Games and Trees in Infinitary Logic: A Survey *

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Abstract

We describe the work and underlying ideas of the Helsinki Logic Group in infinitary logic. The central idea is to use trees and Ehrenfeucht-Fraïssé games to measure differences between uncountable models. These differences can be expressed by sentences of so-called infinitely deep languages. This study has ramified to purely set-theoretical problems related to properties of trees, descriptive set theory in ${}^{\omega_1}\omega_1$, detailed study of transfinite Ehrenfeucht-Fraïssé games, new constructions of uncountable models, non-well-founded induction, infinitely deep languages, non-structure theorems, and stability theory. The aim of this paper is to give an overview of the underlying ideas of this reasearch together with a survey of the main results.

1 Introduction

The so called *finite quantifier* languages $L_{\kappa\omega}$ and their fragments have given rise to a rich and interesting *definability theory.* This theory works particularly nicely on countable structures and in the case $\kappa = \omega_1$. The obvious

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generalisation, the *infinite quantifier* languages $L_{\kappa\lambda}$, have given rise to almost no interesting mathematics at all. In particular, the generalisation $L_{\omega_2\omega_1}$ of $L_{\omega_1\omega}$ has led to no general theory of models of cardinality ω_1.

Hintikka and Rantala 1976 introduced a different approach to generalizing $L_{\kappa\omega}$. They considered so called *constituents* of mathematical structures and were led to the following idea: Rather than allowing transfinite sequences of strings of existential quantifiers and transfinite sequences of universal quantifiers, one should allow transfinite sequences of quantifier and connective alternations. This leads to powerful logics which extend not only the infinitary languages $L_{\kappa\lambda}$ but also extensions of $L_{\kappa\lambda}$ by the usual game-quantifier.

Karttunen realized that while it is essential that the new infinitary expressions of *Hintikka and Rantala 1976* have infinite descending sequences of subformulas, an important distinction is made, if no *uncountable* descending sequences of subformulas are allowed (*Karttunen 1984*). This distinction is of the same nature as the distinction between a game-quantified sentence of $L_{\omega_1 G}$ and its approximations in $L_{\infty\omega}$.

Most of the work on the new infinitary languages has centered around the problem of distinguishing models with infinitary sentences. This problem can be formulated in terms of a transfinite Ehrenfeucht-Fraïssé game. In Section 2 of this paper we describe the relevant notions related to this game. A central concept in this approach to infinitary logic is the concept of a tree with no uncountable branches. These trees are used as measures of similarity of two structures. We find strong parallels between the role of such trees in the study of uncountable models and the role of ordinals in the study of countable models. Section 3 is devoted to a survey of the structure of such trees. Section 4 builds on the contention that the most fundamental mathematical properties of classes of models of cardinality ω_1 are really topological properties of ${}^{\omega_1}\omega_1$ viewed as a generalized Baire space. We survey the basics of descriptive set theory in the space ${}^{\omega_1}\omega_1$. Section 5 gives an account of the analysis of isomorphism-types of uncountable models using trees. Finally, in Section 6 we introduce the infinitely deep languages and survey their basic properties.

We use standard set-theoretic notation. In particular, ZFC denotes the Zermelo-Fraenkel axiom system with the Axiom of Choice, MA denotes Martin's Axiom and CH denotes the Continuum Hypothesis. We refer to *Jech 1978* for any unexplained set-theoretic notation.

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2 The Ehrenfeucht-Fraïssé-game

To see how the new powerful infinitary logics behave and help us study uncountable models, it is not necessary to introduce the languages themselves at all. We can go a long way by studying *Ehrenfeucht-Fraïssé*-games only. This is also in line with the approach of *Hintikka and Rantala 1976*, since consituents are descriptions of positions in Ehrenfeucht-Fraïssé-games. The new feature, analogous to allowing transfinite sequences of quantifier alternations, is that we study Ehrenfeucht-Fraïssé-games of length $> \omega$. We use

$$EF_\alpha(\mathfrak{A}, \mathfrak{B})$$

to denote the Ehrenfeucht-Fraïssé-game of length α between $\mathfrak{A}$ and $\mathfrak{B}$, which we now define. There are two players, called $\exists$ and $\forall$. During a round of the game $\forall$ first picks an element of one of the models and then $\exists$ picks an element of the other model. Let a_i be the element of A and b_i the element of B picked during round i of the game. There are altogether α rounds. Finally, $\exists$ wins the game if the resulting mapping $a_i \mapsto b_i$ is a partial isomorphism and otherwise $\forall$ wins. We say that a player *wins* $EF_\alpha(\mathfrak{A}, \mathfrak{B})$ if he has a winning strategy in it.

A trivial but fundamental observation is:

Lemma 1 *If $\mathfrak{A}$ and $\mathfrak{B}$ have cardinality $\leq \kappa$, then*

1. *$\exists$ wins $EF_\kappa(\mathfrak{A}, \mathfrak{B})$ if and only if $\mathfrak{A} \cong \mathfrak{B}$.*
2. *$\forall$ wins $EF_\kappa(\mathfrak{A}, \mathfrak{B})$ if and only if $\mathfrak{A} \not\cong \mathfrak{B}$.*

Proof. If $f : \mathfrak{A} \cong \mathfrak{B}$, then $\exists$ wins easily by using f to copy the moves of $\forall$ between the models. If, on the other hand $\mathfrak{A} \not\cong \mathfrak{B}$, then $\forall$ lists in his moves systematically all elements of the models. If $\exists$ wins a round of a game like this, an isomorphism has been created between $\mathfrak{A}$ and $\mathfrak{B}$. Since we assume no such exists, $\forall$ is bound to win. $\Box$

One consequence of the above Lemma is that $EF_\kappa(\mathfrak{A}, \mathfrak{B})$ is determined whenever $\mathfrak{A}$ and $\mathfrak{B}$ have cardinality $\leq \kappa$. For models of cardinality $> \kappa$ the game $EF_\kappa(\mathfrak{A}, \mathfrak{B})$ need not be determined, as the following result shows:

Theorem 2 (*Mekler, Shelah and Väänänen 1993*) *There are models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality ω_3 so that the game $EF_{\omega_1}(\mathfrak{A},\mathfrak{B})$ is non-determined. It is consistent relative to the consistency of a measurable cardinal, that $EF_{\omega_1}(\mathfrak{A},\mathfrak{B})$ is determined for all models of cardinality $\leq \omega_2$. It is consistent relative to the consistency of ZFC, that $EF_{\omega_1}(\mathfrak{A},\mathfrak{B})$ is non-determined for some models of cardinality $\leq \omega_2$.*

The question of determinacy of $EF_{\omega_1}(\mathfrak{A},\mathfrak{B})$ has been further studied in *Huuskonen 1991* and *Hyttinen 1992*.

In the case $\kappa = \omega$ we have the notion of a *ranked* game. To see what this means, suppose τ is a winning strategy of $\forall$ in $EF_\omega(\mathfrak{A},\mathfrak{B})$. Every round of the game, $\forall$ playing τ, ends after a finite number of moves at the victory of $\forall$. So we can put an ordinal rank on the moves of $\forall$ and demand that the rank goes down on each move. In this way we get a rank on the triple $(\mathfrak{A},\mathfrak{B},\tau)$. The *Scott rank* of $\mathfrak{A}$ is the smallest α such that if $\mathfrak{B} \not\cong \mathfrak{A}$ then for some winning strategy τ of $\forall$ in $EF_\omega(\mathfrak{A},\mathfrak{B})$, the rank of $(\mathfrak{A},\mathfrak{B},\tau)$ is at most α.

We shall now introduce a similar concept for $EF_\kappa(\mathfrak{A},\mathfrak{B})$. Of course we cannot use ordinals to rank the moves of $\forall$ since the rank may have to decrease transfinitely many times in succession. Instead we take an arbitrary winning strategy τ of $\forall$ and form the tree

$$S_{\mathfrak{A},\mathfrak{B},\tau}$$

of all possible sequences of successor length of moves of $\exists$ against τ so that $\exists$ has not yet lost the game. We get a tree with no branches of length κ and we use this tree itself as a rank for $(\mathfrak{A},\mathfrak{B},\tau)$. The smaller these trees are, when τ varies, the faster $\forall$ can locate a difference between $\mathfrak{A}$ and $\mathfrak{B}$ and beat $\exists$, and the more $\mathfrak{A}$ and $\mathfrak{B}$ are dissimilar. The larger these trees are, the bigger partial isomorphism $\exists$ can build, however fiercely $\forall$ tries block it, and the more $\mathfrak{A}$ and $\mathfrak{B}$ are similar. We arrive at the following idea:

- The trees $S_{\mathfrak{A},\mathfrak{B},\tau}$ for various τ provide a measure of the degree of similarity of $\mathfrak{A}$ and $\mathfrak{B}$.

Rather than taking first a winning strategy of $\forall$ and then the tree of all plays of $\exists$, we may also directly consider winning strategies of $\exists$ in *short* games (*Hyttinen 1987*). Let

$$K_{\mathfrak{A},\mathfrak{B}}$$

be the set of winning strategies of $\exists$ in the games $EF_\alpha(\mathfrak{A},\mathfrak{B})$, where $\alpha<\kappa$ is a successor ordinal. We order the strategies as follows. Suppose σ is a winning strategy of $\exists$ in $EF_\alpha(\mathfrak{A},\mathfrak{B})$ and τ is a winning strategy of $\exists$ in $EF_\beta(\mathfrak{A},\mathfrak{B})$. Then $\sigma\le\tau$ if $\alpha\le\beta$ and τ agrees with σ for the first α moves of $EF_\beta(\mathfrak{A},\mathfrak{B})$. This ordering makes $K_{\mathfrak{A},\mathfrak{B}}$ a tree. If this tree has a branch of length κ, then $\exists$ can follow the strategies on the branch and win $EF_\kappa(\mathfrak{A},\mathfrak{B})$. So the larger the tree $K_{\mathfrak{A},\mathfrak{B}}$ is, the longer $\exists$ can play $EF_\kappa(\mathfrak{A},\mathfrak{B})$ and the more $\mathfrak{A}$ and $\mathfrak{B}$ look alike. On the other hand, the smaller the tree $K_{\mathfrak{A},\mathfrak{B}}$ is, the sooner $\exists$ runs out of possible winning strategies, and the more $\mathfrak{A}$ and $\mathfrak{B}$ look different. In analogy with the trees $S_{\mathfrak{A},\mathfrak{B},\tau}$, we arrive at the following idea:

- The tree $K_{\mathfrak{A},\mathfrak{B}}$ provides a measure of the degree of similarity of $\mathfrak{A}$ and $\mathfrak{B}$.

Starting from the concept of Scott rank, we have introduced two different measures of similarity of structures. Before we can compare these two measures to each other and to other trees, we have to develop tools for comparing trees. The big difference in using (non-well-founded) trees to estimate structural differences, rather than ordinals is that the structure of ordinals is well-understood but the structure of trees is not. This explains why we have to investigate structural properties of the class of all trees before we can proceed in our study of the transfinite Ehrenfeucht-Fraïssé-game.

3 Structure of trees

A *tree* is a partially ordered set with a smallest element (*root*) in which the set of predecessors of every element is well-ordered by the partial ordering.

We can think of ordinals as *well-founded* trees, i.e., trees with no infinite branches. For example, we may identify an ordinal α with the tree B_α of sequences $(\alpha,\alpha_1,\alpha_2,...,\alpha_n)$, where $\alpha_n<...<\alpha_1<\alpha$ and the sequences are ordered by end-extension. It is easy to see that if we assign ordinals to nodes of B_α in such a way that extensions of nodes get smaller ordinals, then α is the smallest ordinal that can be assigned in this process to the root of B_α. In this way we can assign an ordinal $o(T)$ to any well-founded tree T. So there is a nice correspondence between ordinals and well-founded trees. On the other hand, we can think of an ordinal α as a one-branch (non-wellfounded, if $\alpha\ge\omega$) tree. We use α itself to denote this linear tree.

When we move to the *non-well-founded case*, especially to trees with no uncountable branches, several immediate observations can be made:

- Well-founded trees obey *König's lemma*: Every well-founded infinite tree has an infinite level. In the non-well-founded case we have *Aronszajn trees*, i.e., uncountable trees with countable levels but no uncountable branches.

- Trees of height ω obey *Cantor's lemma*: If a tree of height ω has finite levels and uncountably many infinite branches, it has continuum many infinite branches. Trees of height ω_1 may not obey such a law: there may be *Kurepa trees*, i.e., trees of height ω_1, with countable levels, but with exactly ω_2 uncountable branches, while $2^{\omega_1} > \omega_2$.

- *Induction* is possible along a well-founded tree. In the non-well-founded case ordinary induction is out of question. In some cases this can be overcome by assuming the existence of an ω_2-complete normal ideal with a σ-closed dense collection of positive measure sets (*Shelah, Tuuri and Väänänen 1993*). The existence of such an ideal is equiconsistent with the existence of a measurable cardinal. On the other hand, inductive definability can be developed game-theoretically and this approach makes sense also in the non-well-founded case (*Oikkonen and Väänänen 1993*).

- Any two well-founded trees are *comparable* by order-preserving embeddability. This is not so in the non-well-founded case: There are non-comparable trees and the order-structure of the class of all trees is quite involved (*Hyttinen and Väänänen 1990, Mekler and Väänänen 1993*.

- There is a family of ω_1 countable well-founded trees (corresponding to *the second number class*) so that any countable well-founded tree is order-preservingly mappable to some member of the family. The analogous question in the non-well-founded case is undecidable in $ZFC+CH$ (*Mekler and Väänänen 1993*).

These facts have a clear message: there will be manifest differences between the well-founded case (countable models) and the non-well-founded case (uncountable models). The following questions arise:

- Can we isolate some crucial assumptions about trees that decide the particular tree-theoretic questions relevant from the point of view of Ehrenfeucht-Fraïssé-games?

- In what specific and exact ways are properties of uncountable models interwoven with properties of trees?

We order the family of all trees as follows: $T \leq T'$ if there is an order-preserving $f : T \to T'$ (i.e. $x < y$ implies $f(x) < f(y)$). Note that this f need not be one-one. The strict ordering $T < T'$ is defined to hold if $T \leq T'$ and $T' \not\leq T$. Finally, $T \equiv T'$ if $T \leq T'$ and $T' \leq T$. We use σT to denote the tree of all ascending chains from T. Kurepa observed that $T < \sigma T$. With the σ-operation we define a stronger ordering of trees: $T \ll T'$ iff $\sigma T \leq T'$. The following properties of these orderings are fairly easy to prove:

Lemma 3 (*Hyttinen and Väänänen 1990*)

1. *$\sigma T \not\leq T$, i.e., if $T \ll T'$, then $T < T'$.*

2. *$<$ and $\ll$ are transitive relations.*

3. *$T \ll \sigma T$ but there is no T' with $T \ll T' \ll \sigma T$*

4. *The relation $\ll$ is well-founded.*

5. *For well-founded trees both $T < T'$ and $T \ll T'$ are equivalent to $o(T) < o(T')$.*

The reason for introducing the relation $\ll$ is that it comes up very naturally in applications. Also, proving $T \ll T'$ is a handy direct way of achieving $T' \not\leq T$.

The ordering of trees can be defined also in terms of a *comparison game* $G(T, T')$. There are two players $\exists$ and $\forall$. Player $\forall$ starts and moves an element of T'. Then player $\exists$ responds with an element of T. The game goes on, $\forall$ playing elements of T' and $\exists$ playing elements of T, both in a strictly ascending order. The first player unable to move loses.

Lemma 4 (*Hyttinen and Väänänen 1990*) *1. $T' \leq T$ if and only if $\exists$ wins $G(T, T')$.*

2. $T \ll T'$ if and only if $\forall$ wins $G(T, T')$.

We need some operations on trees. Let T and T' be trees. The tree $T \oplus T'$ consists a disjoint union of T and T' identified at the root. So $T \oplus T'$ is the *supremum* of T and T' relative to $\leq$. The tree $T \otimes T'$ consists of pairs (t, t'), where $t \in T$, $t' \in T'$ and t has the same height in T as t' has in T'. The elements of $T \otimes T'$ are ordered coordinatewise. Clearly, $T \otimes T'$ is the *infimum* of T and T' relative to $\leq$. The operations $\bigoplus_{i \in I}$ and $\bigotimes_{i \in I}$ are defined similarly. We can also define "arithmetic" operations on trees. The tree $T + T'$ is obtained from T by adding a copy of T' at the end of each maximal branch of T. With this definition, $B_\alpha + B_\beta \equiv B_{\beta+\alpha}$. The product $T \cdot T'$ consists of triples (g, t, t'), where $t \in T$, $t' \in T'$ and g is a mapping which associates every predecessor of t' with a maximal branch of T. We set $(g, t, t') \leq (g_1, t_1, t'_1)$ if ($t' = t'_1$ and $t \leq t_1$) or ($t' < t'_1$, g coincides with g_1 on predecessors of t' and $t \in g'(t')$). Again, $B_\alpha \cdot B_\beta \equiv B_{\alpha \cdot \beta}$. Intuitively, $T \cdot T'$ is obtained from T' by replacing every element by a copy of T. Since T is likely to have branching, there are different ways of progressing from a node of T' to its successor through the copy of T. This is why the elements of $T \cdot T'$ have the g-component. If we limit the way a branch of $T \cdot T'$ can pass through T', we arrive at the following variant $T \cdot T'$. Let G be a set of maximal branches of T. The tree $T \cdot_G T'$ consists of triples $(g, t, t') \in T \cdot T'$ such that, if $t'' < t'$, then $g(t'') \in G$. The ordering is defined as in $T \cdot T'$.

A phenomenon that is possible in non-well-founded trees, but impossible in well-founded trees, is reflexivity. A tree T is *reflexive* if $T \leq \{s \in T : t \leq_T s\}$ for every $t \in T$. Every tree T can be extended to a reflexive tree in the following way (*Huuskonen 1991, Hyttinen and Tuuri 1991*): Let $R(T)$ be the set of finite sequences $(t_0, ..., t_n)$ of elements of T. We can think of this sequence as a linear ordering which starts with $\{t \in T : t \leq t_0\}$, continues with $\{t \in T : t \leq t_1\}$, then with $\{t \in T : t \leq t_2\}$, etc. until t_n comes in the end. In this way $R(T)$ gets a natural tree-ordering: if s and s' are elements of $R(T)$, then we define $s \leq s'$ to mean that as linear orderings, s is equal to s' or is an initial segment of s'. It is easy to see that $T \leq R(T)$ and that $R(T)$ is reflexive. It is also interesting to note that if T has no branches of length $\kappa > \omega$, then neither has $R(T)$. We can split $R(T)$ into parts that are called *phases* in *Hyttinen and Tuuri 1991*. Namely, if $s = (t_0, ..., t_n) \in R(T)$, we call the number n the phase of s and denote it by $p(n)$. Elements of phase 0 form an isomorphic copy of T. Each element $(t_0, ..., t_n)$ of phase n extends to an isomorphic copy $\{(t_0, ..., t_{n+1}) : t_{n+1} \in T\}$ of T.

We can picture the mutual ordering of the two types of trees that arise from ordinals as follows:

$$B_0 < B_1 < ... < B_\omega < ... < B_{\omega_1} < ... < \omega < \omega + 1 < ... < \omega_1 < ...$$

Note that ω has a proper class $\{B_\alpha : \alpha \in On\}$ of predecessors. The predecessors of ω_1 are all the various trees without uncountable branches. An interesting example is the tree $T_p = (\bigoplus_{\alpha<\omega_1} \alpha) \cdot \omega$, introduced in *Huuskonen 1991*. This tree has the remarkable property that

$$T_p \leq T \text{ or } T \ll T_p$$

for any tree T of height ω_1 (*Huuskonen 1991*). So T_p has a very special place among predecessors of ω_1. The whole picture of the ordering of all trees is quite complicated. We shall now show that some trees are mutually $\leq$-incomparable.

Let $A \subseteq \omega_1$. Recall that A is *closed unbounded* if it is uncountable and contains the supremum of each of its proper initial segments. We say that A is *stationary*, if it meets every closed unbounded subset of ω_1. The complement of a stationary set is *co-stationary*. Finally, a stationary and co-stationary set is called *bistationary*. It is a not-too-hard consequence of the Axiom of Choice that there are bistationary subsets of ω_1 (see e.g. *Jech 1978*). In fact, there are ω_1 disjoint stationary subsets of ω_1 and hence 2^{ω_1} bistationary subsets A_α of ω_1 such that $A_\alpha \setminus A_\beta$ is bistationary whenever $\alpha \neq \beta$. Bistationary sets can be used to construct interesting trees without uncountable branches. If A is a bistationary subset of ω_1, let $T(A)$ be the tree of sequences of elements of A that are ascending, continuous and have a last element.

Lemma 5 (*Hyttinen and Väänänen 1990, Todorčević 1981, 1984)*

1. *If A is bistationary, then $T(A)$ is a tree of height ω_1 with no uncountable branches.*
2. *If A, B and $B \setminus A$ are bistationary, then $T(B) \not\leq T(A)$. If also $A \subset B$, then $T(A) < T(B)$.*
3. *If A and B are bistationary, then $T(A) \not\ll T(B)$.*

4. If T is an Aronszajn tree and A is bistationary, then $T \not\leq T(A)$.

Proof. Every stationary set has closed subsets of all order-types $< \omega_1$. This implies that $T(A)$ has height ω_1. An uncountable branch in $T(A)$ would give rise to a closed unbounded subset of A contrary to the co-stationarity of A. The first claim is proved. For the second claim, suppose $f : T(B) \to T(A)$ is order-preserving. For countable α, let F_α be a function on $T(B)$ so that $F_\alpha(s)$ is some $s' > s$ with $\max(s') > \alpha$. For any countable limit ordinal α, let S_α be a countable subset of $T(B)$ containing $\emptyset$ and closed under every F_β, where $\beta < \alpha$. Let C be the closed unbounded set of countable α such that if $s \in S_\alpha$, then $\max(s) < \alpha$ and $\max(f(s)) < \alpha$. Let $\alpha \in C \cap (B \setminus A)$. Let (s_n) be an ascending sequence in S_α with $\alpha = \sup_n \max(t_n)$. Then $\sup_n \max(f(t_n)) = \alpha$. Since $\alpha \in B \setminus A$, we have a contradiction. The second claim is proved. The third and fourth claims are proved similarly. $\square$

By combining the above lemma and the fact that there are 2^{ω_1} bistationary subsets A_α of ω_1 such that $A_\alpha \setminus A_\beta$ is bistationary whenever $\alpha < \beta$, we get the following result:

Proposition 6 (*Hyttinen and Väänänen 1990*) *There is a set of trees $\{T_\alpha : \alpha < 2^{\omega_1}\}$ such that for all $\alpha < \beta$:*

(1) *T_α has height ω_1 and cardinality 2^ω.*

(2) *T_α has no uncountable branches.*

(3) *T_α and T_β are incomparable by $\leq$.*

The claim remains true if condition (3) above is replaced by one of the following:

(3') $T_\alpha < T_\beta$.

(3") $T_\alpha > T_\beta$.

So there is an explosion in the hierarchy of trees between the trees of countable height and the one-branch tree ω_1. This is in sharp contrast with the situation between trees of finite height and the one-branch tree ω, where we have all the well-founded trees in nice linear order one after another.

Stationarity has also a game-theoretic characterization, which is most helpful for us in the sequel. This characterization is from *Kueker 1977*. If A

is an unbounded subset of ω_1, let G_A be the following game: There are two players $\forall$ and $\exists$. Player $\forall$ starts by playing some countable ordinal α_0. Then $\exists$ plays some bigger countable ordinal α_1. The game goes on players choosing bigger and bigger countable ordinals until an infinite ascending sequence $\alpha_0, \alpha_1, ..., \alpha_n, ...$ is created. Now $\exists$ is declared winner if $\sup_{n<\omega} \alpha_n \in A$. Kueker showed that A contains a closed unbounded set if and only if $\exists$ has a winning strategy in this game. Respectively, $\forall$ has a winning strategy in G_A if and only if the complement of A contains a closed unbounded set. Hence A is bistationary if and only if G_A is non-determined.

If $\exists$ has a winning strategy in G_A, he actually wins the seemingly more difficult game where the players construct an ascending sequence of length ω_1, and $\exists$ wins if all limit points of this sequence are in A. This observation leads to the following *ranked* game: Let T be a tree. The game $G_A(T)$ has at most ω_1 moves. During the game the players $\forall$ and $\exists$ construct an ascending sequence s of length $\leq \omega_1$ of elements of ω_1 as in the game G_A. Whenever $\forall$ moves, he first has to choose an element of T. Moreover, his moves in T have to form an ascending chain. If $\forall$ is not able to make his move in T, the game ends. Player $\exists$ wins if all countable limits of the sequence s are in A.

Lemma 7 (*Hyttinen and Väänänen 1990) Suppose A is bistationary.*

1. *$\forall$ wins $G_A(T)$ if and only if $T(A) \ll T$.*
2. *$\exists$ wins $G_A(T)$ if and only if $T \leq T(A)$.*

Proof. If $\forall$ wins $G(T(A), T)$, he can win $G_A(T)$ by simply translating the sequence of moves of $\exists$ in $G_A(T)$ to a move of $\exists$ in $G(T(A), T)$. For the converse, we assume $\forall$ has a winning strategy τ in $G_A(T)$ and demonstrate how he wins $G(T(A), T)$. At every point of $G_A(T)$ and for all $\alpha < \omega_1$, player $\exists$ has a *counter strategy* of length ζ in $G_A(T)$ which helps him evade defeat during the next ζ moves, provided that $\forall$ plays τ. Let us assume that we are in the middle of $G(T(A), T)$ and the players have so far contributed an ascending chain $a_0 < a_1 < ... < a_\xi < ...$ in $T(A)$ and an ascending chain $t_0 < t_1 < ... < t_\xi < ...$ in T. As an *inductive hypothesis* we assume that the players have played simultaneously the game $G_A(T)$ and in this game $\forall$ has consistently used τ. So the players have contributed in $G_A(T)$ an ascending sequence $\alpha_0 < \alpha_1 < ... < \alpha_\zeta < ....$ Simultaneously $\forall$ has played the above chain $t_0 < t_1 < ... < t_\xi < ...$ in T. Let $\delta = \sup_\xi \max(s_\xi)$. If $\delta \notin A$, player $\exists$ faces a one-move defeat, since τ gives $\forall$ still one move. So let us assume

$\delta \in A$. Le ζ be the smallest ζ such that $\delta + \zeta > \sup_\xi(\alpha_\xi)$. If $\zeta = 0$, we let $\exists$ play δ in $G_A(T)$. Otherwise we let $\exists$ use his counter strategy of length ζ in $G_A(T)$ for the next ζ moves. The emerging T-moves of $\forall$ let $\forall$ play $G(T(A), T)$ for the next ζ moves. The point of this is that when we come to a limit stage, we have $\sup_\xi \max(s_\xi) = \sup_\xi(\alpha_\xi)$. This means that $\exists$ faces a one-move defeat in $G_A(T)$ only if he has the same problem in $G(T(A), T)$. So $\forall$ can go on using τ to guide his playing in $G(T(A), T)$ until $\exists$ is defeated. $\Box$

We have observed that the class of trees with no uncountable branches has ascending chains, descending chains and antichains of cardinality 2^{ω_1}. All these chains arise from the trees $T(A)$, A bistationary. Several questions suggest themselves. Maybe these trees are essentially all there is in this family. Or maybe there is some relatively small number of "representatives" of these trees into which everything else can be reduced. As to the first question, H. Tuuri has pointed out, that if T is the tree of one-one sequences of rationals such that the sequence has a last element, then $T \not\leq T(A)$ (proved like Lemma 5 (3)) and $T(A) \not\leq T$ for bistationary A (as $T(A)$ is non-special by *Todorčević 1984*). So this T is an example of a tree substantially different from the trees $T(A)$.

We approach the question of "representatives" with the notion of a universal family of trees. A family $\mathcal{U}$ of trees is *universal* for a class $\mathcal{V}$ of trees if $\mathcal{U} \subseteq \mathcal{V}$ and

$$\forall T \in \mathcal{V} \exists S \in \mathcal{U}(T \leq S).$$

If we want to find a universal family for the class of all trees with no uncountable branches, there is an obstacle: If the universal family is a set, as it is reasonable to assume, we can apply the σ-operation to its supremum, and obtain a tree which contradicts the universality of the family. So we can only hope to find universal families for restricted classes of trees.

Let $\mathcal{T}_{\omega_1}$ be the class of trees of cardinality ω_1 and with no uncountable branches. If CH holds, then there cannot be a universal family of size $\leq \omega_1$ for $\mathcal{T}_{\omega_1}$, because of the function σ. On the other hand, Hella observed that if $2^\omega = 2^{\omega_1}$, then an upper bound for $\mathcal{T}_{\omega_1}$ is obtained from the full binary tree of height ω by simply extending all its branches by different elements of $\mathcal{T}_{\omega_1}$. The resulting tree has cardinality 2^ω.

Theorem 8 (*Mekler and Väänänen 1993*) *The statement "There is a universal family of cardinality ω_2 for $\mathcal{T}_{\omega_1}$" is independent of ZFC+CH+$2^{\omega_1} \geq \omega_3$.*

We may also ask whether the trees $T(A)$ can be majorized by one single tree. In *Mekler and Shelah 1993* a tree T is called a *Canary tree* if it has cardinality 2^ω, has no uncountable branches, and in any extension of the universe in which no new reals are added and in which some stationary subset of ω_1 is destroyed, T has an uncountable branch. This is equivalent to saying that T has cardinality 2^ω, has no uncountable branches, and satisfies $T(A) \leq T$ for each bistationary A (*Mekler and Väänänen 1993*).

Theorem 9 (*Mekler and Shelah 1993*) *The statement "There is a Canary tree" is independent from $ZFC + GCH$.*

The structure of trees with no uncountable branches is far from being understood even in the light of the above results. More investigation is needed. It is now quite clear that ZFC alone is not sufficient for deciding questions about these trees. The Continuum Hypothesis, for example, makes a big difference. It would be interesting to find new axioms which would fix the structure of trees more or less completely.

4 Topology of the space $\mathcal{N}_1$

There are properties of countable models and infinitary formulas which are so basic that they can be formulated in purely topological terms. To arrive at these one identifies countable models with elements of the *Baire space* $\mathcal{N} = {}^\omega\omega$, whereby classes of countable models are identified with subsets of $\mathcal{N}$. D. Scott established the basic relation between the space $\mathcal{N}$ and $L_{\omega_1\omega}$: An invariant subset of $\mathcal{N}$ is Borel iff it is (in this identification) the class of countable models of a sentence of $L_{\omega_1\omega}$ (*Scott 1965*). R. Vaught developed further the connection between model theoretic properties of $L_{\omega_1\omega}$ and topological properties of the Baire space (*Vaught 1971*).

A characteristic example of this connection is the *undefinability of well-order in* $L_{\omega_1\omega}$, proved in *Lopez–Escobar 1966*, which can be seen as a consequence of the relatively simple topological property of $\mathcal{N}$, that the codes of well-orderings is a non-analytic set. Similarly the interpolation theorem of $L_{\omega_1\omega}$ may be thought of as a logical version of the topological fact that disjoint Σ^1_1 sets can be separated by a Borel set. Finally, the basic topological property of the Baire space, that every closed set is the disjoint union of a countable set and a perfect set, and its elaboration that the cardinality of an analytic set is either $\leq \omega_1$ or 2^ω, appear behind many results of model theory. We have in mind examples such as the result in *Kueker 1968* that the number of automorphisms of a countable structure is ω or 2^ω, and the result in *Morley 1970* that the number of non-isomorphic countable models of a sentence of $L_{\omega_1\omega}$ is either $\leq \omega_1$ or 2^ω. In such cases as the above we feel that the underlying topological fact reveals the *actual mathematical construction* behind the logical result.

We may analogously identify models of cardinality ω_1 with elements of a *generalized Baire space* $\mathcal{N}_1 = {}^{\omega_1}\omega_1$ and raise the question:

- Do topological properties of $\mathcal{N}_1$ help us prove and understand infinitary properties of models of cardinality ω_1 in the spirit of the above results of Scott, Vaught, Lopez-Escobar, Kueker and Morley?

The reduction to topology seems even more important in the case of uncountable models. This is so because we tend to run into statements that are hard to decide on the basis of the standard axioms of set theory, and it is therefore of vital importance to isolate the real mathematical core of each problem. On the other hand the topological space $\mathcal{N}_1$ is much less known than $\mathcal{N}$. In particular descriptive set theory of $\mathcal{N}_1$ has been studied only recently (*Mekler and Väänänen 1993, Väänänen 1991*).

A basic neighborhood of an element $f \in \mathcal{N}_1$ is a set of the form

$$N(f, \alpha) = \{g \in \mathcal{N}_1 : g(\beta) = f(\beta) \text{ for } \beta < \alpha\},$$

where $\alpha < \omega_1$. Note that the intersection of a countable family of basic neighborhoods is still a basic neighborhood, and that there is a dense set of the cardinality of the continuum, namely the set of eventually constant functions. The space $\mathcal{N}_1$ is what Sikorski calls ω_1-*metrizable* space (*Sikorski 1949*).

In this context we are mostly interested in properties of analytic and co-analytic sets of this space. These concepts are defined in the standard way, which we now recall: A set $A \subseteq \mathcal{N}_1$ is *analytic* or Σ^1_1, if there is a closed set $B \subseteq \mathcal{N}_1 \times \mathcal{N}_1$ such that for all f: $f \in A$ if and only if $\exists g((f, g) \in B)$. A set is *co-analytic* or Π^1_1 if its complement is Σ^1_1, and Δ^1_1 if it is both Π^1_1 and Σ^1_1.

The standard example of a co-analytic non-analytic subset of $\mathcal{N}$ is the set of codes of well-orderings of ω. This may be rephrased as the statement that the set of codes of countable trees with no infinite branches is a co-analytic non-analytic subset of $\mathcal{N}$. Analogously, the set of "codes" of trees of cardinality ω_1 with no uncountable branches is a prime candidate for a co-analytic non-analytic subset of $\mathcal{N}_1$. To arrive at this set, we introduce some notation. Let π be a bijection from $\omega_1 \times \omega_1$ onto ω_1. If $f \in \mathcal{N}_1$, let $\leq_f = \{(\alpha, \beta) : f(\pi(\alpha, \beta)) = 0\}$. We may think that f "codes" the binary relation $\leq_f$. Clearly, every binary relation on ω_1 is coded by some $f \in \mathcal{N}_1$ in this way. Let $T_f = (\omega_1, \leq_f)$ and

$$TO = \{f \in \mathcal{N}_1 : T_f \text{ is a tree with no uncountable branches}\}.$$

Lemma 10 (*Mekler and Väänänen 1993)*

1. *The set TO is co-analytic.*
2. *If $A \subseteq TO$ is analytic, then there is a tree W of cardinality $\leq 2^\omega$ with no uncountable branches such that $T_g \leq W$ holds for all $g \in A$.*
3. *If CH holds, TO is non-analytic.*

Proof. The first claim is trivial, so we move to the second claim. If $f \in \mathcal{N}_1$ and $\alpha < \omega_1$, let $\overline{f}(\alpha)$ be the sequence $(f(\beta))_{\beta<\alpha}$. Let R be a closed set such that $f \in A$ holds if and only if $\exists g (f, g) \in R$. Let $U(f)$ be the set of sequences $\overline{g}(\alpha) = (g(\xi))_{\xi<\alpha}$ such that $N((f, g), \alpha) \cap R \neq \emptyset$. Now $U(f)$ is a tree and it is easy to see, that

$$f \in A \iff U(f) \text{ has an uncountable branch.}$$

Let W be the tree of triples $(\overline{f}(\alpha), t, \overline{h}(\alpha))$, where $f \in \mathcal{N}_1$ so that T_f is a tree, t is an element of T_f of height α and $\overline{h}(\alpha) \in U(f)$. Any uncountable branch of W would give rise to an element f of $A \setminus TO$. Hence W cannot have uncountable branches. Suppose now $f \in A$ is arbitrary. Let $(\overline{h}(\alpha))_{\alpha<\omega_1}$ be uncountable branch in $U(f)$. If $t \in T_f$ has height α, let $\phi(t)$ be the triple $(\overline{f}(\alpha), t, \overline{h}(\alpha))$. The mapping ϕ shows that $T_f \leq W$. This ends the proof of the second claim. For the third claim, we assume that TO were analytic, and derive a contradiction. We consider the second claim with the choice $A = TO$. Since we assume CH, we can find $f \in TO$ so that $\sigma(W)$ is isomorphic to T_f. We get the contradiction $\sigma(W) \leq T_f \leq W \ll \sigma(W)$. $\Box$

A subset $C \subseteq \mathcal{N}_1$ is Π^1_1-*complete* if C is co-analytic and for every co-analytic set A there is a continuous mapping ϕ on $\mathcal{N}_1$ such that for all f: $f \in A$ if and only if $\phi(f) \in C$. Assuming CH, the set TO is Π^1_1-complete.

Without CH the set TO need not be Π^1_1-complete:

Proposition 11 (*Mekler and Väänänen 1993*) *If $MA+\neg CH$ holds then TO is Δ^1_1.*[1]

The proof of Lemma 10 can be elaborated to give a more general result. Let A be a co-analytic set. If we assume CH, we can use Π^1_1-completeness of TO to construct a continuous mapping ϕ so that $f \in A$ if and only if $\phi(f) \in TO$. Let

$$A_{\phi,g} = \{f \in \mathcal{N}_1 : \phi(f) \leq T_g\}.$$

The proof of the following result is essentially contained in the proof of Lemma 10.

Proposition 12 (*Mekler and Väänänen 1993*) *Assume CH. Suppose A is co-analytic and ϕ is as above. Then:*

1. *$A_{\phi,g}$ is analytic for each $g \in TO$.*
2. *If $B \subseteq A$ is analytic, then there is a $g \in TO$ such that $B \subseteq A_{\phi,g}$ (Covering Property).*
3. *A is Δ^1_1 if and only if there is a $g \in TO$ such that $T_{\phi(f)} \leq T_g$ for all $f \in A$.*

An interesting analytic subset of $\mathcal{N}_1$ is the set CUB of characteristic functions of subsets of ω_1 which contain a closed unbounded set. Respectively, we have the co-analytic set $STAT$ of characteristic functions of stationary subsets of ω_1. The continuous mapping ϕ associated with this co-analytic set, assuming CH, can be chosen to be the following very natural mapping: If $f \in \mathcal{N}_1$ and $A = \{\alpha : f(\alpha) \neq 0\}$, let $\phi(f)$ be a canonical code of the tree $T(A)$. Now $f \in STAT$ if and only if $\phi(f) \in TO$. Hence, assuming CH, the set $STAT$ is Δ^1_1 if and only if there is an $f \in TO$ such that $T(A) \leq T_f$ for all co-stationary A. In Section 3 we called such a tree a *Canary tree* and we noted (Theorem 9) that the existence of a Canary tree is independent of $ZFC + CH$. The following Proposition follows from Proposition 12:

[1] **Added 2009: The published proof has an error and the status of the claim is open.**

Proposition 13 *The following conditions are equivalent:*

1. *CUB is Δ^1_1.*
2. *$STAT$ is Σ^1_1.*
3. *There is a Canary tree.*

So we cannot decide in $ZFC + CH$ the question whether CUB is Δ^1_1 or not. The best that is known at the moment is that CUB is not Σ^0_3 or Π^0_3 (*Mekler and Väänänen 1993*).

Proposition 14 (*Mekler and Väänänen 1993*) *Assume CH. Let A and B be disjoint analytic sets. There is a Δ^1_1-set C such that $A \subseteq C$ and $C \cap B = \emptyset$. (Separation Property)*[2]

Proof. Suppose ϕ is continuous so that $f \notin B$ if and only if $\phi(f) \in TO$. By the Covering Property there is a $g \in TO$ so that $A \subseteq C$, where $C = (-B)_{\phi,g}$. Clearly $C \cap B = \emptyset$. □

The Separation Property becomes more interesting if we can generate the Δ^1_1-sets via a Borel type hierarchy analogously with the Borel hierarchy of the classical Baire space $\mathcal{N}$. In fact, such a generalized Borel hierarchy, called Borel* hierarchy, can be defined for $\mathcal{N}_1$ (*Mekler and Väänänen 1993*). Then Δ^1_1-subsets of $\mathcal{N}_1$ will be exactly the so called *determined* Borel*-sets (*Tuuri 1992, Mekler and Väänänen 1993*).

The *Cantor-Bendixson Theorem* says that any closed subset of $\mathcal{N}$ can be divided into a perfect part and a scattered part. The perfect part is empty or of the cardinality of the continuum. The scattered part is countable. The corresponding result for analytic sets says that any analytic subset of $\mathcal{N}$ contains a non-empty perfect subset or else has cardinality $\leq \omega_1$. We shall now address the question whether similar results hold for $\mathcal{N}_1$.

It is easy to see that every closed subset of $\mathcal{N}_1$ can be represented as the set of all uncountable branches of a subtree of $\mathcal{N}_1$. So the possible cardinalities of closed subsets of $\mathcal{N}_1$ are limited to the possible numbers of uncountable branches of trees of height ω_1. There are trivial examples of trees where the number of uncountable branches is any number $\leq \omega_1$, 2^ω or 2^{ω_1}. Nothing

[2] **Added 2009: I do not see now why I claim C is Δ^1_1-set.** *Mekler and Väänänen 1993* **makes no such claim.**

more can be said on the basis of ZFC or even $ZFC + CH$, alone. An analysis of the Cantor-Bendixson Theorem for $\mathcal{N}_1$ is contained in *Väänänen 1991.* The implication to the question of cardinality of closed subsets of $\mathcal{N}_1$ is:

Proposition 15 (*Väänänen 1991) The statement "Every closed subset of $\mathcal{N}_1$ has cardinality $\leq \omega_1$ or $= 2^{\omega_1}$" is independent of $ZFC + CH+$ there is an inaccessible cardinal.*

A similar result holds for analytic sets (*Mekler and Väänänen 1993*).

5 Measuring similarity of models

In this Section we return to the idea introduced in Section 2 of using trees to measure similarity of models of cardinality ω_1. For this purpose we introduced the trees $S_{\mathfrak{A},\mathfrak{B},\tau}$ and $K_{\mathfrak{A},\mathfrak{B}}$. We are now ready to compare these trees to each other. Let κ be the common cardinality of $\mathfrak{A}$ and $\mathfrak{B}$ and

$$S_{\mathfrak{A},\mathfrak{B}} = \bigotimes \{S_{\mathfrak{A},\mathfrak{B},\tau} : \tau \text{ is a winning strategy of } \forall \text{ in } EF_\kappa(\mathfrak{A},\mathfrak{B})\}.$$

We let $S_{\mathfrak{A},\mathfrak{B}}$ consist of just one branch of length κ in the special case that $\mathfrak{A} \cong \mathfrak{B}$.

Proposition 16 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures of cardinality κ and of the same vocabulary. Then $K_{\mathfrak{A},\mathfrak{B}} \leq S_{\mathfrak{A},\mathfrak{B}}$. If $K_{\mathfrak{A},\mathfrak{B}}$ is well-founded, then $K_{\mathfrak{A},\mathfrak{B}} \equiv S_{\mathfrak{A},\mathfrak{B}}$.*

Proof. Suppose $\sigma \in K_{\mathfrak{A},\mathfrak{B}}$. If $\mathfrak{A} \cong \mathfrak{B}$, then $S_{\mathfrak{A},\mathfrak{B}}$ has a κ-branch and $K_{\mathfrak{A},\mathfrak{B}} \leq S_{\mathfrak{A},\mathfrak{B}}$ holds trivially. Suppose then τ is a winning strategy of $\forall$ in $EF_\kappa(\mathfrak{A},\mathfrak{B})$. Let $f(\sigma)$ be the sequence of moves in $EF_\kappa(\mathfrak{A},\mathfrak{B})$ when $\forall$ plays τ and $\exists$ plays σ. Clearly, $f(\sigma) \in S_{\mathfrak{A},\mathfrak{B},\tau}$ and f is order-preserving. Suppose then $K_{\mathfrak{A},\mathfrak{B}}$ is well-founded but there is no winning strategy τ of $\forall$ such that $S_{\mathfrak{A},\mathfrak{B},\tau} \leq K_{\mathfrak{A},\mathfrak{B}}$. Note that $\mathfrak{A} \not\cong \mathfrak{B}$, for otherwise $K_{\mathfrak{A},\mathfrak{B}}$ has a branch of length κ. Let $S_{\mathfrak{A},\mathfrak{B},\tau}(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$ be the tree of all possible sequences of successor length of moves of $\exists$ against τ so that $\exists$ has not yet lost the game, and the first n moves of the game have been $(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})$. Let $I(a_0, b_0, \ldots, a_{n-1}, b_{n-1}), n \geq 0$, be the set of such winning strategies τ of $\forall$ in $EF_\kappa(\mathfrak{A},\mathfrak{B})$ that the sequence of first n moves $(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})$ in

$EF_\kappa(\mathfrak{A},\mathfrak{B})$ is consistent with τ. To derive a contradiction, we describe a winning strategy of $\exists$ in $EF_\omega(\mathfrak{A},\mathfrak{B})$. Suppose $\forall$ starts this game with a_0. If there is no b_0 such that for all $\tau \in I(a_0,b_0)$ we have $S_{\mathfrak{A},\mathfrak{B},\tau}(a_0,b_0) \not\leq K_{\mathfrak{A},\mathfrak{B}}$, then there is $\tau \in I()$ such that $S_{\mathfrak{A},\mathfrak{B},\tau} \leq K_{\mathfrak{A},\mathfrak{B}}$, contrary to our assumption. Hence $\exists$ must have a move b_0 with the property that for all $\tau \in I(a_0,b_0)$ we have $S_{\mathfrak{A},\mathfrak{B},\tau}(a_0,b_0) \not\leq K_{\mathfrak{A},\mathfrak{B}}$. Next $\forall$ plays (e.g.) b_1. As above, we may infer that there has to be a move a_1 for $\exists$ so that for all $\tau \in I(a_0,b_0,a_1,b_1)$ we have $S_{\mathfrak{A},\mathfrak{B},\tau}(a_0,b_0,a_1,b_1) \not\leq K_{\mathfrak{A},\mathfrak{B}}$. Going on in this manner yields the required winning strategy of $\exists$ in $EF_\omega(\mathfrak{A},\mathfrak{B})$. $\square$

So if the difference between $\mathfrak{A}$ and $\mathfrak{B}$ is so easy to detect that $K_{\mathfrak{A},\mathfrak{B}}$ is even well-founded, which is the case if $\mathfrak{A} \not\equiv_{L_{\infty\omega}} \mathfrak{B}$, then $K_{\mathfrak{A},\mathfrak{B}} \equiv S_{\mathfrak{A},\mathfrak{B}}$. We shall see below (Proposition 22) that for non-isomorphic models $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \equiv_{L_{\infty\omega}} \mathfrak{B}$, there may be a huge gap between $K_{\mathfrak{A},\mathfrak{B}}$ and $S_{\mathfrak{A},\mathfrak{B}}$.

A basic concept in our closer analysis of similarity of models is the following *approximated* Ehrenfeucht-Fraïssé-game: Let T be a tree. The game $EF_\alpha(\mathfrak{A},\mathfrak{B},T)$ is like $EF_\alpha(\mathfrak{A},\mathfrak{B})$ except that $\forall$ has to go up the tree T move by move. Thus there are two players, $\exists$ and $\forall$. During a round of the game $\forall$ first picks an element of one of the models and an element of T, and then $\exists$ picks an element of the other model. Let a_i be the element of A, b_i the element of B and t_i the element of T picked during round i of the game. There are altogether α rounds. Finally, $\exists$ wins the game if the resulting mapping $a_i \mapsto b_i$ is a partial isomorphism or the sequence of elements t_i does not form an ascending chain in T. Otherwise $\forall$ wins.

Proposition 17 *1. $\exists$ wins $EF_\kappa(\mathfrak{A},\mathfrak{B},T)$ if and only if $T \leq K_{\mathfrak{A},\mathfrak{B}}$.*

2. $\forall$ wins $EF_\kappa(\mathfrak{A},\mathfrak{B},T)$ with strategy τ if and only if $S_{\mathfrak{A},\mathfrak{B},\tau} \ll T$.

Proof. The point here is that while $\forall$ goes up the tree $K_{\mathfrak{A},\mathfrak{B}}$, he reveals longer and longer strategies for $\exists$. Player $\exists$ can simply use these strategies against $\forall$. At limits we envoke the fact that strategies in $K_{\mathfrak{A},\mathfrak{B}}$ are of successor length. The strategy of $\forall$ in $EF_\kappa(\mathfrak{A},\mathfrak{B},\sigma S_{\mathfrak{A},\mathfrak{B},\tau})$ is to play in $\sigma S_{\mathfrak{A},\mathfrak{B},\tau}$ the sequence of previous moves of $\exists$, and otherwise follow τ. $\square$

We call a tree T of height α an *equivalence-tree* of $(\mathfrak{A},\mathfrak{B})$ if $\exists$ wins the game $EF_\alpha(\mathfrak{A},\mathfrak{B},T)$, and a *non-equivalence tree* of $(\mathfrak{A},\mathfrak{B})$ if $\forall$ wins the game $EF_\alpha(\mathfrak{A},\mathfrak{B},T)$. Proposition 17 above implies that $K_{\mathfrak{A},\mathfrak{B}}$ is the largest equivalence tree of $(\mathfrak{A},\mathfrak{B})$. The tree $K_{\mathfrak{A},\mathfrak{B}}$ is unsatisfactory in one respect, though:

there is no reason to believe that it has cardinality $\leq \omega_1$ even if CH is assumed. A tree $T \in \mathcal{T}_{\omega_1}$ is a *Karp tree* of $(\mathfrak{A}, \mathfrak{B})$ if it is an equivalence tree of $(\mathfrak{A}, \mathfrak{B})$ but σT is not. Respectively, a tree $T \in \mathcal{T}_{\omega_1}$ is a *Scott tree* of $(\mathfrak{A}, \mathfrak{B})$ if σT is a non-equivalence tree of $(\mathfrak{A}, \mathfrak{B})$ but T is not.

Theorem 18 (*Hyttinen and Väänänen 1990) Every pair of models* $(\mathfrak{A}, \mathfrak{B})$ *has a Karp tree* T_1 *and a Scott tree* T_2*, and* $T_1 \leq T_2$

The structure of Karp trees and Scott trees of pairs of structures is not fully understood yet. For rather trivial reasons, the families of Karp trees and Scott trees of a given pair of structures are closed under supremums. The following theorem contains some less obvious results that have been obtained about the ordering of Scott or Karp trees of a pair of models.

Theorem 19

1. *There are models* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of cardinality* ω_1 *such that the pair* $(\mathfrak{A}, \mathfrak{B})$ *has* 2^{ω_1} *Scott trees which are mutually non-comparable by* $\leq$*. (Hyttinen and Väänänen 1990)*
2. *There are models* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of cardinality* ω_1 *such that the pair* $(\mathfrak{A}, \mathfrak{B})$ *has two Scott trees the infimum of which is not a Scott tree. (Huuskonen 1991)*
3. *There are models* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of cardinality* ω_1 *such that the pair* $(\mathfrak{A}, \mathfrak{B})$ *has two Karp trees the infimum of which is not a Karp tree. (Huuskonen 1991)*

A tree $T \in \mathcal{T}_{\omega_1}$ is a *universal equivalence tree* of a model $\mathfrak{A}$ of cardinality ω_1 if $\mathfrak{A} \cong \mathfrak{B}$ holds for every $\mathfrak{B}$ of cardinality ω_1 for which T is an equivalence tree of $(\mathfrak{A}, \mathfrak{B})$. If

$$K_{\mathfrak{A}} = \bigoplus \{K_{\mathfrak{A},\mathfrak{B}} : |B| \leq \omega_1, \mathfrak{B} \ncong \mathfrak{A}\}$$

and $T \equiv \sigma K_{\mathfrak{A}}$ with $|T| \leq \omega_1$, then T is a universal equivalence tree of $\mathfrak{A}$. A tree $T \in \mathcal{T}_{\omega_1}$ is a *universal non-equivalence tree* of a model $\mathfrak{A}$ of cardinality ω_1 if $\mathfrak{A} \ncong \mathfrak{B}$ implies T is a non-equivalence tree of $(\mathfrak{A}, \mathfrak{B})$ for every $\mathfrak{B}$ of cardinality ω_1. This is equivalent to the claim that for every $\mathfrak{B} \ncong \mathfrak{A}$ of cardinality ω_1 there is some winning strategy τ of $\forall$ in $EF_\kappa(\mathfrak{A}, \mathfrak{B})$ so that $S_{\mathfrak{A},\mathfrak{B},\tau} \ll T$.

Note that a universal non-equivalence tree is necessarily also a universal equivalence tree. Thus having a universal non-equivalence tree is a stronger property than having a universal equivalence tree. Every countable model has universal non-equivalence trees. This tree is the canonical tree arising from the Scott rank of the model. The concepts of universal non-equivalence tree and universal equivalence tree are attempts to find an analogue of Scott rank for uncountable models.

It is clear that many models of cardinality ω_1 do have universal non-equivalence trees. Let us consider an example. Let T be an ω-stable first order theory with $NDOP$ (or countable superstable with $NDOP$ and $NOTOP$, see *Shelah and Buechler 1989*). By *Shelah 1990* Chapter XIII Section 1, any two $L_{\infty\omega_1}$-equivalent models of T of cardinality ω_1 are isomorphic. There is a back-and-forth characterisation of $L_{\infty\omega_1}$-equivalence which, from the point of view of $\forall$, is a special case of $EF_{\omega\cdot\omega}(\mathfrak{A}, \mathfrak{B})$. Hence every model of T of cardinality ω_1 has a universal non-equivalence tree of height $\leq \omega \cdot \omega$.

Theorem 20 (*Hyttinen and Tuuri 1991*) *Let $\kappa = \kappa^{<\kappa} > \omega$. There is a model $\mathfrak{A}$ of cardinality κ with the following property: For any tree T such that $|T| = \kappa$ and T has no branches of length κ there is a model $\mathfrak{B}$ of cardinality κ so that $\mathfrak{A} \not\cong \mathfrak{B}$ but $\exists$ has a winning startegy in $EF_\kappa(\mathfrak{A}, \mathfrak{B}, T)$. Thus $\mathfrak{A}$ has no universal equivalence tree.*

Proof. Note that $\kappa = \kappa^{<\kappa}$ implies κ is regular. The models $\mathfrak{A}$ and $\mathfrak{B}$ are constructed using the reflexivity operation R introduced in Section 3. Let T_0 be $\kappa^{<\kappa}$ as a tree of sequences of ordinals. We let $\mathfrak{A}$ be the tree-ordered structure $(R(T_0), \leq)$. Let

$$T_1 = ((\bigoplus_{\alpha<\kappa} \alpha) \cdot T) + 1$$

and $T_2 = T_1 \otimes T_0$. Let f be the canonical projection $T_2 \to T_1$. We can extend f to $R(T_2)$ by letting $f((s_0, ..., s_n)) = f(s_n)$. Let $\mathfrak{B}$ be the tree-ordered structure $(R(T_2), \leq)$. Now $\mathfrak{A}$ has branches of length κ but $\mathfrak{B}$ has none, so $\mathfrak{A} \not\cong \mathfrak{B}$. To finish the proof we have to describe the winning strategy of $\exists$ in $EF_\kappa(\mathfrak{A}, \mathfrak{B}, T)$. Because of the special relation between T and T_1, it suffices to show that $\exists$ wins the game $EF'_\kappa(\mathfrak{A}, \mathfrak{B}, T_1)$ which differs from $EF_\kappa(\mathfrak{A}, \mathfrak{B}, T)$ by allowing $\forall$ to play only elements of $\mathfrak{A}$ and $\mathfrak{B}$ the predecessors of which have been played already.

Recall that elements of $R(T_0)$ and $R(T_2)$ come in different phases. An element $(s_0, \ldots, s_n)$ of phase n may have extensions $(s_0, \ldots, s'_n)$ inside phase n but it also has extensions $(s_0, \ldots, s_n, \ldots, s_m)$ of higher phase. During the game elements a_α of $R(T_0)$, elements b_α of $R(T_2)$ and elements t_α of T_1 are played. Here α refers to the round of the game. The strategy of $\exists$ is to play in the obvious way but taking care that he never increases phase by more that 1, and making sure that when $p(b_\alpha) = p(a_\alpha) + 1$, then $f(b_\alpha) \leq t_\alpha$.

Suppose now $\forall$ plays a_α of limit height. There is a chain of predecessors a_β of a_α converging to a_α. The corresponding elements b_β will eventually be inside one phase and because of the "+1" in the definition of T_1, will converge to some element b_α. This is the response of $\exists$.

Suppose then $\forall$ plays a_α of successor height and a_β is the immediate predecessor of a_α. If $p(b_\beta) = p(a_\beta) + 1$, then $f(b_\beta) \leq t_\beta < t_\alpha$, so $f(b_\beta)$ is not maximal in T_1. Then $\exists$ can let b_α be a successor of b_β in $R(T_2)$ so that $p(b_\beta) = p(b_\alpha)$ if and only if $p(a_\beta) = p(a_\alpha)$ and $f(b_\beta) \leq t_\alpha$. If $p(b_\beta) = p(a_\alpha)$, then $f(b_\beta)$ may be maximal in T_1. In that case $\exists$ lets b_α be a successor of b_β in $R(T_2)$ of the next phase. Then $f(b_\alpha)$ is the root of T_1, so $f(b_\alpha) \leq t_\alpha$. Additionally, $\exists$ has to avoid the $< \kappa$ elements played already during the game, but this is not a problem because of the "$\otimes T_0$" part of the definition of T_2.

The case that $\forall$ plays b_α rather than a_α is similar, only easier. □

The models constructed in the above theorem are unstable. This is not an accident, as the following result shows:

Theorem 21 (*Hyttinen and Tuuri 1991*) *(CH) If T is a countable unstable first order theory, then there is a model $\mathfrak{A}$ of T of cardinality ω_1 so that $\mathfrak{A}$ has no universal equivalence tree.*

On the other hand, it is not just the unstable theories that have models with no universal equivalence tree. The paper *Hyttinen and Tuuri 1991* has results about models without universal equivalence tree of certain stable theories. Also, there is a p-group of cardinality ω_1 without universal equivalence tree (*Mekler and Oikkonen 1993*).

The situation is more complicated with universal non-equivalence trees. We know already that models of ω-stable theories with $NDOP$ do have universal non-equivalence trees.

Theorem 22 (A. Mekler) *(CH) Let F be the free abelian group of cardinality $\aleph_1$. Suppose $A \subseteq \omega_1$ is bistationary. There is an $\aleph_1$-free group H so that $\exists$ does not win $EF_{\omega\times 3}(F, H)$, and $\forall$ wins the game $EF_{\omega_1}(F, H, \sigma T(A) + \omega \cdot 2)$ but not the game $EF_{\omega_1}(F, H, T(A) + \omega \cdot 2)$.*

Proof. Let $x_\alpha \in Z^{\omega_1}$ so that $x_\alpha(\beta) = 0$, if $\beta \neq \alpha$, and $x_\alpha(\alpha) = 1$. For limit $\delta < \omega_1$ let $\{\eta_\delta(n) : n < \omega\}$ be an ascending cofinal sequence converging to δ, and

$$z_\delta = \sum_{n=0}^{\infty} 2^n x_{\eta_\delta(n)}.$$

Let F be the free subgroup of Z^{ω_1} generated by the elements $x_\alpha, \alpha < \omega_1$. Let F_α be the free subgroup of Z^{ω_1} generated by the elements $x_\beta, \beta < \alpha$. Let H be the smallest pure subgroup of Z^{ω_1} which contains the elements $x_\alpha, \alpha < \omega_1$ and the elements z_δ, where $\delta \notin A$ is limit. Let H be the smallest pure subgroup of Z^{ω_1} which contains the elements $x_\alpha, \alpha < \omega_1$, and the elements z_δ, where $\delta \notin A \cap \alpha$ is limit.

Claim 1. $\exists$ does not win $EF_{\omega\cdot 3}(F, H)$.

Proof. Suppose $\exists$ has a winning strategy τ in $EF_{\omega\cdot 3}(F, H)$. Let C be the closed unbounded set of α such that as long as all moves of $\forall$ are in $F_\alpha \cup H_\alpha$, also moves of $\exists$ given by τ are. Since A is co-stationary, there is $\delta \in C \setminus A$. Now we let $\forall$ play the first ω moves of $EF_{\omega\cdot 3}(F, H)$ so that an isomorphism f is generated between F_δ and H_δ. During the next ω moves $\forall$ plays so that f is extended to an isomorphism $f' : I \to H_{\delta+1}$, where I is a subgroup of F. Now I/F_δ is free, but $H_{\delta+1}/H_\delta$ is not, for the element $z_\delta + H_\delta$ is infinitely divisible by 2 in $H_{\delta+1}/H_\delta$. Claim 1 is proved.

Claim 2. $\forall$ does not win $EF_{\omega_1}(F, H, T(A) + \omega \cdot 2)$.

Proof. Suppose $\forall$ has a winning strategy τ in $EF_{\omega_1}(F, H, T + \omega \cdot 2)$. If we prove $T(A) \ll T$, the claim follows. In order to prove that $T(A) \ll T$, it is enough to describe a winning startegy of $\forall$ in the game $G_A(T)$. If $x \in H$, let $r(x)$ be the least α for which $x \in H_\alpha$. Define $r(x)$ similarly for $x \in F$. We call $r(x)$ the *rank* of x. The strategy τ gives $\forall$ elements x of the models as well as nodes of the tree T. The ranks of these elements x are the moves of $\forall$ in $G_A(T)$. Sometimes it may be necessary to wait a few moves before τ gives an element of sufficiently high rank. The moves of $\exists$ in $G_A(T)$ are transformed into isomorphisms between subalgebras of F and H. These isomorphisms determine the moves of $\exists$ in $EF_{\omega_1}(F, H, T)$. Finally τ gives $\forall$ a winning move in $EF_{\omega_1}(F, H, T)$. At this point we remark that there must be some reason for $\exists$ to lose. The only conceivable reason is that we have reached a limit stage δ with $\delta \notin A$, and the non-freeness of $H_{\delta+1}/H_\delta$ prevents $\exists$ from continuing successfully. Here $\forall$ needs extra $\omega \cdot 2$ moves to verify the non-freeness. Because of our arrangements, this limit δ is the limit of the moves of $\forall$ and $\exists$ in $G_A(T)$, and thus $\forall$ won $G_A(T)$. Claim 2 is proved.

Claim 3. $\forall$ wins $EF_{\omega_1}(F, H, \sigma T(A) + \omega \cdot 2)$.
Proof. If $x \in H$, let $r'(x)$ be the least limit ordinal in A which is greater than $r(x)$. The strategy of $\forall$ is to play first in F forcing $\exists$ to play eventually elements x with bigger and bigger $r'(x)$. As long as these ordinals $r'(x)$ are in A, everything goes fine. Eventually $\exists$ is bound to converge to a limit ordinal $\delta \notin A$. At this point $\forall$ uses his remaining $\omega \cdot 2$ moves to demonstrate non-freeness of $H_{\delta+1}/H_\delta$. Claim 3 is proved. $\Box$

Note that for F and H as above, the tree $K_{F,H}$ has height $\leq \omega \cdot 3$, but $S_{F,H}$ has height ω_1.

Corollary 23 (*Mekler and Shelah 1993) (CH) There is a universal non-equivalence tree for the free abelian group of cardinality* $\aleph_1$ *if and only if there is a Canary tree.*

Proof. Suppose there is a Canary tree T. We show that $T_1 = \sigma T + \omega \cdot 2$ is a universal non-equivalence tree for F. Suppose H is an abelian group of cardinality $\aleph_1$. We may safely assume H is $\aleph_1$-free, for otherwise $\forall$ wins easily. Hence we may as well assume H arises from a bistationary set A as in the proof above. Now $T(A) \leq T$. By the previous Theorem, $\forall$ wins $EF_{\omega_1}(F, H, T_1)$. Suppose then T is a universal non-equivalence tree of F. To show that T is a Canary tree, let A be bistationary. Let H arise from A as above. Now $\forall$ has a winning strategy τ in $EF_{\omega_1}(F, H, T)$. Let us then work in a generic extension of the universe, where A contains a cub set but no new reals are introduced. In that extension $F \cong H$, but τ still applies to any sequence of moves of $\exists$, whence T contains an uncountable branch. So T is a Canary tree. $\Box$

So the statement that the abelian group F does not have a universal non-isomorphism tree is independent of $ZFC + CH$. This is not an accident, as the following general result demonstrates:

Theorem 24 (*Hyttinen and Tuuri 1991) If ZFC is consistent, then the following statement is consistent with CH: Every countable non-superstable first order theory has a model of cardinality ω_1 without a universal non-equivalence tree.*

If we give up CH, the situation changes again dramatically. In *Hyttinen, Shelah and Tuuri 1993* it is proved consistent relative to the consistency of an inaccessible cardinal, that ($\neg CH$ and) every linear ordering of cardinality

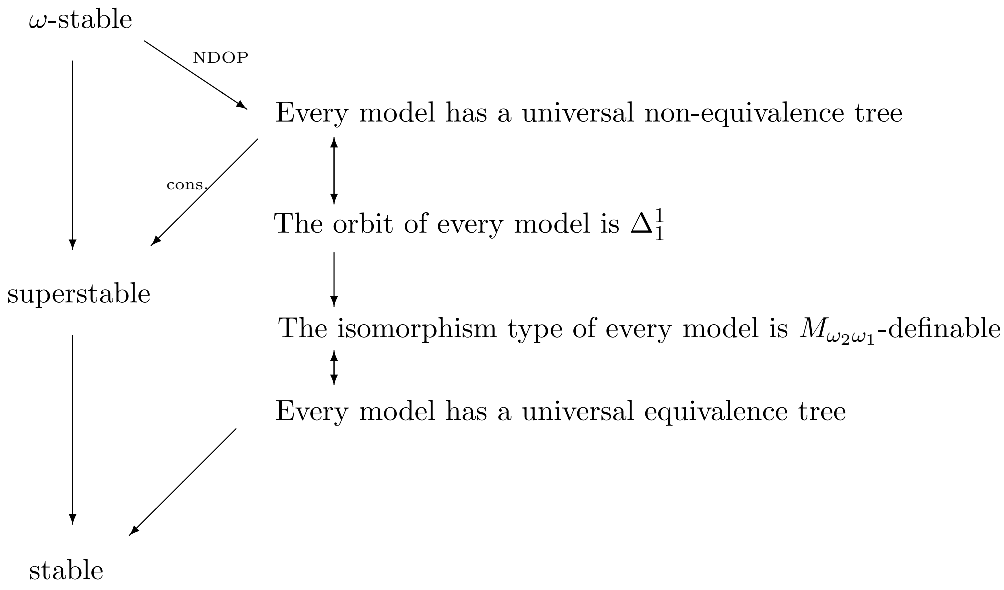

Figure 1: A first order theory and its models of power ω_1.

ω_1 has a universal equivalence tree which is of the form $T+1$, where T has cardinality ω_1.

The orbit $orb(R)$ of a relation on κ is the set $\{S \subseteq \kappa^n : (\kappa, R) \cong (\kappa, S)\}$. D. Scott (*Scott 1965*) proved that the orbit of a relation on ω is a Δ^1_1-subset of $\mathcal{N}$. For orbits of relations on ω_1 the corresponding question is tied up with the problem of the existence of universal equivalence and non-equivalence trees. Implication (2)→(1) in the following Proposition together with a model-theoretic argument for its proof were suggested by H. Tuuri.

Proposition 25 (*Mekler and Väänänen 1993) The following two conditions are equivalent (assuming CH):*

(1) (ω_1, R) *has a universal non-equivalence tree.*

(2) $orb(R)$ *is* Δ^1_1.

Proposition 25 shows that the question, whether a model of cardinality ω_1 can be assigned a tree-invariant via the Ehrenfeucht-Fraïssé game, which is in close relation with stability-properties of the first order theory of the model, has also a topological formulation. Figure 1 displays some relationships between stability theoretic properties of a complete first order theory and infinitary as well as topological properties of its models of cardinality ω_1 (The logic $M_{\omega_2\omega_1}$ will be defined later).

We end this Section with a result which further emphasizes the relationship between properties of trees and properties of models:

Theorem 26 (*Shelah, Tuuri and Väänänen 1993) The following two conditions are equivalent:*

(1) *There is a tree of cardinality and height ω_1 with exactly λ uncountable branches.*

(2) *There is a model of cardinality ω_1 with exactly λ automorphisms.*

Note that the set of uncountable branches of a tree of cardinality and height ω_1 is (up to some identification) a closed subset of $\mathcal{N}_1$. It is consistent relative to the consistency of an inaccessible cardinal, that there are no closed subsets C of $\mathcal{N}_1$ with $\omega_1 < |C| < 2^{\omega_1}$. On the other hand, a Kurepa tree satisfies (1) with $\lambda = \omega_2$ and it is possible to have a Kurepa tree with ω_2 uncountable branches while $2^{\omega_1} > \omega_2$. So there is a lot of freedom for the number of automorphisms of a model of cardinality ω_1. For comparison, recall that the number of automorphisms of a countable model is $\leq \omega$ or $= 2^{\omega}$.

6 Infinitely deep languages

Let $\mathfrak{A}$ be a fixed structure. The property of another structure $\mathfrak{B}$ that $\exists$ wins $EF_\kappa(\mathfrak{A}, \mathfrak{B}, T)$ can be expressed by an infinitary game sentence which imitates the progress of the game $EF_\kappa(\mathfrak{A}, \mathfrak{B}, T)$. These infinitary game sentences are the origin of what we call *infinitely deep languages*. Mathematically speaking, the game with all its problems remains the same whether we write it down as an infinitary game sentence or as an informal description of a set of rules. This is why we have up to now suppressed all syntactic notions. However, the game expressions arising from Ehrenfeucht-Fraïssé games are just very special examples and the underlying more general concept deserves to be made explicit.

Let $\mathfrak{A}$ be a structure of cardinality ω_1. We assume the language of $\mathfrak{A}$ to be finitary and of cardinality $\leq \omega_1$. The universe of $\mathfrak{A}$ is denoted by A. We

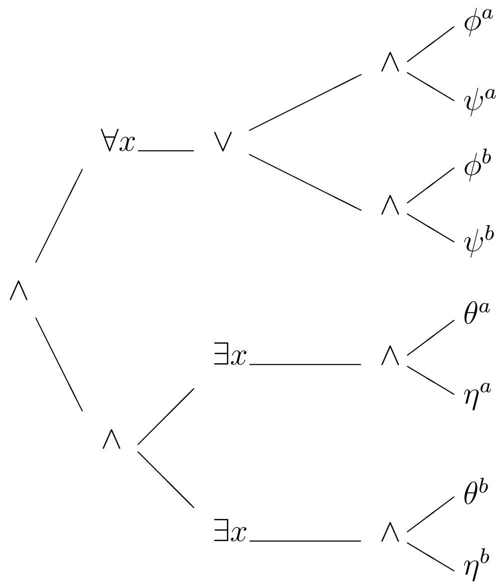

Figure 2: Formula F_A, when $A = \{a, b\}$.

shall define an infinitary formula $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ by describing its syntax-tree. We think of syntax-trees of formulas as labelled trees. Figure 2 is an example of a syntax-tree of a formula, where ϕ^a, ψ^a etc are atomic formulas. In more conventional style this formula is

$$\forall x[(\phi^a \wedge \psi^a) \vee (\phi^b \wedge \psi^b)] \wedge [\exists x(\theta^a \wedge \eta^a) \wedge \exists x(\theta^b \wedge \eta^b)].$$

Figure 3 shows a syntax-tree F_A that we shall use to build up $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$. Note that this formula F_A splits at nodes $\bigvee_{a\in A}$ and $\bigwedge_{a\in A}$ into as many subtrees as there are elements in A. In more conventional style the formula F_A would be

$$\forall x \bigvee_{a\in A} (\phi^a \wedge \psi^a) \wedge \bigwedge_{a\in A} \exists x(\theta^a \wedge \eta^a).$$

The formula $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ is obtained from F_A by letting $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ "repeat" the structure of F_A following the pattern of a given tree T. If T were a one-

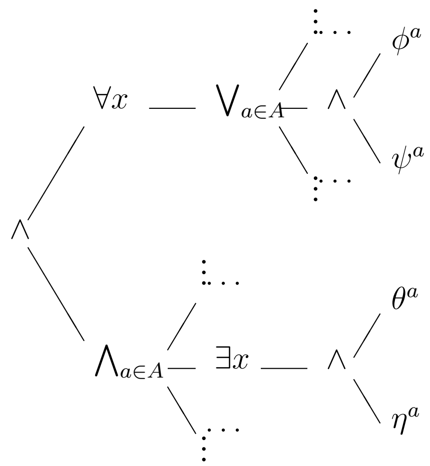

Figure 3: Formula F_A.

branch tree of two elements, then $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ would be

$$\begin{array}{lll}
[\forall x_0 \bigvee_{a_0\in A}(& \phi^{a_0}(x_0) & \wedge \\
 & [\forall x_1 \bigvee_{a_1\in A} & \phi^{a_0a_1}(x_0,x_1)]\wedge \\
 & [\bigwedge_{a_1\in A} \exists x_1 & \phi^{a_0a_1}(x_0,x_1)])]\wedge \\
[\bigwedge_{a_0\in A} \exists x_0(& \phi^{a_0}(x_0) & \wedge \\
 & [\forall x_1 \bigvee_{a_1\in A} & \phi^{a_0a_1}(x_0,x_1)]\wedge \\
 & [\bigwedge_{a_1\in A} \exists x_1 & \phi^{a_0a_1}(x_0,x_1)])],
\end{array}$$

where the formula $\phi^{a_0,a_1}(x_0,x_1)$ is the conjunction of all atomic and negated atomic formulas $\psi(x_0,x_1)$ so that $\mathfrak{A} \models \psi(a_0,a_1)$. The formula $\phi^{a_0}(x_0)$ is defined analogously. In this special case $\mathfrak{B} \models \phi^T_{\mathfrak{A},\vec{a}}(\vec{b})$ clearly means that $\exists$ wins $EF_\omega((\mathfrak{A},\vec{a}),(\mathfrak{B},\vec{b}))$.

We shall now define $\phi^T_{\mathfrak{A},\vec{a}}(\vec{b})$ for more general T. For this end, let T be a tree of height ω_1 in which every node has at most ω_1 successors, there is no branching at limits, and there are no maximal branches of limit length. Let us consider an arbitrary maximal branch C of F_A. The branch C ends in ϕ^a, ψ^a, θ^a or η^a for some $a = a(C) \in A$. Let G be the set of branches C which end in ψ^a or η^a. Let us consider the tree $F_A \cdot_G T$. To make $F_A \cdot_G T$ a syntax-tree, we assign lables $l(g,w,t)$ to nodes (g,w,t) of $F_A \cdot_G T$ as follows. Only nodes $\forall x$, $\exists x, \phi^a, \psi^a, \theta^a$ and η^a of the various copies of F_A are given a label. For other nodes the label is as in the picture of F_A. Suppose we are at a node $(g,\forall x,t)$ of $F_A \cdot_G T$. Let $(t_\xi)_{\xi\leq\alpha}$ be the sequence of $\{s \in T : s \leq t\}$

in ascending order. We let $l(g, \forall x, t) = \forall x_\alpha$. Staying in the same copy of F_A we let $l(g, \exists x, t) = \exists x_\alpha$. If t is maximal in T, we let $l(g, \psi^a, t) = l(g, \eta^a, t) = \forall x_0(x_0 = x_0)$. If t is not maximal in T, we let $l(g, \psi^a, t) = l(g, \eta^a, t) = \wedge$. Let $a_\xi = a(g(t_\xi))$ for $\xi \leq \alpha$. We let $l(g, \phi^a, t)$ and $l(g, \theta^a, t)$ be the conjunction of atomic and negated atomic formulas $\phi(x_\xi)_{\xi\leq\alpha}$ such that $\mathfrak{A} \models \phi(a_\xi)_{\xi\leq\alpha}$. This ends the definition of the labelling of nodes of $F_A \cdot_G T$. The labelled tree $(F_A \cdot_G T, l)$ is our $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$.

The formula $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ can be given semantics by means of the obvious semantic game. The *dual* formula $\psi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ of $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ is obtained by replacing in the lables of $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ everywhere $\wedge$ by $\vee$, $\vee$ by $\wedge$, $\forall$ by $\exists$, $\exists$ by $\forall$ and the labels $l(g, \phi^a, t), l(g, \psi^a, t), l(g, \theta^a, t)$ and $l(g, \eta^a, t)$ by their negations.

The formulas $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ and $\psi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ are taylor-made so that player $\exists$ has a winning strategy in the game $EF_{\omega_1}((\mathfrak{B}, \vec{b}),(\mathfrak{A}, \vec{a}), T)$, if and only if $\mathfrak{B} \models \phi^T_{\mathfrak{A},\vec{a}}(\vec{b})$, and player $\forall$ has a winning strategy in $EF_{\omega_1}((\mathfrak{B}, \vec{b}),(\mathfrak{A}, \vec{a}), T)$ if and only if $\mathfrak{B} \models \psi^T_{\mathfrak{A},\vec{a}}(\vec{b})$. We shall now define a general concept of which formulas $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ and $\psi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ are examples.

A *quasiformula* is a labelled tree (T, l), where T is a tree with no maximal branches of limit length and no branching at limits, and $l(t)$ is

1. a countable conjunction of atomic and negated atomic formulas, if t is maximal in T.

2. $\wedge$ or $\vee$, if t has more than one successor in T.

3. $\exists u$ or $\forall u$, where u is a variable symbol, otherwise.

Definition 27 (*Karttunen 1984) The infinitary language $M_{\omega_2\omega_1}$ consists quasiformulas (T, l), where T is a tree of height ω_1 in which every node has at most ω_1 successors, and there is no u and no branch b of T such that $l(t)$ alternates infinitely many times between the values $\forall u$ and $\exists u$ on b.*

The semantics of $M_{\omega_2\omega_1}$ is defined via a semantic game, exactly as for any game formulas. A formula is *determined* if this semantic game is always determined. The formulas $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ and $\psi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ are clearly examples of formulas of $M_{\omega_2\omega_1}$. These formulas need not be determined, but they are determined in models of cardinality $\leq \omega_1$.

The *quantifier-rank* of a formula (T, l) of $M_{\omega_2\omega_1}$ is the subtree T' of T which consists of nodes t with $l(t) = \forall u$ or $l(t) = \exists u$, where u is a variable symbol. The tree T' may not have a unique root, but relations like $T' \leq T$ still make sense.

The Ehrenfeucht-Fraïssé games $EF_\kappa(\mathfrak{A}, \mathfrak{B}, T)$ have dominated our discussion all the way from the beginning. The special connection between $EF_\kappa(\mathfrak{A}, \mathfrak{B}, T)$ and $M_{\omega_2\omega_1}$ is revealed by the following easy fact:

Proposition 28 (*Karttunen 1984*) *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of the same similarity type and T a tree of height ω_1 in which every node has at most ω_1 successors, there is no branching at limits, and there are no maximal branches of limit length. Then the following two conditions are equivalent:*

(1) *$\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $M_{\omega_2\omega_1}$ of quantifier-rank $\leq T$.*

(2) *Player $\exists$ has a winning strategy in the game $EF_{\omega_1}(\mathfrak{A}, \mathfrak{B}, T)$.*

Note that $M_{\omega_2\omega_1}$ is, up to logical equivalence, closed under conjunctions and disjunctions of length $\leq 2^\omega$ and universal and existential quantification over countable sequences of variables. Although $M_{\omega_2\omega_1}$ is closed under *dual* in the obvious sense, there is no trivial reason for it to be closed under negation, because the relevant semantic games need not be determined, as the example below shows. In fact, Tuuri showed that a sentence of $M_{\omega_2\omega_1}$ has a negation in $M_{\omega_2\omega_1}$ *if and only if* it is definable by a sentence whose semantic game is determined (*Tuuri 1992*).

Example 29 *Let $A \subseteq \omega_1$ be bistationary. Let ϕ_A be the following sentence of $M_{\omega_2\omega_1}$:*

$$\bigwedge_{\alpha_0<\omega_1} \bigvee_{\alpha_1>\alpha_0} \cdots \bigwedge_{\alpha_{2n+2}>\alpha_{2n+1}} \bigvee_{\alpha_{2n+3}>\alpha_{2n+2}} \cdots \phi_{(\alpha_0\ldots\alpha_n\ldots)}$$

where

$$\phi_{(\alpha_0\ldots\alpha_n\ldots)} = \begin{cases} \exists x(x=x) & \text{if } \sup_{n<\omega}\alpha_n \in A \\ \exists x\neg(x=x) & \text{if } \sup_{n<\omega}\alpha_n \notin A \end{cases}$$

Neither ϕ_A nor the dual of ϕ_A is true in any model. In this case the semantic game is non-determined. We still have a negation for ϕ_A in the semantic sense, for example $\exists x(x=x)$.

A $PC(M_{\omega_2\omega_1})$-sentence consists of a sequence of $\leq \omega_1$ existential second-order quantifiers followed by an $M_{\omega_2\omega_1}$-sentence. The existentially quantified predicates are allowed to have any countable ordinal as their arity. The $PC(L_{\omega_2\omega_1})$-sentences are defined analogously. It is easy to see that every $PC(M_{\omega_2\omega_1})$-sentence can be defined by a $PC(L_{\omega_2\omega_1})$-sentence. This observation combined with a standard Skolemization argument gives:

Proposition 30 (*Karttunen 1984*) *Suppose Φ is a $PC(M_{\omega_2\omega_1})$-sentence and $\mathfrak{A}$ is a model of Φ. Then there is a submodel $\mathfrak{B}$ of $\mathfrak{A}$ so that $|\mathfrak{B}| \leq 2^\omega$ and $\mathfrak{B} \models \Phi$.*

Proposition 31 *If CH holds and there is a Kurepa tree, then some sentence of $M_{\omega_2\omega_1}$ does not have a negation.* [3]

Proof. Let us consider the following game G of length $\omega + 1$ introduced in *Heikkilä and Väänänen 1991*: Player $\exists$ plays nodes of a Kurepa tree T in ascending order and $\forall$ plays uncountable branches of T which are elements of a set Z. The task of $\exists$ is to play always a node which is not on any of the branches mentioned so far by $\forall$. We can write a sentence ψ of $M_{\omega_2\omega_1}$ which describes models consisting of a tree T and a set Z and which says that $\exists$ wins G. Let $\mathfrak{A}$ be a model which consists of a Kurepa tree T and the set Z of its uncountable branches. Now $\mathfrak{A} \models \neg\psi$, because a winning strategy of $\exists$ would create an uncountable level to T. If $\neg\psi \in M_{\omega_2\omega_1}$, then there is a submodel $\mathfrak{B}$ of $\mathfrak{A}$ so that $\mathfrak{B} \models \neg\phi$ and $|\mathfrak{B}| \leq \omega_1$. But ψ is true in any submodel of $\mathfrak{A}$ of cardinality $< \omega_2$, because $\exists$ can play elements of one of the ω_2 branches which are not in $\mathfrak{B}$. $\Box$

So, what can we express in the language $M_{\omega_2\omega_1}$? We have already pointed out that the formulas $\phi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ and $\psi^T_{\mathfrak{A},\vec{a}}(\vec{z})$ are in $M_{\omega_2\omega_1}$. This immediately gives the following nice characterisation of rigidity. Recall that a countable model is rigid if and only if all its elements are definable in $L_{\omega_1\omega}$, and a relation on a countable model is invariant if and only if it is definable by a formula of $L_{\omega_1\omega}$.

Proposition 32 *Suppose $\mathfrak{A}$ is a model of cardinality ω_1. The following conditions are equivalent:*

1. *$\mathfrak{A}$ is rigid.*
2. *Every element of $\mathfrak{A}$ is definable by a determined $M_{\omega_2\omega_1}$-formula.*

[3] Recently T. Huuskonen proved this without assuming CH or a Kurepa tree.

Proof. Suppose $\mathfrak{A}$ is rigid. If $b \in A$, we can find a tree T_b with no uncountable branches so that $a \neq b$ if and only if $\mathfrak{A} \models \psi^{T_b}_{\mathfrak{A},b}(a)$. Now

$$\mathfrak{A} \models \forall x(x = a \leftrightarrow \bigwedge_{b \neq a} \psi^{T_b}_{\mathfrak{A},b}(x)).$$

□

Proposition 33 (*Hyttinen 1990*) *The following conditions are equivalent for any relation R on $\mathfrak{A}$:*

(1) *R is invariant (i.e., fixed by all automorphisms of $\mathfrak{A}$).*

(2) *R is definable on $\mathfrak{A}$ by a determined $M_{\omega_2\omega_1}$- formula.*

Proof. Suppose R is invariant. If $b \in R$ and $a \notin R$, we can find a tree $T_{b,a}$ with no uncountable branches such that $\mathfrak{A} \models \psi^{T_{b,a}}_{\mathfrak{A},a}(b)$. Now

$$\mathfrak{A} \models \forall x(x \in R \leftrightarrow \bigvee_{b \in R} \bigwedge_{a \notin R} \psi^{T_{b,a}}_{\mathfrak{A},a}(x)).$$

□

If $\mathfrak{A}$ is a model of cardinality ω_1, let $I(\mathfrak{A})$ denote the class $\{\mathfrak{B} : \mathfrak{B} \cong \mathfrak{A}\}$. That is, $I(\mathfrak{A})$ is the isomorphism type of $\mathfrak{A}$. We say that $I(\mathfrak{A})$ is (*determinedly*) *$M_{\omega_2\omega_1}$-definable* if there is a sentence ϕ in $M_{\omega_2\omega_1}$ so that $I(\mathfrak{A})$ is the class of models of ϕ of cardinality $\leq \omega_1$ (and ϕ is determined in models of power $\leq \omega_1$).

Proposition 34 *Let $\mathfrak{A}$ be a model of cardinality ω_1.*

(1) *$\mathfrak{A}$ has a universal equivalence tree if and only if $I(\mathfrak{A})$ is $M_{\omega_2\omega_1}$-definable.*

(2) *$\mathfrak{A}$ has a universal non-equivalence tree if and only if $I(\mathfrak{A})$ is determinedly $M_{\omega_2\omega_1}$-definable.*

Proof. (1) If T is a universal equivalence tree of $\mathfrak{A}$, then $\phi^T_{\mathfrak{A}}$ defines $I(\mathfrak{A})$ among models of cardinality $\leq \omega_1$. Conversely, assume $\phi = (T, l)$ defines $I(\mathfrak{A})$ among models of cardinality $\leq \omega_1$. To prove that T is a universal equivalence tree of $\mathfrak{A}$, suppose $\exists$ wins $EF_{\omega_1}(\mathfrak{A}, \mathfrak{B}, T)$. Since $\mathfrak{A} \models \phi$, we have by Proposition 28 that $\mathfrak{B} \models \phi$. Hence $\mathfrak{A} \cong \mathfrak{B}$.

(2) If T is a universal non-equivalence tree of $\mathfrak{A}$, then first of all, $\phi^T_{\mathfrak{A}}$ defines $I(\mathfrak{A})$ among models of cardinality $\leq \omega_1$. Moreover, $\phi^T_{\mathfrak{A}}$ is determined in models of cardinality $\leq \omega_1$, for if $\mathfrak{B} \not\models \phi^T_{\mathfrak{A}}$, then $\forall$ wins $EF_{\omega_1}(\mathfrak{A}, \mathfrak{B}, T)$, and hence $\mathfrak{B} \models \psi^T_{\mathfrak{A},\mathfrak{B}}$. Conversely, assume a determined $\phi = (T, l)$ defines $I(\mathfrak{A})$ among models of cardinality $\leq \omega_1$. To prove that T is a universal non-equivalence tree of $\mathfrak{A}$, suppose $\mathfrak{B} \not\cong \mathfrak{A}$. So $\mathfrak{B}$ satisfies the dual of ϕ. Now $\forall$ wins $EF_{\omega_1}(\mathfrak{A}, \mathfrak{B}, T)$ by following ϕ in $\mathfrak{A}$ and the dual of ϕ in $\mathfrak{B}$.□

So whenever we can find a universal equivalence tree for a model $\mathfrak{A}$ of cardinality ω_1, we can find an $M_{\omega_2\omega_1}$-sentence which is an *invariant* of $\mathfrak{A}$, i.e., identifies the isomorphism type of $\mathfrak{A}$. What is the advantage of an $M_{\omega_2\omega_1}$-sentence over a brute force invariant? By a brute force invariant we mean listing all elements and relationships of elements of the model and taking as an invariant the listing which is smallest in the lexicographic ordering of all listings. Both invariants are uncountable objects. But checking the truth of an $M_{\omega_2\omega_1}$-sentence in a given model involves playing a semantic game which can last for countably many rounds only. So there is an important *countable vs. uncountable* distinction between an $M_{\omega_2\omega_1}$-sentence and the brute force invariant. This may be quite relevant if we, for example, extend the universe with forcing that does not add new reals. The truth of $M_{\omega_2\omega_1}$-sentences is preserved while new listings of a model of cardinality ω_1 may have come up. This is exemplified by the lack of a ZFC-provable $M_{\omega_2\omega_1}$-definition of the isomorphism type of the free abelian group of cardinality ω_1: such a definition would contradict the fact that an almost free group can be made free without adding reals. This explains why having an $M_{\omega_2\omega_1}$-sentence as an invariant of a model means we have understood the model better than after merely enumerating the elements and relationships of elements of the model.

Let us now turn to the question, what *cannot* be expressed in $M_{\omega_2\omega_1}$. The most interesting concept undefinable in $L_{\omega_1\omega}$ is the notion of well-ordering. The analogous result for $M_{\omega_2\omega_1}$ is that the class of trees with no uncountable branches is undefinable in $M_{\omega_2\omega_1}$. This fact alone is as central in the study of $M_{\omega_2\omega_1}$ as undefinability of well-order is in the study of $L_{\omega_1\omega}$. The proof we present for this fact is topological. For this it is useful to observe that if Φ is a $PC(M_{\omega_2\omega_1})$-sentence, then the set $\{R \subseteq \omega_1 \ : \ (\omega_1, R) \models \Phi\}$ is a Σ^1_1-subset of $\mathcal{N}_1$.

Proposition 35 (*Hyttinen 1987, Oikkonen 1988) (CH) The class of trees $(T, <)$ of cardinality ω_1 with no uncountable branches is not $PC(M_{\omega_2\omega_1})$-definable.*

Proof. Suppose Φ is a $PC(M_{\omega_2\omega_1})$-sentence whose models are exactly the trees $(T, \leq)$ which have no uncountable branches. Let $A = \{f \in \mathcal{N}_1 : (\omega_1, \leq_f) \models \Phi\}$. Since Φ is $PC(M_{\omega_2\omega_1})$, A is a Σ^1_1-subset of TO. By Proposition 12 there is a tree W of cardinality ω_1 with no uncountable branches so that $T_f \leq W$ for all $f \in A$, contradiction. □

Proposition 36 (*Hyttinen 1987) (CH) For any* $PC(M_{\omega_2\omega_1})$*-sentence* Φ *there is a mapping* $T \mapsto \Phi^T$ *from* $\mathcal{T}_{\omega_1}$ *to* $M_{\omega_2\omega_1}$ *so that*

(1) $\models \Phi \rightarrow \bigwedge\{\Phi^T : T \in \mathcal{T}_{\omega_1}\}$.

(2) $\mathfrak{A} \models \bigwedge\{\Phi^T : T \in \mathcal{T}_{\omega_1}\} \rightarrow \Phi$ *if* $\mathfrak{A}$ *has cardinality* $\leq \omega_1$.

Proof. The analog of the classical game-representation of $PC(L_{\omega_1\omega})$-sentences or Σ^1_1-sets, deriving from Svenonius and Moschovakis, is a game G of length ω_1 of the following kind. If $\mathfrak{A} \models \Phi$, then $\exists$ wins G. If $\mathfrak{A} \not\models \Phi$ and $\mathfrak{A}$ has cardinality $\leq \omega_1$, then $\forall$ wins G. Let G^T be obtained from G by demanding $\forall$ to go move by move up the tree T. If $T \in \mathcal{T}_{\omega_1}$, then the property that $\exists$ wins G^T can be expressed by an $M_{\omega_2\omega_1}$-sentence Φ^T. If $\mathfrak{A} \not\models \Phi$, $\mathfrak{A}$ has cardinality $\leq \omega_1$, and τ is a winning strategy of $\forall$ in G, then τ gives a winning strategy for $\forall$ even in the game G^T, where T is the tree of all possible sequences (of successor length) of moves of $\exists$ against τ such that $\exists$ has not lost yet. □

Proposition 37 (*Hyttinen 1990) (CH) Suppose* Φ *and* Ψ *are* $PC(M_{\omega_2\omega_1})$*-sentences so that* $\Phi \wedge \Psi$ *has no models. Then there is an* $M_{\omega_2\omega_1}$*-sentence* θ *so that* $\Phi \models \theta$ *and* $\Psi \wedge \theta$ *has no models. (Craig Interpolation Theorem for* $M_{\omega_2\omega_1}$*)*

Proof. Let $T \mapsto \Phi^T$ be the mapping given by Proposition 36. If $\Phi^T \wedge \Psi$ has no models for some $T \in \mathcal{T}_{\omega_1}$, we are done. So let us assume $\Phi^T \wedge \Psi$ has a model for all each $T \in \mathcal{T}_{\omega_1}$. By Proposition 30, we may assume these models have cardinality $\leq \omega_1$. But this means that the class of trees $(T, <)$ of cardinality ω_1 with no uncountable branches is $PC(M_{\omega_2\omega_1})$-definable as the class of trees $(T', <')$ of cardinality ω_1 for which there is a tree $(T, <)$, an order-preserving mapping $T' \rightarrow T$, and a model of $\Phi^T \wedge \Psi$. This contradicts Proposition 35. □

A logic $\mathcal{L}$ satisifes the *Souslin-Kleene Interpolation Theorem* if every $PC(\mathcal{L})$-expression, the negation of which is also definable by a $PC(\mathcal{L})$-expression, is actually explicitly definable in $\mathcal{L}$. It is well-known that $L_{\omega_1\omega}$ satisfies the Souslin-Kleene Interpolation Theorem but $L_{\omega_2\omega_1}$ does not.

Theorem 38 (*Hyttinen 1990) (CH) The smallest extension of* $L_{\omega_2\omega_1}$ *to a logic which satisfies the Souslin-Kleene interpolation theorem is the largest fragment of* $M_{\omega_2\omega_1}$ *which is closed under negation.*

One interpretation of Theorem 38 is that $L_{\omega_2\omega_1}$ has implicit expressive power which the syntax of the logic is not able to express explicitly. This emphasizes the naturalness of $M_{\omega_2\omega_1}$ as an extension of $L_{\omega_2\omega_1}$. Various extensions of Craig interpolation theorem for $M_{\omega_2\omega_1}$ have been proved in *Tuuri 1992* and *Oikkonen 19??*.

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