Journal of Mathematics Research; Vol. 15, No. 1; February 2023
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Gamma Derivatives for the Upgraded Mass-Spring Oscillatory System

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Received: November 13, 2022 Accepted: February 1, 2023 Online Published: February 7, 2023

doi:10.5539/jmr.v15n1p57 URL: https://doi.org/10.5539/jmr.v15n1p57

Abstract

Derivatives in trigonometry have always been defined in orthogonal contexts (i.e., where the y-axis is set perpendicular to the x-axis). Within the context of trigonometric, the present work expands the concept of derivative (operating by the principle of 90 degrees phase shift when applicable to sine and cosine functions) to the realm where the y-axis is at a variable angle γ to the x-axis (i.e., non-orthogonal systems). This gives rise to the concept of the *gamma derivative* — which expands the classical derivative to impart phase shifts of γ degrees. Hence, the ordinary derivative (with respect to α) or $d/d\alpha$ is a particular case of the more general *gamma derivative* or $d_\gamma/d_\gamma\alpha$. Formula for the n^{th} gamma derivative of the extended sine and cosine functions are defined. For applied mathematics, the gamma derivatives of the extended sine function $\sin^*(\alpha,\gamma)$ and cosine function $\cos^*(\alpha,\gamma)$ determine the extended governing equation of the energy-coupled mass-spring oscillatory system, and by extended analogy that of the energy-coupled electrical LC (Inductance-Capacitance) circuit.

Keywords: Gamma, trigonometry, derivative, mass, spring, oscillation

1. Introduction

One way to explore the properties of a function is by studying its derivatives. Derivatives are as important as the functions themselves. The particularity of derivatives of trigonometric functions, like sine and cosine, is that they are recurrent. Governing equations involve not only the algebraic combination of such functions, but also often integrate their derivatives. When looking at potential mathematical applications, it is critical when finding solution that these derivatives are known and understood. Both the mass-spring-damper system (Tipler 1998, Rayleigh 1945, Timoshenko 1974), RLC circuit (Rawlins 2000, Hughes 1995), as well as, trigonometry (Euclids et al 1908, Pickover 2012) are taught in universities worldwide (in particular in engineering), and are widely used in the professional world, making the gamma derivative extension to trigonometric functions a subject of general interested (Canadian Ministry of Education 2020).

2. Hypothesis

It is the hypothesis of this article that just as the Pythagoras theorem is a particular case of the more general Law of Cosines (as seen from a trigonometry perspective), an oscillatory system (seen from a physical perspective) that is governed by the Pythagoras theorem should also have an equaly more general representation (that would naturally encompass that aforementioned particular physical system).

3. Theory

The derivatives of sines and cosines are important cornerstones in trigonometry. Hence, the derivatives of their extended counterparts — extended sine function $\sin^*(\alpha,\gamma)$ and extended cosine function $\cos^*(\alpha,\gamma)$ (Teia, 2022c) — will be equally important (if not more), since not only do they encompass the former, but they also open doors to uncharted interpretation of the mathematical dynamics inherent to non-orthogonal systems. One example of an opened door is the recently published generalization of the angle sum and difference identity rules, that now with new upgraded equations encompass also scalene triangles (Teia, 2022d). Before proceeding, we must first examine the definition of derivative used currently in the context of the Pythagoras theorem, and then use that as a template to understand the gamma derivatives within the context of the Law of Cosines. One may be surprised to find that the former (the particular case that is the Pythagoras theorem) is found encompassed in the definition of the later (the general case that is the Law of Cosines). The derivative of sine and cosine functions within the particular context of the Pythagoras theorem translates into a phase shift of 90 degrees (i.e., $\pi/2$ radians) and thus $\alpha \to \alpha + \pi/2$, with the appropriate sign change resulting from the quadrant swap in the system of orthogonal axes (shown in the example in Figure 1a).

The derivative of sine is known to be

$$\frac{d}{d\alpha}\sin(\alpha) = \sin\left(\alpha + \frac{\pi}{2}\right) = \sin(\alpha)\cos\left(\frac{\pi}{2}\right) + \cos(\alpha)\sin\left(\frac{\pi}{2}\right) = \cos(\alpha) \tag{1}$$

Extrapolating, the n^{th} derivative of the sine function becomes

$$\frac{d^n}{d\alpha^n}\sin(\alpha) = \sin\left(\alpha + n\frac{\pi}{2}\right) = \sin(\alpha)\cos\left(n\frac{\pi}{2}\right) + \cos(\alpha)\sin\left(n\frac{\pi}{2}\right) \tag{2}$$

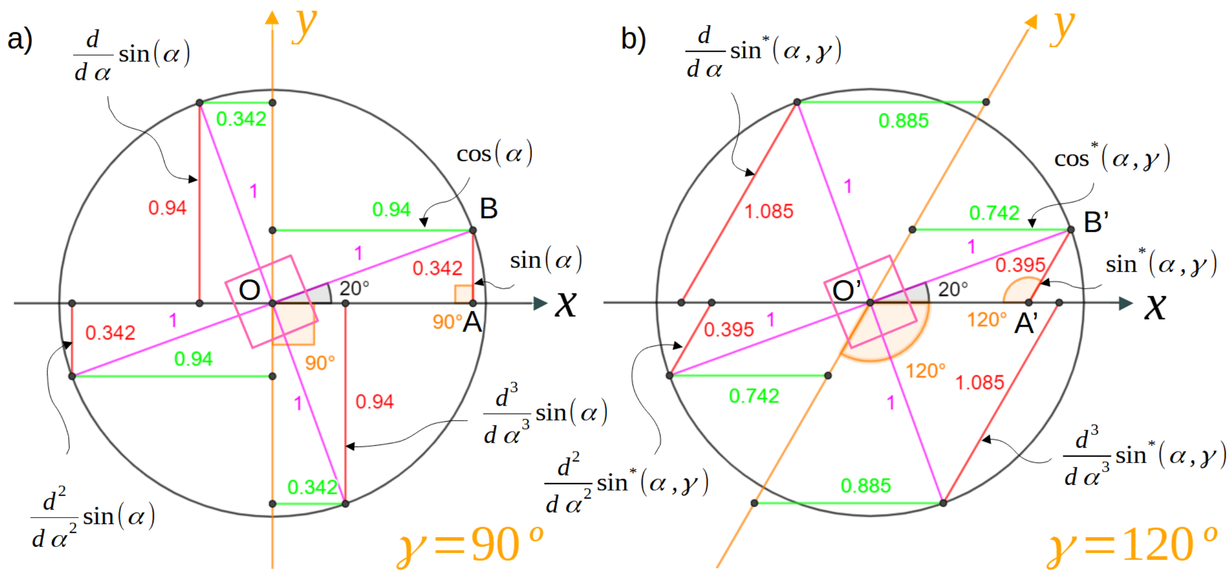

Figure 1. Sine and cosine projections and their classical derivatives for: (a) an orthogonal axes-system and (b) a 120-degree axes-system.

Note that all drawings were created using the open source software Geogebra (Feng, 2013). Continuing, in Eq.(2), the terms with argument n switches between 1, 0, -1 (and then back to 1) serving to iteratively swap the solution between $\sin(\alpha)$ and $\cos(\alpha)$ [almost in a digital fashion, as the radius rotates around the quadrants in 90 degrees intervals]. An expansion of varying n on the classical orthogonal derivative [given by Eq.(2)] is as follows

$$\frac{d^n}{d\alpha^n}\sin(\alpha) = \sin\left(\alpha + n\frac{\pi}{2}\right) = \begin{cases} \frac{d}{d\alpha}\sin(\alpha) = & +\cos(\alpha) \quad : \quad n = 1 \\ \frac{d^2}{d\alpha^2}\sin(\alpha) = & -\sin(\alpha) \quad : \quad n = 2 \\ \frac{d^3}{d\alpha^3}\sin(\alpha) = & -\cos(\alpha) \quad : \quad n = 3 \\ \quad \dots \end{cases} \tag{3}$$

Imagine now that the y-axis is inclined to the x-axis (with respect to the x-axis) in Figure 1a to a non-orthogonal position (where the larger angle between them in Quadrant 4 has been defined in the past by angle γ) in Figure 1b [studies involving non-orthogonal axes have already been published in this journal (Teia 2022c, Teia 2022d)]. In such a case, the (orthogonal) derivative of cosine is no longer sine, and similarly that of sine is no longer the negative of cosine. The recurrence of the functions with their derivatives has been broken, and the reason is simple. The dynamics of the projections of a radius that rotates around in an orthogonal system are governed by the Pythagoras Theorem (i.e., $x^2 + y^2 = z^2$ in Figure 2a), where there is no coupling between the terms x^2 and y^2 (for example in right triangle $\triangle OAB$ in Figure 1a). This coupling term appears mathematically as $-2xycos(\gamma)$ [in particular, as xy for the triangle-driven extended Pythagoras system $\gamma = 120$ deg in Figure 2b (Teia, 2021a), and $-xy$ for the hexagon-driven extended Pythagoras system $\gamma = 60$ deg in Figure 2c (Teia, 2021b)], represented geometrically as the area in yellow (which is added in Figure 2b, and subtracted in Figure 2c).

In a non-orthogonal system of axes, the dynamics are different, and thus the classical sine and cosine no longer operate (and neither do their traditional derivatives). In practice, non-orthogonal system of axes are found in physical systems that employ trigonometric relations based on the scalene triangle that is governed by the Law of Cosines as $x^2 - 2xycos(\gamma) + y^2 = z^2$, of which one example is the laser displacement sensor that measures microscopic displacement/oscillations of surfaces by means of laser light triangulation (MTI Instruments Inc., 2001). Note that the reason why the angle γ — formed between the y-axis and x-axis — is measured in the fourth (and not first) quadrant comes from the (internal) obtuse angle of the scalene triangle formed by the inherent projected extended sine and cosine sides (note that in the example of Figure 1b, the obtuse angle of the scalene triangle $\triangle O'A'B'$ is $\gamma = 120$ degrees). This is consistent with the results presented in previous publications (Teia 2021a, Teia 2021b). As shown in Figure 1b, within a system of axes with $\gamma = 120$ degrees, at no point does one of the n^{th} classical (orthogonal) derivative of the extended cosine function $\cos^*(\alpha, \gamma)$ becomes equal to the one of the n^{th} classical (orthogonal) derivative of the extended sine function $\sin^*(\alpha, \gamma)$. This happens because a classical (orthogonal) derivative is not concordant to a system of axes with $\gamma = 120$ degrees — for matching to occur, one must use a *gamma derivative* concordant with the angle *gamma* of the system (such connection is important when solving differential equations of oscillatory physical systems, as will be shown in Chapter 5). Such non-orthogonal systems have shown to represent geometrical extensions to the Pythagoras Theorem, where the inclination introduces a coupling effect (for example, in scalene triangle $\triangle O'A'B'$ in Figure 1b).

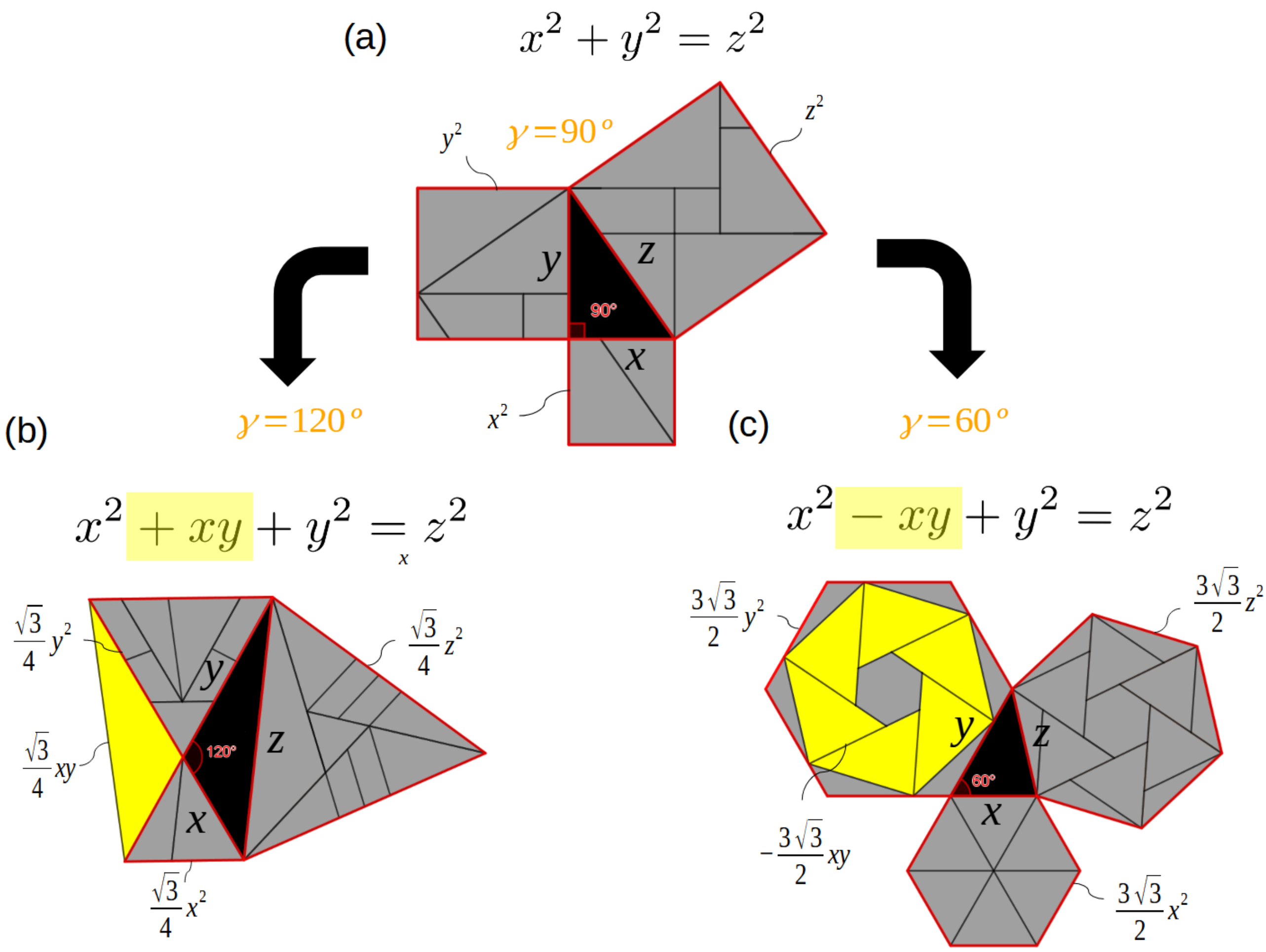

Figure 2. Pythagoras theorem: (a) original, (b) extended with $\gamma = 120$ deg [Teia, 2021a] and (c) extended with $\gamma = 60$ deg [Teia, 2021b].

The perfectly synchronous chain of sine and cosine derivatives in Figure 1a is disturbed in Figure 1b, and their magnitude becomes non-trivial to determine. The derivative of the extended sine and cosine functions – henceforth called the *gamma derivative* — translates into a phase shift of γ degrees (a shift that changes $\alpha \to \alpha + \gamma$) within the general context of the Law of Cosines. Is is important to remember that the general case of the Law of Cosines (axes at any angle) encompasses that of the particular case of the Pythagoras theorem (orthogonal axes), hence it is only logical that the derivative of sines and cosines — expressible as $d/d\alpha$ — is also found in the gamma derivative of extended sines and cosines — expressible as $d_\gamma/d_\gamma\alpha$. Before proceeding, it is important to remember that it was previously shown (Teia, 2022c) that the projected sides of a scalene triangle (possessing a normalized extended hypotenuse $z = 1$) are interrelated by extended expressions (distinguishable from the classical functions by the symbol *) for the sine and cosine — or $y = \sin^*(\alpha, \gamma)$ and $x = \cos^*(\alpha, \gamma)$ — were already proven to be

$$\sin^*(\alpha,\gamma) = \frac{\sin(\alpha)}{\sin(\gamma)} \quad ; \quad \cos^*(\alpha,\gamma) = \cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha) = \frac{\sin(\alpha+\gamma)}{\sin(\gamma)} \tag{4}$$

which inherently satisfy the more general governing equation that is the Law of Cosines. For example, in the case of $\gamma = 120$ degrees (Figure 2), the expressions to determine the normalized (i.e., $z = 1$) sides of the scalene triangle $\triangle O'A'B'$ are for the longer side $O'A' = \cos^*(\alpha, 120)$ and for the shorter side $A'B' = \sin^*(\alpha, 120)$ (which matches Figure 1b). In classical trigonometry, often the sides of a scalene triangle are found by geometrically subdividing each into two right triangles — an approach that requires multiple steps to reach an answer (Tanton, 2015). With *gamma trigonometry*, only one extended sine and cosine function is required to determine each one of the side of the scalene triangle, for any given normalized shape [given by Eq.(4)] (no need for orthogonal partitioning — i.e. referencing a non-orthogonal axes system to an orthogonal one — as the *gamma* effect is already accounted for in the expression). More specifically, these extended functions determine the normalized lengths of the sides x and y of any scalene triangle, when the two internal angles — reference α and obtuse γ — are known. Hence, the particular case of the derivative for $\gamma = 90$ degrees in Eq.(1) transforms into the general case of the gamma derivative (valid for any γ) as

$$\frac{d_\gamma}{d_\gamma\alpha}\sin^*(\alpha,\gamma) = \sin^*(\alpha+\gamma,\gamma) = \frac{\sin(\alpha+\gamma)}{\sin(\gamma)} = \cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha) = \cos^*(\alpha,\gamma) \tag{5}$$

Note that the classical orthogonal derivative is a particular case of Eq.(5) [the general case with the derivative restricted to $\gamma = 90$ degrees], and being applied to the extended sine function gives

$$\frac{d}{d\alpha}\sin^*(\alpha,\gamma) = \sin^*(\alpha+90,\gamma) = \frac{\sin(\alpha+90)}{\sin(\gamma)} = \frac{\cos(\alpha)}{\sin(\gamma)} \tag{6}$$

Extrapolating as before [from Eq.(2)], the n^{th} derivative of the extended sine function becomes

$$\frac{d_\gamma^n}{d_\gamma\alpha^n}\sin^*(\alpha,\gamma) = \frac{\sin(\alpha+n\gamma)}{\sin(\gamma)} = \sin(\alpha)\left[\frac{\cos(n\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(n\gamma)}{\sin(\gamma)}\right] \tag{7}$$

The terms with argument n also switch in sign (due to the quadrant change as the radius rotates) but are different from 1, 0 and -1, thus they neither cancel nor provide the recurrent identity of $\sin(\alpha)$ or $\cos(\alpha)$. These coefficients $\cos(n\gamma)/\sin(\gamma)$ and $\sin(n\gamma)/\sin(\gamma)$ alter both the magnitude and sign of $\sin(\alpha)$ and $\cos(\alpha)$, depending on the value of γ. In expanded form, the gamma derivative for varying n of the extended sine becomes

$$\frac{d_\gamma^n}{d_\gamma\alpha^n}\sin^*(\alpha,\gamma) = \frac{\sin(\alpha+n\gamma)}{\sin(\gamma)} = \begin{cases} \frac{d_\gamma}{d_\gamma\alpha}\sin^*(\alpha,\gamma) = & \sin(\alpha)\left[\frac{\cos(\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(\gamma)}{\sin(\gamma)}\right] = \cos^*(\alpha,\gamma) \quad : \quad n = 1 \\ \frac{d_\gamma^2}{d_\gamma\alpha^2}\sin^*(\alpha,\gamma) = & \sin(\alpha)\left[\frac{\cos(2\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(2\gamma)}{\sin(\gamma)}\right] \quad : \quad n = 2 \\ \frac{d_\gamma^3}{d_\gamma\alpha^3}\sin^*(\alpha,\gamma) = & \sin(\alpha)\left[\frac{\cos(3\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(3\gamma)}{\sin(\gamma)}\right] \quad : \quad n = 3 \\ \ldots \end{cases} \tag{8}$$

Note that the gamma derivative employs a phase shift that can be any angle γ, including 90 degrees [in which case Eq.(5) returns back to Eq.(1)]. Hence, for the case when the relation between the extended sine and cosine is different from 90 degrees, we can no longer use the traditional derivatives given by Eqs.(3), and in the context of the Law of Cosines (governing the side length relation within a scalene triangle), Eqs.(4) are used instead. There are two types of particular cases of gamma derivatives, (1) angularly repetitive and (2) angularly non-repetitive. It all depends if the angle γ between the axes has as a multiple 360 degrees. For example, this happens with the orthogonal case $\gamma = 90$ deg resulting in $4 \times 90 = 360$ (as shown previously in Figure 1a). Another example is when $\gamma = 120$ deg, resulting in $3 \times 120 = 360$ deg (as shown below in Figure 3).

Recurrent derivatives of trigonometric functions can be seen as a tool that allows information from different differentiation levels in an equation to interconnect (for example, the first derivative of displacement from the kinetic energy that connects to the zero derivative of displacement in the potential energy, together forming part of one single energy equation), enabling them to interconnect and form a particular solution. This fact makes recurrent derivatives very handy, when determining solutions to differential equations. Bearing this in mind, it becomes clear that recurrent derivatives are more useful (than non-recurrent derivatives) when addressing problems of differential equations. Focusing on angular repetitive

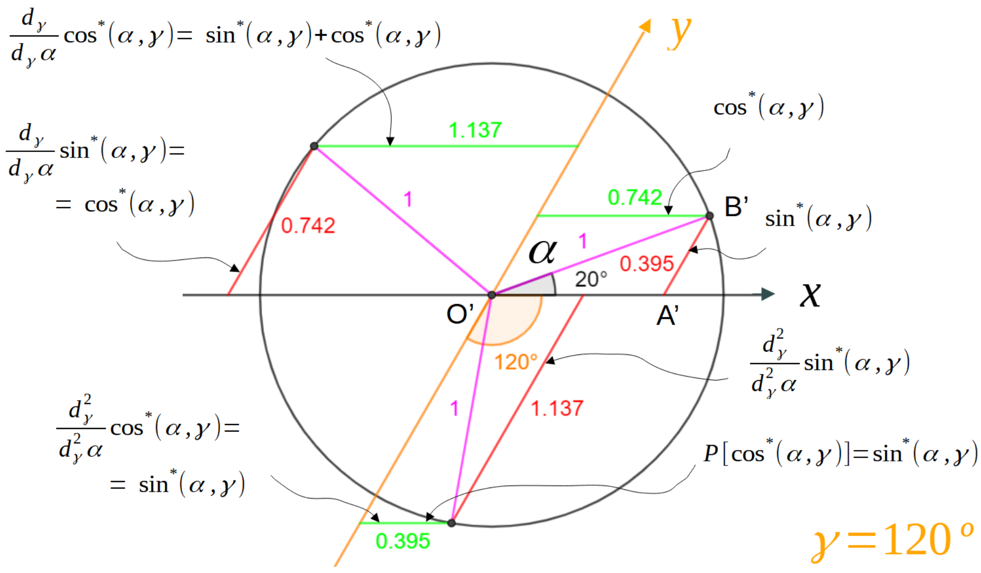

Figure 3. Example of extended sine and cosine functions (α = 20 deg), and their derivatives for a 120-degree axes system.

gamma derivatives, the next aspect that is important to understand is how this recurrence behaves. There are two options, the number of times p that γ adds to form 360 deg can be odd (e.g., γ = 120 deg gives odd $p = 3$ resulting in $3\times120 = 360$ deg) or even (e.g., γ = 90 deg gives even $p = 4$ resulting in $4 \times 90 = 360$ deg). For example, when the angle between the axes is γ = 120 deg, the extended sine returns to itself in the third derivative or $p = 3$. In turn, the extended cosine — being always the first derivative of the extended sine, regardless of α or γ — returns to itself in the fourth derivative or $p + 1 = 4$. As the example in Figure 3 shows, for such a system and a radius angle α = 20 degrees, the value of the extended sine (and its first two derivatives) are

$$\frac{d^n_{120}}{d_{120}\alpha^n}\sin^*(20,120) = \begin{cases} \sin^*(20,120) = \frac{\sin(20)}{\sin(120)} = 0.395 & : \ n=0 \\ \frac{d_{120}}{d_{120}\alpha}\sin^*(20,120) = \sin(20)\left[\frac{\cos(120)}{\sin(120)}\right] + \cos(20)\left[\frac{\sin(120)}{\sin(120)}\right] = 0.742 & : \ n=1 \\ \frac{d^2_{120}}{d_{120}\alpha^2}\sin^*(20,120) = \sin(20)\left[\frac{\cos(240)}{\sin(120)}\right] + \cos(20)\left[\frac{\sin(240)}{\sin(120)}\right] = -1.137 & : \ n=2 \end{cases} \tag{9}$$

which confirm the values measured using the Geogebra software (Feng, 2013) in Figure 4. Generalizing the derivatives of the extended sine to any value of α gives

$$\frac{d^n_{120}}{d_{120}\alpha^n}\sin^*(\alpha,120) = \frac{\sin(\alpha+n\times120)}{\sin(120)} = \begin{cases} \frac{d_{120}}{d_{120}\alpha}\sin^*(\alpha,\gamma) = \sin(\alpha)\left[-\frac{\sqrt{3}}{3}\right] + \cos(\alpha)\left[1\right] = \cos^*(\alpha,120) & : \ n=1 \\ \frac{d^2_{120}}{d_{120}\alpha^2}\sin^*(\alpha,\gamma) = \sin(\alpha)\left[-\frac{\sqrt{3}}{3}\right] + \cos(\alpha)\left[-1\right] & : \ n=2 \\ \frac{d^3_{120}}{d_{120}\alpha^3}\sin^*(\alpha,\gamma) = \sin(\alpha)\left[\frac{2\sqrt{3}}{3}\right] + \cos(\alpha)\left[0\right] = \sin^*(\alpha,120) & : \ n=3 \\ \frac{d^4_{120}}{d_{120}\alpha^4}\sin^*(\alpha,\gamma) = \sin(\alpha)\left[-\frac{\sqrt{3}}{3}\right] + \cos(\alpha)\left[1\right] = \cos^*(\alpha,120) & : \ n=4 \\ \ldots \end{cases} \tag{10}$$

In turn, the derivative of the cosine function $\cos(\alpha)$ is

$$\frac{d}{d\alpha}\cos(\alpha) = \cos\left(\alpha + (1)\frac{\pi}{2}\right) = -\sin(\alpha) = \sin\left(\alpha + (2)\frac{\pi}{2}\right) = \frac{d^2}{d\alpha^2}\sin(\alpha) \tag{11}$$

Hence, the chain of derivatives pertaining to cosine are the same as sine, except for a phase shift of $\pi/2$. While all of this may sound trivial, it lays the foundation for the more general gamma derivative of the extended sine and cosine functions.

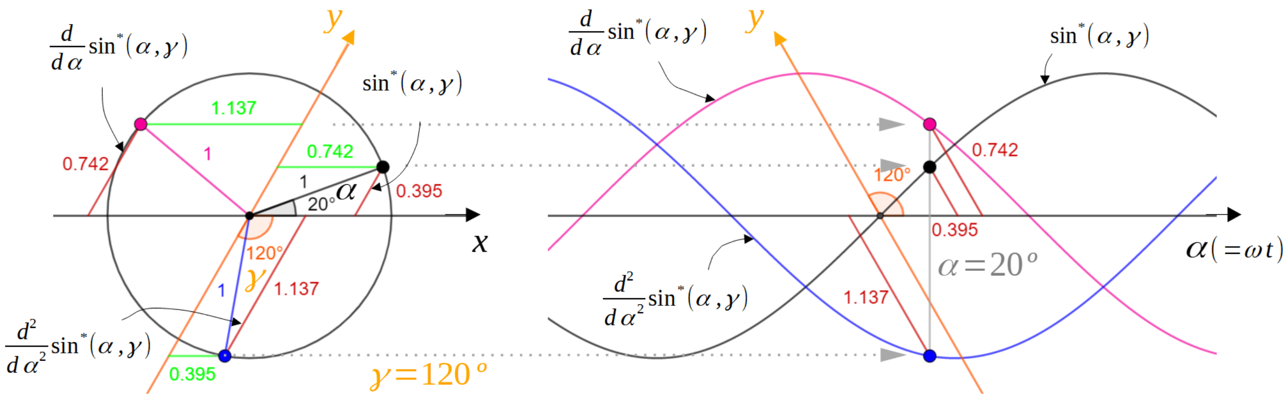

Figure 4. Polar to temporal conversion of extended sine projection and derivatives for a γ = 120-degree axes system.

Extrapolating from above, the n^{th} derivatives of the cosine function are then expressed as

$$\frac{d^n}{d\alpha^n}\cos(\alpha) = \sin\left(\alpha + (n+1)\frac{\pi}{2}\right) = \sin(\alpha)\cos\left((n+1)\frac{\pi}{2}\right) + \cos(\alpha)\sin\left((n+1)\frac{\pi}{2}\right) \tag{12}$$

The gamma derivative of the extended cosine function $\cos^*(\alpha,\gamma)$ is

$$\frac{d_\gamma}{d_\gamma\alpha}\cos^*(\alpha,\gamma) = \frac{\sin(\alpha+2\gamma)}{\sin(\gamma)} = \sin(\alpha)\left[\frac{\cos(2\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(2\gamma)}{\sin(\gamma)}\right] \tag{13}$$

Note that the classical orthogonal derivative is a particular case of Eq.(13) [the general case with the derivative restricted to γ = 90 degrees], and being applied to the extended cosine function gives

$$\frac{d}{d\alpha}\cos^*(\alpha,\gamma) = \frac{d}{d\alpha}\left\{\cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha)\right\} = -\sin(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\cos(\alpha) \tag{14}$$

Similarly as before, the n^{th} gamma derivative of the extended cosine function is expressed as

$$\frac{d^n_\gamma}{d_\gamma\alpha^n}\cos^*(\alpha,\gamma) = \frac{\sin[\alpha+(n+1)\gamma]}{\sin(\gamma)} = \sin(\alpha)\left[\frac{\cos[(n+1)\gamma]}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin[(n+1)\gamma]}{\sin(\gamma)}\right] \tag{15}$$

In expanded form, the gamma derivative for varying n of the extended cosine becomes

$$\frac{d^n_\gamma}{d_\gamma\alpha^n}\cos^*(\alpha,\gamma) = \frac{\sin[\alpha+(n+1)\gamma]}{\sin(\gamma)} = \begin{cases} \frac{d_\gamma}{d_\gamma\alpha}\cos^*(\alpha,\gamma) = & \sin(\alpha)\left[\frac{\cos(2\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(2\gamma)}{\sin(\gamma)}\right] & : \quad n=1 \\ \frac{d^2_\gamma}{d_\gamma\alpha^2}\cos^*(\alpha,\gamma) = & \sin(\alpha)\left[\frac{\cos(3\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(3\gamma)}{\sin(\gamma)}\right] & : \quad n=2 \\ \frac{d^3_\gamma}{d_\gamma\alpha^3}\cos^*(\alpha,\gamma) = & \sin(\alpha)\left[\frac{\cos(4\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(4\gamma)}{\sin(\gamma)}\right] & : \quad n=3 \\ \qquad\ldots \end{cases} \tag{16}$$

Returning to the example in Figure 3, for such a system with an angle of γ = 120 degrees between axes and a radius at an angle of α = 20 degrees, the value of the extended cosine and its first two derivatives are

$$\frac{d^n_{120}}{d_{120}\alpha^n}\cos^*(20,120) = \begin{cases} \cos^*(20,120) = & \frac{\sin(20+120)}{\sin(120)} = 0.742 & : \quad n=0 \\ \frac{d_{120}}{d_{120}\alpha}\cos^*(20,120) = & \sin(20)\left[\frac{\cos(240)}{\sin(120)}\right] + \cos(20)\left[\frac{\sin(240)}{\sin(120)}\right] = -1.137 & : \quad n=1 \\ \frac{d^2_{120}}{d_{120}\alpha^2}\cos^*(20,120) = & \sin(20)\left[\frac{\cos(360)}{\sin(120)}\right] + \cos(20)\left[\frac{\sin(360)}{\sin(120)}\right] = 0.395 & : \quad n=2 \end{cases} \tag{17}$$

which confirm the values measured using the Geogebra software in Figure 5, (shown in green in Figure 5 left — the result

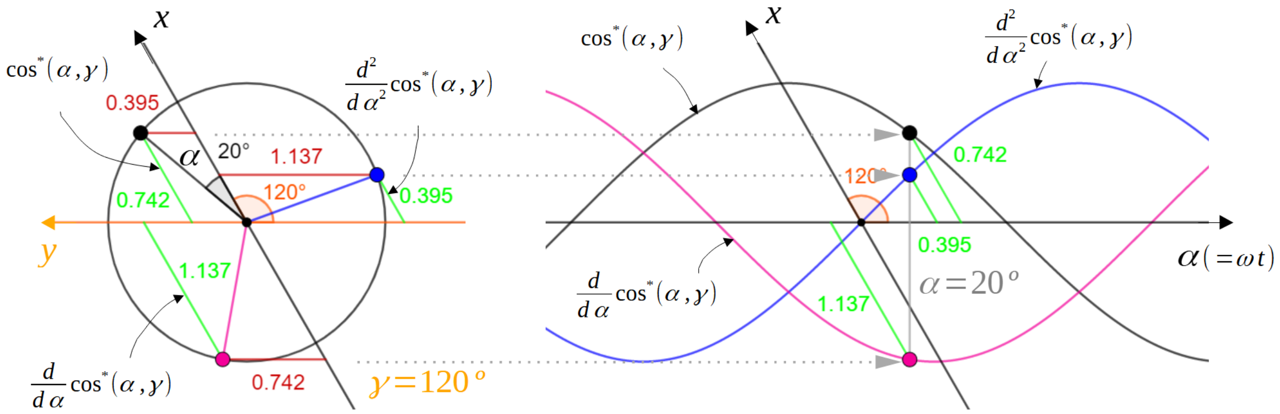

Figure 5. Polar to temporal conversion of extended cosine projection and derivatives for a γ = 120-degree axes system.

of Figure 4 left being rotated by 90 degrees). Generalizing the derivatives of the extended cosine to any value of α gives

$$\frac{d_\gamma^n}{d_\gamma\alpha^n}\cos^*(\alpha,\gamma) = \frac{\sin[\alpha+(n+1)\gamma]}{\sin(\gamma)} = \begin{cases} \frac{d_\gamma}{d_\gamma\alpha}\sin^*(\alpha,\gamma) = \sin(\alpha)\left[\frac{\cos(2\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(2\gamma)}{\sin(\gamma)}\right] & : \; n=1 \\ \frac{d_\gamma^2}{d_\gamma\alpha^2}\sin^*(\alpha,\gamma) = \sin(\alpha)\left[\frac{\cos(3\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(3\gamma)}{\sin(\gamma)}\right] & : \; n=2 \\ \frac{d_\gamma^3}{d_\gamma\alpha^3}\sin^*(\alpha,\gamma) = \sin(\alpha)\left[\frac{\cos(4\gamma)}{\sin(\gamma)}\right] + \cos(\alpha)\left[\frac{\sin(4\gamma)}{\sin(\gamma)}\right] & : \; n=3 \\ \qquad\qquad \dots & \end{cases} \tag{18}$$

It is important to note that by subtracting the angle γ from α (instead of adding) provides the primitive P of $\cos^*(\alpha,\gamma)$, resulting in the extended sine function $\sin^*(\alpha,\gamma)$.

$$P\Big[\cos^*(\alpha,\gamma)\Big] = \cos^*(\alpha-\gamma,\gamma) = \sin^*(\alpha,\gamma) \tag{19}$$

This can be proven by expanding $\cos^*(\alpha-\gamma,\gamma)$ as

$$\cos^*(\alpha-\gamma,\gamma) = \cos(\alpha-\gamma) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha-\gamma) \tag{20}$$

That becomes

$$\cos(\alpha)\cos(\gamma) + \sin(\alpha)\sin(\gamma) + \frac{\cos(\gamma)}{\sin(\gamma)}\Big[\sin(\alpha)\cos(\gamma) - \cos(\alpha)\sin(\gamma)\Big] \tag{21}$$

And simplifies to

$$\cos(\alpha)\cos(\gamma) + \sin(\alpha)\sin(\gamma) + \sin(\alpha)\frac{\cos^2(\gamma)}{\sin(\gamma)} - \cos(\alpha)\cos(\gamma) \tag{22}$$

Which ends up to be

$$\sin(\alpha)\sin(\gamma) + \sin(\alpha)\frac{1-\sin^2(\gamma)}{\sin(\gamma)} = \frac{\sin(\alpha)}{\sin(\gamma)} = \sin^*(\alpha,\gamma) \tag{23}$$

The above reasoning shows that the derivatives of the extended sine function $\sin^*(\alpha,\gamma)$ and cosine function $\cos^*(\alpha,\gamma)$ are not trivially connected to each other for non-orthogonal cases (i.e., when $\gamma \neq 90$ deg).

4. Differentiating the Governing Equation

The Pythagoras theorem $x^2 + y^2 = z^2$ governs the dynamics of the sine $y = \sin(\alpha)$ and cosine $x = \cos(\alpha)$ functions via the sides of a normalized (i.e., $z = 1$) right-angled triangle, resulting in the normalized relation $\cos^2(\alpha) + \sin^2(\alpha) = 1$. In turn, the Law of Cosines given as

$$x^2 - 2xy\cos(\gamma) + y^2 = z^2 \tag{24}$$

is the generalized version that governs the dynamics of the extended sine $y = \sin^*(\alpha, \gamma)$ and cosine $x = \cos^*(\alpha, \gamma)$ functions via the sides of a normalized (i.e., $z = 1$) scalene triangle, possessing a reference angle α and obtuse angle γ, resulting in the normalized relation

$$\cos^*(\alpha, \gamma)^2 - 2\cos^*(\alpha, \gamma)\sin^*(\alpha, \gamma)\cos(\gamma) + \sin^*(\alpha, \gamma)^2 = 1 \tag{25}$$

Differentiating these equation is an important part of the process of determining the equations governing some of the most important (if not the most important) systems in vibration theory, i.e. the mass-spring (and damper) system and the pendulum. These physical systems, and their governing equations with solutions, are a foundation for all studies in science involving oscillations, e.g. theory of sound, mechanical engineering, electrical engineering, planetary physics, to name a few. This will be further explained in Chapter 3. For now, to better understand the differentiation process of the Law of Cosines, let us start by differentiating the Pythagoras Theorem. The derivative of the Pythagoras theorem (assuming z constant) is

$$2xx' + 2yy' = 0 \tag{26}$$

which means that only x and y vary (that is, it is assumed that the hypotenuse of the right- triangle remains fixed, and of equal length, for any change of the internal reference angle α). The above Eq.(26) can be further expanded by replacing the derivatives of sine and cosine resulting in

$$-\cos(\alpha)\sin(\alpha) + \sin(\alpha)\cos(\alpha) = 0 \tag{27}$$

which trivialy simplifies as a valid equality of zeros. While this seems trivial for the particular case of the Pythagoras theorem, it becomes more complex for the general case of the Law of Cosines. In the past, trigonometric functions were successfully employed to determine solutions for Ordinary Differential Equations (ODEs) — a topic already studied in previous publications (Bromwich, 1908). The differentiation of Eq.(24) results in

$$2xx' - 2[xy' + x'y]\cos(\gamma) + 2yy' = 0 \tag{28}$$

Theorem 1 *(Validation of the Differentiated Version of the Law of Cosines). If the extended functions $x = \cos^*(\alpha, \gamma)$, $y = \sin(\alpha, \gamma)$ (the two shorter sides) and $z = 1$ (the longer side, herein expressed as extended hypotenuse) are the lengths of the sides of a scalene triangle of reference angle α and obtuse angle γ, and satisfy the governing equation Law of Cosines, gives as $x^2 - 2xy\cos(\gamma) + y^2 = z^2$ then these same functions, and their derivatives, must satisfy the differentiated version (with respect to α) of the same governing equation, given as*

$$xx' - [xy' + x'y]\cos(\gamma) + yy' = 0 \tag{29}$$

Proof. The extended functions $x = \cos^*(\alpha, \gamma)$ and $y = \sin^*(\alpha, \gamma)$ have proven to satisfy the normalized (i.e., .$z = 1$) Law of Cosines [given by Eq.(24)] [Teia, 2022]. For the differentiated version [given by Eq.(29)], let us start by separating the governing equation into three parts, one for each term, making $xx' - [xy' + x'y]\cos(\gamma) + yy'$ = Part I + II + III. Replacing the extended sine and cosine functions in Eq.(4) and their derivatives [given by Eq.(6) and Eq.(14)], into Part I gives

$$xx' = \sin^*(\alpha, \gamma)\frac{d}{d\alpha}[\sin^*(\alpha, \gamma)] = \frac{\sin(\alpha)}{\sin(\gamma)}\left[\frac{\cos(\alpha)}{\sin(\gamma)}\right] = \frac{1}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha) \tag{30}$$

Expanding explicitly Part II results in

$$xy' + x'y = \sin^*(\alpha,\gamma)\frac{d}{d\alpha}[\cos^*(\alpha,\gamma)] + \frac{d}{d\alpha}[\sin^*(\alpha,\gamma)]\cos^*(\alpha,\gamma) \tag{31}$$

Replacing the definitions from Eq.(4), Eq.(6) and Eq.(14) expands the above as

$$\frac{\sin(\alpha)}{\sin(\gamma)}\Big\{-\sin(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\cos(\alpha)\Big\} + \Big[\frac{\cos(\alpha)}{\sin(\gamma)}\Big]\Big\{\cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha)\Big\} \tag{32}$$

Which expanded gives

$$xy' + x'y = -\frac{\sin^2(\alpha)}{\sin(\gamma)} + \frac{\cos(\gamma)}{\sin^2(\gamma)}\cos(\alpha)\sin(\alpha) + \frac{\cos^2(\alpha)}{\sin(\gamma)} + \frac{\cos(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha) \tag{33}$$

Finaly, Part III yields

$$yy' = \cos^*(\alpha,\gamma)\frac{d}{d\alpha}[\cos^*(\alpha,\gamma)] = \Big\{\cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha)\Big\}\Big[-\sin(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\cos(\alpha)\Big] \tag{34}$$

Which expands to

$$yy' = -\cos(\alpha)\sin(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\Big\{\cos^2(\alpha) - \sin^2(\alpha)\Big\} + \frac{\cos^2(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha) \tag{35}$$

Replacing Eq.(30), Eq.(33) and Eq.(35) back into the differentiated governing equation [in Eq.(29)] gives

$$\begin{aligned}\frac{1}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha) - \Big[-\frac{\sin^2(\alpha)}{\sin(\gamma)} + \frac{\cos(\gamma)}{\sin^2(\gamma)}\cos(\alpha)\sin(\alpha) + \frac{\cos^2(\alpha)}{\sin(\gamma)} + \frac{\cos(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha)\Big]\cos(\gamma) - \\ -\cos(\alpha)\sin(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\Big\{\cos^2(\alpha) - \sin^2(\alpha)\Big\} + \frac{\cos^2(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha)\end{aligned} \tag{36}$$

which expands to

$$\begin{aligned}\frac{1}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\sin^2(\alpha) - \frac{\cos^2(\gamma)}{\sin^2(\gamma)}\cos(\alpha)\sin(\alpha) - \frac{\cos(\gamma)}{\sin(\gamma)}\cos^2(\alpha) - \frac{\cos^2(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha) - \\ -\cos(\alpha)\sin(\alpha) + \frac{\cos(\gamma)}{\sin(\gamma)}\Big\{\cos^2(\alpha) - \sin^2(\alpha)\Big\} + \frac{\cos^2(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha)\end{aligned} \tag{37}$$

cancelling the terms with $\cos^2(\gamma)/\sin^2(\gamma)$ and $\cos(\gamma)/\sin(\gamma)$ substantially simplifies the expression to

$$\frac{1}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha) - \frac{\cos^2(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha) - \cos(\alpha)\sin(\alpha) \tag{38}$$

which re-writes and concludes as

$$\Big\{\frac{1}{\sin^2(\gamma)} - \frac{\cos^2(\gamma)}{\sin^2(\gamma)} - 1\Big\}\sin(\alpha)\cos(\alpha) = \Big\{\frac{\sin^2(\gamma)}{\sin^2(\gamma)} - 1\Big\}\sin(\alpha)\cos(\alpha) = 0 \tag{39}$$

As expected from Eq.(29), the end result is zero. This completes the proof.

5. Applied Mathematics

5.1 Classical Mass-Spring Oscillatory System

In a physical system that is oscillating, a particular derivative corresponds to a given level of information. For example, in the case of a mass-spring oscillatory system (Figure 6a), the governing equation can be defined as the sum of energies (i.e., kinetic $K = 1/2m\dot{s}^2$ and potential energy $U = 1/2ks^2$ being equal to a total T) [Rayleigh 1945, Timoshenko et al 1974] — it is the zero derivative level of information, as it were — or this can be differentiated to give a sum of forces — subsequentially, the first derivative level of information. Moreover, within each level there can be terms governed by different derivatives — for instance, in an energy balance there is a displacement s and velocity term $\dot{s}$ (dersivative of displacement). Likewise, at a sum of force level, these terms transform into a relation between acceleration $\ddot{s}$ and velocity $\dot{s}$. This is to be distinguished from x, y and z, which are the sides of the right triangle governing the displacement of the mass-spring system, which (as we will see below) are a function of s and $\dot{s}$. Overall, it is well highlighted then the need for trigonometric functions that are recurrent with differentiation, as they establish links between those different derivatives within, and in between different levels of a governing equation. The equation that governs the oscillation of a mass-spring system, in an energy balance perspective, is

$$K + U = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}ks^2 = T \qquad \Longleftrightarrow \qquad x^2 + y^2 = z^2 \tag{40}$$

The mass-spring oscillatory system has various embodiments in engineering [as for example the suspension of a car or train (Sully, 1988)], and can appear in various physcial shapes (see for example the often studied problem of a clamped vibrating membrane or diaphragm (Teia 2009, Leissa 1969), which is vitaly used in the operation of microphones (Eargle, 2005) and loud speakers (Merhaut, 1981)]. Here, the new coupled version (Figure 6b) will be compared against the classical mass-spring (Figure 6a) in the following section 5.2. Usually there is also a resistive element (in Figure 6a), such as a damper, that accounts for the energy extraction from the system — converting it into a mass-spring-damper system. For simplicity, this is not considered for now, and can always be added as part of a future follow-on research effort. The unusual situation here is that kinetic energy has the same cycle as the potential energy, except that it is offset by 90 degrees, i.e., having a derivative term. This offset is often regarded to as a time delay as the displacement derivative is in time. However, from a Pythagoras theorem perspective, this offset is not really in time, as as the relation between the sides of a triangle occur for a specific triangle, i.e, a fixed geometrical domains at a certain point in time. In this optics, at any given time, the two energies are related by an angular shift in their cycle. This is not the same as saying that one energy will match the other after a certain period or phase in time. This is the same as relating the side of one triangle with that of another triangle at a different point in time.

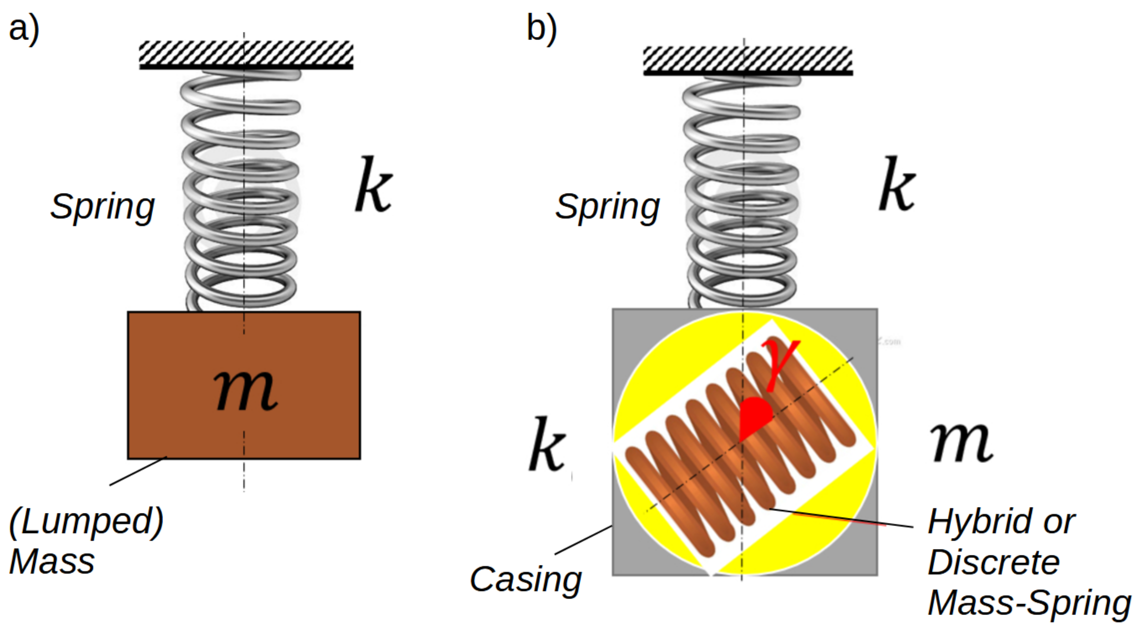

Figure 6. Oscillatory system of mass (inertia) and spring [stiffness]: (a) classical and (b) coupled.

This distinction is important, for while in the traditional case (Pythagoras theorem) the relation between energy states for a particular time (i.e., $d/d\alpha$ of 90 degrees) can be regarded as the same relation between any one of the energies between a position of time and one later that corresponds to the same cycle shift (i.e., d/dt), in the general case (Law of Cosines) this is not true. That is, the phase relation between the potential and kinetic energy is not 90 degrees (it is γ) due to the coupling effect, while the relation between the energy state from a point in time to that later as determined by the time derivative is 90 degrees.

$$K + U = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}ks^2 = T \qquad \equiv \qquad \frac{1}{2}m\frac{ds}{d\alpha}^2 + \frac{1}{2}ks^2 = T \tag{41}$$

Which is equivalent to say

$$x^2 + y^2 = z^2 \tag{42}$$

where $y = \sqrt{m/2}\dot{s}$, $x = \sqrt{k/2}s$ and $z = \sqrt{T}$ (is considered constant). This equivalency generates the bridge between the mathematics (and geometry) of the Pythagoras theorem with that of the physical behaviour of a mass-spring system. This then is used as a template to obtain the same equivalence between the more general Law of Cosine and the new coupled mass-spring system. Start by differentiating Eq.(42), which gives

$$xx' + yy' = 0 \qquad \equiv \qquad x\frac{dx}{d\alpha} + y\frac{dy}{d\alpha} = 0 \tag{43}$$

Figure 7 shows the the three-dimensional trajectory of a radius, as it revolves around a temporal domain/axis (linearly interconnected to the its own angular position α). The same angles [as in Figure 3] are presented, but in a geometrical domain of the Pythagoras theorem (i.e., for a system axes with an angle $\gamma = 90$ deg). It is a fortunate fact that both the time domain is perpendicular or orthogonal (uncoupled) to the Pythagorean domain, and the side (or axis) y is also perpendicular or orthogonal to the side (or axis) x within the Pythagorean domain (governing the shape of the right triangles within). This means that in both cases orthogonal derivatives d/dt and $d/d\alpha$ are employed (where sine and cosine functions relate to their derivatives by 90 degree rotation) [Figure 7b]. Fortunate because, if the y-axis was not perpendicular to the x-axis, the classical orthogonal derivative could not be employed, and the solution would require the application of the *gamma derivative*.

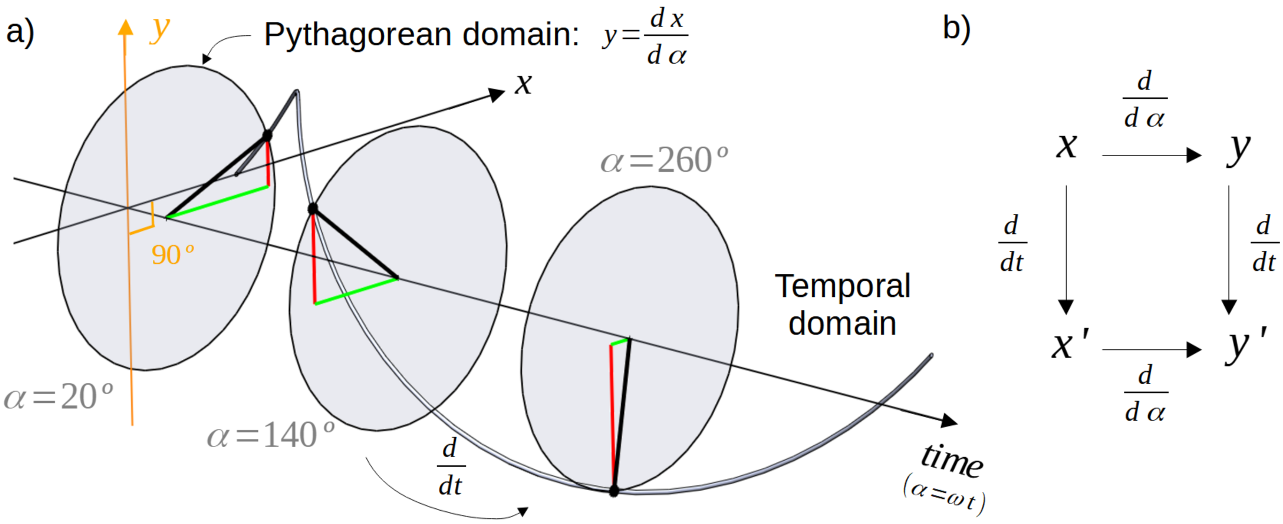

Figure 7. Extended cosine projections $\cos^*(\alpha, \gamma)$ and their derivatives for a $\gamma = 120$-degree axes system.

Substituting explicitly $x' = dx/dt$ (temporal domain) and implicitly $y = dx/d\alpha$ (Pythagorean domain), gives Eq.(43) in the form

$$x\dot{x} + \dot{x}\ddot{x} = 0 \qquad \equiv \qquad x\frac{dx}{d\alpha} + \frac{dx}{d\alpha}\frac{d^2x}{d\alpha^2} = 0 \tag{44}$$

However, from a perspective of the mass-spring system, this is not enough, as we need to account for the physical properties. Hence, as defined previously with Eq.(42) and Eq.(42), replacing the previous terms $y = \sqrt{m/2}\dot{s}$ and $x = \sqrt{k/2}s$ in Eq.(43) gives

$$\sqrt{\frac{k}{2}}s\sqrt{\frac{k}{2}}\dot{s} + \sqrt{\frac{m}{2}}\dot{s}\sqrt{\frac{m}{2}}\ddot{s} = 0 \qquad \rightarrow \qquad ks\dot{s} + m\dot{s}\ddot{s} = 0 \tag{45}$$

Both Eq.(26) and Eq.(45) are the same, since $x = \sqrt{m}\dot{s}$ and $y = \sqrt{k}s$. Cancelling the mutually common coefficient $\dot{s}$ reduces the equation to its more familiar format as

$$m\ddot{s} + ks = 0 \tag{46}$$

which states that during an oscillation the inertia force from the mass $m\ddot{s}$ is always balanced by the opposing force generated by the spring ks, where the later increases linearly with displacement from the point of equilibrium of the spring (Hooke's Law) [Tipler 1998].

5.2 Coupled Mass-Spring Oscillatory System

As explained before in Figure 2a, the Pythagoras theorem is a particular case where there is no coupling between information in the x-axis and y-axis. Thus, the general case of the Law of Cosines, not only encompasses this particular one, but in its vast majority a coupling occurs the x-axis and y-axis, highlighted as the yellow area in Figure 2b (for $\gamma = 120$ deg) and Figure 2c (for $\gamma = 60$ deg). Since the classical mass-spring system is governed by the uncoupled dynamics of the Pythagoras theorem, then it is logical to assume there ia coupled version of the mass-spring system that follows the dynamics of the Law of Cosines. This *coupling effect* is modelled by replacing the infinitely stiff (rigid) mass by a finitely stiff mass distributed in the shape of a spring, that is henceforth called the *hybrid or discrete mass-spring* (Figure 6b). *Hybrid* because it is neither a mass (here recognized as a rectangular block with mass m) nor a spring (here recognized as a flexible coil with a certain stiffness k, and no mass). *Discrete* because the coupling effect emerges from the fact that the mass is not lumped, and is indeed distributed along the length of the spring. This hybrid mass-spring is encased in an ideally massless casing attached to the main spring, that then links to a fixed reference plane or surface. The γ effect, highlighted as the mains difference between the Pythagoras theorem and the Law of Cosines, is modeled by rotating the axis of displacement of the hybrid mass-spring with respect to the axis of displacement of the main spring. Note that when the angle $\gamma = 90$ degrees, the two axes are perpendicular, at which point the hybrid mass-spring behaves only as a mass, and the coupled system behaves as a classical mass-spring system. Modifying the classical mass-spring system (Figure 7a) to a coupled mass-spring counterpart (Figure 8a) changed the geometrical domain from Pythagorean to the Law of Cosines (here examplified with the case of $\gamma = 120$ deg), while the temporal domain ($\alpha = \omega t$) remained the same. The differentiated version (assuming z constant) of the Law of Cosines as $x^2 - 2xy\cos(\gamma) + y^2 = z^2$ [given previously in Eq.(29)] is repeated below for convenience

$$xx' - [xy' + x'y]\cos(\gamma) + yy' = 0 \qquad \equiv \qquad x\frac{dx}{dt} - \left[x\frac{dy}{dt} + \frac{dx}{dt}y\right]\cos(\gamma) + y\frac{dy}{dt} = 0 \tag{47}$$

Considering the equality $y = d_\gamma x/d_\gamma t$ (where $d_\gamma/d_\gamma t$ is the gamma derivative discussed in Chapter 2), the above differential equation can be re-written as a function of only one undefined variable x, or

$$x\frac{dx}{dt} - \left[x\frac{d}{dt}\left(\frac{d_\gamma x}{d_\gamma t}\right) + \frac{dx}{dt}\left(\frac{d_\gamma x}{d_\gamma t}\right)\right]\cos(\gamma) + \left(\frac{d_\gamma x}{d_\gamma t}\right)\left[\frac{d}{dt}\left(\frac{d_\gamma x}{d_\gamma t}\right)\right] = 0 \tag{48}$$

Figure 8 shows the same projections [as in Figure 3] in a geometrical domain of Law of Cosines [to examplify, this system of axes has an agle $\gamma = 120$ deg], evolving in a temporal domain. Remember that $y = dx/dt$ is only valid for orthogonal differentiation (hence, only adequate for governing equations of the type of the Pythagoras theorem), while $y = d_\gamma x/d_\gamma t$ is valid for any system of axes (orthogonal and non-orthogonal) (hence, adequate for governing equations of the type of the Law of Cosines). If we assume temporarily that $s = x$ and $\dot{s}_\gamma = d_\gamma x/d_\gamma t = y$, Eq.(48) would simplify to

$$s\dot{s} - [s\ddot{s}_\gamma + \dot{s}\dot{s}_\gamma]\cos(\gamma) + \dot{s}_\gamma\ddot{s}_\gamma = 0 \tag{49}$$

But to be more complete, in the context of the coupled mass-spring system, we know from previously that $x = \sqrt{k}s$ and $y = \sqrt{m}\dot{s}_\gamma$. Replacing these in Eq.(47) gives the same as Eq.(49), but complemented with the mass and stiffness coefficients, resulting in

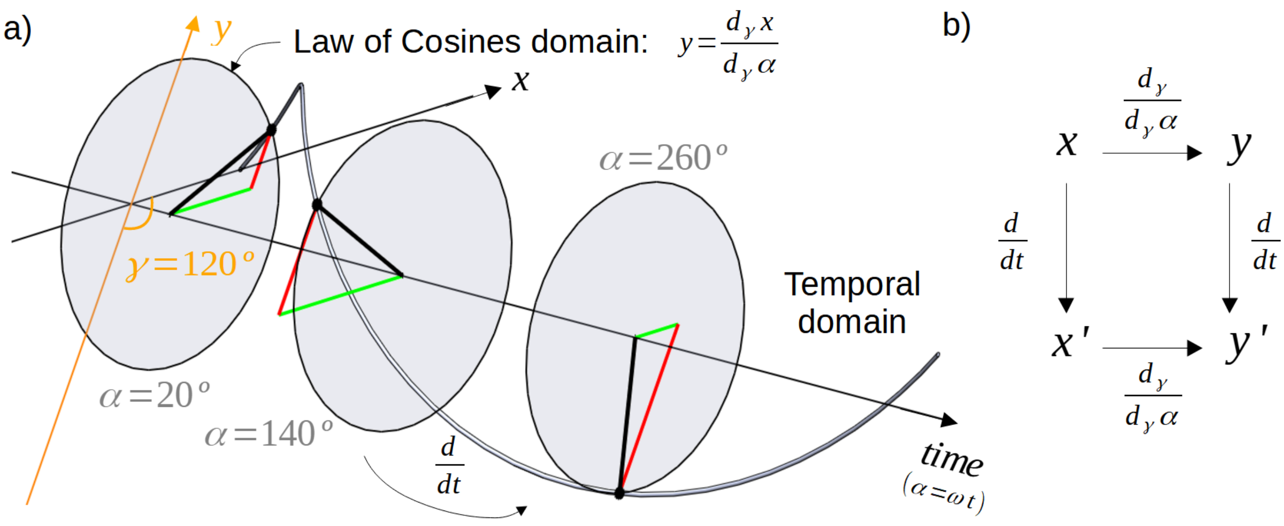

Figure 8. Extended cosine projections $\cos^*(\alpha,\gamma)$ and their derivatives for a γ = 120-degree axes system.

$$ks\dot{s} - \sqrt{mk}\Big[s\ddot{s}_\gamma + \dot{s}\dot{s}_\gamma\Big]\cos(\gamma) + m\dot{s}_\gamma\ddot{s}_\gamma = 0 \tag{50}$$

Integrating Eq.(50) gives a new energy balance equation for the coupled mass-spring system as

$$\frac{1}{2}ks^2 - \frac{1}{2}\sqrt{mk}\Big[s\dot{s}_\gamma\Big]\cos(\gamma) + \frac{1}{2}m\dot{s}_\gamma^2 = T \tag{51}$$

Note that for the uncoupled version when γ = 90 degrees, the general case of the above energy equation [Eq.(51)] returns back to the well-known particular case of $1/2ks^2 + 1/2m\dot{s}^2 = T$ [given previously by Eq.(42)]. The middle term in Eq.(51) is completely new, and is neither completely kinetic energy nor completely potential energy, but it is in fact an energy that is in a state of coupling between the two — i.e., inertia and stiffness effects express themselves superimposed, as indicated by the term $\sqrt{mk}$. Physically, this reflects on a component that is neither a mass nor a spring, but a hybrid of the two — i.e., a spring with distributed mass along its length (as shown in Figure 6b). This *discrete or hybrid* mass-spring has a content where both inertia and stiffness change with its length (i.e., a longer hybrid mass-spring results in higher stiffness and higher mass). Let us continue with the determination of the natural frequency of the coupled mass-spring system, here assuming implicitly an oscillation with an angular rotation of $\alpha = \omega t$ (i.e., the derivatives introduce the coefficient ω into the various terms, but for convenience, the nomenclature remains angle α within the sine and cosine functions). Now, the gamma derivative of the extended sine function $x = \sin^*(\alpha,\gamma)$ was shown previously [in Eq.(5), but also accounting for $\alpha = \omega t$] to be

$$\frac{d_\gamma x}{d_\gamma t} = \frac{d_\gamma}{d_\gamma t}\sin^*(\alpha,\gamma) = \omega\cos^*(\alpha,\gamma) \tag{52}$$

Replacing this in Eq.(48) and expanding gives

$$\sin^*(\alpha,\gamma)\frac{d}{dt}[\sin^*(\alpha,\gamma)] - \left\{\frac{d}{dt}[\sin^*(\alpha,\gamma)\cos^*(\alpha,\gamma)] + \frac{d}{dt}[\sin^*(\alpha,\gamma)]\cos^*(\alpha,\gamma)\right\}\cos(\gamma) +$$
$$+\omega\cos^*(\alpha,\gamma)\frac{d}{dt}[\omega\cos^*(\alpha,\gamma)] = 0 \tag{53}$$

The first term expands further including $\alpha = \omega t$ as

$$\sin^*(\alpha,\gamma)\frac{d}{dt}[\sin^*(\alpha,\gamma)] = \frac{\sin(\alpha)}{\sin(\gamma)}\left[\frac{\omega\cos(\alpha)}{\sin(\gamma)}\right] = \frac{\omega}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha) \tag{54}$$

And the last term expands similarly further as

$$\omega\cos^*(\alpha,\gamma)\frac{d}{dt}[\omega\cos^*(\alpha,\gamma)] = \omega\Big\{\cos(\alpha)+\frac{\cos(\gamma)}{\sin(\gamma)}\sin(\alpha)\Big\}\omega^2\Big[-\sin(\alpha)+\frac{\cos(\gamma)}{\sin(\gamma)}\cos(\alpha)\Big] \tag{55}$$

The advantage here is that the algebra that follows draws parallelism to the one executed previously during the mathematical proof of the differentiated Law of Cosines (Theorem 1). Hence, replacing the terms Eq.(54-55) in Eq.(53) is an exercise already done previously, and to save time and work, we can draw a parallelism and start already at Eq.(36) [repeated below for convenience]

$$\frac{1}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha)-\Big[-\frac{\sin^2(\alpha)}{\sin(\gamma)}+\frac{\cos(\gamma)}{\sin^2(\gamma)}\cos(\alpha)\sin(\alpha)+\frac{\cos^2(\alpha)}{\sin(\gamma)}+\frac{\cos(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha)\Big]\cos(\gamma)- \tag{56}$$

$$-\cos(\alpha)\sin(\alpha)+\frac{\cos(\gamma)}{\sin(\gamma)}\Big\{\cos^2(\alpha)-\sin^2(\alpha)\Big\}+\frac{\cos^2(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha) \tag{57}$$

whilst remembering to multiply each term by the respective mass m and stiffness k coefficients, as done previsouly when passing from Eq.(49) to Eq.(50), results in

$$\frac{\omega k}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha)-\omega^2\sqrt{mk}\Big[-\frac{\sin^2(\alpha)}{\sin(\gamma)}+\frac{\cos(\gamma)}{\sin^2(\gamma)}\cos(\alpha)\sin(\alpha)+\frac{\cos^2(\alpha)}{\sin(\gamma)}+\frac{\cos(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha)\Big]\cos(\gamma)-$$
$$-\omega^3 m\cos(\alpha)\sin(\alpha)+\omega^3 m\frac{\cos(\gamma)}{\sin(\gamma)}\Big\{\cos^2(\alpha)-\sin^2(\alpha)\Big\}+\omega^3 m\frac{\cos^2(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha)=0 \tag{58}$$

This expands further to

$$\frac{\omega k}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha)+\omega^2\sqrt{mk}\frac{\cos(\gamma)}{\sin(\gamma)}\sin^2(\alpha)-2\omega^2\sqrt{mk}\frac{\cos^2(\gamma)}{\sin^2(\gamma)}\cos(\alpha)\sin(\alpha)-\omega^2\sqrt{mk}\frac{\cos(\gamma)}{\sin(\gamma)}\cos^2(\alpha)-$$
$$-\omega^3 m\cos(\alpha)\sin(\alpha)+\omega^3 m\frac{\cos(\gamma)}{\sin(\gamma)}\Big\{\cos^2(\alpha)-\sin^2(\alpha)\Big\}+\omega^3 m\frac{\cos^2(\gamma)}{\sin^2(\gamma)}\sin(\alpha)\cos(\alpha)=0 \tag{59}$$

As before, grouping terms for $\sin(\alpha)\cos(\alpha)$ and for $\cos^2(\alpha)-\sin^2(\alpha)$ gives

$$\Big(\frac{\omega k}{\sin^2(\gamma)}-2\omega^2\sqrt{mk}\frac{\cos^2(\gamma)}{\sin^2(\gamma)}-\omega^3 m+\omega^3 m\frac{\cos^2(\gamma)}{\sin^2(\gamma)}\Big)\sin(\alpha)\cos(\alpha)+\Big(-\omega^2\sqrt{mk}\frac{\cos(\gamma)}{\sin(\gamma)}+\omega^3 m\frac{\cos(\gamma)}{\sin(\gamma)}\Big)\Big\{\cos^2(\alpha)-\sin^2(\alpha)\Big\}=0 \tag{60}$$

This leads to two interconnected non-trivial possibilities that must be satisfied at the same time: firstly, the terms multiplying $\cos^2(\alpha)-\sin^2(\alpha)$ must be zero

$$-\omega^2\sqrt{mk}\frac{\cos(\gamma)}{\sin(\gamma)}+\omega^3 m\frac{\cos(\gamma)}{\sin(\gamma)}=0 \qquad \Rightarrow \qquad \omega=\sqrt{\frac{k}{m}} \tag{61}$$

the end result gives the same natural frequency to the couple mass-spring system than that commonly known for the classical uncoupled version. The reason for this is discussed at the end of this algebraic process. Secondly, the other set of terms multiplying $\sin(\alpha)\cos(\alpha)$ must also be zero

$$\frac{\omega k}{\sin^2(\gamma)}-2\omega^2\sqrt{mk}\frac{\cos^2(\gamma)}{\sin^2(\gamma)}-\omega^3 m+\omega^3 m\frac{\cos^2(\gamma)}{\sin^2(\gamma)}=0 \tag{62}$$

It simplifies to

$$k - 2\omega\sqrt{mk}\cos^2(\gamma) - \omega^2 m \sin^2(\gamma) + \omega^2 m \cos^2(\gamma) = 0 \tag{63}$$

And can be re-arranges as

$$\omega^2 m[\cos^2(\gamma) - \sin^2(\gamma)] - 2\omega\sqrt{mk}\cos^2(\gamma) + k = 0 \tag{64}$$

Diving all terms by mass m gives

$$\omega^2[\cos^2(\gamma) - \sin^2(\gamma)] - 2\omega\sqrt{\frac{k}{m}}\cos^2(\gamma) + \frac{k}{m} = 0 \tag{65}$$

This is a quadratic equation of the form $A\omega^2 + B\omega + C = 0$, whose roots have the general form

$$\omega = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{66}$$

with

$$A = \cos^2(\gamma) - \sin^2(\gamma) \quad ; \quad B = -2\sqrt{\frac{k}{m}}\cos^2(\gamma) \quad ; \quad C = \frac{k}{m} \tag{67}$$

Replacing these above gives

$$\omega = \frac{2\sqrt{\frac{k}{m}}\cos^2(\gamma) \pm \sqrt{\Big[-2\sqrt{\frac{k}{m}}\cos^2(\gamma)]^2 - 4\Big[\cos^2(\gamma) - \sin^2(\gamma)\Big]\Big[\frac{k}{m}\Big]}}{2\Big[\cos^2(\gamma) - \sin^2(\gamma)\Big]} \tag{68}$$

Some algebra will be required to simplify this expression. Start by grouping the $\sqrt{k/m}$ term, which results in

$$\omega = \sqrt{\frac{k}{m}}\left\{\frac{2\cos^2(\gamma) \pm \sqrt{\Big[-2\cos^2(\gamma)\Big]^2 - 4\Big[\cos^2(\gamma) - \sin^2(\gamma)\Big]}}{2\Big[\cos^2(\gamma) - \sin^2(\gamma)\Big]}\right\} \tag{69}$$

Cancelling the factor ×2 on denominator and numerator, simplify this to

$$\omega = \sqrt{\frac{k}{m}}\left\{\frac{\cos^2(\gamma) \pm \sqrt{\cos^4(\gamma) - \Big[\cos^2(\gamma) - \sin^2(\gamma)\Big]}}{\cos^2(\gamma) - \sin^2(\gamma)}\right\} \tag{70}$$

The coefficient between braces is not trivially simplified algebraically, hence a different approach is used. Assume that the ratio between braces actually reduces to unity. This reformulates the ratio as follows

$$\cos^2(\gamma) - \sin^2(\gamma) = \cos^2(\gamma) \pm \sqrt{\cos^4(\gamma) - \Big[\cos^2(\gamma) - \sin^2(\gamma)\Big]} \tag{71}$$

Cancelling the $\cos^2(\gamma)$ on either side, while subsequentially squaring the rest (on either side)

$$\sin^4(\gamma) = \cos^4(\gamma) - \Big[\cos^2(\gamma) - \sin^2(\gamma)\Big] \tag{72}$$

That re-arranges to

$$\sin^4(\gamma) - \cos^4(\gamma) = -\cos^2(\gamma) + \sin^2(\gamma) \tag{73}$$

Here, the left hand side can be expressed as a product of terms as follows

$$\Big[\sin^2(\gamma)+\cos^2(\gamma)\Big]\Big[\sin^2(\gamma)-\cos^2(\gamma)\Big]=-\cos^2(\gamma)+\sin^2(\gamma) \tag{74}$$

Cancelling $\sin^2(\gamma)-\cos^2(\gamma)$ on both side concludes in the true equality

$$\sin^2(\gamma)+\cos^2(\gamma)=1 \tag{75}$$

This confirms that, the fractional term in Eq.(70) between braces reduces to unity, and the equation itself reduces to

$$\omega=\sqrt{\frac{k}{m}} \tag{76}$$

which is the same as the previous result in Eq.(61) — i.e., equal to the natural frequency of the classical orthogonal mass-spring system. It is believed that the reason for this is that, even though the coupled mass-spring system has different acceleration and decelleration phases in its period (due to the coupling) than the classical mass-spring system, the net effect in terms of time ellapsed at the end of each cycle remains the same (i.e, same period of oscillation, that translates into the same natural frequency). After all, the total energy in the system remains the same, it is only how it behaves that changes. Whilst proof of this is beyond the scope of this publication, one argument that could justify this effect is the fact that the total energy shifting from potential to kinetic remains the same for both coupled and uncoupled systems — it is just the exchange from one form to the other, that occurs at a different pace. An analogy that may serve to illustrate better how such a couter intuitive effect can take place is the period of a pendulum, and its independent relation to its mass. In essence, the coupling effect on the upgraded mass-spring system is equivalent to the variation of the mass of the pendulum, in that there is no impact in the natural frequency or period of the system. In essence, this is the conclusion from the above algebraic process. Follow-on research will investigate this behaviour further (including finding solutions for these equations, i.e. particular integrals and complementary functions), with the final results being published in a potential future article.

5.3 Main Advantage

Vibration theory teaches us that energy tends to migrate and occupy (in different proportions) the various degrees of freedom or modes of oscillation of a system, wether it be discrete or continuous. Here, a new degree of freedom concerning the coupling of two energy states has been found in classical oscillatory models, leading to a step towards a more realistic modelling and understanding of how vibrational systems behave in reality. Even a real mass-spring oscilatory device cannot achieve an ideal separation between stiffness effects (spring) and inertial effects (mass), as for instance any spring has mass. Moving from the particular case of a system governed by the Pythagoras theorem, to a general case of a system governed by the Law of Cosines, adds another layer of realism into what an actual oscillatory system looks like. This new degree of freedom opens doors into how we design classical mass-spring devices, such as for instance those found in automobiles to smooth our ride and provide better vehicle controlability, or those found in washing machines to control the relative displacement of the rotating drum assembly from the static surrounding casing. Other examples are in better control and attenuation of the vibrational deployment of solar arrays in satellites. Quantum mechanic oscillators are extended versions of the classical vibrational systems, and could benefit from a better understanding on how different states of energy in an electron cloud couple with each other via the present coupled spring-mass system. Overall, with time, it is expected that the present upgrade will invariably replace the classical approach, as a new baseline against which to study more complex vibrational systems.

5.4 Analogy to the LC Electrical Circuit

A system equivalent to the couple mass-spring system in electrical engineering is the LC (inductance-capacitance) circuit, which is used as a foundation (more in particular its resistance including variant, the RLC circuit) to build larger and more complex circuits (like those employed in telecommunications).

$$L\frac{d^2i}{dt^2}+\frac{1}{C}i=0 \qquad \Longleftrightarrow \qquad m\ddot{s}+ks=0 \tag{77}$$

where i is the electrical current, L is the inductance (equivalent to mass m in a mass-spring system, i.e. $L\equiv m$) and C the capacitance (equivalent to the inverse of stiffness k in a mass-spring system, i.e. $1/C\equiv k$). Differentiation occurs with respect to time, and for the sake of convenience appears explicitly. The circuit is composed of a capacitor and an inductor connected in series as a closed-loop circuit, and it is the electrical analogous to the mass and spring mechanical oscillator. Just as an oscillation expresses the shift of energy from inertia (kinetic) to stiffness (potential) [and vice versa] in a mass-spring system, in an LC circuit the shift of energy occurs between induction (equivalent kinetic) to capacitance (equivalent potential). The electrical version of the governing equation [adapted from Eq.(50), but with the terms on either side of the middle one swapped, to match Eq.(77) above] is accordingly as

$$L\left(\frac{d_\gamma i}{d_\gamma t}\right)\left[\frac{d}{dt}\left(\frac{d_\gamma i}{d_\gamma t}\right)\right] - \sqrt{\frac{L}{C}}\left[i\frac{d}{dt}\left(\frac{d_\gamma i}{d_\gamma t}\right) + \frac{di}{dt}\left(\frac{d_\gamma i}{d_\gamma t}\right)\right]\cos(\gamma) + \frac{1}{C}i\frac{di}{dt} = 0 \tag{78}$$

The aforementioned *discrete or hybrid* mass-spring for the upgraded mass-spring oscillatory sytem has an electrical equivalent, or a *discrete or hybrid* capacitor-inductor, that is part of the upgraded LC circuit. As the physical system, this hybrid capacitor-inductor component needs — from a mathematical perspective — to exhibit a content where both capacitance and induction change with its length. While this is beyond the scope of this article, it is worth mentioning that one way to achieve this gradual change in properties is to add small micro incremental capacitors and inductors in both parallel and series to result in a macro hybrid component, exhibiting both characteristics simultaneously. This will be explained in a following article.

5.5 Adding Resistance

Mechanical resistance from a damper can be added to the mass-spring system, and electrical resistance can be added to the LC circuit. The above part of applied mathematics is important to hold into perspective, as the extended sine and cosine with the Law of Cosines will replace and upgrade these physical and electrical equations, opening room for a broader scope of application of this fundamental theory of oscillation, including further practical technological advantages. This is to be added later, mechanically to form the upgraded mass-spring-damper system, and electrically to form the upgraded RLC (Resistance-Inductance-Capacitance) circuit, both being topics for future research and follow-on publications.

4. Conclusion

Gamma derivatives are the generalization to non-orthogonal cases of classical derivatives applied within trigonometry. Classical orthogonal derivatives $d/d\alpha$ of fundamental functions such as $sin(\alpha)$ and $cos(\alpha)$ follow exclusively (in polar coordinates) the dynamics of right-angled triangles. In this article, the expression of such derivatives is expanded to encompass non-orthogonal cases (where the axes between systems are not 90 degrees), that inherently govern (in polar coordinates) the dynamics of scalene triangles. Just as classical derivatives represent 90 degree phase shifts in sine and cosine functions, the gamma derivative represents γ degree phase shifts in the already published *extended sine and cosine functions* (that measure the non-orthogonal normalized projected sides of a scalene triangle). Just as normal derivatives are employed mathematically to govern equations of physical systems (such as the mass-spring system and electrical LC circuit, when converting a balance of energy to a balance of forces), and thus following the relation defined by the Pythagoras theorem, so will the *gamma derivative* become a more generic tool to derive governing equations for more generalized versions of the same systems following the relation of the Law of Cosines. In this line, it is proven that just as extended sine and cosine functions satisfy the Law of Cosines (presented in a prior publication), so do the derivatives of these functions also satisfy the differentiated version of this equation — it is shown that this is an important mathematical tool, critical in deriving the upgraded versions of the aforementioned governing equations of physical systems. During this investigation, an important point that is defined is the dissociation of the Pythagorean (geometrical or spatial) domain from the temporal domain. Deriving the governing equation of the mass-spring system implies classical orthogonal derivatives on both domains. When the geometric domain is generalized to be governed by the Law of Cosines, the gamma derivative is applicable — which is (in most cases) different from the orthogonal derivative commonly applied in the temporal domain. Hence, the derivation process of the governing equation is altered accordingly, resulting in a more generalized expression. The classical mass-spring system — fundamental to physics and mechanical engineering — is upgraded to encompass coupling effects, which emerge naturally when moving from the particular case of a Pythagoras theorem driven physical system to those governed by the Law of Cosines. This upgraded mass-spring system sees a coupling of stiffness with inertial effects, and a new *hybrid* spring with discretized mass along its length, replaces the traditional infinitely stiff mass to account for the aforementioned coupling effect. The LC (Inductance-Capacitance) circuit inherents analogous similarity to the physical (and hence mathematical) dynamics of the mass-spring system (i.e., the governing equations are equivalent), hence the findings of this article also extend to the field of electrical engineering. Building on this research, the following future step will be to investigate the addition of resistance to the system (an aspect that affects both the mechanical and electrical expressions), which will upgrade the aforementioned systems into their more commonly used variants — the mass-spring-damper system in physics and the RLC (Resitance-Inductance-Capacitance) system in electrical engineering. Moreover, given its oscillatory nature, the findings on this article are also expected to have repercussions in the field of acoustics. The ultimate advantage of this work emerges from the realization that in real applications of physics and engineering, there is no such thing as an ideal right angle. In that sense, the presented upgraded theory in oscillations encompasses this non-orthogonality as a reality within vibration problems — a rule rather than an exception in our technological world.

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