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GAMMA PRIOR DISTRIBUTION SELECTION FOR BAYESIAN ANALYSIS OF FAILURE RATE AND RELIABILITY

by

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ABSTRACT

We assume that the phenomenon under study is such that the time-to-failure may be modeled by an exponential distribution with failure rate λ. For Bayesian analyses of the assumed model, the family of gamma distributions provides conjugate prior models for λ. Thus, an experimenter needs to select a particular gamma model to conduct a Bayesian reliability analysis. The purpose of this report is to present a methodology that can be used to translate engineering information, experience, and judgment into a choice of a gamma prior distribution.

The proposed methodology assumes that the practicing engineer can provide percentile data relating to either the failure rate or the reliability of the phenomenon being investigated. For example, the methodology will select the gamma prior distribution which conveys an engineer's belief that the failure rate λ simultaneously satisfies the probability statements, $P(\lambda<1.0x10^{-3}) = 0.50$ and $P(\lambda<1.0x10^{-5}) = 0.05$. That is, we use two percentiles provided by an engineer to determine a gamma prior model which agrees with the specified percentiles. For those engineers who prefer to specify reliability percentiles rather than the failure rate percentiles illustrated above, we can use the induced negative-log gamma prior distribution which satisfies the probability statements, $P(R(t_0)<0.99) = 0.50$ and $P(R(t_0)<0.99999) = 0.95$ for some operating time t_0. Also, the report includes graphs for selected percentiles which assist an engineer in applying the procedure.

I. INTRODUCTION

A widely used assumption in reliability analyses is that the time-to-failure variable T is exponentially distributed with failure rate λ. That is, the time-to-failure variable T has density function

$$f(t|\lambda) = \begin{cases} \lambda e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

When conducting a Bayesian reliability analysis, conjugate prior models [5] for the failure rate λ are given by the family of gamma density functions

$$h(\lambda) = \begin{cases} \dfrac{\lambda^{\alpha-1} e^{-\lambda/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, & \lambda > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

A particular Bayesian analysis requires that values be assigned to the prior parameters, α and β. The purpose of this report is to present two methods by which engineering experiences, judgments, and beliefs can be used to assign values to α and β. One method relies on expertise and knowledge concerning the failure rate λ while the other technique assumes information about the reliability, $R(t) = e^{-\lambda t}$. Both methods require an engineer to provide two percentile values which are used to solve a pair of simultaneous equations to determine values for α and β. Since the solutions do not exist in closed form, we present graphs which aid the engineer in applying the procedure.

The subsequent development is divided into three sections and three appendices. Section II presents the development based on failure rate percentiles. The technique using reliability percentiles is given in Section III. Section IV provides discussion of the procedures which considers some of the consequences that follow from the assumption of a gamma prior model. Appendix A provides a table and graphs for applying the results in Section II. Tables and graphs for application of the results in Section III are given in Appendix B. Justification for mathematical results given in Section IV is provided in Appendix C.

II. TECHNIQUE FOR FAILURE RATE PERCENTILES

In this section, we suppose that an engineer can best summarize his experiences, judgments, and beliefs about the performance of an item by making statements about the failure rate λ. The type of information desired is called a percentile. The pth percentile, say λ_p, is that value of λ such that the probability that λ is less then λ_p is p. In symbols we use $h(\lambda)$ in Eq. (1) to write

$$P(\lambda < \lambda_p) = \int_0^{\lambda_p} h(\lambda)d\lambda = p. \tag{2}$$

In practice there exists a set of values for (α,β) which satisfies Eq. (2) with any given value of p. Therefore, the method we propose requires that the engineer provide two distinct percentiles that generate a pair of simultaneous equations. Explicitly we ask that the engineer provide two percentiles of λ, say λ_1 and λ_2, such that $\lambda_1 < \lambda_2$ and

$$\begin{aligned} P(\lambda < \lambda_1) &= p_1 , \\ P(\lambda < \lambda_2) &= p_2 . \end{aligned} \tag{3}$$

Clearly, the specifications of λ_1 and λ_2 are made with reference to the probabilities, p_1 and p_2, where $p_1 < p_2$. The simultaneous solution of Eq. (3) will select the pair of values for (α,β) which determine the gamma prior that summarizes the engineer's information.

A specific outline of the methodology is as follows:

Step 1: The engineer specifies the values for λ_1, λ_2, p_1, and p_2 which best represent the totality of his experiences, judgments and beliefs about the failure rate λ. These values provide Eq. (3).

Step 2: A search procedure is used to determine α and β for Eq. (1) which simultaneously satisfy the conditions of Step 1.

The table and graphs in Appendix A present values of α and β for selected choices of λ_0 and p_0 where

$$P(\lambda < \lambda_0) = p_0 \; . \qquad (4)$$

By overlaying transparencies of the two graphs which present the specific choices of λ_1, λ_2, p_1, and p_2 of interest, we can determine graphically the desired values of α and β.

Example: Suppose we are studying the reliability of an item for which the available engineering information indicates that the failure rate λ is such that

$$P(\lambda < 1.0 \times 10^{-5}) = 0.05 \text{ and}$$

$$P(\lambda < 1.0 \times 10^{-3}) = 0.50 \; .$$

By overlaying transparencies of the graphs in Figs. A4 and A7 we determine that $\alpha = 0.505$ and $\beta = 0.004$ in Eq. (1) provide a gamma prior distribution which possesses the percentile properties given by the stated conditions. The selected prior is described by

$$h(\lambda) = \frac{\lambda^{-0.495} \, e^{-\lambda/0.004}}{(0.004)^{0.505} \, \Gamma(0.505)} \; , \; \lambda > 0 \; ,$$

and is graphically represented in Fig. 1.

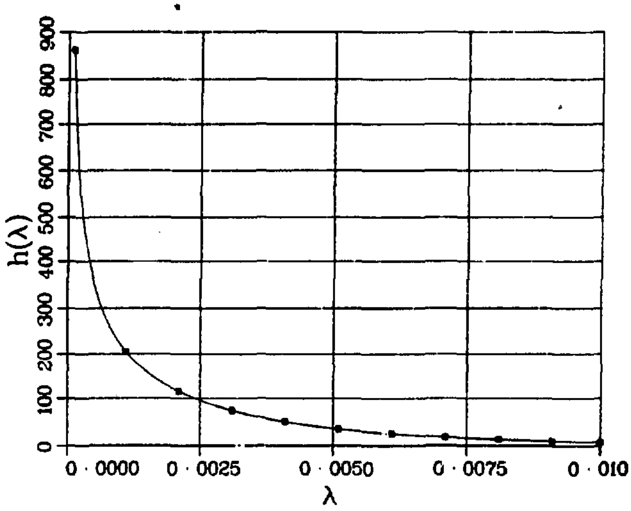

Fig. 1. A gamma prior distribution with $\alpha = 0.505$ and $\beta = 0.004$.

III. TECHNIQUE FOR RELIABILITY PERCENTILES

An alternative approach to the failure rate method in the preceding section is to use percentiles of the reliability. This procedure would be preferable to the previous method when the engineer providing the percentile information is more knowledgeable about reliability properties than failure rate characteristics of the item under study.

To begin the development we transform the gamma density for λ (Eq. (1)) into a negative-log gamma distribution for the reliability, $R = R(t) = e^{-\lambda t}$. The resulting density is

$$p(r) = \begin{cases} \dfrac{(-\ln r)^{\alpha-1}\, r^{\frac{1}{t\beta}-1}}{(t\beta)^{\alpha}\,\Gamma(\alpha)}, & 0 < r < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

The density in Eq. (5) has been used by Springer and Thompson (1965, 1967), Mann (1970), and Mastran and Singpurwalla (1974). Locks (1973) provides a discussion of the negative-log gamma distribution. Thus, the prior distribution induced on the reliability by assuming a conjugate gamma prior for the failure rate occurs frequently in the literature. Our purpose here is to present a methodology that allows a reliability engineer to use available information as a tool for selecting values of α and β to be used in a Bayesian reliability analysis.

From Eq. (5), it is clear that there is a different prior on R for each choice of time t. Yet for analysis, it is convenient to reparameterize the density in Eq. (5) by using $\gamma = t\beta$. With that change, the density of interest becomes

$$k(r) = \begin{cases} \dfrac{(-\ln r)^{\alpha-1}\, r^{\frac{1}{\gamma}-1}}{\gamma^{\alpha}\,\Gamma(\alpha)}, & 0 < r < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

The pth percentile of the reliability at time t is $R_p = R_p(t)$. In symbols we write

$$P(R < R_p) = \int_0^{R_p} k(r)\, dr = p \; .$$

Once the engineer supplies two percentiles with respect to a reference time, say t_0, we can set up two simultaneous equations whose solution provides values for α and γ. The desired values of (α,β) are then given by α and $\beta = \gamma/t_0$. An outline of the method is as follows:

Step 1: The engineer provides a reference time t_0.

Step 2: With respect to t_0, the engineer specifies two percentiles R_1 and R_2 for p_1 and p_2, respectively, which best summarizes his experiences, judgments, and beliefs about the reliability and are such that

$$P(R < R_1) = p_1 \text{ and } P(R < R_2) = p_2 \; .$$

Step 3: A search procedure is used to determine α and γ that satisfy the probability statements in Step 2 for the density in Eq. (6).

Step 4: Solve for $\beta = \gamma/t_0$. The values so determined for (α,β) are the selected parameters for either the gamma prior on the failure rate (Eq. (1)) or the negative-log gamma prior on the reliability (Eq. (5)).

Appendix B presents tables and graphs giving values of α and γ which satisfy

$$P(R < R_0) = p_0 \qquad (7)$$

for selected values of R_0 and p_0. By overlaying transparencies of the two graphs which contain the values of $R_1(t_0)$, $R_2(t_0)$, p_1, and p_2 used in Step 2 of the procedure, we determine the values of α and γ. The value of β is given by γ/t_0 where t_0 is the reference time provided by the engineer in Step 1.

Example: Suppose a reliability engineer believes that the reliability of a motor is such that for t_0 = 100 hrs:

$$P(R(100) < 0.99) = 0.50 \; ,$$

$$P(R(100) < 0.99999) = 0.95 \; .$$

By overlaying transparencies of the graphs in Figs. B7 and B10, we determine that $\alpha = 0.35$ and $\gamma = 0.10$. Therefore, $\alpha = 0.35$ and $\beta = 0.001$ in Eq. (5) yield the selected negative-log gamma prior

$$p(r) = \begin{cases} \dfrac{(-\ln r)^{-0.65} \, r^{\frac{1000}{t} - 1}}{\left(\frac{t}{1000}\right)^{0.35} \Gamma(0.35)} , & 0 < r < 1 , \\ 0, & \text{otherwise} . \end{cases}$$

Recall that we get a different prior for each choice of mission time t. A graph of the selected prior is given for t = 50, 100, 500, 4000 in Fig. 2.

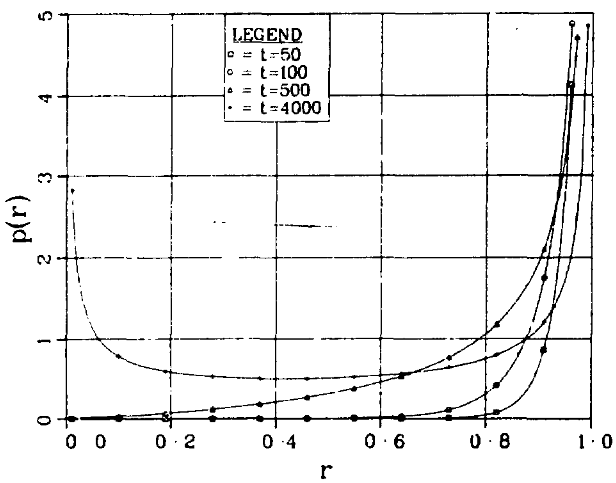

Figure 2. A negative-log gamma prior for $\alpha = 0.35$, $\beta = 0.001$, and t = 50,100, 500, 4000.

IV. DISCUSSION

In this section we discuss some of the mathematical properties of the prior distributions selected by the percentile methods of Sections II and III. Mathematical justification of the properties is provided in the appendices.

4.1 The Conjugate Gamma Prior on λ

By differentiating $h(\lambda)$ in Eq. (2) with respect to λ [2], we find that $h(\lambda)$ is a unimodal function of λ with a mode at $\lambda = \beta(\alpha-1)$ when $\alpha > 1$. For $\alpha = 1$, we have an exponential distribution with parameter β. For $0 < \alpha < 1$, $h(\lambda)$ is either L-shaped or decreasing. These properties are summarized in Table I.

TABLE I

SHAPE PROPERTIES OF A CONJUGATE GAMMA PRIOR ON λ

$\beta > 0$	L-shaped or decreasing	Exponential	Unimodal with mode at $\lambda = \beta(\alpha-1)$
	$0 < \alpha < 1$	$\alpha = 1$	$\alpha > 1$

Since error bounds on engineering estimates of failure rates are frequently large, the useful gamma prior distributions are often those with $0 < \alpha < 1$. That is, the selected gamma prior models have density functions which are heavily concentrated on values of λ near zero. With $\alpha = 0.505$ and $\beta = 0.004$, the gamma prior in Fig. 1 exhibits a concentration of density near zero.

The precision on some incomplete gamma subroutines deteriorates as α approaches zero. Then precise calculation of the necessary gamma percentiles for small values of α becomes difficult. To address that problem we can use the theorem of Appendix C.1. Let λ_p denote the pth percentile of a gamma prior with parameters α and β. Then

$$p = \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)} \int_0^{\lambda_p} \lambda^{\alpha-1} e^{-\lambda/\beta}\, d\lambda = \frac{1}{\Gamma(\alpha)} \int_0^{\frac{\lambda_p}{\beta}} x^{\alpha-1} e^{-x}\, dx \, .$$

From the theorem, $\lambda_p/\beta \sim p^{\frac{1}{\alpha}}$, as defined in Appendix C.1. Thus, $\lambda_p \sim \beta p^{\frac{1}{\alpha}}$ for small values of α. This result can be used to extend the table in Appendix A to smaller values of α.

4.2 The Negative-Log Gamma Prior on R

By differentiating k(r) in Eq. (6) with respect to r (see Appendix C.2), we find that the possible shapes of k(r) can be summarized as in Table II.

TABLE II

SHAPE PROPERTIES OF A NEGATIVE-LOG GAMMA PRIOR ON R

	$0 < \alpha < 1$	$\alpha = 1$	$1 < \alpha$
$1 < \gamma$	U-shaped with antimode at $r = \exp\left[\frac{\gamma(\alpha-1)}{\gamma-1}\right]$	L-shaped or decreasing	L-shaped or decreasing
$\gamma = 1$	J-shaped or increasing	Uniform	L-shaped or decreasing
$0 < \gamma < 1$	J-shaped or increasing	J-shaped or increasing	Unimodal with mode at $r = \exp\left[\frac{\gamma(\alpha-1)}{\gamma-1}\right]$

Our interest, as stated in Section III, is to select a pair of values (α,β) for the negative-log gamma prior in Eq. (5). However, since $\gamma = t\beta$, it follows that γ increases as t increases for a fixed value of β. Thus, we have a different prior on the reliability for each choice of t. That property was illustrated in Fig. 2 for $\alpha = 0.35$, $\beta = 0.001$, and t = 50, 100, 500, 4000. The portion of Table II frequently encountered in reliability analyses is for $0 < \alpha < 1$. Now for $0 < \alpha < 1$ and t small enough so that $0 < \gamma = t\beta \leq 1$, the prior distribution p(r) is an increasing function of r. But, when t is such that $\gamma = t\beta > 1$ and $0 < \alpha < 1$, p(r) is U-shaped with its minimum (antimode) at $r = \exp\left[\frac{\gamma(\alpha-1)}{\gamma-1}\right]$. Thus, the negative-log gamma prior on the reliability is concentrated near one in early life (small t). As t increases to large values (late life), the distribution becomes U-shaped. That is, values of r near either zero or one are more likely to occur than other values. Thus, in "old-age" the prior predicts the item will be either quite reliable or quite unreliable. The density function in Fig. 2 for t = 4000 illustrates that point.

As for the gamma prior on λ, precision problems may be encountered when computing the percentiles of Eq. (6) for small α values. The theorem in Appendix C.1 again assists us by providing good approximations to the reliability percentiles as follows. Let R_p be the pth percentile in Eq. (6) for parameters α and γ. Then

$$p = \int_0^{R_p} k(r)\, dr = \frac{1}{\Gamma(\alpha)} \int_{\frac{-\ln R_p}{\gamma}}^{\infty} x^{\alpha-1} e^{-x}\, dx$$

or

$$1-p = \frac{1}{\Gamma(\alpha)} \int_0^{\frac{-\ln R_p}{\gamma}} x^{\alpha-1} e^{-x}\, dx \ .$$

From the theorem in Appendix C.1

$$\frac{-\ln R_p}{\gamma} \sim (1-p)^{1/\alpha} \ .$$

To apply the result we act as if the "$\sim$" is an equality even though it is only an asymptotic result.

To examine the performance of the approximation we refer to Table B10 in Appendix B. Let $\alpha = 0.05$, $p = 0.95$, and $\gamma = 1.795 \times 10^{23}$. Then the approximation gives $R_0 = 0.9983$ in place of 0.9990 in the table. Similarly, for $R_0 = 0.999$, $\alpha = 0.05$, and $p = 0.95$, we find $\gamma = 1.049 \times 10^{23}$ in place of 1.795×10^{23} in the table. Thus the approximation is of reasonable quality even for moderately large values of α.

ACKNOWLEDGMENTS

The authors acknowledge the assistance of David Kahaner of Los Alamos Scientific Laboratory in developing the theorem in Appendix C.1. The necessary computation of gamma percentiles was facilitated by use of codes developed by Amos and Daniel (1972) of Sandia Laboratories, Albuquerque.

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APPENDIX A

TABLE AND GRAPHS OF α AND β FOR GAMMA PRIOR DISTRIBUTIONS

Table A1 gives β values which satisfy Eq. (4) for p = 0.005, 0.01, 0.025, 0.05, 0.10, 0.25, 0.50, 0.75, 0.90, 0.95, 0.975, 0.99, 0.995, a selected set of α values, and $\lambda_0 = 1.0 \times 10^{-6}$. We use the fact that β is a scale parameter to obtain values of β which correspond to failure rate percentiles different from $\lambda_0 = 1.0 \times 10^{-6}$. For given values of p_0 and α in Eq. (4), the ratio λ_0/β is constant. Therefore, in Table A1 for α and p_0, multiplication of the β value by $(\lambda_s/1.0 \times 10^{-6})$ yields the β value corresponding to a p_0th percentile of λ_s.

Example: Let $p_0 = 0.05$ and $\alpha = 0.25$. Then

$$\beta = 2.3705 \times 10^{-1} \text{ for } \lambda_0 = 1.0 \times 10^{-6} .$$

Thus, for $\lambda_s = 1.0 \times 10^{-9}$

$$\beta = (1.0 \times 10^{-9}/1.0 \times 10^{-6})(2.3705 \times 10^{-1}) = 2.3705 \times 10^{-4} .$$

That is, the 5th percentile of a gamma distribution with $\alpha = 0.25$ and $\beta = 2.3705 \times 10^{-1}$ is 1.0×10^{-6} while the 5th percentile of a gamma distribution with $\alpha = 0.25$ and $\beta = 2.3705 \times 10^{-4}$ is 1.0×10^{-9}.

In Figs. A1-A13 we have graphed α and β for selected values of λ_0. The results are derived from Table A1 as illustrated by the above example. To provide better resolution in the graphs, a logarithmic scale is used for β. The notation in Table A1 and Figs. A1-A13 is defined as follows: nEm = n x 10^m.

λ_0=1.E-06

α/p_0	0.005	0.01	0.025	0.05	0.10	0.25	0.50
0.05	1.7941E+40	1.7110E+34	1.8812E+26	1.7941E+20	1.7110E+14	1.8812E+06	1.7941E+00
0.10	1.6861E+17	1.6466E+14	1.7266E+10	1.6861E+07	1.6466E+04	1.7266E+00	1.6852E-03
0.15	3.4743E+09	3.4198E+07	7.6043E+04	7.4851E+02	7.3676E+00	1.6342E-02	1.6039E-04
0.20	4.9039E+05	1.5325E+04	1.5692E+02	4.9039E+00	1.5324E-01	1.5684E-03	4.8201E-05
0.25	2.3705E+03	1.4815E+02	3.7928E+00	2.3705E-01	1.4815E-02	3.7848E-04	2.2897E-05
0.30	6.7096E+01	6.6568E+00	3.1390E-01	3.1143E-02	3.0890E-03	1.4493E-04	1.3674E-05
0.35	5.2164E+00	7.1991E-01	5.2517E-02	7.2474E-03	9.9958E-04	7.2231E-05	9.3136E-06
0.40	7.6286E-01	1.3485E-01	1.3646E-02	2.4117E-03	4.2574E-04	4.2436E-05	6.8928E-06
0.45	1.7005E-01	3.6443E-02	4.7561E-03	1.0187E-03	2.1779E-04	2.7822E-05	5.3958E-06
0.50	5.0929E-02	1.2732E-02	2.0365E-03	5.0863E-04	1.2666E-04	1.9698E-05	4.3962E-06
0.55	1.8910E-02	5.3621E-03	1.0129E-03	2.8678E-04	8.0862E-05	1.4752E-05	3.6895E-06
0.60	8.2511E-03	2.5985E-03	5.6379E-04	1.7715E-04	5.5370E-05	1.1524E-05	3.1675E-06
0.65	4.0765E-03	1.4030E-03	3.4217E-04	1.1740E-04	4.0014E-05	9.3027E-06	2.7684E-06
0.70	2.2208E-03	8.2466E-04	2.2231E-04	8.2215E-05	3.0170E-05	7.7062E-06	2.4544E-06
0.75	1.3083E-03	5.1887E-04	1.5252E-04	6.0182E-05	2.3535E-05	6.5186E-06	2.2018E-06
0.80	8.2145E-04	3.4506E-04	1.0939E-04	4.5667E-05	1.8874E-05	5.6093E-06	1.9946E-06
0.85	5.4355E-04	2.4018E-04	8.1371E-05	3.5698E-05	1.5487E-05	4.8962E-06	1.8218E-06
0.90	3.7575E-04	1.7367E-04	6.2405E-05	2.8605E-05	1.2954E-05	4.3254E-06	1.6758E-06
0.95	2.6953E-04	1.2967E-04	4.9107E-05	2.3405E-05	1.1012E-05	3.8605E-06	1.5508E-06
1.00	1.9950E-04	9.9499E-05	3.9498E-05	1.9496E-05	9.4912E-06	3.4761E-06	1.4427E-06
1.10	1.1807E-04	6.2654E-05	2.6967E-05	1.4134E-05	7.2964E-06	2.8808E-06	1.2654E-06
1.20	7.5829E-05	4.2357E-05	1.9493E-05	1.0736E-05	5.8174E-06	2.4448E-06	1.1262E-06
1.30	5.1865E-05	3.0250E-05	1.4726E-05	8.4552E-06	4.7720E-06	2.1140E-06	1.0142E-06
1.40	3.7280E-05	2.2558E-05	1.1519E-05	6.8528E-06	4.0042E-06	1.8557E-06	9.2219E-07
1.50	2.7886E-05	1.7417E-05	9.2680E-06	5.6843E-06	3.4225E-06	1.6494E-06	8.4532E-07
1.60	2.1548E-05	1.3834E-05	7.6302E-06	4.8058E-06	2.9701E-06	1.4814E-06	7.8016E-07
1.70	1.7104E-05	1.1250E-05	6.4030E-06	4.1280E-06	2.6105E-06	1.3422E-06	7.2424E-07
1.80	1.3895E-05	9.3305E-06	5.4602E-06	3.5937E-06	2.3195E-06	1.2252E-06	6.7574E-07
1.90	1.1489E-05	7.8687E-06	4.7202E-06	3.1644E-06	2.0801E-06	1.1258E-06	6.3329E-07
2.00	9.6623E-06	6.7315E-06	4.1287E-06	2.8140E-06	1.8804E-06	1.0403E-06	5.9582E-07

α/p_0	0.75	0.90	0.95	0.975	0.99	0.995
0.05	5.3858E-04	1.3103E-05	3.7604E-06	1.7631E-06	9.1943E-07	6.4741E-07
0.10	2.8324E-05	3.7572E-06	1.7228E-06	1.0226E-06	6.2953E-07	4.7743E-07
0.15	9.9143E-06	2.2482E-06	1.2113E-06	7.8584E-07	5.1798E-07	4.0590E-07
0.20	5.5910E-06	1.6532E-06	9.7038E-07	6.6177E-07	4.5407E-07	3.6302E-07
0.25	3.8369E-06	1.3326E-06	8.2637E-07	5.8254E-07	4.1087E-07	3.3313E-07
0.30	2.9163E-06	1.1302E-06	7.2868E-07	5.2624E-07	3.7687E-07	3.1050E-07
0.35	2.3560E-06	9.8942E-07	6.5705E-07	4.8348E-07	3.5378E-07	2.9245E-07
0.40	1.9810E-06	8.8508E-07	6.0170E-07	4.4950E-07	3.3332E-07	2.7750E-07
0.45	1.7127E-06	8.0415E-07	5.5729E-07	4.2159E-07	3.1615E-07	2.6482E-07
0.50	1.5114E-06	7.3922E-07	5.2064E-07	3.9810E-07	3.0144E-07	2.5333E-07
0.55	1.3545E-06	6.8576E-07	4.8972E-07	3.7795E-07	2.8861E-07	2.4416E-07
0.60	1.2290E-06	6.4082E-07	4.6318E-07	3.6039E-07	2.7727E-07	2.3554E-07
0.65	1.1261E-06	6.0240E-07	4.4007E-07	3.4490E-07	2.6714E-07	2.2780E-07
0.70	1.0400E-06	5.6911E-07	4.1971E-07	3.3110E-07	2.5802E-07	2.2076E-07
0.75	9.6703E-07	5.3992E-07	4.0159E-07	3.1868E-07	2.4972E-07	2.1434E-07
0.80	9.0426E-07	5.1407E-07	3.8534E-07	3.0744E-07	2.4214E-07	2.0843E-07
0.85	8.4966E-07	4.9099E-07	3.7064E-07	2.9718E-07	2.3516E-07	2.0297E-07
0.90	8.0171E-07	4.7022E-07	3.5728E-07	2.8778E-07	2.2871E-07	1.9790E-07
0.95	7.5925E-07	4.5142E-07	3.4505E-07	2.7911E-07	2.2273E-07	1.9317E-07
1.00	7.2135E-07	4.3429E-07	3.3381E-07	2.7109E-07	2.1715E-07	1.8874E-07
1.10	6.5654E-07	4.0421E-07	3.1381E-07	2.5667E-07	2.0703E-07	1.8067E-07
1.20	6.0309E-07	3.7858E-07	2.9650E-07	2.4406E-07	1.9807E-07	1.7348E-07
1.30	5.5819E-07	3.5642E-07	2.8134E-07	2.3290E-07	1.9007E-07	1.6701E-07
1.40	5.1989E-07	3.3705E-07	2.6792E-07	2.2292E-07	1.8285E-07	1.6114E-07
1.50	4.8681E-07	3.1993E-07	2.5593E-07	2.1394E-07	1.7629E-07	1.5579E-07
1.60	4.5793E-07	3.0468E-07	2.4514E-07	2.0580E-07	1.7030E-07	1.5087E-07
1.70	4.3247E-07	2.9098E-07	2.3536E-07	1.9837E-07	1.6480E-07	1.4634E-07
1.80	4.0986E-07	2.7861E-07	2.2646E-07	1.9156E-07	1.5972E-07	1.4214E-07
1.90	3.8961E-07	2.6736E-07	2.1830E-07	1.8528E-07	1.5502E-07	1.3823E-07
2.00	3.7138E-07	2.5709E-07	2.1080E-07	1.7948E-07	1.5064E-07	1.3459E-07

Table A1. Values of β for p_0 = 0.005, 0.01, 0.025, 0.05, 0.10, 0.25, 0.50, 0.75, 0.90, 0.95, 0.975, 0.99, 0.995 and a selected set of α which solve Eq. (4) when $\lambda_0 = 1.0 \times 10^{-6}$.

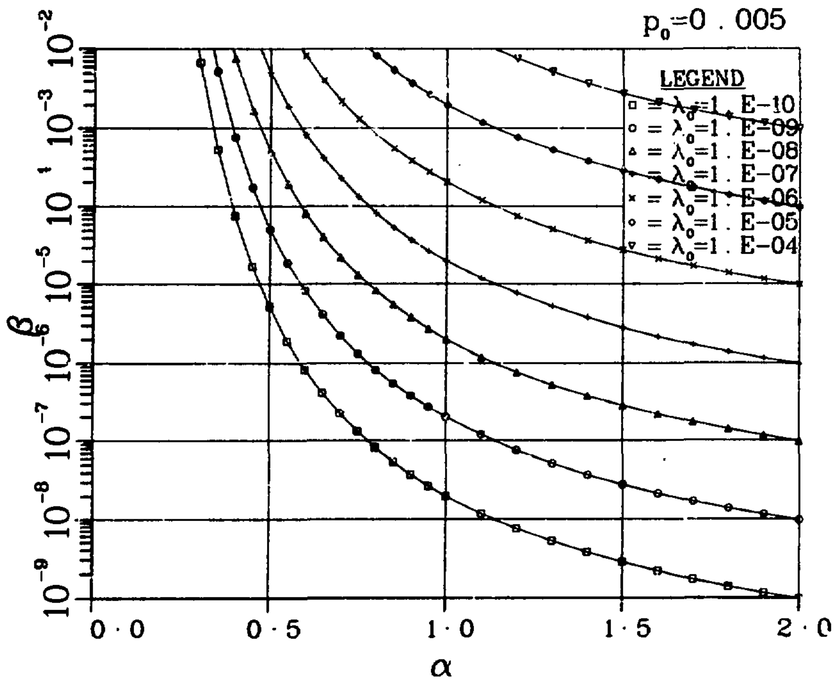

Figure A1. A graph of Table A1 for p_0 = 0.005 and a selected set of λ_0 values.

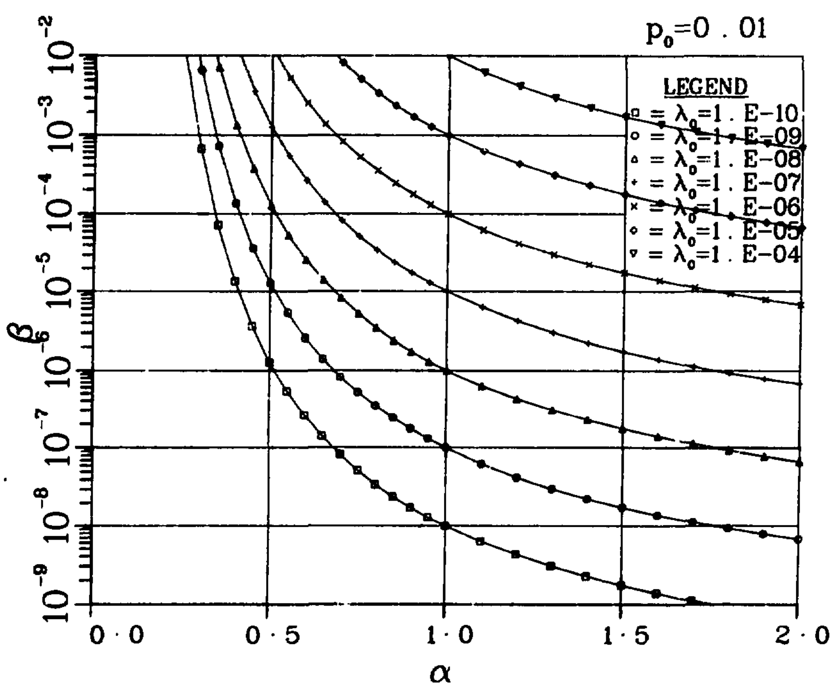

Figure A2. A graph of Table A1 for p_0 = 0.01 and a selected set of λ_0 values.

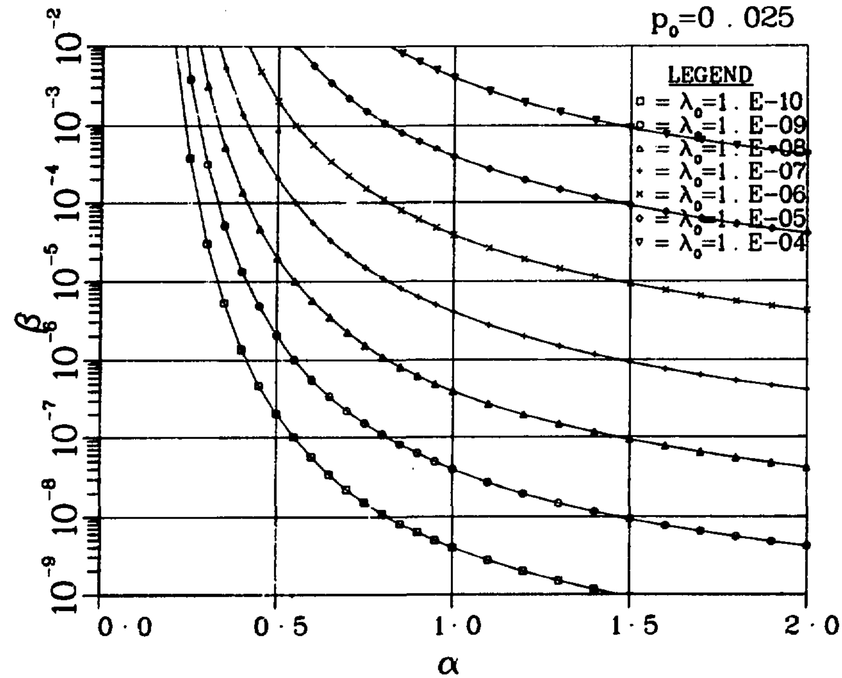

Figure A3. A graph of Table A1 for $p_0 = 0.025$ and a selected set of λ_0 values.

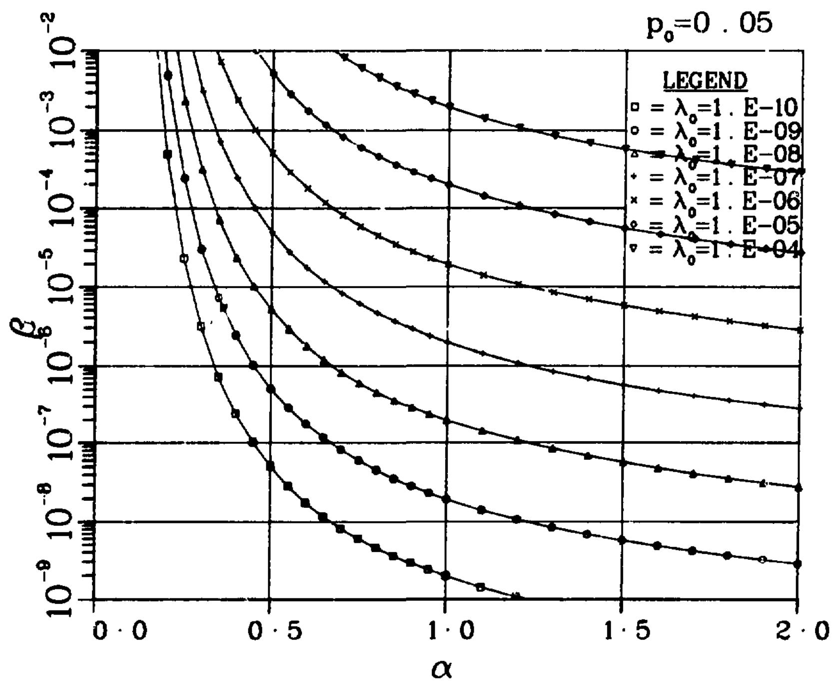

Figure A4. A graph of Table A1 for $p_0 = 0.05$ and a selected set of λ_0 values.

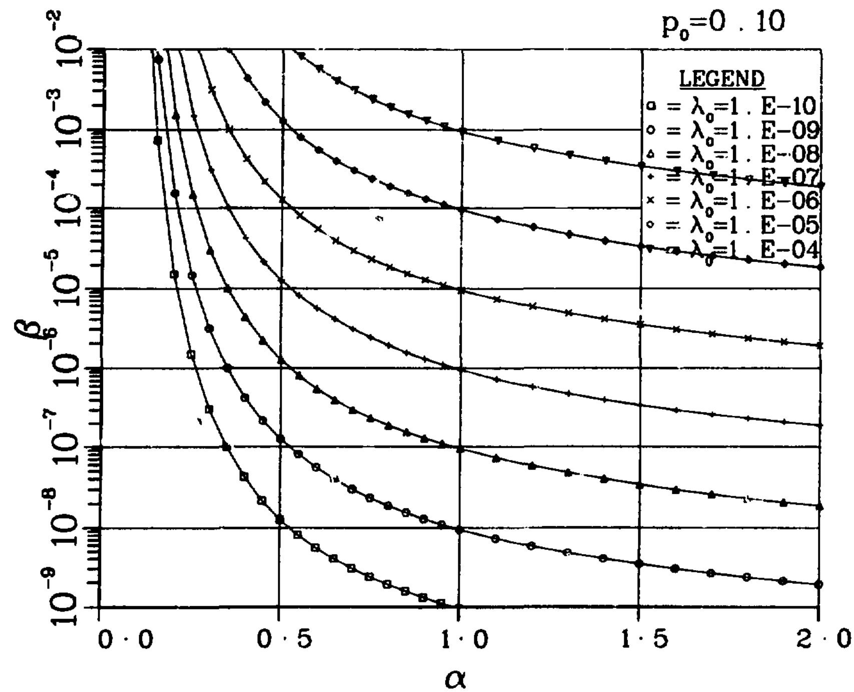

Figure A5. A graph of Table A1 for p_0 = 0.10 and a selected set of λ_0 values.

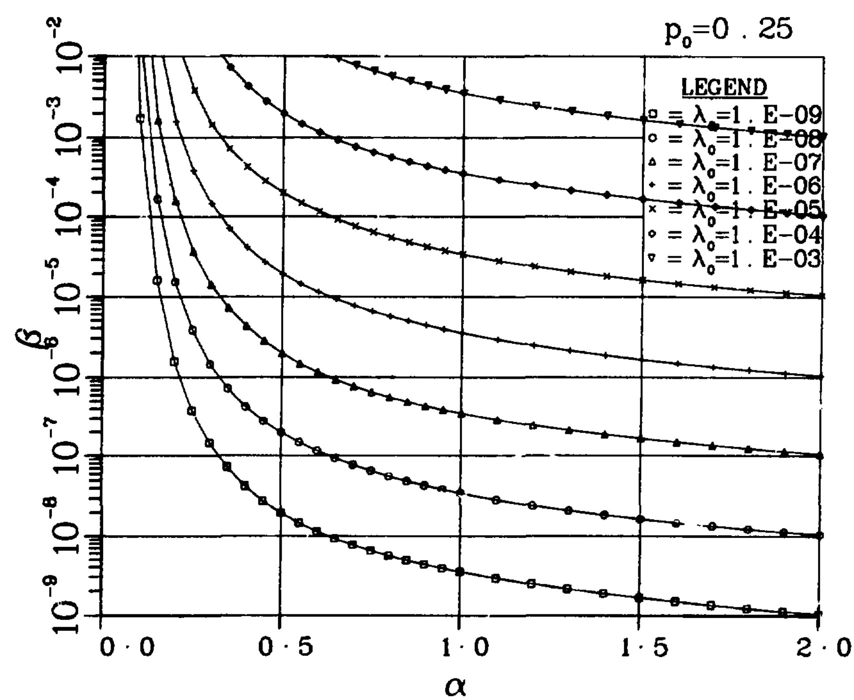

Figure A6. A graph of Table A1 for p_0 = 0.25 and a selected set of λ_0 values.

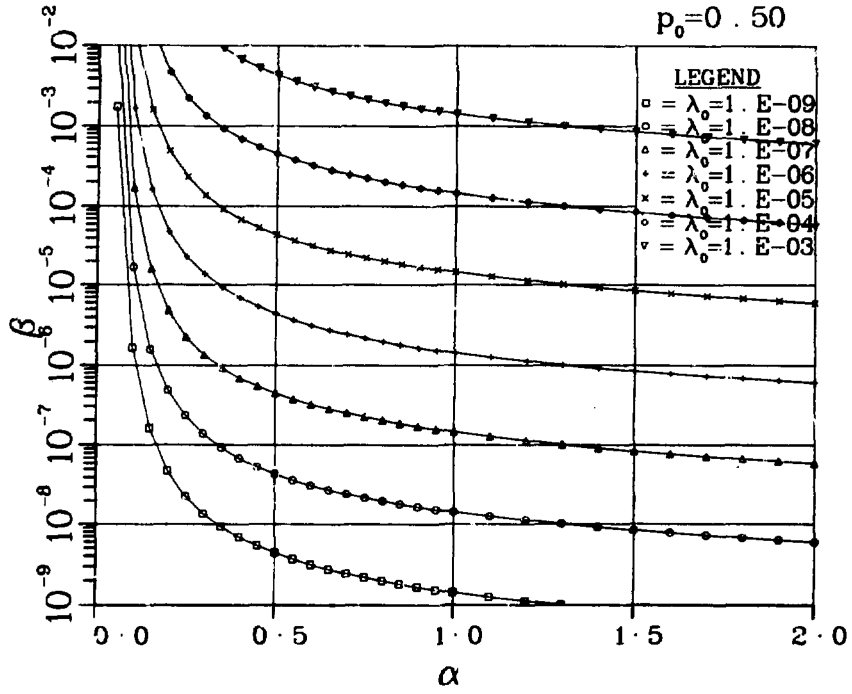

Figure A7. A graph of Table A1 for p_0 = 0.50 and a selected set of λ_0 values.

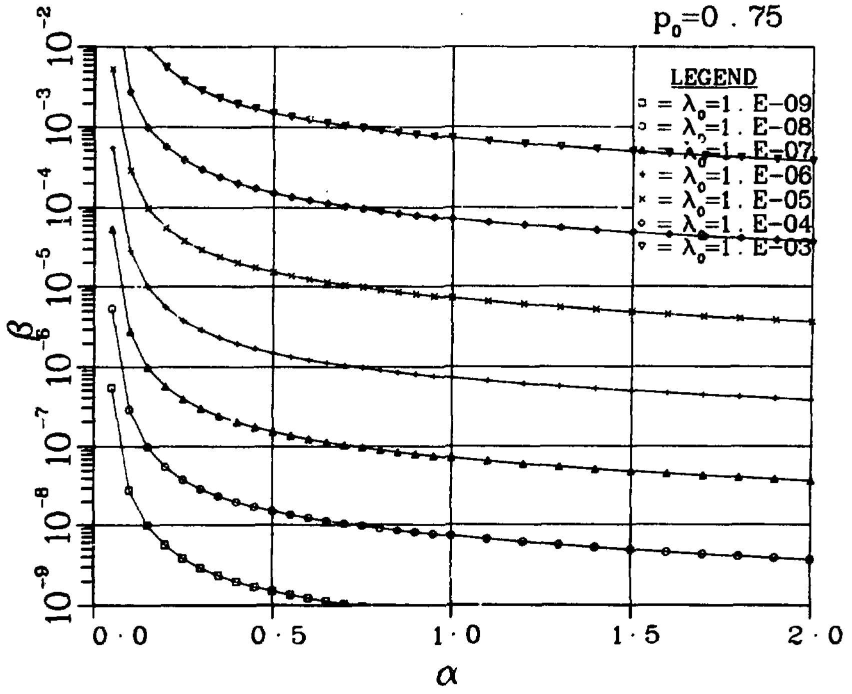

Figure A8. A graph of Table A1 for p_0 = 0.75 and a selected set of λ_0 values.

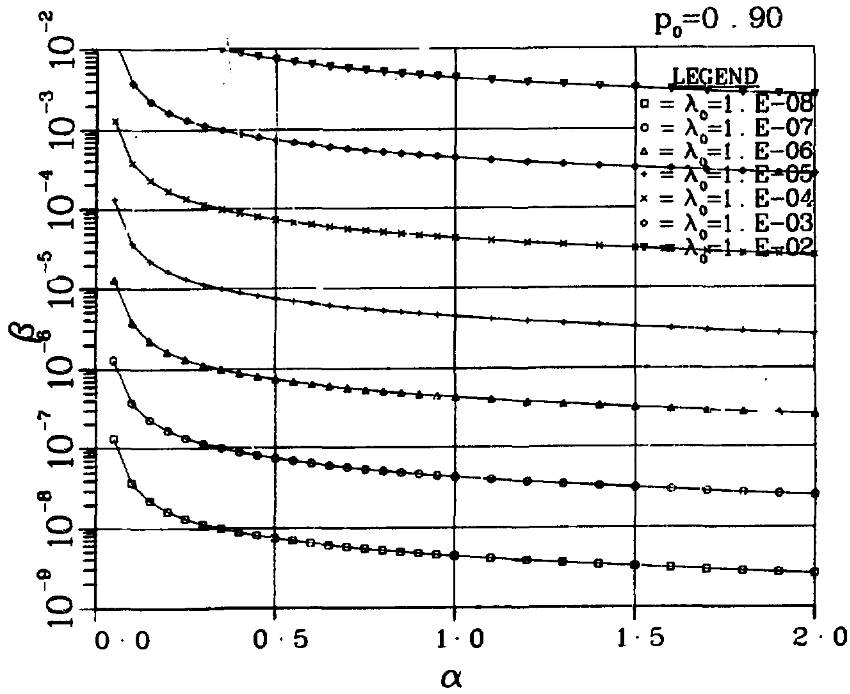

Figure A9. A graph of Table A1 for p_0 = 0.90 and a selected set of λ_0 values.

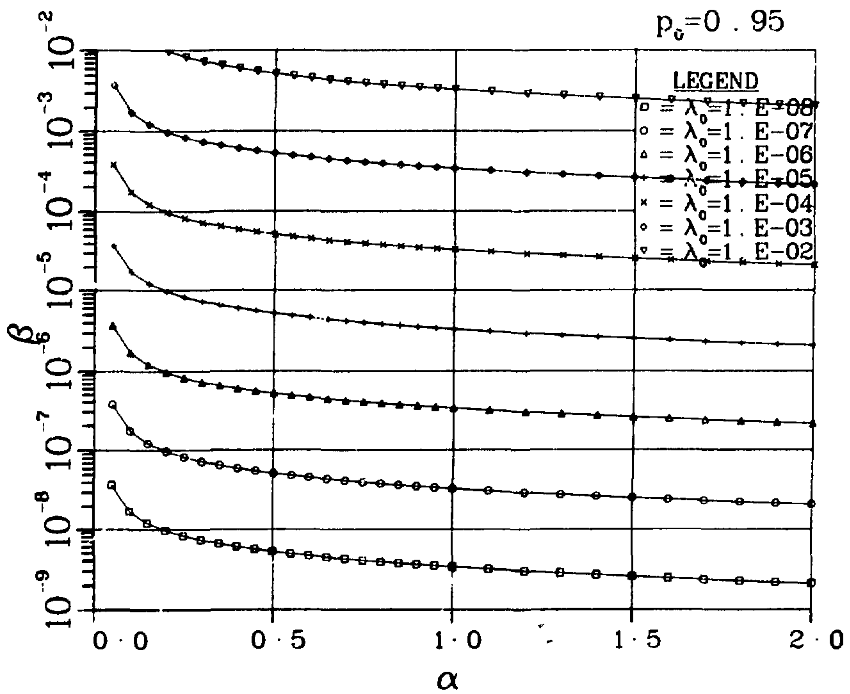

Figure A10. A graph of Table A1 for p_0 = 0.95 and a selected set of λ_0 values.

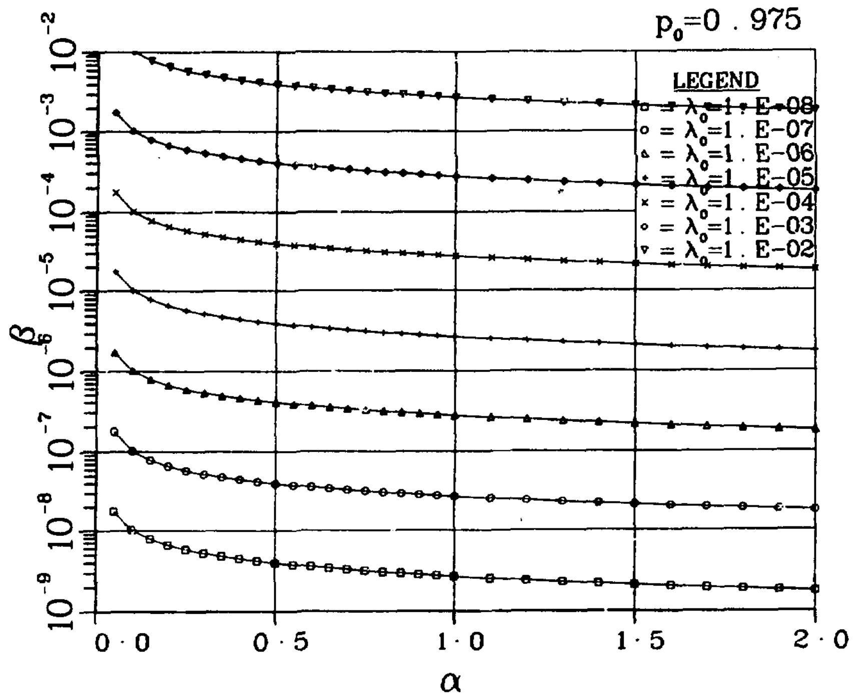

Figure A11. A graph of Table A1 for p_0 = 0.975 and a selected set of λ_0 values.

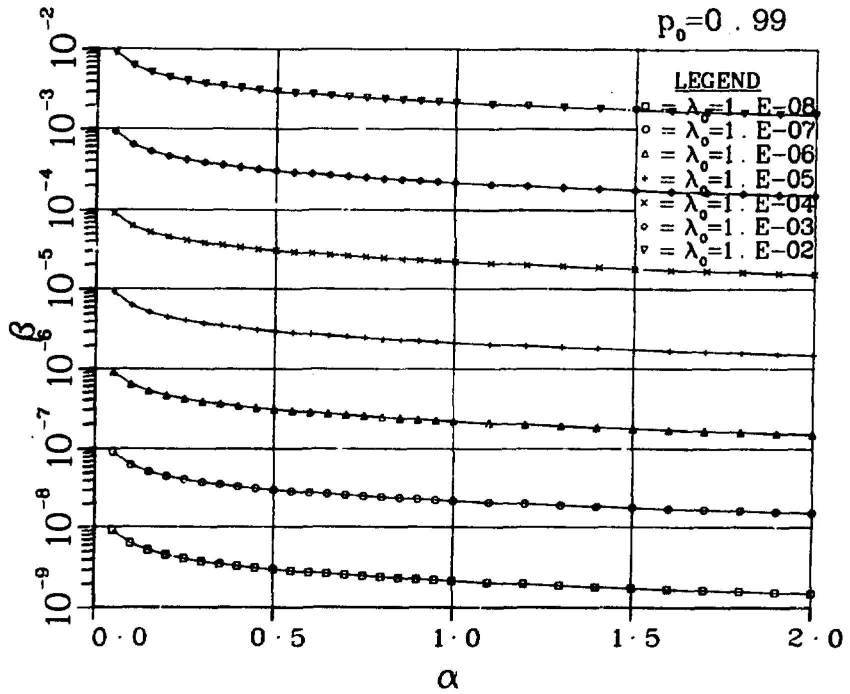

Figure A12. A graph of Table A1 for p_0 = 0.99 and a selected set of λ_0 values.

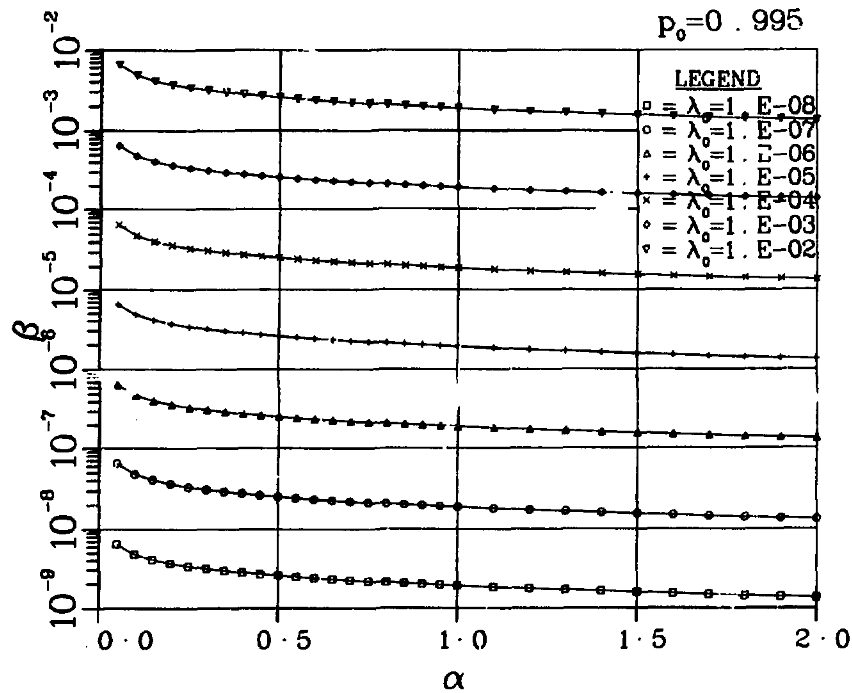

Figure A13. A graph of Table A1 for p_0 = 0.995 and a selected set of λ_0 values.

APPENDIX B

TABLES AND GRAPHS OF α AND γ FOR NEGATIVE LOG-GAMMA PRIOR DISTRIBUTION

$p_0 = 0.005$

$\alpha \backslash R_0$	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0.05	4.4875E-01	3.3071E-01	2.3091E-01	1.4446E-01	6.8211E-02	3.3208E-02	6.5066E-03
0.10	3.3093E-01	2.4388E-01	1.7029E-01	1.0653E-01	5.0302E-02	2.4489E-02	4.7963E-03
0.15	2.8135E-01	2.0735E-01	1.4478E-01	9.0575E-02	4.2766E-02	2.0820E-02	4.0795E-03
0.20	2.5162E-01	1.8544E-01	1.2948E-01	8.1005E-02	3.8248E-02	1.8620E-02	3.6484E-03
0.25	2.3091E-01	1.7017E-01	1.1882E-01	7.4336E-02	3.5099E-02	1.7087E-02	3.3481E-03
0.30	2.1522E-01	1.5861E-01	1.1075E-01	6.9287E-02	3.2715E-02	1.5927E-02	3.1207E-03
0.35	2.0271E-01	1.4939E-01	1.0431E-01	6.5257E-02	3.0812E-02	1.5000E-02	2.9392E-03
0.40	1.9235E-01	1.4176E-01	9.8979E-02	6.1923E-02	2.9238E-02	1.4234E-02	2.7890E-03
0.45	1.8356E-01	1.3527E-01	9.4453E-02	5.9092E-02	2.7901E-02	1.3583E-02	2.6615E-03
0.50	1.7594E-01	1.2966E-01	9.0533E-02	5.6639E-02	2.6743E-02	1.3020E-02	2.5510E-03
0.55	1.6924E-01	1.2472E-01	[illegible]	5.4482E-02	2.5724E-02	1.2524E-02	2.4539E-03
0.60	1.6327E-01	1.2032E-01	[illegible]	5.2560E-02	2.4817E-02	1.2082E-02	[illegible]
0.65	1.5790E-01	1.1636E-01	8.1249E-02	5.0831E-02	2.4001E-02	1.1684E-02	2.2894E-03
0.70	1.5302E-01	1.1277E-01	7.8741E-02	4.9262E-02	2.3260E-02	1.1324E-02	2.2188E-03
0.75	1.4857E-01	1.0949E-01	7.6449E-02	4.7828E-02	2.2583E-02	1.0994E-02	2.1542E-03
0.80	1.4447E-01	1.0647E-01	7.4342E-02	4.6510E-02	2.1960E-02	1.0691E-02	2.0948E-03
0.85	1.4069E-01	1.0368E-01	7.2394E-02	4.5291E-02	2.1385E-02	1.0411E-02	2.0399E-03
0.90	1.3717E-01	1.0109E-01	7.0584E-02	4.4159E-02	2.0850E-02	1.0151E-02	1.9889E-03
0.95	1.3389E-01	9.8674E-02	6.8897E-02	4.3104E-02	2.0352E-02	9.9081E-03	1.9414E-03
1.00	1.3082E-01	9.6413E-02	6.7319E-02	4.2116E-02	1.9886E-02	9.6811E-03	1.8969E-03
1.10	1.2523E-01	9.2290E-02	6.4440E-02	4.0315E-02	1.9035E-02	9.2671E-03	1.8158E-03
1.20	1.2024E-01	8.8616E-02	6.1874E-02	3.8710E-02	1.8277E-02	8.8981E-03	1.7435E-03
1.30	1.1576E-01	8.5311E-02	5.9567E-02	3.7266E-02	1.7596E-02	8.5662E-03	1.6785E-03
1.40	1.1169E-01	8.2314E-02	5.7474E-02	3.5957E-02	1.6978E-02	8.2654E-03	1.6195E-03
1.50	1.0798E-01	7.9579E-02	5.5565E-02	3.4763E-02	1.6414E-02	7.9908E-03	1.5657E-03
1.60	1.0458E-01	7.7069E-02	5.3812E-02	3.3666E-02	1.5896E-02	7.7387E-03	1.5163E-03
1.70	1.0143E-01	7.4754E-02	5.2196E-02	3.2655E-02	1.5418E-02	7.5062E-03	1.4708E-03
1.80	9.8524E-02	7.2609E-02	5.0698E-02	3.1718E-02	1.4976E-02	7.2908E-03	1.4286E-03
1.90	9.5816E-02	7.0613E-02	4.9304E-02	3.0846E-02	1.4564E-02	7.0904E-03	1.3893E-03
2.00	9.3289E-02	6.8751E-02	4.8004E-02	3.0032E-02	1.4180E-02	6.9034E-03	1.3526E-03

Table B1. Table of γ values for $p_0 = 0.005$ and a selected set of values for α and R_0 in Eq.(7).

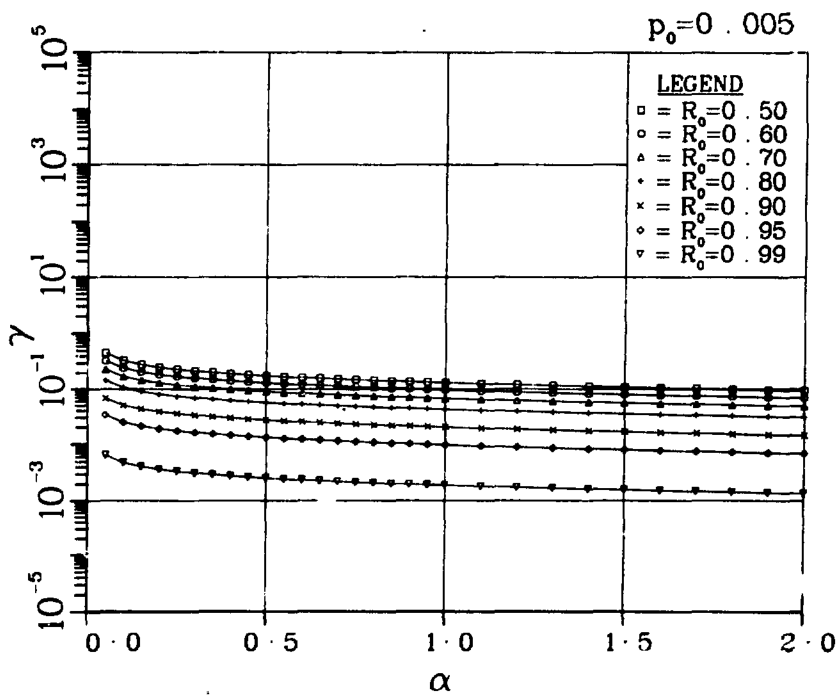

Figure B1. A graph of Table B1.

$p_0=0.01$

$\alpha \backslash R_0$	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0.05	6.3730E-01	4.6967E-01	3.2794E-01	2.0517E-01	9.6872E-02	4.7161E-02	9.2406E-03
0.10	4.3636E-01	3.2158E-01	2.2454E-01	1.4048E-01	6.6328E-02	[illegible]	[illegible]
0.15	3.5903E-01	2.6460E-01	1.8475E-01	1.1558E-01	5.4574E-02	2.6569E-02	5.2058E-03
0.20	3.1474E-01	2.3195E-01	1.6196E-01	1.0132E-01	4.7841E-02	2.3291E-02	4.5636E-03
0.25	2.8479E-01	2.0988E-01	1.4655E-01	9.1682E-02	4.3289E-02	2.1075E-02	4.1293E-03
0.30	2.6261E-01	1.9354E-01	1.3513E-01	8.4543E-02	3.9918E-02	1.9434E-02	3.8078E-03
0.35	2.4522E-01	1.8072E-01	1.2619E-01	7.8945E-02	3.7275E-02	1.8147E-02	3.5556E-03
0.40	2.3104E-01	1.7027E-01	1.1889E-01	7.4379E-02	3.5119E-02	1.7097E-02	3.3500E-03
0.45	2.1914E-01	1.6150E-01	1.1276E-01	7.0548E-02	3.3310E-02	1.6217E-02	3.1775E-03
0.50	2.0894E-01	1.5398E-01	1.0751E-01	6.7264E-02	3.1760E-02	1.5462E-02	3.0295E-03
0.55	2.0005E-01	1.4743E-01	1.0294E-01	6.4401E-02	3.0408E-02	1.4804E-02	2.9006E-03
0.60	1.9219E-01	1.4164E-01	9.8895E-02	6.1871E-02	2.9213E-02	1.4222E-02	2.7867E-03
0.65	1.8517E-01	1.3646E-01	9.5284E-02	5.9611E-02	2.8146E-02	1.3703E-02	2.6849E-03
0.70	1.7884E-01	1.3180E-01	9.2028E-02	5.7574E-02	2.7185E-02	1.3234E-02	2.5931E-03
0.75	1.7309E-01	1.2756E-01	8.9070E-02	5.5724E-02	2.6311E-02	1.2809E-02	2.5098E-03
0.80	1.6784E-01	1.2369E-01	8.6364E-02	5.4031E-02	2.5512E-02	1.2420E-02	2.4336E-03
0.85	1.6300E-01	1.2013E-01	8.3876E-02	5.2475E-02	2.4777E-02	1.2062E-02	2.3635E-03
0.90	1.5853E-01	1.1683E-01	8.1577E-02	5.1036E-02	2.4097E-02	1.1731E-02	2.2987E-03
0.95	1.5438E-01	1.1377E-01	7.9441E-02	4.9700E-02	2.3467E-02	1.1424E-02	2.2385E-03
1.00	1.5051E-01	1.1092E-01	7.7451E-02	4.8455E-02	2.2879E-02	1.1138E-02	2.1824E-03
1.10	1.4350E-01	1.0576E-01	7.3843E-02	4.6198E-02	2.1813E-02	1.0619E-02	2.0807E-03
1.20	1.3729E-01	1.0118E-01	7.0648E-02	4.4199E-02	2.0869E-02	1.0160E-02	1.9907E-03
1.30	1.3174E-01	9.7091E-02	6.7792E-02	4.2412E-02	2.0026E-02	9.7491E-03	1.9102E-03
1.40	1.2674E-01	9.3403E-02	6.5217E-02	4.0801E-02	1.9265E-02	9.3788E-03	1.8377E-03
1.50	1.2220E-01	9.0054E-02	6.2879E-02	3.9338E-02	1.8574E-02	9.0426E-03	1.7718E-03
1.60	1.1804E-01	8.6994E-02	6.0742E-02	3.8002E-02	1.7943E-02	8.7353E-03	1.7116E-03
1.70	1.1423E-01	8.4184E-02	5.8780E-02	3.6774E-02	1.7363E-02	8.4531E-03	1.6563E-03
1.80	1.1071E-01	8.1590E-02	5.6969E-02	3.5641E-02	1.6828E-02	8.1927E-03	1.6053E-03
1.90	1.0745E-01	7.9187E-02	5.5291E-02	3.4591E-02	1.6333E-02	7.9513E-03	1.5580E-03
2.00	1.0442E-01	7.6951E-02	5.3729E-02	3.3614E-02	1.5871E-02	7.7268E-03	1.5140E-03

Table B2. Table of γ values for p_0 = 0.01 and a selected set of values for α and R_0 in Eq.(7).

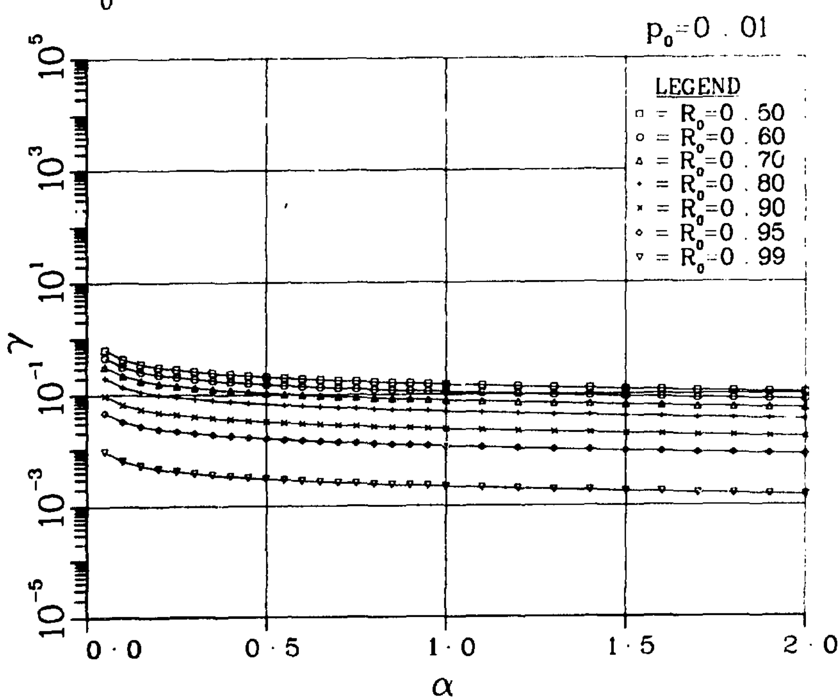

Figure B2. A graph of Table B2.

$p_0 = 0.025$

$\alpha \backslash R_0$	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0.05	1.2221E+00	9.0065E-01	6.2886E-01	3.9343E-01	1.8576E-01	9.0437E-02	1.7720E-02
0.10	7.0881E-01	5.2237E-01	3.6473E-01	2.2819E-01	1.0774E-01	5.2452E-02	1.0277E-02
0.15	5.4470E-01	4.0143E-01	2.8029E-01	1.7535E-01	8.2796E-02	4.0308E-02	7.8979E-03
0.20	4.5870E-01	3.3805E-01	2.3604E-01	1.4767E-01	6.9724E-02	3.3944E-02	6.6510E-03
0.25	4.0379E-01	2.9758E-01	2.0778E-01	1.2999E-01	6.1377E-02	2.9880E-02	5.8547E-03
0.30	3.6476E-01	2.6882E-01	1.8770E-01	1.1743E-01	5.5445E-02	2.6993E-02	5.2889E-03
0.35	3.3512E-01	2.4697E-01	1.7245E-01	1.0789E-01	5.0940E-02	2.4799E-02	4.8591E-03
0.40	3.1157E-01	2.2961E-01	1.6032E-01	1.0030E-01	4.7359E-02	2.3056E-02	4.5176E-03
0.45	2.9222E-01	2.1536E-01	1.5037E-01	9.4074E-02	4.4419E-02	2.1625E-02	4.2371E-03
0.50	2.7594E-01	2.0336E-01	1.4199E-01	8.8833E-02	4.1944E-02	2.0420E-02	4.0010E-03
0.55	2.6197E-01	1.9306E-01	1.3480E-01	8.4336E-02	3.9821E-02	1.9386E-02	3.7985E-03
0.60	2.4980E-01	1.8410E-01	1.2854E-01	8.0419E-02	3.7971E-02	1.8486E-02	3.6220E-03
0.65	2.3907E-01	1.7619E-01	1.2302E-01	7.6963E-02	3.6339E-02	1.7691E-02	3.4664E-03
0.70	2.2950E-01	1.6913E-01	1.1809E-01	7.3882E-02	3.4885E-02	1.6983E-02	3.3276E-03
0.75	2.2089E-01	1.6279E-01	1.1367E-01	7.1112E-02	3.3577E-02	1.6346E-02	3.2029E-03
0.80	2.1310E-01	1.5705E-01	1.0966E-01	6.8603E-02	3.2392E-02	1.5769E-02	3.0898E-03
0.85	2.0599E-01	1.5181E-01	1.0600E-01	6.6314E-02	3.1311E-02	1.5243E-02	2.9868E-03
0.90	1.9947E-01	1.4700E-01	1.0264E-01	6.4216E-02	3.0320E-02	1.4761E-02	2.8923E-03
0.95	1.9346E-01	1.4258E-01	9.9551E-02	6.2281E-02	2.9407E-02	1.4316E-02	2.8051E-03
1.00	1.8790E-01	1.3848E-01	9.6689E-02	6.0491E-02	2.8562E-02	1.3905E-02	2.7245E-03
1.10	1.7791E-01	1.3112E-01	9.1549E-02	5.7275E-02	2.7043E-02	1.3166E-02	2.5796E-03
1.20	1.6917E-01	1.2467E-01	8.7050E-02	5.4460E-02	2.5714E-02	1.2519E-02	2.4529E-03
1.30	1.6143E-01	1.1897E-01	8.3068E-02	5.1969E-02	2.4538E-02	1.1946E-02	2.3407E-03
1.40	1.5452E-01	1.1387E-01	7.9511E-02	4.9744E-02	2.3487E-02	1.1434E-02	2.2404E-03
1.50	1.4829E-01	1.0929E-01	7.6307E-02	4.7739E-02	2.2541E-02	1.0974E-02	2.1502E-03
1.60	1.4265E-01	1.0513E-01	7.3403E-02	4.5922E-02	2.1683E-02	1.0556E-02	2.0683E-03
1.70	1.3750E-01	1.0133E-01	7.0753E-02	4.4265E-02	2.0900E-02	1.0175E-02	1.9937E-03
1.80	1.3278E-01	9.7853E-02	6.8324E-02	4.2745E-02	2.0183E-02	9.8256E-03	1.9252E-03
1.90	1.2843E-01	9.4648E-02	6.6086E-02	4.1345E-02	1.9522E-02	9.5038E-03	1.8622E-03
2.00	1.2441E-01	9.1683E-02	6.4016E-02	4.0050E-02	1.8910E-02	9.2061E-03	1.8038E-03

Table B3. Table of γ values for p_0 = 0.025 and a selected set of values for α and R_0 in Eq.(7).

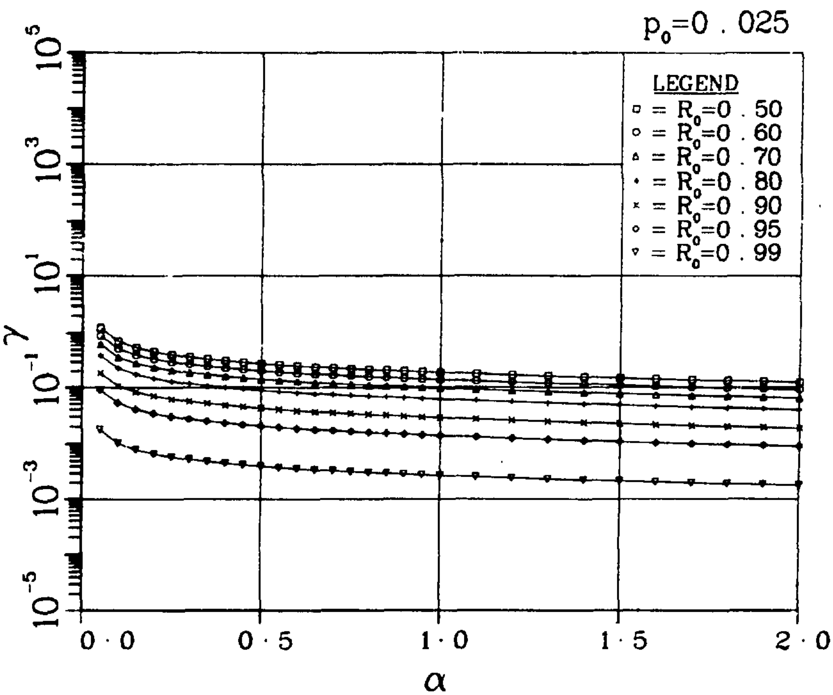

Figure B3. A graph of Table B3.

$p_0 = 0.05$

$\alpha \backslash R_0$	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0.05	2.6065E+00	1.9209E+00	1.3412E+00	8.3910E-01	3.9619E-01	1.9288E-01	3.7793E-02
0.10	1.1942E+00	8.8007E-01	6.1450E-01	3.8444E-01	1.8152E-01	8.8370E-02	1.7315E-02
0.15	8.39[illegible]01	6.1874E-01	4.3203E-01	2.7028E-01	1.2762E-01	6.2129E-02	1.2174E-02
0.20	6.72[illegible]-01	4.9569E-01	3.4611E-01	2.1653E-01	1.0224E-01	4.9774E-02	9.7526E-03
0.25	5.7279E-01	4.2213E-01	2.9474E-01	1.8440E-01	8.7066E-02	4.2387E-02	8.3053E-03
0.30	5.0508E-01	3.7223E-01	2.5990E-01	1.6260E-01	7.6774E-02	3.7376E-02	7.3235E-03
0.35	4.5543E-01	3.3564E-01	2.3435E-01	1.4662E-01	6.9227E-02	3.3702E-02	6.6036E-03
0.40	4.1707E-01	3.0736E-01	2.1461E-01	1.3427E-01	6.3395E-02	3.0863E-02	6.0473E-03
0.45	3.8628E-01	2.8468E-01	1.9877E-01	1.2436E-01	5.8716E-02	2.8585E-02	5.6009E-03
0.50	3.6088E-01	2.6595E-01	1.8570E-01	1.1618E-01	5.4854E-02	2.6705E-02	5.2326E-03
0.55	3.3945E-01	2.5016E-01	1.7467E-01	1.0928E-01	5.1597E-02	2.5119E-02	4.9216E-03
0.60	3.2105E-01	2.3660E-01	1.6520E-01	1.0335E-01	4.8800E-02	2.3758E-02	4.6551E-03
0.65	3.0503E-01	2.2480E-01	1.5696E-01	9.8198E-02	4.6366E-02	2.2572E-02	4.4228E-03
0.70	2.9092E-01	2.1440E-01	1.4970E-01	9.3655E-02	4.4220E-02	2.1528E-02	4.2182E-03
0.75	2.7836E-01	2.0514E-01	1.4324E-01	8.9612E-02	4.2312E-02	2.0599E-02	4.0361E-03
0.80	2.6709E-01	1.9684E-01	1.3744E-01	8.5985E-02	4.0599E-02	1.9765E-02	3.8727E-03
0.85	2.5691E-01	1.8933E-01	1.3220E-01	8.2706E-02	3.9051E-02	1.9011E-02	3.7251E-03
0.90	2.4764E-01	1.8251E-01	1.2743E-01	7.9724E-02	3.7643E-02	1.8326E-02	3.5907E-03
0.95	2.3917E-01	1.7626E-01	1.2307E-01	7.6995E-02	3.6354E-02	1.7699E-02	3.4679E-03
1.00	2.3138E-01	1.7052E-01	1.1906E-01	7.4487E-02	3.5170E-02	1.7122E-02	3.3549E-03
1.10	2.1751E-01	1.6030E-01	1.1193E-01	7.0024E-02	3.3063E-02	1.6096E-02	3.1539E-03
1.20	2.0552E-01	1.5146E-01	1.0575E-01	6.6162E-02	3.1240E-02	1.5209E-02	2.9799E-03
1.30	1.9501E-01	1.4372E-01	1.0035E-01	6.2779E-02	2.9642E-02	1.4431E-02	2.8276E-03
1.40	1.8571E-01	1.3686E-01	9.5559E-02	5.9784E-02	2.8228E-02	1.3742E-02	2.6926E-03
1.50	1.7740E-01	1.3073E-01	9.1283E-02	5.7108E-02	2.6965E-02	1.3127E-02	2.5722E-03
1.60	1.6992E-01	1.2522E-01	8.7434E-02	5.4701E-02	2.5828E-02	1.2574E-02	2.4637E-03
1.70	1.6314E-01	1.2023E-01	8.3948E-02	5.2520E-02	2.4798E-02	1.2073E-02	2.3655E-03
1.80	1.5697E-01	1.1568E-01	8.0772E-02	5.0533E-02	2.3860E-02	1.1616E-02	2.2760E-03
1.90	1.5132E-01	1.1151E-01	7.7863E-02	4.8713E-02	2.3000E-02	1.1197E-02	2.1940E-03
2.00	1.4611E-01	1.0769E-01	7.5187E-02	4.7038E-02	2.2210E-02	1.0813E-02	2.1186E-03

Table B4. Table of γ values for p_0 = 0.05 and a selected set of values for α and R_0 in Eq.(7).

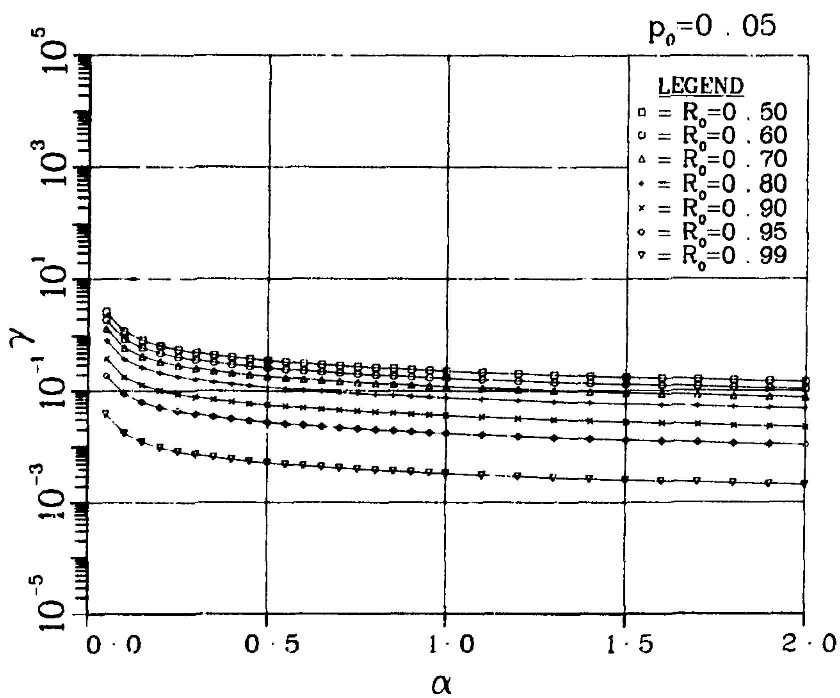

Figure B4. A graph of Table B4.

p_0 =0.10

$\alpha \backslash R_0$	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0.05	9.0825E+00	6.6935E+00	4.6736E+00	2.9239E+00	1.3806E+00	6.7211E-01	1.3169E-01
0.10	2.6043E+00	1.9193E+00	1.3401E+00	8.3840E-01	3.9586E-01	1.9272E-01	3.7761E-02
0.15	1.5583E+00	1.1484E+00	8.0188E-01	5.0167E-01	2.3687E-01	1.1532E-01	2.2595E-02
0.20	1.1459E+00	8.4448E-01	5.8964E-01	3.6889E-01	1.7418E-01	8.4796E-02	1.6615E-02
0.25	9.2371E-01	6.8074E-01	4.7532E-01	2.9737E-01	1.4041E-01	6.8355E-02	1.3393E-02
0.30	7.8338E-01	5.7733E-01	4.0311E-01	2.5219E-01	1.1908E-01	5.7971E-02	1.1359E-02
0.35	6.8581E-01	5.0542E-01	3.5290E-01	2.2078E-01	1.0425E-01	5.0751E-02	9.9440E-03
0.40	6.1349E-01	4.5212E-01	3.1569E-01	1.9750E-01	9.3252E-02	4.5399E-02	8.8953E-03
0.45	5.5739E-01	4.1078E-01	2.8682E-01	1.7944E-01	8.4726E-02	4.1247E-02	8.0820E-03
0.50	5.1239E-01	3.7761E-01	2.6366E-01	1.6495E-01	7.7885E-02	3.7917E-02	7.4294E-03
0.55	4.7533E-01	3.5030E-01	2.4459E-01	1.5302E-01	7.2252E-02	3.5175E-02	6.8921E-03
0.60	4.4418E-01	3.2735E-01	2.2856E-01	1.4299E-01	6.7517E-02	3.2870E-02	6.4404E-03
0.65	4.1755E-01	3.0772E-01	2.1486E-01	1.3442E-01	6.3470E-02	3.0899E-02	6.0544E-03
0.70	3.9448E-01	2.9072E-01	2.0299E-01	1.2699E-01	5.9962E-02	2.9192E-02	5.7197E-03
0.75	3.7424E-01	2.7580E-01	1.9258E-01	1.2048E-01	5.6886E-02	2.7694E-02	5.4264E-03
0.80	3.5533E-01	2.6260E-01	1.8336E-01	1.1471E-01	5.4163E-02	2.6368E-02	5.1666E-03
0.85	3.4033E-01	2.5081E-01	1.7512E-01	1.0956E-01	5.1731E-02	2.5184E-02	4.9346E-03
0.90	3.2593E-01	2.4020E-01	1.6772E-01	1.0493E-01	4.9543E-02	2.4119E-02	4.7259E-03
0.95	3.1290E-01	2.3060E-01	1.6101E-01	1.0073E-01	4.7562E-02	2.3155E-02	4.5369E-03
1.00	3.0103E-01	2.2185E-01	1.5490E-01	9.6910E-02	4.5757E-02	2.2276E-02	4.3648E-03
1.10	2.8018E-01	2.0648E-01	1.4417E-01	9.0197E-02	4.2588E-02	2.0733E-02	4.0625E-03
1.20	2.6241E-01	1.9339E-01	1.3503E-01	8.4477E-02	3.9887E-02	1.9419E-02	3.8048E-03
1.30	2.4705E-01	1.8207E-01	1.2713E-01	7.9534E-02	3.7553E-02	1.8282E-02	3.5822E-03
1.40	2.3362E-01	1.7217E-01	1.2022E-01	7.5210E-02	3.5511E-02	1.7288E-02	3.3874E-03
1.50	2.2176E-01	1.6343E-01	1.1411E-01	7.1390E-02	3.3708E-02	1.6410E-02	3.2154E-03
1.60	2.1118E-01	1.5564E-01	1.0867E-01	6.7986E-02	3.2101E-02	1.5628E-02	3.0621E-03
1.70	2.0169E-01	1.4864E-01	1.0379E-01	6.4931E-02	3.0658E-02	1.4925E-02	2.9245E-03
1.80	1.9312E-01	1.4232E-01	9.9372E-02	6.2169E-02	2.9354E-02	1.4291E-02	2.8001E-03
1.90	1.8532E-01	1.3657E-01	9.5361E-02	5.9660E-02	2.8169E-02	1.3714E-02	2.6871E-03
2.00	1.7820E-01	1.3133E-01	9.1697E-02	5.7368E-02	2.7087E-02	1.3187E-02	2.5838E-03

Table B5. Table of γ values for p_0 = 0.10 and a selected set of values for α and R_0 in Eq.(7).

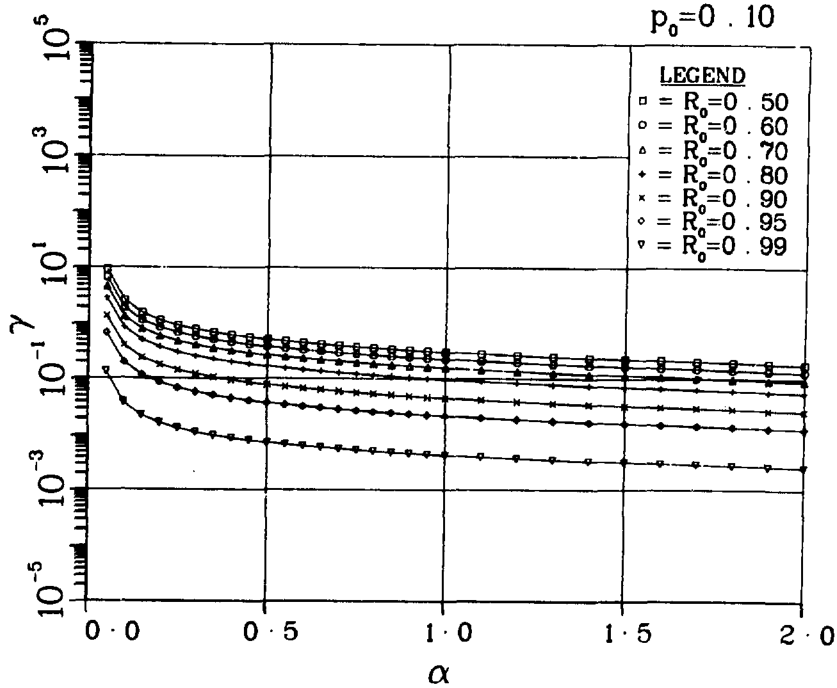

Figure B5. A graph of Table B5.

$p_0 = 0.25$

$\alpha \backslash R_0$	0.7000	0.8000	0.9000	0.9500	0.9900	0.9990	0.9999
0.05	1.9210E+02	1.2018E+02	5.6745E+01	2.7626E+01	5.4129E+00	5.3885E-01	5.3861E-02
0.10	1.0102E+01	6.3202E+00	2.9842E+00	1.4528E+00	2.8466E-01	2.8338E-02	2.8325E-03
0.15	3.5362E+00	2.2123E+00	1.0446E+00	5.0854E-01	9.9642E-02	9.9192E-03	9.9148E-04
0.20	1.9942E+00	1.2476E+00	5.8907E-01	2.8678E-01	5.6191E-02	5.5938E-03	5.5915E-04
0.25	1.3685E+00	8.5618E-01	4.0426E-01	1.9681E-01	3.8562E-02	3.8388E-03	3.8371E-04
0.30	1.0402E+00	6.5076E-01	3.0726E-01	1.4959E-01	2.9310E-02	2.9178E-03	2.9165E-04
0.35	8.4034E-01	5.2573E-01	2.4823E-01	1.2085E-01	2.3679E-02	2.3572E-03	2.3562E-04
0.40	7.0656E-01	4.4204E-01	2.0871E-01	1.0161E-01	1.9909E-02	1.9819E-03	1.9811E-04
0.45	6.1088E-01	3.8218E-01	1.8045E-01	8.7850E-02	1.7213E-02	1.7136E-03	1.7128E-04
0.50	5.3907E-01	3.3725E-01	1.5924E-01	7.7523E-02	1.5190E-02	1.5121E-03	1.5114E-04
0.55	4.8316E-01	3.0227E-01	1.4272E-01	6.9483E-02	1.3614E-02	1.3553E-03	1.3547E-04
0.60	4.3836E-01	2.7425E-01	1.2949E-01	6.3041E-02	1.2352E-02	1.2296E-03	1.2291E-04
0.65	4.0164E-01	2.5127E-01	1.1864E-01	5.7759E-02	1.1317E-02	1.1266E-03	1.1261E-04
0.70	3.7095E-01	2.3208E-01	1.0958E-01	5.3347E-02	1.0453E-02	1.0406E-03	1.0401E-04
0.75	3.4492E-01	2.1579E-01	1.0189E-01	4.9602E-02	9.7190E-03	9.6752E-04	9.6708E-05
0.80	3.2253E-01	2.0178E-01	9.5273E-02	4.6382E-02	9.0881E-03	9.0471E-04	9.0430E-05
0.85	3.0305E-01	1.8960E-01	8.9521E-02	4.3582E-02	8.5394E-03	8.5009E-04	8.4970E-05
0.90	2.8595E-01	1.7890E-01	8.4469E-02	4.1123E-02	8.0575E-03	8.0211E-04	8.0175E-05
0.95	2.7080E-01	1.6942E-01	7.9994E-02	3.8944E-02	7.6307E-03	7.5963E-04	7.5928E-05
1.00	2.5729E-01	1.6096E-01	7.6002E-02	3.7000E-02	7.2498E-03	7.2171E-04	7.2138E-05
1.10	2.3417E-01	1.4650E-01	6.9174E-02	3.3676E-02	6.5985E-03	6.5687E-04	6.5657E-05
1.20	2.1511E-01	1.3458E-01	6.3542E-02	3.0934E-02	6.0612E-03	6.0339E-04	6.0312E-05
1.30	1.9909E-01	1.2456E-01	5.8811E-02	2.8631E-02	5.6100E-03	5.5847E-04	5.5822E-05
1.40	1.8543E-01	1.1601E-01	5.4776E-02	2.6667E-02	5.2251E-03	5.2015E-04	5.1992E-05
1.50	1.7363E-01	1.0863E-01	5.1291E-02	2.4970E-02	4.8926E-03	4.8706E-04	4.8684E-05
1.60	1.6333E-01	1.0218E-01	4.8248E-02	2.3489E-02	4.6024E-03	4.5816E-04	4.5795E-05
1.70	1.5425E-01	9.6504E-02	4.5566E-02	2.2183E-02	4.3465E-03	4.3269E-04	4.3250E-05
1.80	1.4619E-01	9.1457E-02	4.3183E-02	2.1023E-02	4.1192E-03	4.1006E-04	4.0988E-05
1.90	1.3897E-01	8.6940E-02	4.1050E-02	1.9985E-02	3.9157E-03	3.8981E-04	3.8963E-05
2.00	1.3246E-01	8.2872E-02	3.9129E-02	1.9049E-02	3.7325E-03	3.7157E-04	3.7140E-05

Table B6. Table of γ values for $p_0 = 0.25$ and a selected set of values for α and R_0 in Eq.(7).

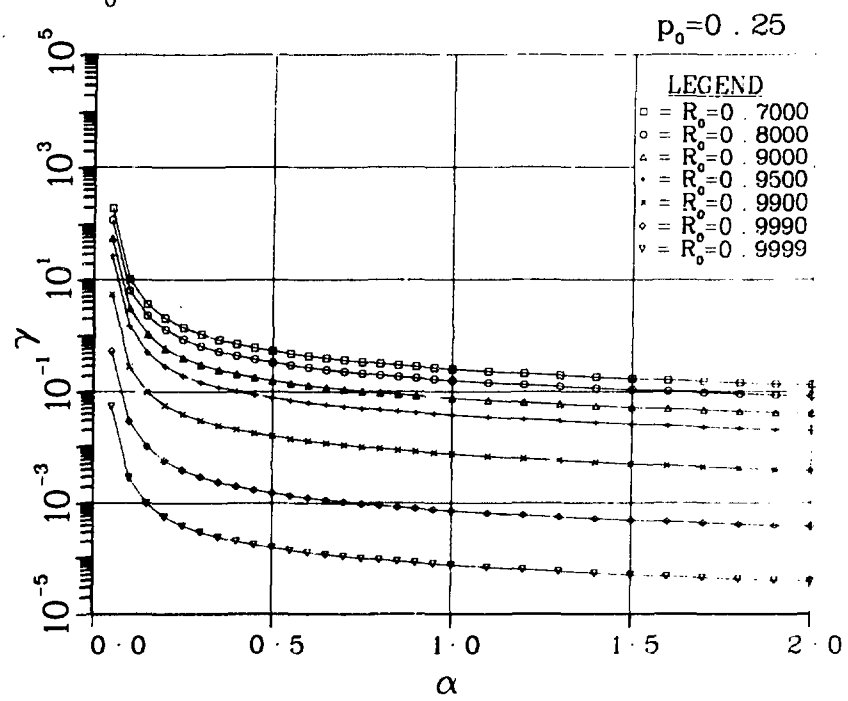

Figure B6. A graph of Table B6.

$p_0 = 0.50$

$\alpha \backslash R_0$	0.7000	0.8000	0.9000	0.9500	0.9900	0.9990	0.9999
0.05	6.3990E+05	4.0034E+05	1.8903E+05	9.2024E+04	1.8031E+04	1.7950E+03	1.7942E+02
0.10	6.0108E+02	3.7605E+02	1.7756E+02	8.6441E+01	1.6937E+01	1.6861E+00	1.6853E-01
0.15	5.7207E+01	3.5790E+01	1.6899E+01	8.2269E+00	1.6120E+00	1.6047E-01	1.6040E-02
0.20	1.7192E+01	1.0756E+01	5.0785E+00	2.4724E+00	4.8444E-01	4.8225E-02	4.8204E-03
0.25	8.1668E+00	5.1093E+00	2.4124E+00	1.1745E+00	2.3012E-01	2.2908E-02	2.2898E-03
0.30	4.8772E+00	3.0513E+00	1.4407E+00	7.0139E-01	1.3743E-01	1.3681E-02	1.3675E-03
0.35	3.3219E+00	2.0783E+00	9.8129E-01	4.7773E-01	9.3605E-02	9.3183E-03	9.3141E-04
0.40	2.4585E+00	1.5381E+00	7.2623E-01	3.5356E-01	6.9275E-02	6.8963E-03	6.8932E-04
0.45	1.9246E+00	1.2040E+00	5.6851E-01	2.7677E-01	5.4230E-02	5.3985E-03	5.3961E-04
0.50	1.5680E+00	9.8099E-01	4.6319E-01	2.2550E-01	4.4183E-02	4.3984E-03	4.3964E-04
0.55	1.3160E+00	8.2330E-01	3.8873E-01	1.8925E-01	3.7081E-02	3.6914E-03	3.6897E-04
0.60	1.1298E+00	7.0682E-01	3.3373E-01	1.6247E-01	3.1835E-02	3.1691E-03	3.1677E-04
0.65	9.8741E-01	6.1774E-01	2.9168E-01	1.4200E-01	2.7823E-02	2.7698E-03	2.7685E-04
0.70	8.7544E-01	5.4769E-01	2.5860E-01	1.2590E-01	2.4668E-02	2.4557E-03	2.4546E-04
0.75	7.8534E-01	4.9132E-01	2.3199E-01	1.1294E-01	2.2129E-02	2.2029E-03	2.2019E-04
0.80	7.1143E-01	4.4508E-01	2.1015E-01	1.0231E-01	2.0046E-02	1.9956E-03	1.9947E-04
0.85	6.4980E-01	4.0653E-01	1.9195E-01	9.3448E-02	1.8310E-02	1.8227E-03	1.8219E-04
0.90	5.9770E-01	3.7394E-01	1.7656E-01	8.5955E-02	1.6842E-02	1.6766E-03	1.6758E-04
0.95	5.5312E-01	3.4605E-01	1.6339E-01	7.9544E-02	1.5586E-02	1.5515E-03	1.5509E-04
1.00	5.1457E-01	3.2193E-01	1.5200E-01	7.4001E-02	1.4700E-02	1.4434E-03	1.4428E-04
1.10	4.5133E-01	2.8236E-01	1.3332E-01	6.4906E-02	1.2718E-02	1.2660E-03	1.2654E-04
1.20	4.0169E-01	2.5131E-01	1.1866E-01	5.7767E-02	1.1319E-02	1.1268E-03	1.1263E-04
1.30	3.6174E-01	2.2631E-01	1.0686E-01	5.2022E-02	1.0193E-02	1.0147E-03	1.0143E-04
1.40	3.2892E-01	2.0578E-01	9.7163E-02	4.7302E-02	9.2684E-03	9.2265E-04	9.2224E-05
1.50	3.0150E-01	1.8863E-01	8.9063E-02	4.3359E-02	8.4957E-03	8.4574E-04	8.4536E-05
1.60	2.7826E-01	1.7409E-01	8.2198E-02	4.0017E-02	7.8408E-03	7.8055E-04	7.8019E-05
1.70	2.5832E-01	1.6161E-01	7.6306E-02	3.7149E-02	7.2788E-03	7.2460E-04	7.2427E-05
1.80	2.4102E-01	1.5079E-01	7.1196E-02	3.4661E-02	6.7914E-03	6.7608E-04	6.7577E-05
1.90	2.2588E-01	1.4131E-01	6.6724E-02	3.2483E-02	6.3648E-03	6.3361E-04	6.3332E-05
2.00	2.1252E-01	1.3295E-01	6.2776E-02	3.0562E-02	5.9882E-03	5.9612E-04	5.9585E-05

Table B7. Table of γ values for p_0 = 0.50 and a selected set of values for α and R_0 in Eq.(7).

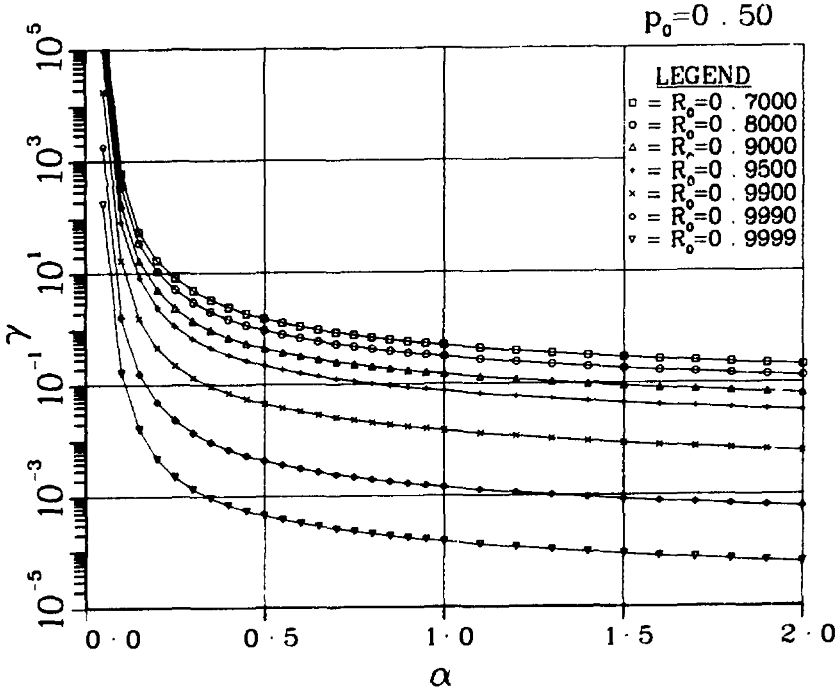

Figure B7. A graph of Table B7.

p_0 =0.75

$\alpha \backslash R_0$	0.7000	0.8000	0.9000	0.9500	0.9900	0.9990	0.9999
0.05	6.7099E+11	4.1979E+11	1.9821E+11	9.6495E+10	1.8907E+10	1.8822E+09	1.8813E+08
0.10	6.1584E+05	3.8528E+05	1.8192E+05	8.8563E+04	1.7353E+04	1.7275E+03	1.7267E+02
0.15	5.8431E+03	3.6556E+03	1.7260E+03	8.4030E+02	1.6465E+02	1.6390E+01	1.6383E+00
0.20	5.5941E+02	3.4998E+02	1.6525E+02	8.0448E+01	1 5763E+01	1.5692E+00	1.5685E-01
0.25	1.3499E+02	8.4454E+01	3.9876E+01	1.9413E+01	3.8038E+00	3.7867E-01	3.7849E-02
0.30	5.1693E+01	3.2341E+01	1.5270E+01	7.4340E+00	1.4566E+00	1.4500E-01	1.4494E-02
0.35	2.5763E+01	1.6118E+01	7.6103E+00	3.7050E+00	7.2595E-01	7.2267E-02	7.2235E-03
0.40	1.5136E+01	9.4693E+00	4.4711E+00	2.1767E+00	4.2650E-01	4.2457E-02	4.2438E-03
0.45	9.9233E+00	6.2082E+00	2.9313E+00	1.4271E+00	2.7962E-01	2.7836E-02	2.7823E-03
0.50	7.0259E+00	4.3956E+00	2.0754E+00	1.0104E+00	1.9798E-01	1.9708E-02	1.9699E-03
0.55	5.2616E+00	3.2918E+00	1.5543E+00	7.5667E-01	1.4826E-01	1.4759E-02	1.4753E-03
0.60	4.1105E+00	2.5716E+00	1.2142E+00	5.9113E-01	1.1583E-01	1.1530E-02	1.1525E-03
0.65	3.3180E+00	2.0758E+00	9.8013E-01	4.7717E-01	9.3495E-02	9.3073E-03	9.3031E-04
0.70	2.7486E+00	1.7196E+00	8.1193E-01	3.9528E-01	7.7450E-02	7.7101E-03	7.7066E-04
0.75	2.3250E+00	1.4546E+00	6.8680E-01	3.3436E-01	6.5514E-02	6.5218E-03	6.5189E-04
0.80	2.0007E+00	1.2517E+00	5.9100E-01	2.8772E-01	5.6375E-02	5.6121E-03	5.5096E-04
0.85	1.7464E+00	1.0926E+00	5.1587E-01	2.5114E-01	4.9208E-02	4.8986E-03	4.8964E-04
0.90	1.5428E+00	9.6519E-01	4.5573E-01	2.2187E-01	4.3472E-02	4.3276F-03	4.3256E-04
0.95	1.3769E+00	8.6145E-01	4.0675E-01	1.9802E-01	3.8799E-02	3.8624E-03	3.8607E-04
1.00	1.2398E+00	7.7566E-01	3.6624E-01	1.7830E-01	3.4936E-02	3.4778E-03	3.4762E-04
1.10	1.0275E+00	6.4284E-01	3.0353E-01	1.4777E-01	2.8953E-02	2.8823E-03	2.8810E-04
1.20	8.7199E-01	5.4554E-01	2.5758E-01	1.2540E-01	2.4571E-02	2.4460E-03	2.4449E-04
1.30	7.5400E-01	4.7172E-01	2.2273E-01	1.0843E-01	2.1246E-02	2.1150E-03	2.1141E-04
1.40	6.6190E-01	4.1410E-01	1.9552E-01	9.5157E-02	1.8651E-02	1.8567E-03	1.8558E-04
1.50	5.8831E-01	3.6806E-01	1.7379E-01	8.4605E-02	1.6577E-02	1.6503E-03	1.6495E-04
1.60	5.2837E-01	3.3056E-01	1.5608E-01	7.5984E-02	1.4888E-02	1.4821E-03	1.4814E-04
1.70	4.7872E-01	2.9950E-01	1.4141E-01	6.8844E-02	1.3489E-02	1.3428E-03	1.3422E-04
1.80	4.3701E-01	2.7340E-01	1.2909E-01	6.2846E-02	1.2314E-02	1.2258E-03	1.2253E-04
1.90	4.0154E-01	2.5121E-01	1.1861E-01	5.7745E-02	1.1314E-02	1.1263E-03	1.1258E-04
2.00	3.7104E-01	2.3213E-01	1.0960E-01	5.3359E-02	1.0455E-02	1.0408E-03	1.0403E-04

Table B8. Table of γ values for p_0 = 0.75 and a selected set of values for α and R_0 in Eq.(7).

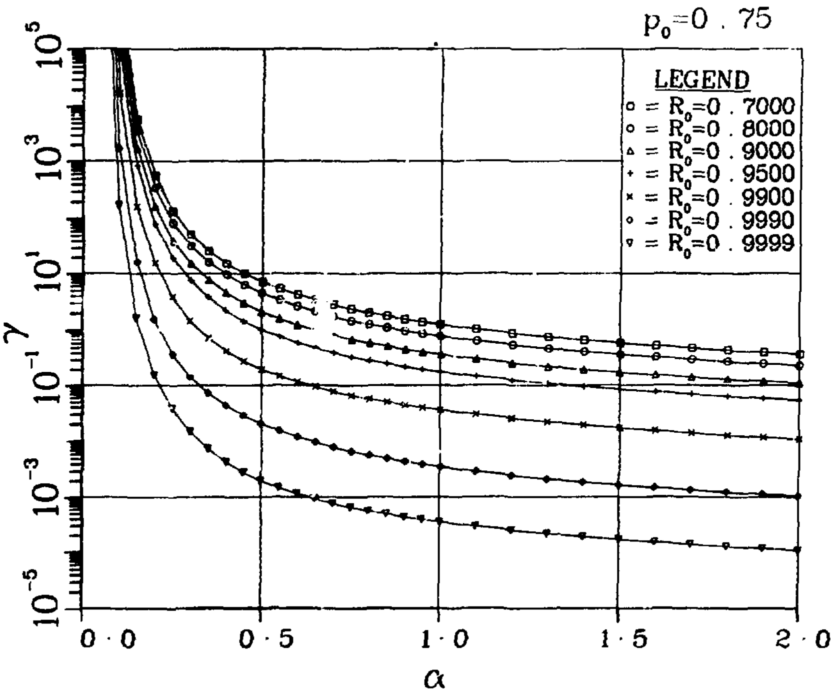

Figure B8. A graph of Table B8.

$p_0 = 0.90$

$\alpha \backslash R_0$	0.900000	0.950000	0.990000	0.999000	0.999900	0.999990	0.999999
0.05	1.8027E+19	8.7761E+18	1.7196E+18	1.7118E+17	1.7111E+16	1.7110E+15	1.7110E+14
0.10	1.7349E+09	8.4461E+08	1.6549E+08	1.6474E+07	1.6467E+06	1.6466E+05	1.6466E+04
0.15	7.7626E+05	3.7791E+05	7.4047E+04	7.3713E+03	7.3680E+02	7.3677E+01	7.3676E+00
0.20	1.6146E+04	7.8604E+03	1.5402E+03	1.5332E+02	1.5325E+01	1.5325E+00	1.5324E-01
0.25	1.5609E+03	7.5989E+02	1.4889E+02	1.4822E+01	1.4815E+00	1.4815E-01	1.4815E-02
0.30	3.2546E+02	1.5845E+02	3.1046E+01	3.0906E+00	3.0892E-01	3.0891E-02	3.0890E-03
0.35	1.0532E+02	5.1272E+01	1.0046E+01	1.0001E+00	9.9963E-02	9.9958E-03	9.9958E-04
0.40	4.4856E+01	2.1837E+01	4.2788E+00	4.2595E-01	4.2576E-02	4.2574E-03	4.2574E-04
0.45	2.2946E+01	1.1171E+01	2.1888E+00	2.1790E-01	2.1780E-02	2.1779E-03	2.1779E-04
0.50	1.3345E+01	6.4966E+00	1.2729E+00	1.2672E-01	1.2666E-02	1.2666E-03	1.2666E-04
0.55	8.5197E+00	4.1477E+00	8.1269E-01	8.0903E-02	8.0866E-03	8.0863E-04	8.0862E-05
0.60	5.8338E+00	2.8401E+00	5.5648E-01	5.5397E-02	5.5372E-03	5.5370E-04	5.5370E-05
0.65	4.2159E+00	2.0524E+00	4.0215E-01	4.0034E-02	4.0016E-03	4.0014E-04	4.0014E-05
0.70	3.1787E+00	1.5475E+00	3.0322E-01	3.0185E-02	3.0172E-03	3.0170E-04	3.0170E-05
0.75	2.4796E+00	1.2072E+00	2.3653E-01	2.3546E-02	2.3536E-03	2.3535E-04	2.3535E-05
0.80	1.9886E+00	9.6813E-01	1.8969E-01	1.8884E-02	1.8875E-03	1.8874E-04	1.8874E-05
0.85	1.6318E+00	7.9440E-01	1.5565E-01	1.5495E-02	1.5488E-03	1.5487E-04	1.5487E-05
0.90	1.3648E+00	6.6445E-01	1.3019E-01	1.2960E-02	1.2955E-03	1.2954E-04	1.2954E-05
0.95	1.1602E+00	5.6483E-01	1.1067E-01	1.1017E-02	1.1012E-03	1.1012E-04	1.1012E-05
1.00	1.0000E+00	4.8684E-01	9.5390E-02	9.4960E-03	9.4917E-04	9.4913E-05	9.4912E-06
1.10	7.6875E-01	3.7425E-01	7.3331E-02	7.3000E-03	7.2967E-04	7.2964E-05	7.2964E-06
1.20	6.1292E-01	2.9839E-01	5.8467E-02	5.8203E-03	5.8177E-04	5.8174E-05	5.8174E-06
1.30	5.0278E-01	2.4477E-01	4.7960E-02	4.7744E-03	4.7722E-04	4.7720E-05	4.7720E-06
1.40	4.2189E-01	2.0539E-01	4.0244E-02	4.0062E-03	4.0044E-04	4.0042E-05	4.0042E-06
1.50	3.6059E-01	1.7555E-01	3.4397E-02	3.4242E-03	3.4226E-04	3.4225E-05	3.4225E-06
1.60	3.1293E-01	1.5235E-01	2.9850E-02	2.9716E-03	2.9702E-04	2.9701E-05	2.9701E-06
1.70	2.7505E-01	1.3390E-01	2.6237E-02	2.6119E-03	2.6107E-04	2.6106E-05	2.6106E-06
1.80	2.4438E-01	1.1898E-01	2.3312E-02	2.3207E-03	2.3196E-04	2.3195E-05	2.3195E-06
1.90	2.1916E-01	1.0669E-01	2.0905E-02	2.0811E-03	2.0802E-04	2.0801E-05	2.0801E-06
2.00	1.9812E-01	9.6450E-02	1.8898E-02	1.8813E-03	1.8805E-04	1.8804E-05	1.8804E-06

Table B9. Table of γ values for $p_0 = 0.90$ and a selected set of values for α and R_0 in Eq.(7).

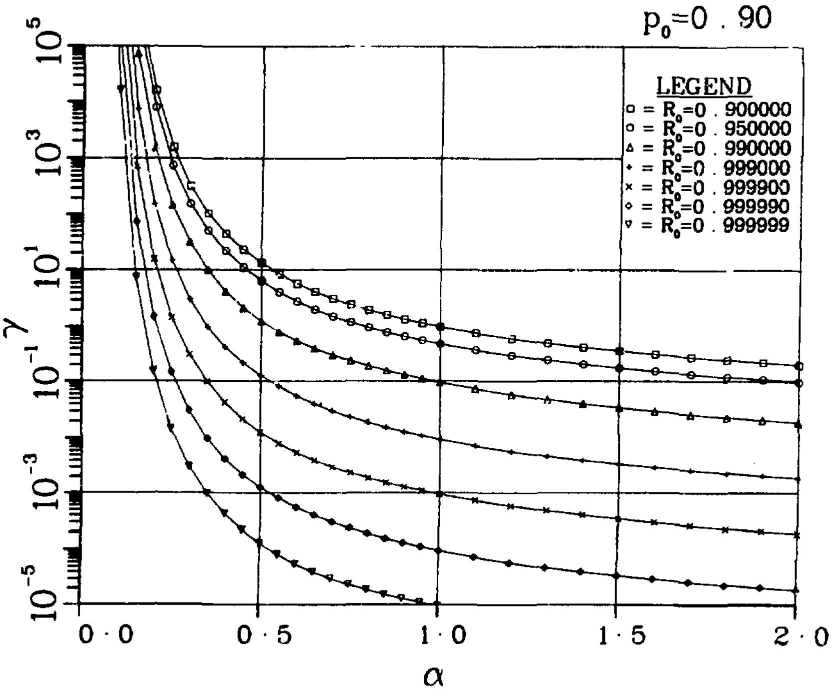

Figure B9. A graph of Table B9.

$p_0=0.95$

$\alpha \backslash R_0$	0.900000	0.950000	0.990000	0.999000	0.999900	0.999990	0.999999
0.05	1.8903E+25	9.2024E+24	1.8031E+24	1.7950E+23	1.7942E+22	1.7941E+21	1.7941E+20
0.10	1.7765E+12	8.6488E+11	1.6946E+11	1.6870E+10	1.6862E+09	1.6861E+08	1.6861E+07
0.15	7.8863E+07	3.8393E+07	7.5227E+06	7.4888E+05	7.4854E+04	7.4851E+03	7.4851E+02
0.20	5.1667E+05	2.5153E+05	4.9285E+04	4.9063E+03	4.9041E+02	4.9039E+01	4.9039E+00
0.25	2.4975E+04	1.2159E+04	2.3824E+03	2.3717E+02	2.3706E+01	2.3705E+00	2.3705E-01
0.30	3.2812E+03	1.5974E+03	3.1299E+02	3.1158E+01	3.1144E+00	3.1143E-01	3.1143E-02
0.35	7.6359E+02	3.7174E+02	7.2838E+01	7.2510E+00	7.2477E-01	7.2474E-02	7.2474E-03
0.40	2.5409E+02	1.2370E+02	2.4238E+01	2.4129E+00	2.4118E-01	2.4117E-02	2.4117E-03
0.45	1.0734E+02	5.2255E+01	1.0239E+01	1.0193E+00	1.0188E-01	1.0188E-02	1.0187E-03
0.50	5.3589E+01	2.6089E+01	5.1119E+00	5.0888E-01	5.0865E-02	5.0863E-03	5.0863E-04
0.55	3.0216E+01	1.4710E+01	2.8823E+00	2.8693E-01	2.8680E-02	2.8679E-03	2.8678E-04
0.60	1.8665E+01	9.0868E+00	1.7804E+00	1.7724E-01	1.7716E-02	1.7715E-03	1.7715E-04
0.65	1.2369E+01	6.0216E+00	1.1799E+00	1.1746E-01	1.1740E-02	1.1740E-03	1.1740E-04
0.70	8.6622E+00	4.2171E+00	8.2629E-01	8.2256E-02	8.2219E-03	8.2216E-04	8.2215E-05
0.75	6.3408E+00	3.0869E+00	6.0484E-01	6.0212E-02	6.0185E-03	6.0182E-04	6.0182E-05
0.80	4.8115E+00	2.3424E+00	4.5897E-01	4.5690E-02	4.5670E-03	4.5667E-04	4.5667E-05
0.85	3.7611E+00	1.8310E+00	3.5877E-01	3.5715E-02	3.5699E-03	3.5698E-04	3.5698E-05
0.90	3.0138E+00	1.4672E+00	2.8749E-01	2.8619E-02	2.8606E-03	2.8605E-04	2.8605E-05
0.95	2.4660E+00	1.2005E+00	2.3523E-01	2.3417E-02	2.3407E-03	2.3406E-04	2.3405E-05
1.00	2.0541E+00	1.0000E+00	1.9594E-01	1.9505E-02	1.9497E-03	1.9496E-04	1.9496E-05
1.10	1.4892E+00	7.2498E-01	1.4205E-01	1.4141E-02	1.4135E-03	1.4134E-04	1.4134E-05
1.20	1.1311E+00	5.5068E-01	1.0790E-01	1.0741E-02	1.0736E-03	1.0736E-04	1.0736E-05
1.30	8.9084E-01	4.3370E-01	8.4978E-02	8.4594E-03	8.4556E-04	8.4552E-05	8.4552E-06
1.40	7.2201E-01	3.5150E-01	6.8873E-02	6.8562E-03	6.8531E-04	6.8528E-05	6.8528E-06
1.50	5.9890E-01	2.9157E-01	5.7129E-02	5.6871E-03	5.6846E-04	5.6843E-05	5.6843E-06
1.60	5.0634E-01	2.4650E-01	4.8299E-02	4.8082E-03	4.8060E-04	4.8058E-05	4.8058E-06
1.70	4.3493E-01	2.1174E-01	4.1488E-02	4.1301E-03	4.1282E-04	4.1280E-05	4.1280E-06
1.80	3.7863E-01	1.8433E-01	3.6117E-02	3.5955E-03	3.5938E-04	3.5937E-05	3.5937E-06
1.90	3.3341E-01	1.6231E-01	3.1804E-02	3.1660E-03	3.1646E-04	3.1644E-05	3.1644E-06
2.00	2.9649E-01	1.4434E-01	2.8282E-02	2.8154E-03	2.8142E-04	2.8140E-05	2.8140E-06

Table B10. Table of γ values for $p_0 = 0.95$ and a selected set of values for α and R_0 in Eq.(7).

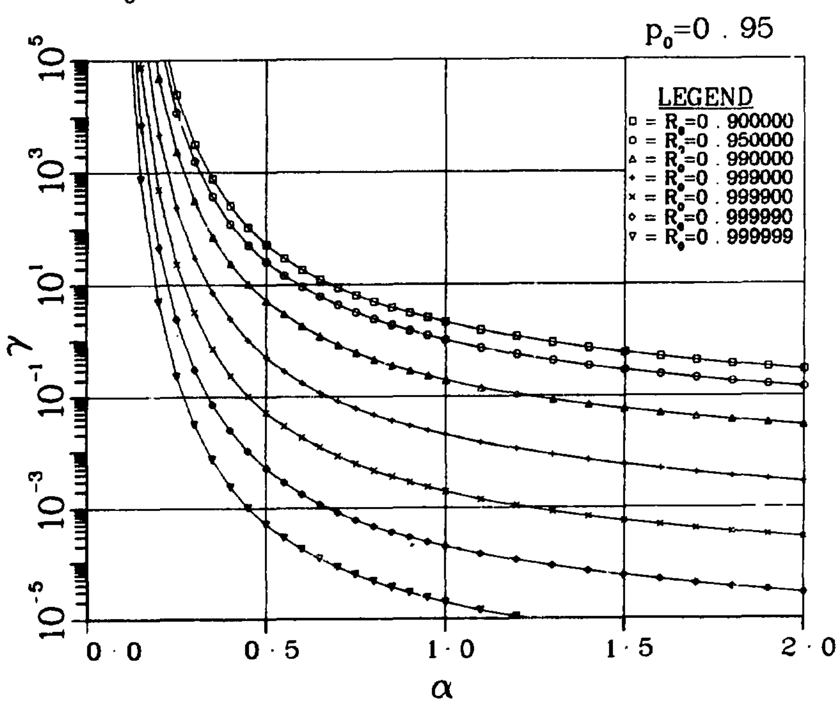

Figure B10. A graph of Table B10.

$p_0 = 0.975$

$\alpha \backslash R_0$	0.900000	0.950000	0.990000	0.999000	0.999900	0.999990	0.999999
0.05	1.9821E+31	9.6495E+30	1.8907E+30	1.8822E+29	1.8813E+28	1.8812E+27	1.8812E+26
0.10	1.8192E+15	8.8563E+14	1.7353E+14	1.7275E+13	1.7267E+12	1.7266E+11	1.7266E+10
0.15	8.0120E+09	3.9005E+09	7.6426E+08	7.6082E+07	7.6047E+06	7.6044E+05	7.6044E+04
0.20	1.6534E+07	8.0491E+06	1.5771E+06	1.5700E+05	1.5693E+04	1.5692E+03	1.5692E+02
0.25	3.9961E+05	1.9454E+05	3.8119E+04	3.7947E+03	3.7930E+02	3.7928E+01	3.7928E+00
0.30	3.3073E+04	1.6101E+04	3.1548E+03	3.1406E+02	3.1392E+01	3.1391E+00	3.1390E-01
0.35	5.5333E+03	2.6938E+03	5.2792E+02	5.2544E+01	5.2520E+00	5.2518E-01	5.2518E-02
0.40	1.4377E+03	6.9993E+02	1.3714E+02	1.3653E+01	1.3646E+00	1.3646E-01	1.3646E-02
0.45	5.0111E+02	2.4396E+02	4.7801E+01	4.7585E+00	4.7564E-01	4.7561E-02	4.7561E-03
0.50	2.1457E+02	1.0446E+02	2.0468E+01	2.0375E+00	2.0366E-01	2.0365E-02	2.0365E-03
0.55	1.0672E+02	5.1957E+01	1.0180E+01	1.0134E+00	1.0130E-01	1.0129E-02	1.0129E-03
0.60	5.9401E+01	2.8919E+01	5.6663E+00	5.6407E-01	5.6382E-02	5.6379E-03	5.6379E-04
0.65	3.6052E+01	1.7551E+01	3.4390E+00	3.4235E-01	3.4219E-02	3.4218E-03	3.4218E-04
0.70	2.3422E+01	1.1403E+01	2.2342E+00	2.2242E-01	2.2232E-02	2.2231E-03	2.2231E-04
0.75	1.6070E+01	7.8233E+00	1.5329E+00	1.5260E-01	1.5253E-02	1.5252E-03	1.5252E-04
0.80	1.1525E+01	5.6108E+00	1.0994E+00	1.0944E-01	1.0939E-02	1.0939E-03	1.0939E-04
0.85	8.5733E+00	4.1738E+00	8.1780E-01	8.1411E-02	8.1375E-03	8.1371E-04	8.1371E-05
0.90	6.5750E+00	3.2010E+00	6.2719E-01	6.2436E-02	6.2408E-03	6.2406E-04	6.2405E-05
0.95	5.1739E+00	2.5189E+00	4.9354E-01	4.9132E-02	4.9109E-03	4.9107E-04	4.9107E-05
1.00	4.1615E+00	2.0260E+00	3.9697E-01	3.9518E-02	3.9500E-03	3.9498E-04	3.9498E-05
1.10	2.8412E+00	1.3832E+00	2.7103E-01	2.6980E-02	2.6968E-03	2.6967E-04	2.6967E-05
1.20	2.0538E+00	9.9985E-01	1.9591E-01	1.9503E-02	1.9494E-03	1.9493E-04	1.9493E-05
1.30	1.5515E+00	7.5533E-01	1.4800E-01	1.4733E-02	1.4727E-03	1.4726E-04	1.4726E-05
1.40	1.2137E+00	5.9087E-01	1.1577E-01	1.1525E-02	1.1520E-03	1.1520E-04	1.1519E-05
1.50	9.7649E-01	4.7539E-01	9.3147E-02	9.2727E-03	9.2685E-04	9.2681E-05	9.2680E-06
1.60	8.0392E-01	3.9138E-01	7.6686E-02	7.6340E-03	7.6306E-04	7.6302E-05	7.6302E-06
1.70	6.7462E-01	3.2843E-01	6.4352E-02	6.4062E-03	6.4033E-04	6.4030E-05	6.4030E-06
1.80	5.7529E-01	2.8007E-01	5.4876E-02	5.4629E-03	5.4604E-04	5.4602E-05	5.4602E-06
1.90	4.9732E-01	2.4211E-01	4.7440E-02	4.7226E-03	4.7204E-04	4.7202E-05	4.7202E-06
2.00	4.3500E-01	2.1177E-01	4.1494E-02	4.1307E-03	4.1289E-04	4.1287E-05	4.1287E-06

Table B11. Table of γ values for $p_0 = 0.975$ and a selected set of values for α and R_0 in Eq.(7).

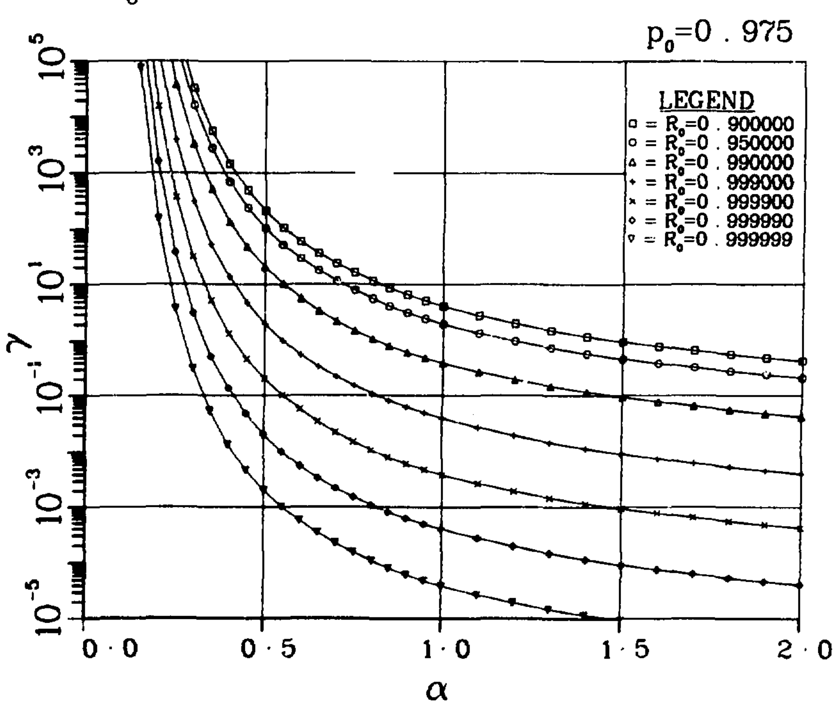

Figure B11. *A graph of Table B11.*

$p_0 = 0.99$

α\R₀	0.900000	0.950000	0.990000	0.999000	0.999900	0.999990	0.999999
0.05	1.8027E+39	8.7761E+38	1.7196E+38	1.7118E+37	1.7111E+36	1.7110E+35	1.7110E+34
0.10	1.7349E+19	8.4461E+18	1.6549E+18	1.6474E+17	1.6467E+16	1.6466E+15	1.6466E+14
0.15	3.6031E+12	1.7541E+12	3.4370E+11	3.4215E+10	3.4199E+09	3.4198E+08	3.4198E+07
0.20	1.6146E+09	7.8603E+08	1.5402E+08	1.5332E+07	1.5325E+06	1.5325E+05	1.5325E+04
0.25	1.5610E+07	7.5993E+06	1.4890E+06	1.4823E+05	1.4816E+04	1.4816E+03	1.4815E+02
0.30	7.0136E+05	3.4145E+05	6.6903E+04	6.6601E+03	6.6571E+02	6.6568E+01	6.6568E+00
0.35	7.5851E+04	3.6927E+04	7.2354E+03	7.2027E+02	7.1995E+01	7.1992E+00	7.1991E-01
0.40	1.4208E+04	6.9171E+03	1.3553E+03	1.3492E+02	1.3486E+01	1.3486E+00	1.3485E-01
0.45	3.8397E+03	1.8693E+03	3.6627E+02	3.6462E+01	3.6445E+00	3.6444E-01	3.6443E-02
0.50	1.3414E+03	6.5305E+02	1.2796E+02	1.2738E+01	1.2732E+00	1.2732E-01	1.2732E-02
0.55	5.6495E+02	2.7504E+02	5.3891E+01	5.3648E+00	5.3624E-01	5.3621E-02	5.3621E-03
0.60	2.7378E+02	1.3329E+02	2.6116E+01	2.5998E+00	2.5986E-01	2.5985E-02	2.5985E-03
0.65	1.4782E+02	7.1962E+01	1.4100E+01	1.4037E+00	1.4030E-01	1.4030E-02	1.4030E-03
0.70	8.6886E+01	4.2299E+01	8.2881E+00	8.2507E-01	8.2470E-02	8.2466E-03	8.2466E-04
0.75	5.4669E+01	2.6615E+01	5.2149E+00	5.1913E-01	5.1890E-02	5.1888E-03	5.1887E-04
0.80	3.6355E+01	1.7699E+01	3.4679E+00	3.4523E-01	3.4507E-02	3.4506E-03	3.4506E-04
0.85	2.5306E+01	1.2320E+01	2.4139E+00	2.4030E-01	2.4019E-02	2.4018E-03	2.4018E-04
0.90	1.8298E+01	8.9080E+00	1.7454E+00	1.7375E-01	1.7368E-02	1.7367E-03	1.7367E-04
0.95	1.3662E+01	6.6512E+00	1.3032E+00	1.2973E-01	1.2968E-02	1.2967E-03	1.2967E-04
1.00	1.0483E+01	5.1036E+00	1.0000E+00	9.9549E-02	9.9504E-03	9.9500E-04	9.9499E-05
1.10	6.6012E+00	3.2137E+00	6.2969E-01	6.268[illegible]E-02	6.2657E-03	6.2654E-04	6.2654E-05
1.20	4.4628E+00	2.1726E+00	4.2570E-01	4.2378E-02	4.2359E-03	4.2357E-04	4.2357E-05
1.30	3.187[illegible]E+00	1.5516E+00	3.0402E-01	3.0265E-02	3.0251E-03	3.0250E-04	3.0250E-05
1.40	2.3767E+00	1.1571E+00	2.2671E-01	2.2569E-02	2.2559E-03	2.2558E-04	2.2558E-05
1.50	1.8350E+00	8.9336E-01	1.7504E-01	1.7425E-02	1.7418E-03	1.7417E-04	1.7417E-05
1.60	1.4576E+00	7.0961E-01	1.3904E-01	1.3841E-02	1.3835E-03	1.3834E-04	1.3834E-05
1.70	1.185[illegible]E+00	5.7705E-01	1.1307E-01	1.1256E-02	1.1251E-03	1.1250E-04	1.1250E-05
1.80	9.8307E-01	4.7859E-01	9.3775E-02	9.3352E-03	9.3310E-04	9.3305E-05	9.3305E-06
1.90	8.2906E-01	4.0361E-01	7.9084E-02	7.8727E-03	7.8691E-04	7.8688E-05	7.8688E-06
2.00	7.0924E-01	3.4528E-01	6.7654E-02	6.7349E-03	6.7319E-04	6.7316E-05	6.7315E-06

Table B12. Table of γ values for $p_0 = 0.99$ and a selected set of values for α and R_0 in Eq.(7).

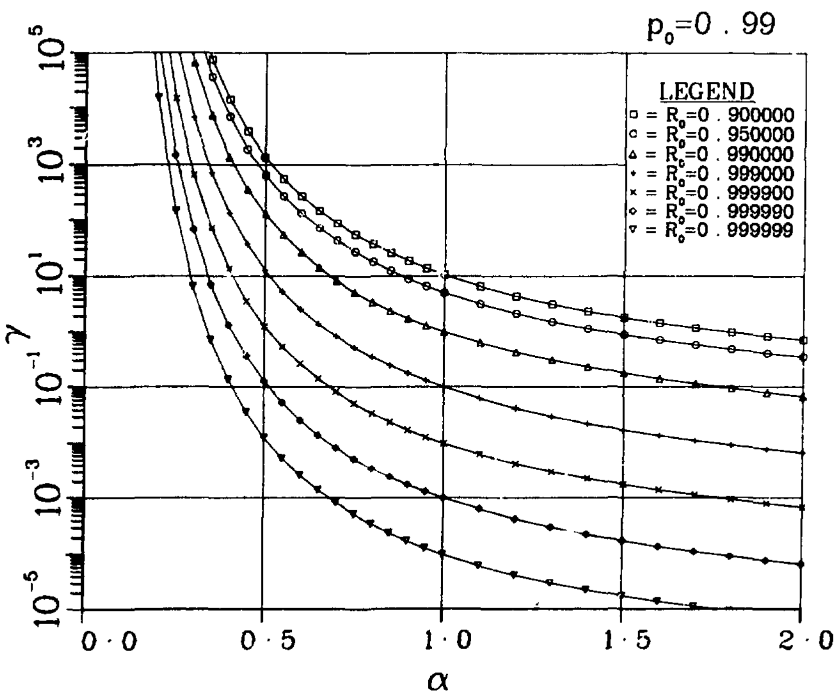

Figure B12. A graph of Table B12.

p_0=0.995

α\R0	0.900000	0.950000	0.990000	0.999000	0.999900	0.999990	0.999999
0.05	1.8903E+45	9.2024E+44	1.8031E+44	1.7950E+43	1.7942E+42	1.7941E+41	1.7941E+40
0.10	1.7765E+22	8.6488E+21	1.6946E+21	1.6870E+20	1.6862E+19	1.6861E+18	1.6861E+17
0.15	3.6605E+14	1.7821E+14	3.4917E+13	3.4760E+12	3.4744E+11	3.4743E+10	3.4743E+09
0.20	5.1667E+10	2.5153E+10	4.9285E+09	4.9063E+08	4.9041E+07	4.9039E+06	4.9039E+05
0.25	2.4975E+08	1.2159E+08	2.3824E+07	2.3717E+06	2.3706E+05	2.3705E+04	2.3705E+03
0.30	7.0693E+06	3.4416E+06	6.7434E+05	6.7130E+04	6.7100E+03	6.7097E+02	6.7096E+01
0.35	5.4960E+05	2.6756E+05	5.2426E+04	5.2190E+03	5.2166E+02	5.2164E+01	5.2164E+00
0.40	8.0375E+04	3.9129E+04	7.6670E+03	7.6324E+02	7.6289E+01	7.6286E+00	7.6286E-01
0.45	1.7917E+04	8.7225E+03	1.7091E+03	1.7014E+02	1.7006E+01	1.7005E+00	1.7005E-01
0.50	5.3659E+03	2.6123E+03	5.1185E+02	5.0954E+01	5.0931E+00	5.0929E-01	5.0929E-02
0.55	1.9924E+03	9.6997E+02	1.9006E+02	1.8920E+01	1.8911E+00	1.8910E-01	1.8910E-02
0.60	8.6934E+02	4.2323E+02	8.2927E+01	8.2553E+00	8.2515E-01	8.2512E-02	8.2511E-03
0.65	4.2951E+02	2.0910E+02	4.0971E+01	4.0786E+00	4.0767E-01	4.0766E-02	4.0765E-03
0.70	2.3399E+02	1.1391E+02	2.2320E+01	2.2219E+00	2.2209E-01	2.2208E-02	2.2208E-03
0.75	1.3785E+02	6.7110E+01	1.3149E+01	1.3090E+00	1.3084E-01	1.3084E-02	1.3083E-03
0.80	8.6549E+01	4.2135E+01	8.2559E+00	8.2186E-01	8.2149E-02	8.2146E-03	8.2145E-04
0.85	5.7268E+01	2.7880E+01	5.4628E+00	5.4382E-01	5.4357E-02	5.4355E-03	5.4355E-04
0.90	3.9590E+01	1.9274E+01	3.7765E+00	3.7594E-01	3.7577E-02	3.7576E-03	3.7575E-04
0.95	2.8397E+01	1.3825E+01	2.7088E+00	2.6966E-01	2.6954E-02	2.6953E-03	2.6953E-04
1.00	2.1019E+01	1.0233E+01	2.0050E+00	1.9960E-01	1.9951E-02	1.9950E-03	1.9950E-04
1.10	1.2440E+01	6.0564E+00	1.1867E+00	1.1813E-01	1.1808E-02	1.1807E-03	1.1807E-04
1.20	7.9894E+00	3.8895E+00	7.6211E-01	7.5867E-02	7.5833E-03	7.5830E-04	7.5829E-05
1.30	5.4646E+00	2.6603E+00	5.2126E-01	5.1891E-02	5.1868E-03	5.1866E-04	5.1865E-05
1.40	3.9278E+00	1.9122E+00	3.7468E-01	3.7299E-02	3.7282E-03	3.7280E-04	3.7280E-05
1.50	2.9380E+00	1.4303E+00	2.8026E-01	2.7899E-02	2.7887E-03	2.7886E-04	2.7886E-05
1.60	2.2703E+00	1.1052E+00	2.1656E-01	2.1558E-02	2.1549E-03	2.1548E-04	2.1548E-05
1.70	1.8020E+00	8.7730E-01	1.7190E-01	1.7112E-02	1.7104E-03	1.7104E-04	1.7104E-05
1.80	1.4629E+00	7.1221E-01	1.3955E-01	1.3892E-02	1.3886E-03	1.3885E-04	1.3885E-05
1.90	1.2105E+00	5.8931E-01	1.1547E-01	1.1495E-02	1.1490E-03	1.1489E-04	1.1489E-05
2.00	1.0180E+00	4.9561E-01	9.7110E-02	9.6672E-03	9.6628E-04	9.6624E-05	9.6623E-06

Table B13. Table of γ values for p_0 = 0.995 and a selected set of values for α and R_0 in Eq.(7).

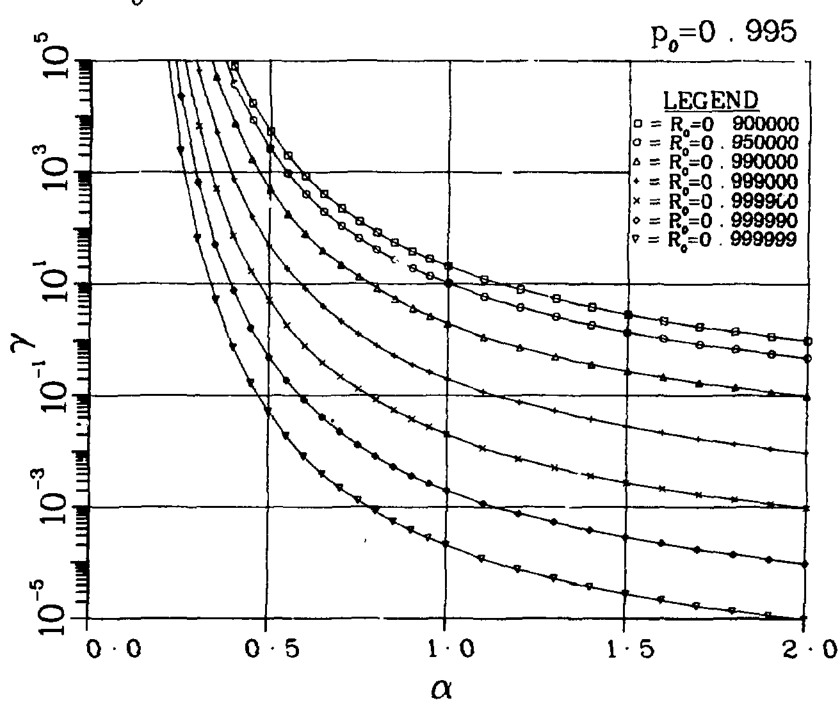

Figure B13. A graph of Table B13.

APPENDIX C

MATHEMATICAL RESULTS

C.1. A theorem on the asymptotic behavior of gamma percentiles.

Theorem: If, as $\alpha \to 0^+$, $K(\alpha)$ is defined by

$$\frac{1}{\Gamma(\alpha)} \int_0^{K(\alpha)} e^{-x} x^{\alpha-1} dx = p ,$$

and if $0 < p < 1$, then

$$K(\alpha) \sim p^{\frac{1}{\alpha}} ,$$

where $f(\alpha) \sim g(\alpha)$ means $\lim_{\alpha \to \alpha_0} \frac{f(\alpha)}{g(\alpha)} = 1$

Proof: Consider

$$F(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^K e^{-x} x^{\alpha-1} dx$$

for small α and fixed $0 < K < \infty$. Now

$$F(\alpha) = \frac{\int_0^K e^{-x} x^{\alpha-1} dx}{\int_0^K e^{-x} x^{\alpha-1} dx + \int_K^\infty e^{-x} x^{\alpha-1} dx}$$

and

(i) $\lim_{\alpha \to 0^+} \int_0^K e^{-x} x^{\alpha-1} dx = +\infty$

(ii) there is an M such that, for $0 < \alpha < \alpha_1$, $\int_K^\infty e^{-x} x^{\alpha-1} dx < M$.

Therefore, $F(\alpha) \to 1$ as $\alpha \to 0^+$. This implies, if $K(\alpha)$ satisfies

$$\frac{1}{\Gamma(\alpha)} \int_0^{K(\alpha)} e^{-x} x^{\alpha-1} \, dx = p \ ,$$

then

$$\lim_{\alpha \to 0^+} K(\alpha) = 0.$$

Then, as $\alpha \to 0^+$,

$$p = \frac{1}{\Gamma(\alpha)} \int_0^{K(\alpha)} e^{-x} x^{\alpha-1} \, dx \sim \frac{1}{\Gamma(\alpha)} \int_0^{K(\alpha)} x^{\alpha-1} \, dx$$

$$= [\alpha \ \Gamma(\alpha)]^{-1} \ [K(\alpha)]^{\alpha}$$

$$= [\Gamma(\alpha+1)]^{-1} \ [K(\alpha)]^{\alpha}$$

$$\sim [K(\alpha)]^{\alpha} \ .$$

Therefore, the conclusion, $K(\alpha) \sim p^{1/\alpha}$, holds.

C.2. Differentiation of a negative-log gamma density.

Let

$$k(r) = \frac{(-\ln r)^{\alpha-1} \ r^{\frac{1}{\gamma}-1}}{\gamma^{\alpha} \ \Gamma(\alpha)} , \quad 0 < r < 1 \ .$$

Differentiation gives

$$\frac{\partial k(r)}{\partial r} = \frac{(-\ln r)^{\alpha-2} \ r^{\frac{1}{\gamma}-2}}{\Gamma(\alpha) \ \gamma^{\alpha+1}} \ [(\gamma-1) \ln r - (\alpha-1)\gamma] \ .$$

Now for $0 < r < 1$,

$$\frac{\partial k(r)}{\partial r} \begin{cases} > 0 \iff (\gamma-1) \ln r > (\alpha-1)\gamma \\ = 0 \iff (\gamma-1) \ln r = (\alpha-1)\gamma \\ < 0 \iff (\gamma-1) \ln r < (\alpha-1)\gamma \ . \end{cases}$$

The nine possible cases for different combinations of α and γ values are summarized in Table II of the report text.

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