b}_i^t(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{i-1})\mathbf{X}_i\}$ be the set of capital returns induced by all such sequential portfolios. Then $\mathbf{S}$ is a convex family.

PROOF FOR EXAMPLE 3. Suppose the portfolio sequence $\{\mathbf{b}_i^{(1)}\}_{i=1}^n$ generates $S^{(1)}$ and $\{\mathbf{b}_i^{(2)}\}_{i=1}^n$ generates $S^{(2)}$. How does one prove that $S = \lambda S^{(1)} + (1 - \lambda)S^{(2)}$ can also be generated by a sequential portfolio? Two obvious guesses fail. For example, one might try $\tilde{\mathbf{b}}_i = \lambda \mathbf{b}_i^{(1)} + (1 - \lambda)\mathbf{b}_i^{(2)}$, $i = 1, 2, \cdots n$. This doesn't work. Neither does the device of choosing $\{\mathbf{b}_i^{(1)}\}$ with probability λ and $\{\mathbf{b}_i^{(2)}\}$ with probability $1 - \lambda$. However, if one divides the initial 1 unit capital into an amount λ to be invested according to $\{\mathbf{b}_i^{(1)}\}_{i=1}^n$ and an amount $1 - \lambda$ to be invested according to $\{\mathbf{b}_i^{(2)}\}_{i=1}^n$, pooling the money "on paper" only at time n, the result is the desired $S = \lambda S^{(1)} + (1 - \lambda)S^{(2)}$. This can be viewed as a sequential portfolio—a value weighted average of $b_i^{(1)}$ and $b_i^{(2)}$. Thus the family $\mathbf{S}$ is convex.

If $-\infty < \sup_{S \in \mathbf{S}} E \ln S < \infty$ is achieved for some S^* in $\mathbf{S}$, then necessarily $E \ln (S/S^*) \leq 0$, for all $S \in \mathbf{S}$. We now establish that this characterization of S^* is equivalent to $E(S/S^*) \leq 1$, for all $S \in \mathbf{S}$, if $\mathbf{S}$ is a convex family. This equivalence is central to the subsequent theory. Note that the following theorem does not require $-\infty < E \ln S^* < \infty$.

THEOREM 2. *If $\mathbf{S}$ is a convex family, then S^* satisfies*

$$E \ln (S/S^*) \leq 0, \qquad \text{for all} \qquad S \in \mathbf{S}, \tag{3.1}$$

if and only if S^* satisfies

$$E(S/S^*) \leq 1, \qquad \text{for all} \qquad S \in \mathbf{S}. \tag{3.2}$$

PROOF. The implication $(3.2) \Rightarrow (3.1)$ follows from Jensen's inequality: $E \ln (S/S^*) \leq \ln E(S/S^*) \leq 0$.

To prove the converse, suppose that S^* satisfies (3.1) and that (3.2) is violated for some S_1 in $\mathbf{S}$, i.e.,

$$ES_1/S^* > 1. \tag{3.3}$$

We form the convex combination

$$S_\lambda = \lambda S_1 + \bar{\lambda} S^*, \qquad 0 \leq \lambda \leq 1, \qquad \bar{\lambda} = 1 - \lambda.$$

Of course, $S_\lambda \in \mathbf{S}$ and $ES_\lambda/S^* > 1$, for $0 < \lambda \leq 1$.

Now consider

$$E \ln S_\lambda/S^* = E \ln (\lambda S_1/S^* + \bar{\lambda}) = E \ln (1 + \lambda(S_1/S^* - 1)). \tag{3.4}$$

We shall show that this is >0, contradicting (3.1). Define $Y = (S_1/S^*) - 1$ and $Y_M = \min\{Y, M\}$. Since, by hypothesis, $EY > 0$, there exists a real number $M_0 \geq 2$ such that $EY_{M_0} > 0$. Using a Taylor series expansion, we have

$$\ln (S_\lambda/S^*) \geq \ln (1 + \lambda Y_{M_0}) = \lambda Y_{M_0} - \frac{a^2}{2}, \tag{3.5}$$

for some a between λ and λY_{M_0}. But $Y_{M_0} \leq M_0$. Thus from (3.5),

$$E \ln S_\lambda/S^* \geq \lambda E(Y_{M_0}) - \frac{\lambda^2 M_0^2}{2}.$$

Finally, since $EY_{M_0} > 0$, it is possible to choose $\lambda > 0$ sufficiently small so that

$$E \ln (S_\lambda/S^*) > 0. \tag{3.6}$$

This contradicts (3.1), thereby proving Theorem 2.

At this point it is perhaps wise to specialize the convex family $\mathbf{S}$ in order to make some concrete assertions about the original stock market problem. We do this in the following sequence of corollaries.

COROLLARY 1. *If* $-\infty < \sup_{S\in\mathbf{S}} E \ln S < \infty$ *is achieved for some* $S^* \in \mathbf{S}$, *then* S^* *satisfies* (3.1) *and* (3.2) *in Theorem* 2.

Consequently the investment S^* maximizing expected log return outperforms all other investments in the sense that $E(S/S^*) \le 1$, for all $S \in \mathbf{S}$.

COROLLARY 2. *If* $\mathbf{S} = \{S_{\mathbf{b}}: \mathbf{b} \ge 0, \sum b_i = 1\}$ *where* $S_{\mathbf{b}} = \mathbf{b}^t\mathbf{X}$, *and* $\mathbf{b}^*$ *achieves* $-\infty < \sup E \ln S < \infty$, *then*

$$E\frac{X_i}{\mathbf{b}^{*t}\mathbf{X}} \le 1, \qquad \textit{for all } i, \tag{3.7}$$

with $b_i^* = 0$ *if*

$$E\frac{X_i}{\mathbf{b}^{*t}\mathbf{X}} < 1. \tag{3.8}$$

REMARK. These are the Kuhn-Tucker conditions. See Bell and Cover (1980), Finkelstein and Whitley (1981), Breiman (1961) and Thorp (1969).

PROOF. We note that $\mathbf{S}$ is a convex family and that $\mathbf{S}$ is the convex hull of $\{X_i\}_{i=1}^{\infty}$. Let $S^* = \mathbf{b}^{*t}\mathbf{X}$.

We first show that (3.7) implies (3.2), since

$$E\frac{X_i}{\mathbf{b}^{*t}\mathbf{X}} \le 1, \qquad \text{for all } i,$$

implies, for any portfolio $\mathbf{b} \in \mathbf{B}$,

$$1 \ge \sum b_i E\frac{X_i}{\mathbf{b}^{*t}\mathbf{X}} = E\frac{\sum b_i X_i}{\sum b_i^* X_i} = E\frac{S}{S^*}. \tag{3.9}$$

Conversely, (3.2) implies (3.7) trivially, by setting $S = X_i$ in (3.2). Finally, (3.2) implies (3.8), since, if $b_{i_0}^* > 0$ and $E(X_{i_0}/\mathbf{b}^{*t}\mathbf{X}) < 1$, then

$$1 = E\frac{S^*}{S^*} = \sum b_i^* E\frac{X_i}{\mathbf{b}^{*t}\mathbf{X}} < 1, \tag{3.10}$$

because $\sum b_i^* E(X_i/\mathbf{b}^{*t}\mathbf{X})$ is a convex combination of terms ≤ 1, with positive mass $b_{i_0}^*$ on a term with expectation strictly less than 1. The contradiction in (3.10) establishes that (3.2) implies (3.7) and (3.8), proving the corollary.

Summarizing, if $\mathbf{S} = \{\sum_{i=1}^{\infty} b_i X_i\}$ is the set of all random capitals resulting from portfolios of the stocks $\{X_i\}_{i=1}^{\infty}$, and if $-\infty < \sup_{S\in\mathbf{S}} E \ln S < \infty$, then the following three characterizations of S^* are equivalent:

(1) S^* maximizes $E \ln S$,

(2) $ES/S^* \le 1$, for all $S \in \mathbf{S}$,

(3) $EX_i/S^* \le 1$, for all i, and $b_i^* = 0$, if $EX_i/S^* < 1$.

As a final note, consider the partial ordering $S_1 \ge S_2$ iff $E(S_2/S_1) \le 1$, for $S_1, S_2 \in \mathbf{S}$ a convex family. We have shown this partially ordered set has a maximal element S^*.

Now consider the partial ordering induced by expected logarithms, i.e., $S_1 \geq S_2$ iff $E \ln S_2 \leq E \ln S_1$. This ordering is transitive and has a maximal element S^{**}. Although the orderings induced by $E(S_1/S_2)$ and $E \ln S_1 - E \ln S_2$ are different, they have the same maximal element $S^* = S^{**}$. This may answer to some extent the concerns of Samuelson (1969) about the difficulties (based on intransitivity) with the notion of finding the "best" portfolio.

4. The ϕ-Game for the Stock Market

We now show that the log optimal portfolio $\mathbf{b}^*$ has short-term robustness properties in that it simultaneously solves many competitive stock market games.

Recall the stock market ϕ-game of §1. Players 1 and 2 each start with one unit of capital. A strategy for player i consists of a choice of a "fair" distribution function $G_i(w)$, $G_i(0^-) = 0$, $\int w dG_i(w) \leq 1$, and a choice of portfolio $\mathbf{b}_i \in \mathbf{B}$. He then exchanges his unit of capital for the fair random variable $W_i \sim G_i(w)$, and distributes the result W_i of this gamble across the stocks according to portfolio $\mathbf{b}_i$. We assume that W_1, W_2, and $\mathbf{X}$ are independent r.v.'s. The payoff to player 1 for the game is defined to be

$$E\phi(W_1\mathbf{b}_1^t\mathbf{X}/W_2\mathbf{b}_2^t\mathbf{X}) = \int \phi(w_1\mathbf{b}_1^t\mathbf{x}/w_2\mathbf{b}_2^t\mathbf{x})dG_1(w_1)dG_2(w_2)dF(\mathbf{x}). \quad (4.1)$$

We now solve for the optimal strategies in this game.

Let $W_1^* \sim G_1^*$, $W_2^* \sim G_2^*$ denote the minimax strategies for the primitive ϕ-game, and let $\mathbf{v}_\phi$ denote the value of this game given by

$$\mathbf{v}_\phi = \inf_{W_2 \in \mathbf{W}} \sup_{W_1 \in \mathbf{W}} E\phi(W_1/W_2), \quad (4.2)$$

where $\mathbf{W}$ is the set of all fair r.v.'s $W \geq 0$, $EW \leq 1$. Let $\mathbf{b}^*$ maximize $E \ln \mathbf{b}^t\mathbf{X}$, and let $S^* = \mathbf{b}^{*t}\mathbf{X}$.

THEOREM 3. *Let $\phi(t)$ be a monotonic nondecreasing function. Then the two-person zero-sum game with payoff $E\phi(W_1\mathbf{b}_1^t\mathbf{X}/W_2\mathbf{b}_2^t\mathbf{X})$ has a value $\mathbf{v}_\phi$ and optimal strategies*

$$W_1^* \sim G_1^*, \qquad \mathbf{b}_1^* = \mathbf{b}^*,$$
$$W_2^* \sim G_2^*, \qquad \mathbf{b}_2^* = \mathbf{b}^*, \quad (4.3)$$

where $\mathbf{b}^*$ is log optimal, and G_1^*, G_2^* solve the primitive ϕ-game.

REMARK. The optimal strategies for the stock market ϕ-game factor into two parts:

(1) the game-theoretic randomization W_1^*, W_2^*, designed solely to win the primitive ϕ-game where no subsequent market investment is allowed, and

(2) a deterministic choice of portfolio $\mathbf{b}_1^* = \mathbf{b}_2^* = \mathbf{b}^*$, identical for both players, chosen independently of the payoff criterion ϕ. This choice $\mathbf{b}^*$ is defined by its log optimality.

PROOF. We observe that for any $W_2 \in \mathbf{W}$ and $S_2 \in \mathbf{S}$,

$$W_2S_2/S^* \geq 0, \qquad \text{and} \quad (4.4)$$

$$E(W_2S_2/S^*) = (EW_2)E(S_2/S^*) \leq 1, \quad (4.5)$$

by Theorem 2. Thus the random variable $(W_2S_2/S^*) \in \mathbf{W}$. This allows us to write

$$E\phi(W_1^*S^*/W_2S_2) = E\phi(W_1^*/(W_2S_2/S^*)). \quad (4.6)$$

But by the definition of the value of the game

$$E\phi(W_1^*/W) \geq E\phi(W_1^*/W_2^*) = \mathbf{v}_\phi, \quad (4.7)$$

for all $W \in \mathbf{W}$, yielding

$$E\phi(W_1^*S^*/W_2S_2) \geq \mathbf{v}_\phi. \quad (4.8)$$

Similarly, since $W_1S_1/S^* \in \mathbf{W}$,

$$E\phi(W_1S_1/W_2^*S^*) = E\phi((W_1S_1/S^*)/W_2^*) \le E\phi(W_1^*/W_2^*) = \mathbf{v}_\phi. \tag{4.9}$$

Consequently, $W_1 = W_1^*$, $\mathbf{b}_1 = \mathbf{b}^*$, $W_2 = W_2^*$, $\mathbf{b}_2 = \mathbf{b}^*$ achieve the value $\mathbf{v}_\phi$ of the game.

5. Multistage Market Games

In the previous theorem, the stock market game provided only one investment opportunity. We now consider n investment periods, with compounding of the investment each time and reallocation of the capital across the stocks based on the past. Suppose we are to invest sequentially in a market process $\{\mathbf{X}_i\}_{i=1}^n$, $\mathbf{X}_i \in \mathbf{R}_+^m$, with known joint distribution $(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \sim F(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$. Again fair randomization is allowed.

We now show that log optimal portfolios remain optimal in this multistage game against time dependent stocks.

Player 1 selects any sequential portfolio $\mathbf{b}_1, \mathbf{b}_2(\mathbf{X}_1), \ldots, \mathbf{b}_n(\mathbf{X}_1, \ldots, \mathbf{X}_{n-1})$ and fair randomization $W_1 \sim G_1(w)$. This results in capital $W_1S_n = W_1 \prod_{i=1}^n \mathbf{b}_i^t\mathbf{X}_i$. Similarly, Player 2 selects a sequential portfolio $\mathbf{b}'_1, \mathbf{b}'_2(\mathbf{X}_1), \ldots, \mathbf{b}'_n(X_1, \ldots, X_{n-1})$ and fair randomization $W_2 \sim G_2(w)$, resulting in capital $W_2S'_n = W_2 \prod_{i=1}^n \mathbf{b}_i'^t\mathbf{X}_i$. The payoff to player 1 is $\mathbf{E}\phi(W_1S_n/W_2S'_n)$. It is assumed that W_1, W_2, $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ are independent random variables. Of course, the $\mathbf{X}_i$'s can be dependent.

THEOREM 4. *The optimal portfolio strategies for the n-stage market game are given, for both players* 1 *and* 2, *by* $\mathbf{b}_k^*(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{k-1})$, $k = 1, \ldots, n$, *where* $\mathbf{b}_k^*$ *maximizes the conditional expected log return* $E(\ln \mathbf{b}^t\mathbf{X}_k \mid \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{k-1})$. *The optimal randomization is given by* $W_1^* \sim G_1^*$, $W_2^* \sim G_2^*$, *where* G_1^*, G_2^* *solve the primitive* ϕ*-game with payoff* $E\phi(W_1/W_2)$ *and corresponding value* $\mathbf{v}_\phi$. *The value of the n-stage market game is* $\mathbf{v}_\phi$.

PROOF. From the proof of Example 3 in §3, we know that the set $\mathbf{S}$ of all S_n generated by sequential portfolios is a convex family. Thus, if S_n^* maximizes $E \ln S_n$ over all $S_n \in \mathbf{S}$, then $ES_n/S_n^* \le 1$, for all $S_n \in \mathbf{S}$, which is all that is needed to apply the minimax argument of Theorem 3.

It remains only to find S_n^* and the associated log optimal sequential portfolio. We observe

$$\begin{aligned} E \ln S_n &= E \ln \prod_{k=1}^n \mathbf{b}_k^t\mathbf{X}_k \\ &= \sum_{k=1}^n E(\ln \mathbf{b}_k^t X_k). \\ &= \sum_{k=1}^n E(E \ln \mathbf{b}_k^t\mathbf{X}_k \mid \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{k-1})). \end{aligned}$$

The maximum of each term $E(E \ln \mathbf{b}_k^t X_k \mid \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{k-1}))$ over $\mathbf{b}_k(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{k-1})$ is achieved by the maximum conditional expected log return portfolio $b_k^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1})$ achieving max $E(\ln \mathbf{b}^t\mathbf{X}_k \mid \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{k-1})$. This sequential portfolio maximizes $E \ln S_n$.

6. Example: Posterior Randomization Based on Relative Capital

Now let us allow the players to observe each other's progress over many rounds of investment.

How do competitive investment decisions change? Do the investors jockey for position in a non log optimal way? We shall allow an initial fair randomization by each player. Both players then observe the outcomes $W_1^{(0)}$, $W_2^{(0)}$. A portfolio choice $\mathbf{b}_1^{(1)}$, $\mathbf{b}_2^{(1)}$ is then announced and $\mathbf{X}_1$ is revealed to both players. Another round of fair randomization is allowed and portfolios $\mathbf{b}_1^{(2)}$, $\mathbf{b}_2^{(2)}$ are announced, and so forth. The stock market process $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n \in \mathbf{R}_+^m$ can be dependent.

After a tortuous development of conditionally game theoretic optimal play, we shall find that the obvious first guess at optimal play is minimax—use the conditionally expected log optimal portfolio from §5 at each stage and finish with a fair randomization $W_{i\phi}^*$, all without regard to the opponent's fortunes. Knowledge of one's opponent's progress allows sharper play but is not necessary to achieve the value of the game.

We know (Bell and Cover 1980) that if two gamblers with equal capital try to outgamble one another in the zero sum game with payoff $P\{W_1 \geq W_2\}$, then the value of the game is $\frac{1}{2}$ and the optimal strategies for both players are to choose W_i according to a uniform distribution over the interval [0, 2]. What happens if player 1 starts with u units and player 2 with 1 unit? Now the payoff becomes $P(uW_1 \geq W_2)$. Assume first that $u > 1$. Then the optimal strategies can be shown to be $W_1^* \sim$ Uniform [0, 2], and

$$W_2^* \sim F_2(w) = \begin{cases} 0, & w < 0, \\ 1 - \dfrac{1}{u} + \dfrac{w}{2u^2}, & 0 \leq w \leq 2u, \\ 1, & w \geq 2u. \end{cases} \tag{6.1}$$

Thus W_2^* is a mixture of a Uniform $[0, 2u]$ r.v. and the degenerate r.v. $W \equiv 0$, where the mix is chosen to satisfy the constraint $EW_2 = 1$. (If $u < 1$, then the players switch distributions.)

All of this suggests a similar problem in the stock market game with payoff $E\phi(W_1S_1/W_2S_2)$. Suppose investors 1 and 2 choose their respective portfolio strategies $\mathbf{b}_1$ and $\mathbf{b}_2$. Then the stock vector $\mathbf{X}$ and the resulting capitals $S_1 = \mathbf{b}_1^t\mathbf{X}$ and $S_2 = \mathbf{b}_2^t\mathbf{X}$ are revealed. At this point each player is allowed to exchange his capital S_i for a fair r.v. S_iW_i, where $W_i \in \mathbf{W}$, and $\mathbf{W}$ is the set of all nonnegative r.v.'s with mean no greater than one. But since the purpose is to play the ϕ-game, each player chooses his distribution for W_i based on the knowledge of the relative capital S_1/S_2. How should they play? Formally, the game is as follows.

The conditionally randomized market game. Player 1 chooses a portfolio $\mathbf{b}_1 \in \mathbf{B}$ and an indexed set of fair random variables $W_1(\mathbf{x}, S_1, S_2) \in \mathbf{W}$, where $\mathbf{x} \in \mathbf{R}^m$, $S_1, S_2 \in \mathbf{R}$. Player 2 simultaneously chooses $\mathbf{b}_2 \in \mathbf{B}$, and an indexed set of r.v.'s $W_2(\mathbf{x}, S_1, S_2) \in \mathbf{W}$. The payoff to player 1 is

$$E\phi\left(\frac{\mathbf{b}_1^t\mathbf{X}W_1(\mathbf{X}, \mathbf{b}_1^t\mathbf{X}, \mathbf{b}_2^t\mathbf{X})}{\mathbf{b}_2^t\mathbf{X}W_2(\mathbf{X}, \mathbf{b}_1^t\mathbf{X}, \mathbf{b}_2^t\mathbf{X})}\right) = \int \phi\left(\frac{\mathbf{b}_1^t\mathbf{x}w_1}{\mathbf{b}_2^t\mathbf{x}w_2}\right) dF(\mathbf{x})dG_1(w_1 \mid \mathbf{x}, \mathbf{b}_1^t\mathbf{x}, \mathbf{b}_2^t\mathbf{x})dG_2(w_2 \mid \mathbf{x}, \mathbf{b}_1^t\mathbf{x}, \mathbf{b}_2^t\mathbf{x}). \tag{6.2}$$

Before solving this market ϕ-game, we digress to analyze the primitive ϕ-game with unequal starting capitals.

Let $\mathrm{v}_\phi(u) = \inf_{W_2 \in \mathbf{W}} \sup_{W_1 \in \mathbf{W}} E\phi(uW_1/W_2)$. In particular, $\mathbf{v}_\phi(1) = \mathbf{v}_\phi$, as previously defined. We next argue that $\mathbf{v}_\phi(u)$ is concave in u. This is reasonable, since player 1 can exchange u for u_1 and u_2 with fair randomization and then play the game $\phi(u_1W_1/W_2)$ or $\phi(u_2W_1/W_2)$ optimally. This is captured in the following lemma.

LEMMA. *For any ϕ: $\mathbf{R}_+ \to \mathbf{R}$, ϕ nondecreasing, $\mathbf{v}_\phi(u)$ is a concave nondecreasing function of u.*

Now to put the stock market back into the game. Let $S_1 = \mathbf{b}_1^t\mathbf{X}$, $S_2 = \mathbf{b}_2^t\mathbf{X}$.

THEOREM 5. *The conditionally randomized ϕ-game has value $\mathbf{v}_\phi$ and optimal strategies*

$$\mathbf{b}_1^* = \mathbf{b}_2^* = \mathbf{b}^*, \qquad W_1^*(\mathbf{x}, S_1, S_2) = W_{1\phi}^* \sim G_{1\phi}^*(w), \qquad W_2^*(\mathbf{x}, S_1, S_2) = W_{2\phi}^* \sim G_{2\phi}^*(w), \tag{6.3}$$

where $\mathbf{b}^*$ is log optimal, and $W_{1\phi}^*$, $W_{2\phi}^*$ are optimal in the primitive ϕ-game with equal starting capitals. Thus unconditional randomization is sufficient to achieve the value of the game.

PROOF. It is enough to show that the strategies above satisfy the saddlepoint conditions (see (2.1)):

$$E\phi\left(\frac{W_1(\mathbf{x}, S_1, S^*)S_1}{W_{2\phi}^* S^*}\right) \le E\phi\left(\frac{W_{1\phi}^*}{W_{2\phi}^*}\right) = \mathbf{v}_\phi \le E\phi\left(\frac{W_{1\phi}^* S^*}{W_2(\mathbf{x}, S^*, S_2)S_2}\right), \tag{6.4}$$

for all $W_1(\mathbf{x}, S_1, S^*)$, $W_2(\mathbf{x}, S^*, S_2) \in \mathbf{W}$.

For the first inequality we verify that

$$E\left(\frac{W_1(\mathbf{X}, S_1, S^*)S_1}{S^*}\right) = E\left(\frac{S_1}{S^*} E(W_1(\mathbf{X}, S_1, S^* \mid \mathbf{X}, S_1, S^*))\right) \le E\frac{S_1}{S^*} \le 1, \tag{6.5}$$

because of the conditional fairness of $W_1(\cdot, \cdot, \cdot)$ and the conditions for S^* developed in Theorem 2. Also, $W_{1\phi}^*$ and $W_1(\mathbf{X}, S_1, S^*)S_1/S_2^*$ are independent and nonnegative. Thus by the saddlepoint condition for primitive ϕ-games,

$$E\phi\left(\frac{W_1(\mathbf{X}, S_1, S^*)S_1/S^*}{W_{2\phi}^*}\right) \le \mathbf{v}_\phi, \tag{6.6}$$

and the first inequality is established.

The second inequality follows from a similar verification for $W_{1\phi}^*$ and $W_2(\mathbf{X}, S^*, S_2)S_2/S^*$ with reference again to the saddlepoint conditions for primitive ϕ-games.

It is a curious fact that unconditional randomization $W_{1\phi}^*$, $W_{2\phi}^*$ for either player achieves as much game-theoretically as conditional randomization $W_1(\mathbf{x}, S_1, S_2)$, $W_2(\mathbf{x}, S_1, S_2)$. Isn't it foolish for player i to ignore the relative capital S_1/S_2 when he uses his final randomization to outperform the other player? Yes and no. One reason is that, if $\mathbf{b}_1 = \mathbf{b}^*$, $\mathbf{b}_2 = \mathbf{b}^*$, then $S_1/S_2 = S^*/S^* = 1$, the ratio of player capital will be 1, and the players are thrown into the primitive ϕ-game where $W_{1\phi}^*$, $W_{2\phi}^*$ are optimal. So conditional randomization is not needed if both use the log optimal portfolio $\mathbf{b}^*$. Apparently, it follows as a result of the mathematics that any deviation from S^* by one of the players intended to set up a better competitive position for the subsequent randomization hurts the player. This is reflected in the fact that $E(S/S^*) \le 1$, for all S; i.e., S/S^* is a subfair random variable and composing it with a fair randomization leaves it subfair.

One final point. Although conditional randomization is not necessary to achieve the value of the game, it is true that use of conditionally minimax randomization $W_1^*(\mathbf{X}, S_1, S_2)$ and $W_2^*(\mathbf{X}, S_1, S_2)$ is sharper than use of unconditionally minimax randomization $W_{1\phi}^*$, $W_{2\phi}^*$. Both strategies are minimax, but the conditional randomization dominates the unconditional randomization. Both achieve $\mathbf{v}_\phi$, but conditional randomization is superior when either player deviates from the log optimal S^*. In short, the unconditional randomization $W_{i\phi}^*$ is minimax but not admissible.

7. Conclusions

First consider payoff functions $\phi(S_1/S_2)$, like S_1/S_2 satisfying the conditions of Theorem 1. Apparently, any investor competing with another investor according to such a payoff criterion will achieve the value of the game by choosing the conditional expected log optimal portfolio at each investment opportunity. No randomization is required. This is true for all such ϕ, arbitrary time dependent market processes, and for every stopping time n. Moreover, if conditions on ϕ are relaxed to include all nondecreasing functions, then the same sequential portfolio selections, followed now by appropriate fair randomization, is minimax. In short, the investor does not need to know ϕ or n in order to choose his portfolio at each time.

Here is a possible reason for the robustness of log optimal portfolios. Since the ratio of capitals S_n/S_n^* at time n, where S_n^* is the capital induced by the conditionally log optimal portfolio and S_n is the capital induced by any other sequential portfolio, obeys

$$\frac{S_n}{S_n^*} \geq 0, \qquad E\frac{S_n}{S_n^*} \leq 1,$$

we see S_n/S_n^* belongs to the set of all fair random variables. In this sense, S_n is always within "fair reach" of S_n^*. So it is not surprising that log optimal portfolios behave well in the competitive investment game.[1]

[1] Dr. Bell's research was partially supported by grants from the Department of Health and Human Services and from the Rand Corporation.

Professor Cover's work was partially supported by NSF ECS82-11568.

References

ALGOET, P. AND T. M. COVER, "Asymptotic Optimality and Asymptotic Equipartition Properties of Log-optimum Investment," *Ann. Probab.*, 16, (1988) (to appear).

BELL, R. AND T. COVER, "Competitive Optimality of Logarithmic Investment," *Math. Oper. Res.*, 5, 2 (May 1980), 161–166.

BREIMAN, L., "Optimal Gambling Systems for Favorable Games," *Fourth Berkeley Sympos., Vol.* 1, 1961, 65–78.

COVER, T., "An Algorithm for Maximizing Expected Log Investment Return," *IEEE Trans. Information Theory*, IT-30, 2 (March 1984), 369–373.

——— AND D. GLUSS, "Empirical Bayes Stock Market Portfolios," *Adv. in Appl. Math.*, 7 (1986), 170–181.

FINKELSTEIN, M. AND R. WHITLEY, "Optimal Strategies for Repeated Games," *Adv. in Appl. Probab.*, 13 (1981), 415–428.

SAMUELSON, P. A., "General Proof that Diversification Pays," *J. Financial Quantitative Anal. II*, (March 1967), 1–13.

———, "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Rev. Economics and Statist.*, (August 1969), 239–246.

THORP, EDWARD, "Optimal Gambling Systems for Favorable Games," *Rev. Internat. Statist. Inst.*, 37, 3 (1969), 273–293.

References

Optimal Strategies for Repeated Games
Mark Finkelstein; Robert Whitley
Advances in Applied Probability, Vol. 13, No. 2. (Jun., 1981), pp. 415-428.
Stable URL:
http://links.jstor.org/sici?sici=0001-8678%28198106%2913%3A2%3C415%3AOSFRG%3E2.0.CO%3B2-U