

Gamow vectors and Supersymmetric Quantum Mechanics

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Gamow solutions are used to transform self-adjoint energy operators by means of factorization (supersymmetric) techniques. The transformed non-hermitian operators admit a discrete real spectrum which is occasionally extended by a single complex eigenvalue associated to normalized eigensolutions. These new Hamiltonians are not pseudo-hermitian operators and also differ from those obtained by means of complex-scaling transformations. As an example, Coulomb-like potentials are studied.

Keywords: Factorization method, Gamow vectors, Non-hermitian Hamiltonians

El método de factorización es extendido al caso complejo para construir Hamiltonianos no Hermitianos con espectro real. Algunos de los nuevos Hamiltonianos admiten además un eigenvalor complejo con eigenfunción normalizada. Las funciones de transformación usadas son funciones de Gamow. Los nuevos Hamiltonianos no son pseudo-hermitianos y son diferentes también de aquellos obtenidos con el método de dilatación compleja. Se presenta el caso de potenciales Coulombianos como ejemplo.

Descriptores: Método de factorización, vectores de Gamow, Hamiltonianos no Hermitianos

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1. Introduction

Complex energies were studied for the first time in a paper of Gamow concerning the alpha decay (1928) [1]. In a simple picture, a given nucleus is composed in part by alpha particles (${}^4_2\text{He}$ nuclei) which interact with the rest of the nucleus via an attractive well (obeying the presence of nuclear forces) plus a potential barrier (due, in part, to repulsive electrostatic forces). The former interaction constrains the particles to be bounded while the second holds them inside the nucleus. The alpha particles have a small (non-zero) probability of tunneling to the other side of the barrier instead of remaining confined to the interior of the well. Outside the potential region, they have a finite lifetime. Thus, alpha particles in a nucleus should be represented by *quasi-stationary* states. For such states, if at time $t = 0$ the probability of finding the particle inside the well is unity, in subsequent moments the probability will be a slowly decreasing function of time (see *e.g.* Secs. 7 and 8 of Ref. 2).

In his paper of 1928, Gamow studied the escape of alpha particles from the nucleus via the tunnel effect. In order to describe eigenfunctions with exponentially decaying time evolution, Gamow introduced energy eigenfunctions ψ_G belonging to complex eigenvalues $Z_G = E_G - i\Gamma_G$, $\Gamma_G > 0$. The real part of the eigenvalue was identified with the energy of the system and the imaginary part was associated with the inverse of the lifetime. Such ‘decaying states’ were the first application of quantum theory to nuclear physics.

Three years later, in 1931, Fock showed that the law of decay of a quasi-stationary state depends only on the energy distribution function $\omega(E)$ which, in turn, is meromorphic [2]. According to Fock, the analytical expression of $\omega(E)$ is rather simple and has only two poles $E = E_0 \pm i\Gamma$, $\Gamma > 0$ (see Eq. (8.13) of Ref. 2). A close result was derived by Breit and Wigner in 1936. They studied the cross sec-

tion of slow neutrons and found that the related energy distribution reaches its maximum at E_R with a half-maximum width Γ_R . A resonance is supposed to take place at E_R and to have ‘half-value breath’ Γ_R [3]. The resonances can be defined as eigensolutions ψ_R of the Hamiltonian with complex eigenvalue $z_R = E_R - i\Gamma_R/2$. This complex number also corresponds to a first-order pole of the S matrix [4] (for more details see *e.g.* [5]). However, as the Hamiltonian is a Hermitian operator, then (in the Hilbert space \mathcal{H}) there can be no eigenstate having a strict complex exponential dependence on time. In other words, decaying states are an approximation within the conventional quantum mechanics framework. This fact is usually taken to motivate the study of the rigged (equipped) Hilbert space $\tilde{\mathcal{H}}$ [6]. The mathematical structure of $\tilde{\mathcal{H}}$ lies on the nuclear spectral theorem introduced by Dirac in a heuristic form [7] and studied in formal rigor by Maurin [8] and Gelfand and Vilenkin [9].

Some other approaches extend the framework of quantum theory so that quasi-stationary states can be defined in a precise form. For example, the complex-scaling method [10–12] (see also [13]) embraces the transformation $H \rightarrow SHS^{-1} = H_\theta$, where S is the complex-scaling operator $S = e^{-\theta rp}$, $[r, p] = i$, such that $Sf(r) = f(re^{i\theta})$. This transformation converts the description of resonances by non-integrable Gamow states into one by square integrable states (A relevant aspect of the method is that it is possible to construct a resolution to the identity [14]). Thus, the complex-scaled resonance eigenfunctions are θ -dependent so they can be normalized. Moreover, as the complex eigenvalues are θ -independent, the resonance phenomenon is just associated with the discrete part of the complex-scaled Hamiltonian [15] (but see [13]).

In this paper we show that Gamow (decaying) eigensolutions can be used to transform Hermitian Hamiltonians into non-self adjoint energy operators with purely real spectrum

or admitting a single extra complex eigenvalue with square-integrable wavefunction. The new Hamiltonians could be profitable as testing operators in diverse approaches including complex-scaling and pseudo-hermitian [16] transformations. As we shall see, it is not necessary to work in a equipped Hilbert space framework because the Gamow solutions will be used merely as mathematical tools. Moreover, the exponential growing of the Gamow solutions for large distances will be primordial in order to get well-behaved complex potentials. The mechanism we are going to use is the factorization method in a ‘complex’ version [17]. As usual, the procedure and results can be interpreted in terms of supersymmetric quantum mechanics.

The next section introduces general expressions for transforming spherically symmetric potentials in terms of appropriate Gamow vectors. It is shown that new complex potentials are derived so that their discrete spectrum is real. The Coulomb potential is managed as example. The Sec. 3 shows how the approach can be generalized to include an extra single complex eigenvalue into the initial discrete spectrum. The related eigensolution is then shown to be of finite norm. Finally, Sec. 4 is devoted to the concluding remarks.

2. Supersymmetric Gamow transformations

2.1. The complex factorization

Let us consider the time-independent Schrödinger equation for a spherically symmetric potential $V(r)$. After separation of angular variables, the equation reduces to a differential equation involving only the radial variable:

$$H_\ell \psi(r, \ell) = E \psi(r, \ell), \quad (1)$$

which can always be integrated numerically. The reduced Hamiltonian reads

$$H_\ell \equiv -\frac{d^2}{dr^2} + V_\ell(r) = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + V(r), \quad (2)$$

where the effective potential $V_\ell(r)$ has the domain $D_V = [0, +\infty)$ and the units of energy and coordinates have been properly chosen.

The nature of the energy spectrum of H_ℓ may be deduced from the asymptotic behaviour of the solutions $\psi(r, \ell)$ which are regular at the origin. If $V(r)$ approaches zero asymptotically faster than $1/r$: $\lim_{r \rightarrow \infty} rV(r) = 0$, then the energy spectrum contains two parts: (a) Negative discrete values $E_1(\ell), E_2(\ell), \dots$. To each of them corresponds a radial wavefunction of finite norm. (b) Unbound continuous positive spectrum, with solutions regular at the origin but indefinitely oscillating in the asymptotic region. On the other hand, if $V(r)$ approaches zero as $1/r$ when $r \rightarrow \infty$, the essential result concerning the nature of the spectrum persists [18]. We shall concentrate on the discrete spectrum by assuming that a complete set of normalized wavefunctions $\psi_n(r, \ell) \in \mathcal{H}$ has been given for each $V(r)$, otherwise H_ℓ would not be an observable.

We look for a complex-type factorization [17] of the Hamiltonian (2):

$$H_\ell = AB + \epsilon \quad (3)$$

with factorization constant $\mathbb{C} \ni \epsilon = \epsilon_1 + i\epsilon_2$; $\epsilon_1, \epsilon_2 \neq 0 \in \mathbb{R}$ and a couple of not mutually adjoint first order operators

$$A := -\frac{d}{dr} + \beta, \quad B := \frac{d}{dr} + \beta \quad (4)$$

where β is a complex-valued function fulfilling the Riccati equation

$$-\beta'(r) + \beta^2(r) + \epsilon = V_\ell(r). \quad (5)$$

This last equation is easily solved by means of the logarithmic transformation $\beta(r) = -(d/dr) \ln u(r)$, with $u(r)$ the eigensolution of H_ℓ belonging to the complex eigenvalue $\epsilon \equiv -k^2$, $\mathbb{C} \ni k = k_1 + ik_2$; $k_1, k_2 \in \mathbb{R}$.

Remark that $H_\ell^\dagger = B^\dagger A^\dagger + \bar{\epsilon} = H_\ell$ (the bar stands for complex conjugation) because the Hamiltonian is assumed to be self-adjoint in the Hilbert space \mathcal{H} . A relevant aspect of the complex factorization (3)-(5) is that the reverse ordering of the factors gives rise to non-hermitian second order differential operators:

$$BA + \epsilon = H_\ell + 2\beta'(r) := h_\ell. \quad (6)$$

Conventional factorizations assume *a priori* $A = B^\dagger$ and real ϵ (see *e.g.* [19]). In counterdistinction, complex factorization is more in the spirit of the ‘refined factorizations’ reported recently [20] (see also [21]). The following intertwining relationships hold

$$h_\ell B = B H_\ell, \quad H_\ell A = A h_\ell \quad (7)$$

which permit to determine the solutions $\Psi \propto B\varphi$ of $h_\ell \Psi = \lambda \Psi$, $\lambda \in \mathbb{C}$, by giving the solutions φ of $H_\ell \varphi = \lambda \varphi$. The operator A reverses the action of B . In the supersymmetric language, H_ℓ and h_ℓ are understood as supersymmetric partners while $\beta(r)$ is the superpotential (see *e.g.* [22] and references quoted therein).

In general, we want to keep the physical interpretation of Ψ as connected with the probability density $\rho(r) = |\Psi(r)|^2$ in \mathcal{H} (The dependence of Ψ on ℓ will be always implicitly considered). Hence, we look for functions

$$\Psi \propto B\varphi = \frac{W(u, \varphi)}{u} \quad (8)$$

which are square-integrable in \mathcal{H} (the symbol $W(\cdot, \cdot)$ stands for the wronskian of the involved functions). Of course, this last condition is not imperative in Eq. (8). For instance, one could extend the initial boundedness condition $|\psi(r, \ell)|^2 < \infty$ to better admit another kind of normalization in order to generalize selfadjointness (*e.g.*, in the picture of a equipped Hilbert space \mathcal{H}). But, in this way, the physical interpretation of either $\psi(r, \ell)$ or $\Psi(r)$ as wavefunctions is less

clear (one dimension plane waves, for example, are known to be not in $L^2(\mathbb{R})$ but having a probability density which is everywhere finite in the Dirac sense. In other words, the plane waves could be understood as energy Dirac vectors in \mathcal{H} . However, if we apply realistic vanishing boundary conditions at $x = 0$ and $x = L$, or L -periodic boundary conditions, the plane waves can be normalized in the conventional form. Thus, ‘free particles’ are but an abstraction from the actual quantum world).

As it could be expected, the set of eigenvectors (8) is uncommon in \mathcal{H} : though they can be normalized, their elements are not mutually orthogonal [17] (An optional bi-orthogonal basis has been recently discussed in [23]). These vectors are natural in the spaces with an indefinite metric as studied in the Pontrjagin–Krein formalism [24] (see also [25]).

2.2. Gamow transformations

Let us show how the Gamow solutions can be used as transformation functions $u(r)$ in Eq. (8). First, following Gadella–de la Madrid, we define a Gamow function as a solution of the time-independent Schrödinger equation with complex eigenvalue and purely outgoing boundary conditions [26]. Thus, if $u(r)$ is such that $u(r = 0) = 0$, $u(r \rightarrow +\infty) \sim e^{-k_1 r}$ ($k_1 < 0$), and solves (1), (2) with $E = \epsilon \in \mathbb{C}$, then $u(r)$ is a Gamow solution (Observe that ϵ does not necessarily correspond to the poles of the S matrix!). In the context of the alpha decay, the condition $u(r = 0) = 0$ describes the ‘creation’ of alpha particles inside the nucleus and obeys the fact that there cannot be any transmission into the region $r < 0$ because the effective potential is infinite there (*i.e.*, this condition avoids the incoming probabilities and is related with the adjointness of the Hamiltonian [26]). On the other hand, the outgoing boundary condition ensures the decay rate obeyed by the particles after tunneling the electrostatic barrier.

Let us take $\text{Re}(\epsilon) \equiv E_R = k_2^2 - k_1^2 > 0$ in $\epsilon = (k_2^2 - k_1^2) - 2ik_1k_2$. Thus $|k_2| > |k_1|$. We can distinguish two general cases:

- 1) $k_1 < 0, k_2 < 0$. Here $\epsilon^- = E_R - i\Gamma^-/2$, with $\Gamma^- = 4k_1k_2 > 0$, is associated with the decaying part of the solution $U(t)|\phi_{\epsilon^-}\rangle = e^{-itE_R}e^{-t\Gamma^-/2}|\phi_{\epsilon^-}\rangle$.
- 2) $k_1 < 0$ and $k_2 > 0$. The complex energy $\epsilon^+ = E_R + i\Gamma^+/2$, with $\Gamma^+ = 4|k_1|k_2 > 0$, is associated with the growing part of the solution $U(t)|\phi_{\epsilon^+}\rangle = e^{-itE_R}e^{t\Gamma^+/2}|\phi_{\epsilon^+}\rangle$.

In both cases the roles are interchanged under complex conjugation. Now, if ϵ^\pm correspond to the poles z_R^\pm of the S matrix, then the lifetime $\tau = 1/\Gamma_R^\pm$ decreases as the energy increases. Thus, for small widths (large lifetime) the energy resonances are close to the real axis and the Gamow vectors could be considered as bounded states for certain physical phenomena. On the other hand, as Γ_R^\pm increases, the resonances move away from the real axis and the Gamow vectors are far to be considered as representative of bound states.

Now, let us analyze in detail the Eq. (8). Our goal is to characterize the spectrum of h_ℓ as well as its eigenfunctions in terms of the analytical behaviour of $\varphi(r, \ell)$ and the boundary conditions of $u(r)$.

A direct calculation shows that $u(r) \propto r^{\ell+1}$ satisfies $u(r = 0) = 0$. Thereby, Eq. (8) reads

$$\Psi(r \ll 1) \sim \varphi'(r \ll 1) - \frac{\ell+1}{r}\varphi(r \ll 1). \quad (9)$$

It is clear that $\Psi(r)$ will be regular at the origin if φ is such that $\varphi(r \ll 1) \sim r^s$, $s \geq 1$. In other words, if φ is regular at the origin then $\Psi(r = 0) = 0$.

The purely outgoing boundary condition, in turn, is equivalent to the following expression (see [26] p 630):

$$\lim_{r \rightarrow \infty} \frac{d}{dr} \ln u(r) = - \lim_{r \rightarrow \infty} \beta(r) = -k. \quad (10)$$

Hence, Eq. (8) reduces to

$$\lim_{r \rightarrow \infty} \Psi \propto \lim_{r \rightarrow \infty} \varphi' + k \lim_{r \rightarrow \infty} \varphi. \quad (11)$$

As the solution φ grows indefinitely as one of either $e^{\pm\kappa r}$, $\kappa = \sqrt{-\lambda}$, we can identify the following cases:

- I) For a (denumerably infinite) set of negative discrete values $\lambda \in \{E_n(\ell)\}$, the solution φ in (11) behaves as $\varphi \sim e^{-\kappa r}$, $\kappa > 0$. Thus $\Psi \sim (k - \kappa)e^{-\kappa r}$, $\kappa > 0$.
- II) If $\lambda > 0$, then $\varphi \sim \sin(\kappa r - \ell\pi/2 + \delta_\ell)$, with δ_ℓ the phase shift. Thus, φ is an acceptable eigensolution of H_ℓ for any $\lambda > 0$ and represents an unbound state [18]. Hence, if $\lambda > 0$ then $\Psi(r)$ indefinitely oscillates when $r \rightarrow \infty$.
- III) If $\lambda \in \mathbb{C}$ then Eq. (11) gives $\Psi_\pm \sim (\pm\kappa + k)e^{\pm\kappa r}$. Moreover, if $\lambda = \epsilon$ (equivalently $\kappa = k$) then $\Psi_- = 0$ and $\Psi_+ \sim 2ke^{k r}$. The former solution is rather trivial as $W(u, u) = 0$ in Eq. (8). Now, as $k_1 < 0$, it seems that Ψ_+ could satisfy $\lim_{r \rightarrow +\infty} |\Psi_+| = 0$. However, in such a case, φ_+ should also satisfy both conditions $\varphi_+(0) = 0$ and $\varphi_+ \propto e^{k r}$, $k_1 < 0$, which is not possible since λ is complex and H_ℓ is a selfadjoint operator in $H_\ell\varphi_+ = \lambda\varphi_+$. A similar situation arises for any complex number λ different from ϵ .

In summary, for Gamow transformation functions in (8), if $\varphi \in L^2(\mathbb{R}^+)$ then $\Psi \in L^2(\mathbb{R}^+)$. Furthermore, Eq. (8) does not produce eigenfunctions of the non-hermitian Hamiltonian h_ℓ belonging to complex eigenvalues. Thereby, the complete discrete spectrum $\sigma_d(H_\ell)$ of the initial Hamiltonian H_ℓ is inherited to the Gamow transformed Hamiltonian h_ℓ . In order to exhaust our analysis, let us consider the complex factorization (6). It is easy to verify that the kernel of A provides an eigenfunction $\xi_\epsilon(r)$ of h_ℓ belonging to $\epsilon \in \mathbb{C}$. Thus, $\xi_\epsilon \propto 1/u$ fulfills $h_\ell\xi_\epsilon = \epsilon\xi_\epsilon$. However, as u is a Gamow vector, ξ_ϵ diverges at the origin as $r^{-\ell-1}$. In other words, ξ_ϵ is out of \mathcal{H} and it is deprived of a physical meaning. The same

situation arises by considering the two-dimensional kernel of the product BA . Hence, there are no more square-integrable solutions of h_ℓ and the discrete spectrum $\sigma_d(h_\ell)$ is just the same as $\sigma_d(H_\ell)$.

We have then constructed a non-hermitian Hamiltonian h_ℓ which is strictly isospectral to the initial spherically symmetric Hermitian Hamiltonian H_ℓ . A simple calculation shows that the global behaviour of the new potential $v_\ell(r) = V_\ell(r) - 2u'(r)/u(r)$ is as follows

$$v_\ell(r) = \begin{cases} V_{\ell+1}(r) & r \sim 0 \\ 0 & r \rightarrow \infty. \end{cases} \quad (12)$$

Thus, for small distances, a particle with energy $E_n(\ell)$ interacts with the field as having a quantum number $\ell + 1$. In the asymptotic region the particle behaves as free of interaction. On the other hand, the intermediate region could be interpreted as 'opaque' in the sense that the particle interacts with a series of wells and barriers which alternate their positions in the real and imaginary parts of $v_\ell(r)$ (see the discussion on the optical bench given in [27]). The next section elucidates the applications of the method by transforming the Coulomb potential.

2.3. Non-hermitian Hamiltonians with hydrogen-like spectrum

If the radial potential in (2) is the Coulomb one $V(r) = -2/r$, the convenient Gamow vectors are given by the expression (see Fig. 1):

$$u(r) = r^{\ell+1} e^{-kr} {}_1F_1(\ell + 1 - 1/k, 2\ell + 2, 2kr) \quad (13)$$

with ${}_1F_1(a, c, z)$ the Kummer's function. The units of energy and coordinates are respectively taken as $\mathcal{E} = Ze^2/2r_B$ ($= Z 13.5$ eV) and $r_B = \hbar^2/Ze^2m$ ($= 0.529 \times 10^{-8}/Z$ cm). The solutions (13) have been explicitly derived in [17].

Once these Gamow vectors have been used as transformation functions in (6), the non-hermitian potential $v_\ell(r)$ resembles a cardioid curve as depicted in the complex plane (see

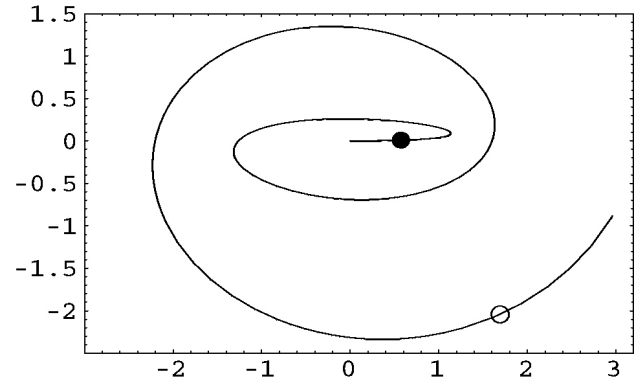
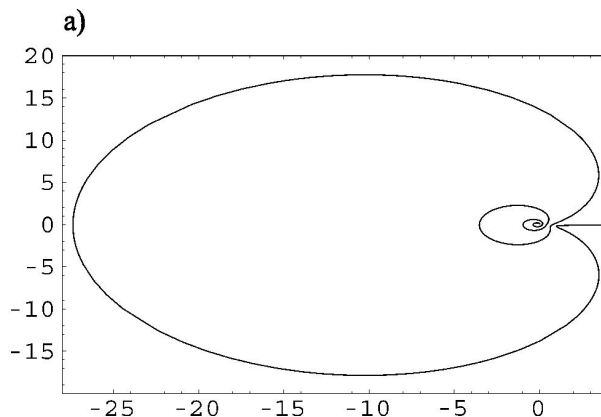


FIGURE 1. The Argand-Wessel diagram of the hydrogen's Gamow vector $u(r)$, from zero to 20 Bohr radii r_B , with $\ell = 1$ and $\epsilon = -0.2604 + i0.104(\times 13.5\text{ev})$. Horizontal scale stands for the real part. The disk is at $r = 1r_B$ and the circle at $r = 19r_B$.

Fig. 2). Notice that $v_\ell(r)$ becomes almost real for small distances and goes to $+\infty$ on the real branch for $r = 0$. On the other hand, this potential goes to zero when $r \rightarrow +\infty$. The imaginary part of $v_\ell(r)$ becomes relevant for intermediate distances (*i.e.* at distances which are between 2 and 6 Bohr radii, for the parameters considered in the figure).

Finally, Fig. 3 depicts the potential $V_{\ell+1}(r)$ as well as the real part of $v_\ell(r)$. Observe the presence of barriers and wells in the intermediate distances. A similar situation occurs for the imaginary part of $v_\ell(r)$. These 'partial potentials' induce local 'resonance' effects which are not present in the Hermitian potential $V_{\ell+1}(r)$. Thus, the spatial distribution of the wave-packets corresponding to $v_\ell(r)$ differ from that of the wave-packets of $V_{\ell+1}(r)$ at the same energy.

3. Generalized Gamow transformations

As it has been shown in the precedent section, though the Gamow vectors $u(r)$ could have a definite physical meaning as resonant states of H_ℓ , we consider them merely as transformation functions to construct the non-hermitian Hamilto-

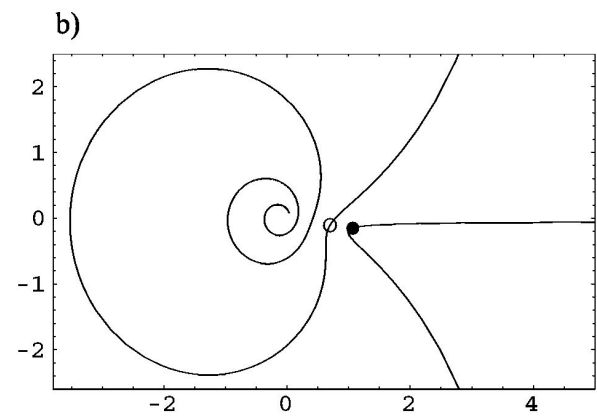


FIGURE 2. The Argand-Wessel diagram of the non-hermitian potential (12) for the same parameters as in Fig. 1. This potential has a discrete spectrum identical with that of the hydrogen atom. The right-hand side figure is a detail of the cardioid-type one. The disk is at $r = 2r_B$ and the circle at $r = 6r_B$.

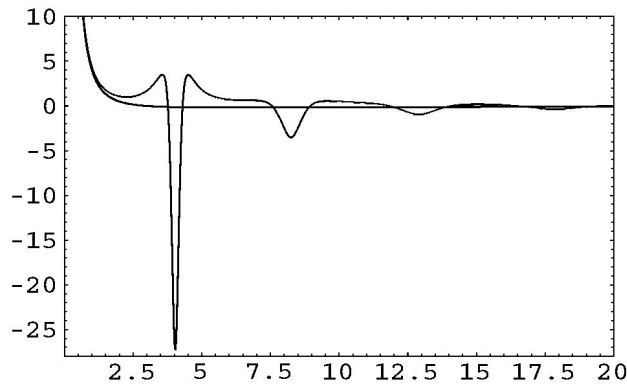


FIGURE 3. The real part (continuous curve) of the cardioid-type potential in Fig. 2 contrasted with the effective Coulomb potential $V_{\ell=2}(r)$.

nians h_ℓ . In general, all the unphysical (not square-integrable) solutions of the Schrödinger equation are useful to construct new Hamiltonians admitting real spectra and square-integrable eigenfunctions [28,29]. In particular, if the factorization constant ϵ is a real number, the conventional factorization operators $A = B^\dagger$ are automatically recovered.

Now, we extend the previous results by opening the chance to incorporate complex eigenvalues with square-integrable solutions in the spectra of the transformed Hamiltonians. First, notice that the general solution of (1) is, for small distances, a linear combination of two particular solutions: $r^{\ell+1}$ and $r^{-\ell}$. The second one is usually rejected because it is singular. Moreover, in the context of alpha decay, a vector $u(r=0) \neq 0$ does not describe the ‘creation’ of alpha particles. We shall relax the Gamow condition at the origin to include the solution $r^{-\ell}$ but preserving the purely outgoing condition e^{-kr} . Let us remark that a ‘generalized’ Gamow vector $\omega(r)$, satisfying these new conditions, still is unphysical in the sense that it is not square-integrable in \mathcal{H} .

The relevance of our generalization lies on the fact that expressions (1)–(8) still hold if $\omega(r)$ is taken as the transformation function. Equation (9), on the other hand, is slightly

modified:

$$\Psi(r \ll 1) \sim \varphi'(r \ll 1) + \frac{\ell}{r} \varphi(r \ll 1). \quad (14)$$

Hence, the same conclusion is obtained: if $\varphi \in L^2(\mathbb{R})$ then $\Psi \in L^2(\mathbb{R})$. However, for complex eigenvalues of $h_\ell(r)$, the kernel of A provides the eigensolution $\Psi_\epsilon \propto 1/\omega$, which can be normalized in \mathcal{H} and satisfies $h_\ell \Psi_\epsilon = \epsilon \Psi_\epsilon$. It is easy to check that the corresponding complex conjugate $\bar{\Psi}_\epsilon$ is neither in the kernel of A nor that of BA (see Eq. (6)). In counterdistinction, if $\varphi(r)$ is eigensolution of H_ℓ belonging to ϵ , then $\bar{\varphi}(r)$ belongs to $\bar{\epsilon}$ as H_ℓ is selfadjoint.

Therefore, the discrete spectrum of h_ℓ is now given by $\sigma_d(H_\ell) \cup \{\epsilon\}$. On the other hand, the new potential $v_\ell(r) = V_\ell(r) - 2\omega'(r)/\omega(r)$ behaves in this case as

$$v_\ell(r) = \begin{cases} V_{\ell-1}(r) & r \sim 0 \\ 0 & r \rightarrow \infty \end{cases} \quad (15)$$

with a similar interpretation as for (12) but changing $\ell+1$ by $\ell-1$.

Figures 4 and 5 show respectively the behaviour of the generalized Gamow vector $\omega(r)$, the wavefunction $\Psi_\epsilon(r)$ and the new non-hermitian potential $v_\ell(r)$ for the Coulomb case $V(r) = -2/r$. The related transformation function is [17]:

$$\omega(r) = r^{\ell+1} e^{-kr} [{}_1F_1(\ell+1-1/k, 2\ell+2, 2kr) + \xi U(\ell+1-1/k, 2\ell+2, 2kr)] \quad (16)$$

where ξ is a complex constant and $U(a, c, z)$ is the logarithmic hypergeometric function.

4. Concluding remarks

The Gamow (decaying) eigensolutions have been shown to be appropriate transformation functions in the framework of supersymmetric quantum mechanics. Non-hermitian Hamiltonians, which are supersymmetric partners of spherically symmetric self-adjoint energy operators, have been construc-

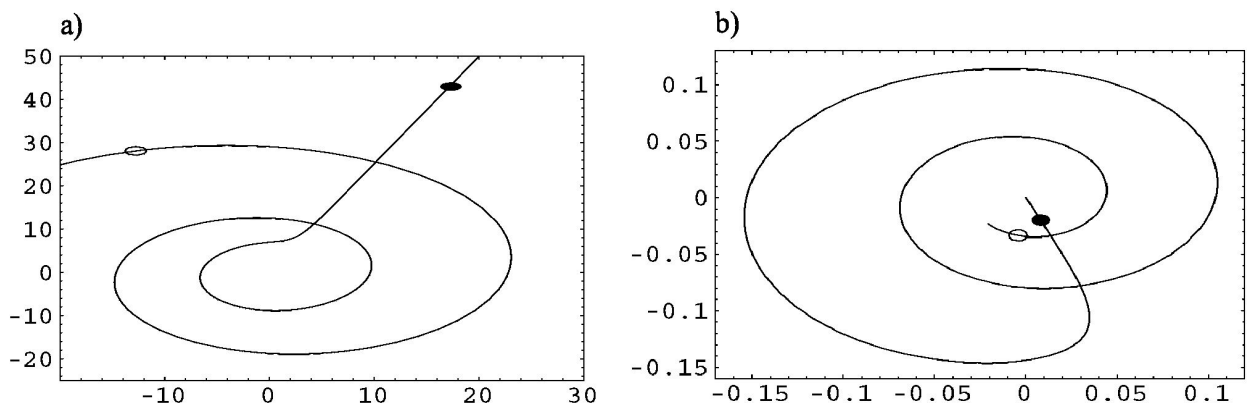


FIGURE 4. (a) The generalized hydrogen's Gamow vector $\omega(r)$, plotted from zero to 20 Bohr radii and the same parameters as in Fig. 1. The disk is at $r = 0.05r_B$ and the circle at $r = 19.5r_B$. (b) The square-integrable wavefunction Ψ_ϵ belonging to the complex eigenvalue $\epsilon = -0.2604 + i0.104(\times 13.5\text{ev})$. The disk is at $r = 0.05r_B$ and the circle at $r = 19r_B$.

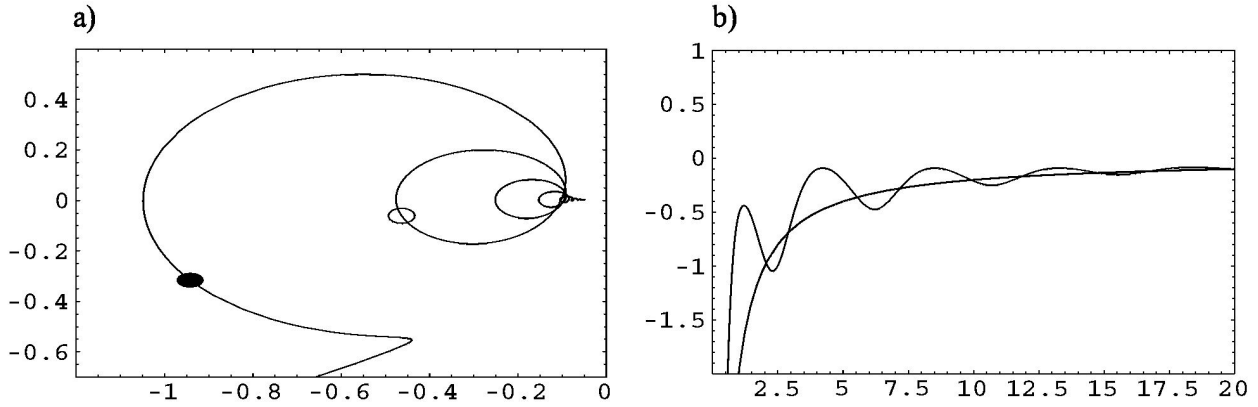


FIGURE 5. (a) The non-hermitian potential (15) constructed via the generalized Gamow vector of Fig. 4a. Here we have plotted from zero to 40 Bohr radii. The disk is at $r = 2r_B$ and the circle at $r = 6r_B$. This potential has the same spectrum as the hydrogen atom extended by an extra complex eigenvalue at ϵ with the square-integrable wavefunction of Fig. 4b. (b) The corresponding real part (continuous curve) contrasted with the Coulomb potential $V_{\ell=0}(r)$.

ted so that they admit purely real spectrum with normalized wavefunctions. For other of these new non-hermitian operators an extra complex eigenvalue with square-integrable eigensolution is present.

At first sight, our results could be connected with those derived in the complex-scaling method. However, the Gamow-transformed potential $v_\ell(r) = V_\ell(r) - 2\omega'(r)/\omega(r)$ is not as simple as the complex-scaled potential $V_\theta = e^{i2\theta}V_\ell(re^{i\theta})$. In general, an intertwined Hamiltonian $hB = BH$, $HA = Ah$ (factorized in a refined way [20]: $H = AB + \epsilon$, $h = BA + \epsilon$) could correspond to a complex-scaled Hamiltonian $h_\theta S = SH$ if $h_\theta = h$. Thus, there must exist a couple of differential operators $M = AS$ and $N = BS^{-1}$, such that $[H, M] = [h, N] = 0$. A particular case has been recently reported [30] (see also [31]) by considering conventional factorization operators $a^\pm = \mp(d/dr) + \alpha(r)$, $\alpha: \mathbb{R} \mapsto \mathbb{R}$, real factorization constants \mathcal{E} , and the squeezing operator $S = U_r = e^{i(\lambda/2)\{r,p\}}$, $[r, p] = i$. The so derived ‘scaled intertwined’ Hamiltonian h_λ has a real potential $v_\lambda(r) = e^{2\lambda}V(e^\lambda r) - \alpha'(r)$ and real discrete spectrum $\sigma_d(h_\lambda) = \{e^{2\lambda}\mathcal{E}, e^{2\lambda}E_n\}_{n \in \mathbb{N}}$, where $e^{2\lambda}\mathcal{E}$ is the ground state energy and $E_n \in \sigma_d(H)$. This procedure allows to deform the excited energy levels of h_λ but leaving unaffected the ground state \mathcal{E}_0 : $\sigma_d(h_\lambda) \mapsto \{\mathcal{E}_0, (\mathcal{E}_0/\mathcal{E})E_n\}$. This is a remarkable profile of the factorization which is rarely considered in the literature. Thus, it seems that complex-scaled Hamiltonians h_θ could be successfully con-

structed as an application of the technique reported in [30]. Work in this direction is in progress.

On the other hand, the possible connection of our results with other approaches as the PT -symmetry [32] or the pseudo-hermitian transformation [16] has been discussed in a previous work [17].

Finally, we have presented the case of first order, not mutually adjoint, intertwining operators A, B . However, the method can be iterated at will by considering h_ℓ as the new initial Hamiltonian. The n th iterated result can be also obtained by means of intertwining operators of n th order. In particular, the second order case can be properly used to obtain self-adjoint Hamiltonians with spectrum identical to the initial one [17, 33]. It is also possible to show that second order transformations can produce non-hermitian operators with real spectrum extended by two extra complex eigenvalues ϵ and $\bar{\epsilon}$. These results will be published elsewhere.

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