Garhy-Generated Family of Distributions with Application

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Abstract

This paper introduces a new family of continuous distributions called a Garhy generated family of distributions. Some mathematical properties of this family are discussed. The derived properties are hold to any proper distribution in this family. Some special sub-models in the new family are derived. General explicit expressions for the quantile function, ordinary and incomplete moments, generating function and order statistics are obtained. The estimation of the model parameters is discussed by using maximum likelihood and the potentiality of the extended family is illustrated with one application to real data.
Keywords: Kumaraswamy distribution; Exponetiated distribution; Moments; quantile function, Maximum likelihood estimation.

1. Introduction

Many statistical distributions have been extensively used and applied for modeling data in several areas such as engineering, actuarial, medical sciences, demography, etc. However, in many situations, the classical distributions are not suitable for describing and predicting real world phenomena. For that reason, attempts have been made to define new techniques for creating new distributions by introducing additional shape parameter(s) to baseline model and at the same time provide great flexibility in modeling data in practice
The extended distributions have attracted the attention of many authors to expand new models because the computational and analytical facilities available in programming software such as R, Maple, and Mathematica can easily tackle the problems involved in computing special functions in these extended distributions.
Several mathematical properties of the extended distributions may easily be explored using mixture forms of the exponentiated-H (exp-H for short) distributions. The addition of parameters has been proved useful in exploring skewness and tail generated family. The well-known generators are the following: Beta Generalized (Beta-H), introduced by Singh et al. (1988) is a rich class of generalized distributions. This class has captured a considerable attention over the last few years. Eugene et al. (2002) introduced a new class of distributions generated from the beta distribution. The cumulative distribution function (cdf) for beta-generated distributions has the form

$$G(x)=\frac{1}{B(a,b)}\int_0^{H(x)} w^{a-1}(1-w)^{b-1}dw\,, \qquad (1)$$

where, $a>0$ and $b>0$ are two additional parameters and $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Eugene et al. (2002) defined and studied the beta normal distribution by taking $H(x)$ to be the cdf of normal distribution. Following the procedure of Eugene et al. (2002), many other authors have been defined and studied a number of the beta-generated distributions, using various forms of known $H(x)$. Sepanski and Kong (2008) applied the Beta-H distribution to model the size distribution of income. This distribution has been studied in literature for various forms of H .
Kumaraswamy (1980) introduced a two-parameter distribution on the interval $(0,1)$ which bears his name. It's cumulative distribution function is given by

$$G(x)=1-\left[1-x^{a}\right]^{b}, \quad 0<x<1, \qquad (2)$$

where $a>0$ and $b>0$ are shape parameters. The cdf (2) compares extremely favorably in terms of simplicity with the beta cdf which is given by the incomplete beta function ratio. The corresponding probability density function (pdf) is

$$g(x) = abx^{a-1}\left[1-x^{a}\right]^{b-1}. \tag{3}$$

The pdf of the Kumaraswamy (Kw) has the same basic shape properties of the beta distribution: $a>1$ and $b>1$ (unimodal) ; $a<1$ and $b<1$ (uniantimodel); $a>1$ and $b\le 1$ (increasing) ; $a\le 1$ and $b>1$ (decreasing); $a=1$ and $b=1$ (constant). It does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution been widely appreciated. Jones (2009) explored the background and genesis of this distribution and, more importantly, made clear some similarities and differences between the beta and Kw distributions. Cordeiro and de Castro (2011) replaced the classical beta generator distribution with the Kumaraswamy's distribution and introduced the Kumaraswamy generated family. They derived some mathematical properties of a new model, called the $Kw-H$ distribution, which stems from the following general construction: If H denotes the baseline cumulative distribution function of a random variable, then a generalized class of distributions can be defined by

$$G(x) = 1-[1-H(x)^{a}]^{b} \qquad a,b>0. \tag{4}$$

Where $a>0$ and $b>0$ are two additional shape parameters which aim to govern skewness and tail weight of the generated distribution. An attractive feature of this distribution is that the two parameters a and b can afford greater control over the weights in both tails and in its centre. The corresponding pdf is

$$g(x) = abh(x)H(x)^{a-1}\left[1-H(x)^{a}\right]^{b-1}. \tag{5}$$

The density family (5) has many of the same properties of the class of beta $-H$ distributions (see Eugene et al. (2002)), but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. Equivalently, as occurs with the beta $-H$ family of distributions, special $Kw-H$ distributions can be generated as follows: The $Kw-$Weibull (Cordeiro et al. (2010)), $Kw-$generalized gamma (Pascoa et al. (2011)), $Kw-$Birnbaum-Saunders (Saulo et al. (2012)), and $Kw-$ Gumbel (Cordeiro et al. (2012)) distributions are obtained by taking $H(x)$ to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively, among several others. Hence, each new $Kw-H$ distribution can be generated from a specified H distribution. General results for the $Kw-H$ distribution were studied by Nadarajah et al. (2012).

Recently, the $G^{\alpha}-$ distributions (or exponentiated distributions) have been shown to have a wide domain of applicability, in particular in modeling and analysis of life time data. The exponentiated distributions have been widely studied in statistics and numerous authors have developed various classes of these distributions. Mudholkar et al. (1995) proposed the exponentiated Weibull distribution. It's properties have been studied in more detail by Mudholkar and Hutson (1996). Gupta et al. (1998) introduced and developed the general class of exponentiated distributions. They defined and studied the exponentiated exponential distribution. Gupta and Kundu (1999) introduced the exponentiated exponential distribution as a generalization of the standard exponential distribution. Nadarajah and Kotz (2006) proposed, based on the same idea, four more exponentiated type distributions to extend the standard gamma, standard Weibull, standard Gumbel and standard Fr´echet distributions. More recently, Lemonte and Cordeiro (2011) introduced the exponentiated generalized inverse Gaussian distribution.

In this paper, a new extension of the $Kw-H$ family of distributions called exponentiated Kumaraswamy denoted by (Garhy) $(G-H)$ family of distributions is proposed. The rest of this article is organized as follows. In Section 2, the Garhy family of distributions is defined and some of it's properties are provided. Some general mathematical properties of the family are discussed in Section 3. Distribution of kth order statistics is discussed in Section 4. Some members of Garhy distributions are discussed in Section 5. In Section 6, the estimation of the model parameters is performed by the method of maximum likelihood. An illustrative application based on real data is investigated in Section 7. Finally, concluding remarks are addressed in Section 8.

2. The Garhy-Generated Family

In this section, the new class of distributions, called Garhy $(G-H)$ generated family of distributions is introduced. The cdf of this family is defined by adding shape parameter β to the cdf (4) as follows

$$F(x)=[1-(1-H(x)^a)^b]^\beta \qquad a,b,\beta>0, x>0, \tag{6}$$

where $a,b>0$, and $\beta>0$ are three shape parameters. The cdf (6) provides a wider family of continuous distributions. The pdf corresponding to (6) is given by

$$f(x)=ab\beta h(x)H(x)^{a-1}(1-H(x)^a)^{b-1}[1-(1-H(x)^a)^b]^{\beta-1} \quad a,b,\beta>0, x>0. \tag{7}$$

Hereafter, a random variable X with pdf (7) is denoted by $X \sim G-H$.
Note that:

1. For $\beta=1$, the $G-H$ distribution reduces to $Kw-H$ which is obtained by Cordeiro and de Castro (2011).
2. For $a=b=1$ the $G-H$ distribution reduces to exponentiated distributions $(E-H)$ which is obtained by Gupta et al. (1998).

The survival, hazard and reversed hazard functions are obtained, respectively, as follows

$$\overline{F}(x)=1-[1-(1-H(x)^a)^b]^\beta \qquad a,b,\beta>0,\ x>0,$$

$$R(x)=\frac{f(x)}{\overline{F}(x)}=\frac{ab\beta h(x)H(x)^{a-1}(1-H(x)^a)^{b-1}[1-(1-H(x)^a)^b]^{\beta-1}}{1-[1-(1-H(x)^a)^b]^\beta},$$

and,

$$\tau(x)=\frac{f(x)}{F(x)}=\frac{ab\beta h(x)H(x)^{a-1}(1-H(x)^a)^{b-1}}{1-(1-H(x)^a)^b}.$$

3. Statistical Properties

In this section, we provide some of the important properties of $G-H$ generated family of distributions.

3.1 Quantile and median

Quantile functions are generally in common use in statistics. The quantile function of $G-H$ distributions, say $Q(u)=F^{-1}(u)$, is straightforward to be computed by inverting (6) as follows

$$H(x_q)-[1-(1-q^{\frac{1}{\beta}})^{\frac{1}{b}}]^{\frac{1}{a}}=0, \tag{8}$$

where, $x_q=Q(u), u$ is the uniform distribution on $(0,1)$. Therefore, by solving numerically the nonlinear equation (8), the generated random number from $G-H$ random variable X will be obtained. Also, the median can be derived from (8) be setting $q=0.5$, that is, the median is given through the following relation

$$H(x_q)-[1-(1-(0.5)^{\frac{1}{\beta}})^{\frac{1}{b}}]^{\frac{1}{a}}=0.$$

3.2 Expansion for distribution and density functions

In this subsection some representations of cdf and pdf for Garhy family of distributions will be presented. The useful mathematical relation will be given below.

It is well-known that, if $\beta>0$ is a real non integer and $|z|<1$, the generalized binomial theorem is written as follows

$$\left(1-z\right)^{\beta-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}z^{i}. \tag{9}$$

Then, by applying the binomial theorem (9) in (6), the distribution function of $G-H$ distribution becomes

$$\begin{aligned}F(x)&=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta}{i}\left(1-H(x)^{a}\right)^{bi},\\&=\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{\beta}{i}\binom{bi}{j}H(x)^{aj},\\&=\sum_{i,j=0}^{\infty}w_{i,j}H(x)^{aj},\end{aligned}$$

where $w_{i,j}=(-1)^{i+j}\binom{\beta}{i}\binom{bi}{j}$.

Also, using the binomial expansion (9) in (7), the probability density function of $G-H$ distribution becomes

$$\begin{aligned}f(x)&=ab\beta h(x)H(x)^{a-1}\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}(1-H(x)^{a})^{b(i+1)-1}\\&=h(x)\sum_{i,j=0}^{\infty}\eta_{i,j}\,H(x)^{a(j+1)-1},\end{aligned} \tag{10}$$

where, $\eta_{i,j}=ab\beta(-1)^{i+j}\binom{\beta-1}{i}\binom{b(i+1)-1}{j}$.

Then pdf (10) reduces to

$$f(x)=\sum_{i,j=0}^{\infty}\tau_{i,j}h_{a(j+1)-1}(x), \tag{11}$$

where,

$$h_{a(j+1)-1}(x)=(a(j+1)-1)h(x)H(x)^{a(j+1)-1} \text{ and } \tau_{i,j}=\frac{\eta_{i,j}}{a(j+1)-1}.$$

For β is an integer the index i in the previous sum stops at $\beta-1$.

3.3 Moments

A formula of the rth moment of X can be obtained from (10) as $\mu_r^{'}=\sum_{i,j=0}^{\infty}\tau_{i,j}E(Z_{i,j}^{r})$ where $Z_{i,j}$ denotes the $G-H$ distribution with power parameter $a(j+1)-1$. Since the inner quantities in (11) are absolutely integrable, the incomplete moments and moment generating function of X can be written as

$$I_X(y)=\int_{-\infty}^{y}x^{r}f(x)dx=\sum_{i,j=0}^{\infty}\tau_{i,j}I_{i,j}(y),$$

where $I_{i,j}(y)=\int_{-\infty}^{y}x^{r}h_{a(j+1)-1}(x)dx$ and

$$M_X(t)=\sum_{i,j=0}^{\infty}\tau_{i,j}E(e^{tZ_{i,j}}).$$

4. Order Statistics

Order statistics are among the most fundamental tools in non-parametric statistics and inference. They play an important rule in the problems of estimation and hypothesis tests in a variety of ways. For a given random sample $X_1,...,X_n$ from the $G-H$ distribution, the pdf $f_{X_{(k)}}(x_{(k)})$ of the kth order is obtained by using the following

$$f_{X_{(k)}}(x_{(k)})=\frac{n!}{(k-1)!(n-k)!}f(x_{(k)})\left[F(x_{(k)})\right]^{k-1}\left[1-F(x_{(k)})\right]^{n-k}$$

$$f_{X_{(k)}}(x_{(k)})=\frac{n!}{(k-1)!(n-k)!}\sum_{u=0}^{n-k}(-1)^u\binom{n-k}{u}f(x_{(k)})F(x_{(k)})^{u+k-1}. \qquad (12)$$

By substituting cdf (6) in pdf(12), then

$$f_{X_{(k)}}(x_{(k)})=\frac{n!}{(k-1)!(n-k)!}\sum_{u=0}^{n-k}(-1)^u\binom{n-k}{u}f(x_{(k)})\left[1-\left(1-H\left(x_{(k)}\right)^a\right)^b\right]^{\beta(u+k-1)}, \qquad (13)$$

by applying the binomial theorem (9) in (13), then the pdf $f_{X_{(k)}}(x_{(k)})$ of the kth order statistic from $G-H$ distribution becomes

$$f_{X_{(k)}}(x_{(k)})=\frac{n!}{(k-1)!(n-k)!}\sum_{i,j=0}^{\infty}\sum_{u=0}^{n-k}(-1)^{i+j+u}\binom{n-k}{u}$$
$$\times\binom{\beta(u+k-1)}{i}\binom{bi}{j}H(x_{(k)})^{aj}f(x_{(k)}) \, . \qquad (14)$$

Where $h(.)$ and $H(.)$ are the density and cumulative functions of the $G-H$ distribution, respectively.

5. Some Special Sub-Models

In this section, we discuss some special distributions which will be derived from $G-H$ family. The density function (7) will be most tractable when the cdf $H(x)$ and the pdf $h(x)$ have simple analytic expressions.

5.1 G-uniform distribution

Suppose that the parent distribution is uniform in the interval $0<x<S<\infty$ as a first example, where $H(x)=\frac{x}{s}$. Therefore, the $G-$uniform distribution, say $GUD(a,b,\beta,s)$ has the following cdf , pdf by direct substituting $H(x)=\frac{x}{s}$, in (6) and (7)as follows

$$F(x)=[1-(1-(\frac{x}{s})^a)^b]^\beta \qquad a,b,\beta>0 \quad , \quad 0<x<s,$$

$$f(x)=\frac{ab\beta x^{a-1}}{s^a}(1-(\frac{x}{s})^a)^{b-1}[1-(1-(\frac{x}{s})^a)^b]^{\beta-1}, \quad a,b,\beta>0 \;,\; 0<x<s.$$

Furthermore, the survival and hazard rate functions are given, respectively, as follows

$$\overline{F}(x)=1-[1-(1-(\frac{x}{s})^a)^b]^\beta \qquad a,b,\beta>0 \quad , \quad 0<x<s,$$

and

$$R(x)=\frac{ab\beta x^{a-1}(1-(\frac{x}{s})^a)^{b-1}[1-(1-(\frac{x}{s})^a)^b]^{\beta-1}}{s^a\{1-[1-(1-(\frac{x}{s})^a)^b]^\beta\}}, \quad a,b,\beta>0 \;,\; 0<x<s.$$

5.2 G-BurrXII distribution

Let us consider the parent Burr XII distribution with pdf and cdf given by $h(x)=c\sigma\mu^{-c}x^{c-1}[1+(\frac{x}{\mu})^{c}]^{-\sigma-1}$ $\quad c,\mu,\sigma>0$ and $\mathrm{H}(x)=1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}$, respectively. Then the $G-BurrXII$ distribution, denoted by $GBurrXIID(a,b,\beta,c,\mu,\sigma)$ has the following cdf, pdf , survival and the hazard rate functions

$$F(x)=[1-(1-\{1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\}^{a})^{b}]^{\beta} \qquad a,b,\beta,c,\mu,\sigma>0 \quad , \quad x>0,$$

$$f(x)=ab\beta c\sigma\mu^{-c}x^{c-1}[1+(\frac{x}{\mu})^{c}]^{-\sigma-1}(1-[1+(\frac{x}{\mu})^{c}]^{-\sigma})^{a-1}(1-(1-[1+(\frac{x}{\mu})^{c}]^{-\sigma})^{a})^{b-1}$$
$$\times[1-(1-\{1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\}^{a})^{b}]^{\beta-1},$$

$$\overline{F}(x)=1-[1-(1-\{1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\}^{a})^{b}]^{\beta} \qquad a,b,\beta,c,\mu,\sigma>0 \quad , \quad x>0,$$

and

$$R(x)=ab\beta c\sigma\mu^{-c}x^{c-1}[1+(\frac{x}{\mu})^{c}]^{-\sigma-1}(1-[1+(\frac{x}{\mu})^{c}]^{-\sigma})^{a-1}(1-(1-[1+(\frac{x}{\mu})^{c}]^{-\sigma})^{a})^{b-1}$$
$$\times[1-(1-(1-[1+(\frac{x}{\mu})^{c}]^{-\sigma})^{a})^{b}]^{\beta-1}\ \{1-[1-(1-(1-[1+(\frac{x}{\mu})^{c}]^{-\sigma})^{a})^{b}]^{\beta}\}^{-1},$$

where $x>0,\mu>0$ is scale parameter, $a,b,\beta,c,\sigma>0$ are shape parameters.

5.3 G-Weibull distribution

In this subsection, the pdf and cdf of $G-$Weibull is derived from the G family of distributions.
If the random variable X follows the Weibull distribution with scale parameter $\lambda>0$ and shape parameter $\gamma>0$, then the cdf of X is $H(x)=1-e^{-\lambda x^{\gamma}}$ for $x>0$. The cdf , pdf , survival and the hazard rate functions of the $G-$Weibull distribution, say $GWD(a,b,\beta,\lambda,\gamma)$, take the following forms

$$F(\mathrm{x})=[1-(1-(1-e^{-\lambda x^{\gamma}})^{a})^{b}]^{\beta} \qquad a,b,\beta,\lambda,\gamma>0 \quad , \quad x>0,$$

$$f(x)=ab\beta\lambda xe^{-\lambda x^{\gamma}}(1-e^{-\lambda x^{\gamma}})^{a-1}(1-(1-e^{-\lambda x^{\gamma}})^{a})^{b-1}$$
$$\times[1-(1-(1-e^{-\lambda x^{\gamma}})^{a})^{b}]^{\beta-1} \qquad a,b,\beta,\lambda,\gamma>0 \quad , \quad x>0,$$

$$\overline{F}(x)=1-[1-(1-(1-e^{-\lambda x^{\gamma}})^{a})^{b}]^{\beta} \qquad a,b,\beta,\lambda,\gamma>0 \quad , \quad x>0,$$

and

$$R(x)=\frac{ab\beta\lambda xe^{-\lambda x^{\gamma}}(1-e^{-\lambda x^{\gamma}})^{a-1}(1-(1-e^{-\lambda x^{\gamma}})^{a})^{b-1}[1-(1-(1-e^{-\lambda x^{\gamma}})^{a})^{b}]^{\beta-1}}{1-[1-(1-(1-e^{-\lambda x^{\gamma}})^{a})^{b}]^{\beta}}$$

respectively. Note that if $\gamma=1$ the $G-$Weibull distribution reduces to $G-$exponential distribution.

5.4 G- quasi Lindley distribution

Quasi Lindley distribution is introduced by Rama and Mishra (2013). The cdf, pdf , survival and the hazard rate functions for Garhy quasi Lindley distribution ($GQLD$) are obtained from (6)and(7) by taking $H(.)$ and

$h(.)$ to be the cdf and pdf of the quasi Lindley $QL(\theta, p)$ distribution, where, $H(x) = 1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}]$ and $h(x) = \frac{\theta}{p+1}(p + \theta x)e^{-\theta x}(1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}])$

Hence,

$$F(x) = [1 - (1 - (1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}])^a)^b]^\beta, \; a, b, \beta, \theta > 0 \;, p > -1, \; x > 0,$$

$$f(x) = \frac{ab\beta\theta}{p+1}(p + \theta x)e^{-\theta x}(1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}])^{a-1}(1 - (1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}])^a)^{b-1}$$
$$\times [1 - (1 - (1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}])^a)^b]^{\beta - 1} a, b, \beta, \theta > 0 \;\; p > -1 \; x > 0,$$

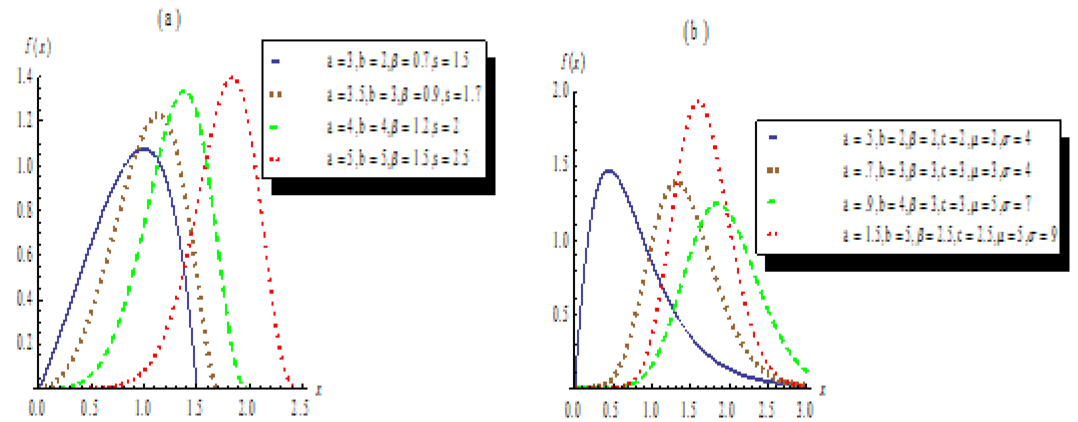
$$\overline{F}(x) = 1 - [1 - (1 - (1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}])^a)^b]^\beta, \; a, b, \beta, \theta > 0 \;, p > -1, \; x > 0,$$

and

$$R(x) = \frac{ab\beta\theta}{p+1}(p + \theta x)e^{-\theta x}(1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}])^{a-1}[1 - (1 - (1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}])^a)^b]^{\beta - 1}$$
$$\times (1 - (1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}])^a)^{b-1}\{1 - [1 - (1 - (1 - e^{-\theta x}[1 + \frac{\theta x}{p+1}])^a)^b]^\beta\}^{-1}$$

respectively. For $p = \theta$ the $G-$Lindley distribution will be obtained.

Considering the above different distributions derived from $G-$family, plots of pdf, cdf , survival function and hazard rate functions for some parameter values are displayed in Figures 1, 2, 3 and 4 respectively.



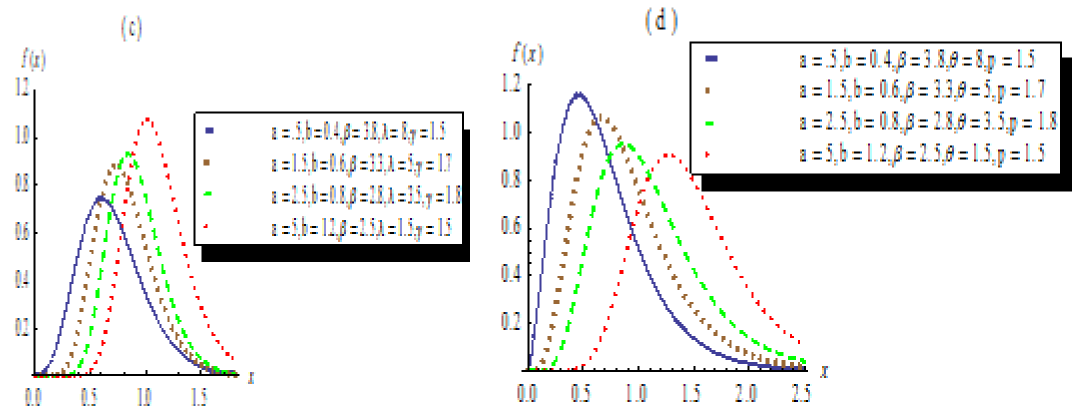


Figure 1: (a) $GUD\left(a,b,\beta,s\right)$, (b) $GBurrXIID\left(a,b,\beta,c,\mu,\sigma\right)$ (c) $GWD\left(a,b,\beta,\lambda,\gamma\right)$ and $G\,\mathrm{QLD}\left(a,b,\beta,\theta,p\right)$ density functions.

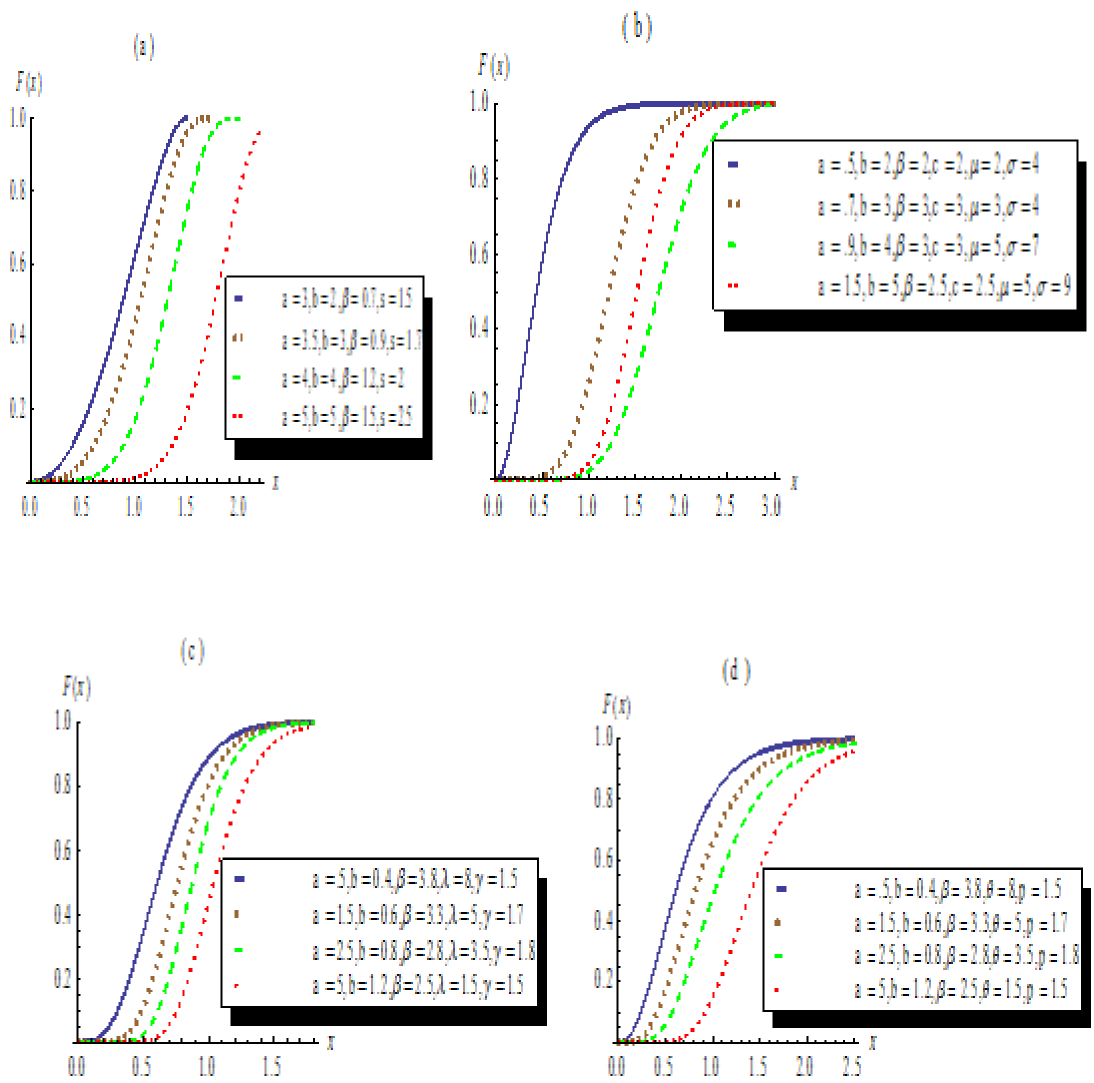


Figure 2: (a) $GUD\left(a,b,\beta,s\right)$, (b) $GBurrXIID\left(a,b,\beta,c,\mu,\sigma\right)$ (c) $GWD\left(a,b,\beta,\lambda,\gamma\right)$ and

$G\,\mathrm{QLD}\left(a,b,\beta,\theta,p\right)$ distribution functions

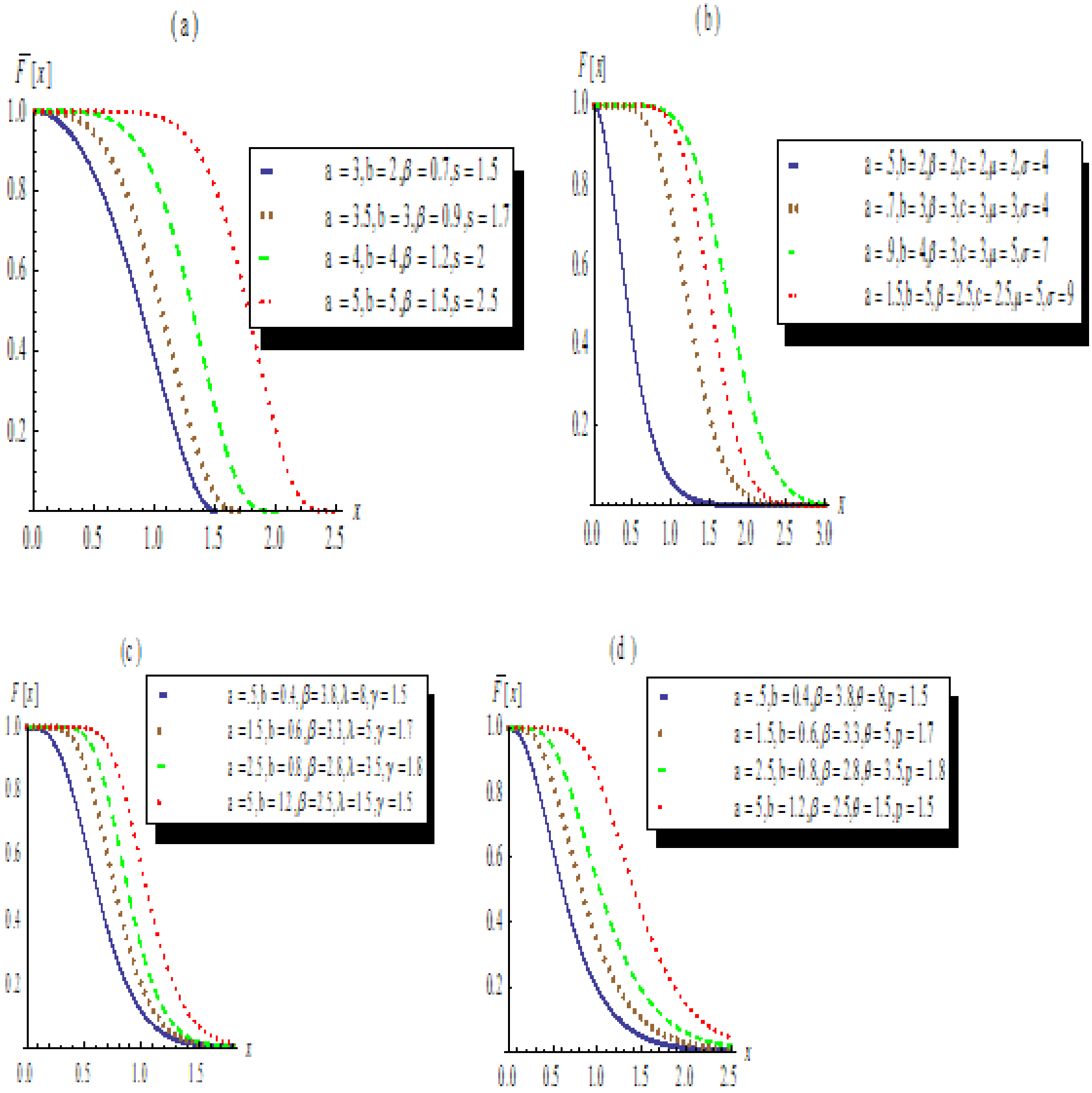


Figure 3 : (a) $GUD\left(a,b,\beta,s\right)$, (b) $GBurrXIID\left(a,b,\beta,c,\mu,\sigma\right)$ (c) $GWD\left(a,b,\beta,\lambda,\gamma\right)$ and $G\,\mathrm{QLD}\left(a,b,\beta,\theta,p\right)$ survival functions

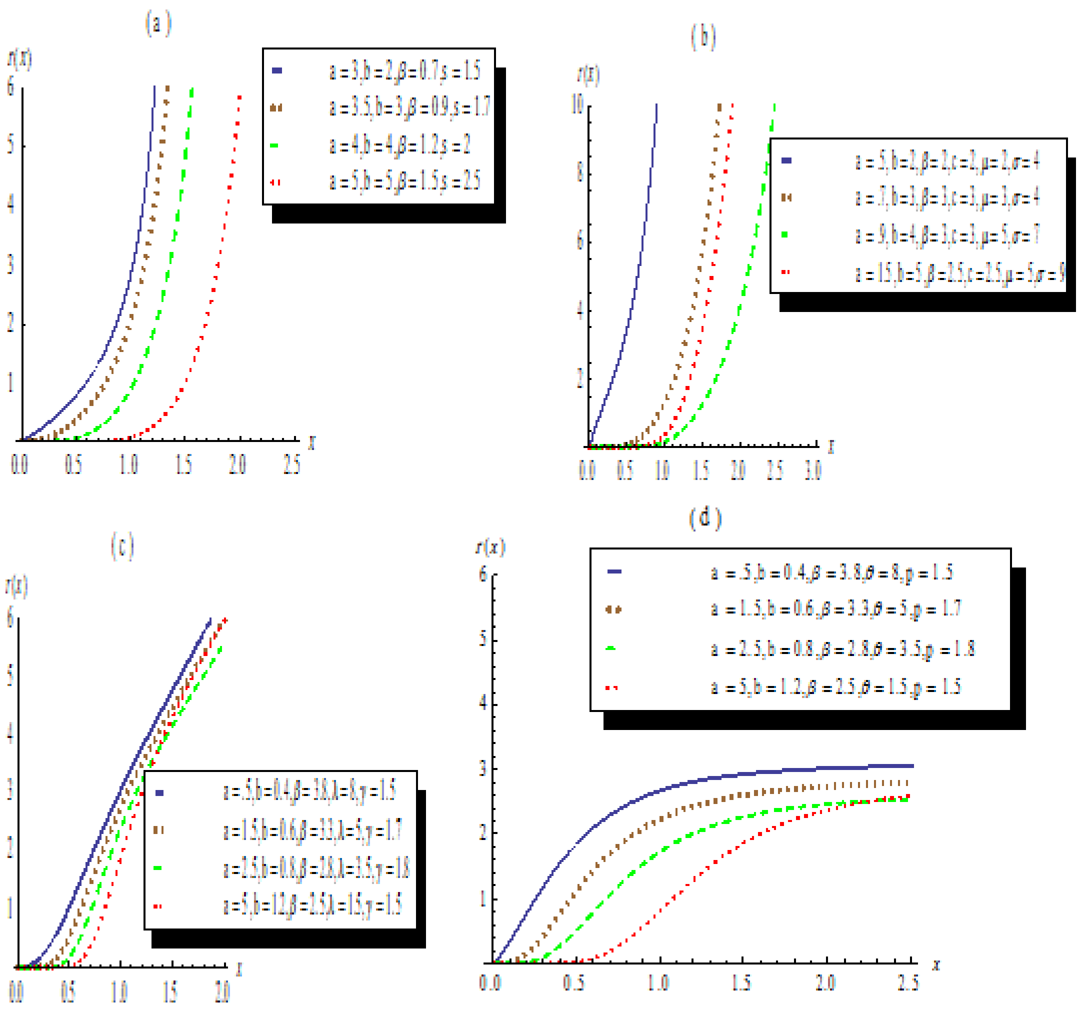


Figure 4(a) $GUD\left(a,b,\beta,s\right)$, (b) $GBurrXIID\left(a,b,\beta,c,\mu,\sigma\right)$ (c) $GWD\left(a,b,\beta,\lambda,\gamma\right)$ and $G\,\text{QLD}\left(a,b,\beta,\theta,p\right)$ hazard rate functions

6. Maximum Likelihood Estimation

The maximum likelihood estimators (MLEs) of the model parameters of the new family of distributions based on complete random samples are determined. Let $X_1,\dots,X_n$ be the observed values from the $G-H$ distribution with parameters a,b,β and ζ. Let $\Theta=(a,b,\beta,\zeta)^T$ be the $p\times 1$ parameter vector. The total log-likelihood function for the vector of parameters Θ can be expressed as

$$\ln L(\Theta)=n\ln a+n\ln b+n\ln\beta+\sum_{i=1}^{n}\ln h(x_i,\zeta)+(a-1)\sum_{i=1}^{n}\ln H(x_i,\zeta)$$
$$+(b-1)\sum_{i=1}^{n}\ln[1-H(x_i,\zeta)^a]+(\beta-1)\sum_{i=1}^{n}\ln[1-[1-H(x_i,\zeta)^a]^b]$$

The elements of the score function $U(\Theta)=(U_a,U_b,U_\beta,U_\zeta)$ are given by

$$U_a = \frac{n}{a} + \sum_{i=1}^{n} \ln H(x_i,\zeta) - (b-1)\sum_{i=1}^{n} \frac{H(x_i,\zeta)^a \ln H(x_i,\zeta)}{\left[1-H(x_i,\zeta)^a\right]}$$

$$+b(\beta-1)\sum_{i=1}^{n} \frac{[1-H(x_i,\zeta)^a]^{b-1} H(x_i,\zeta)^a \ln H(x_i,\zeta)}{1-[1-H(x_i,\zeta)^a]^b},$$

$$U_b = \frac{n}{b} + \sum_{i=1}^{n} \ln[1-H(x_i,\zeta)^a] - (\beta-1)\sum_{i=1}^{n} \frac{[1-H(x_i,\zeta)^a]^b \ln[1-H(x_i,\zeta)^a]}{1-[1-H(x_i,\zeta)^a]^b},$$

$$U_\beta = \frac{n}{\beta} + \sum_{i=1}^{n} \ln[1-[1-H(x_i,\zeta)^a]^b],$$

$$U_{\zeta_k} = \sum_{i=1}^{n} \frac{h_k^{?}(x_i,\zeta)}{h(x_i,\zeta)} + (a-1)\sum_{i=1}^{n} \frac{H_k(x_i,\zeta)}{H(x_i,\zeta)}$$

$$-a(b-1)\sum_{i=1}^{n} \frac{H(x_i,\zeta)^{a-1} H_k^{?}(x_i,\zeta)}{1-H(x_i,\zeta)^a}$$

$$+ab(\beta-1)\sum_{i=1}^{n} \frac{[1-H(x_i,\zeta)^a]^{b-1} H(x_i,\zeta)^{a-1} H_k^{?}(x_i,\zeta)}{1-[1-H(x_i,\zeta)^a]^b}.$$

Setting U_a, U_b, U_β and U_ζ equal to zero and solving the equations simultaneously yields the MLEs $\Theta = (\widehat{a}, \widehat{b}, \beta, \zeta)$ of $\Theta = (a, b, \beta, \zeta)^T$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms.

For interval estimation of the parameters, the 4×4 observed information matrix $I(\Theta) = \{I_{u,v}\}$ for $(u, v = a, b, \beta, \zeta)$, whose elements are given in appendix. Under the regularity conditions, the known asymptotic properties of the maximum likelihood method ensure that: $\sqrt{n}(\Theta - \Theta) \xrightarrow{d} N(0, I^{-1}(\Theta))$ as $n \to \infty$, where $\xrightarrow{d}$ means the convergence in distribution, with mean zero and covariance matrix $I^{-1}(\Theta)$ then, the $100(1-\alpha)\%$ confidence interval for $\Theta = (a, b, \beta\ \zeta)$ is given as follows

$$\Theta \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\Theta)}$$

where $Z_{\alpha/2}$ is the standard normal at $\alpha/2$ is significance level and $\operatorname{var}(.)$ denote the diagonal elements of $I^{-1}(\Theta)$ corresponding to the model parameters.

7. Application

In this section a data analysis will be provided below to see how the new model works in practice. The data have been obtained from (Aarset 1987). It represents the lifetimes of 50 devices.

0.1 0.2 1 1 1 1 1 2 3 6 7 11 12 18 18 18 18 18 21 32 36 40 45 46 47 50 55 60 63 63 67 67 67 67 72 75 79 82 82 83 84 84 84 85 85 85 85 85 86 86.

For the selected data, we fit the Garhy Lindley (GL) distribution defined in (4). Its fitting also compared with the widely known Kumaraswamy Lindley (KwL)(see Çakmakyapan, and Kadlar), exponentiated Lindley (EL) (see Nadarajah et al., 2011) and Lindley (L)(see Lindley, 1958) models with the following corresponding densities:

$$f_{GL}(x)=\frac{ab\beta\theta^2}{\theta+1}(1+x)e^{-\theta x}(1-e^{-\theta x}[1+\frac{\theta x}{\theta+1}])^{a-1}(1-(1-e^{-\theta x}[1+\frac{\theta x}{\theta+1}])^{a})^{b-1}$$
$$\times[1-(1-(1-e^{-\theta x}[1+\frac{\theta x}{\theta+1}])^{a})^{b}]$$

$$f_{KwL}(x)=\frac{ab\theta^2}{\theta+1}(1+x)e^{-\theta x}(1-e^{-\theta x}[1+\frac{\theta x}{\theta+1}])^{a-1}(1-(1-e^{-\theta x}[1+\frac{\theta x}{\theta+1}])^{a})^{b-1}$$

$$f_{EL}(x)=\frac{a\theta^2}{\theta+1}(1+x)e^{-\theta x}(1-e^{-\theta x}[1+\frac{\theta x}{\theta+1}])^{a-1}$$

$$f_{L}(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$$

In order to compare the four distribution models, we consider criteria like $-2\ln L$, Akaike information criterion (AIC), corrected Akaike information criterion $(CAIC)$, Bayesian information criterion (BIC), and Hannan- Quinn information criterion $(HQIC)$ for the chosen data set. The formula for these criteria is as follows

$$AIC=2k-2\ln L\ ,\ CAIC=AIC+\frac{2k\left(k+1\right)}{n-k-1}$$
$$BIC=k\ln(n)-2\ln L \quad \text{and} \quad HQIC=2k\ln\left[\ln\left(n\right)\right]-2\ln L,$$

where k is the number of parameters in the statistical model, and n is the sample size.

The best distribution corresponds to smaller values of $-2\ln L$, AIC, $CAIC$ and $HQIC$

Table 1: Maximum likelihood estimates and information criteria of the models based on above data set

Model	Estimates	$-2\ln L$	AIC	BIC	$CAIC$	$HQIC$
GL	$\hat{a}=0.102$ $\hat{b}=0.574$ $\hat{\theta}=0.054$ $\hat{\beta}=3.076$	474.259	482.259	489.907	490.37	485.172
EL	$\hat{a}=0.585$ $\hat{\theta}=0.042$	485.44	489.44	493.264	493.504	490.896
L	$\hat{\theta}=0.043$	502.861	504.861	506.773	506.902	505.589
KwL	$\hat{a}=0.983$ $\hat{b}=1.906$ $\hat{\theta}=0.027$	503.731	509.731	515.467	515.818	511.915

.

Table 1 shows the values of $AIC, CAIC, BIC$ and $HQIC$ for the real data set. The values in Table 1 indicate that the Garhy Lindley distribution is a strong competitor to other distributions used here for fitting data set.

8. Concluding Remarks

In this paper, we propose the new Garhy generated family of distributions. Some of it's structural properties including an expansion for the density function, explicit expressions for the ordinary and incomplete moments, quantile function and order statistics have been derived. The maximum likelihood method has been employed for estimating the model parameters. We fit one of the special models of the proposed family to real data set to demonstrate the usefulness of the new family. The Garhy Lindley distribution provides consistently better fits than other competing models.

Appendix

The elements of the observed Fisher information matrix $I(\Theta)$, are given by

$$I(\Theta) = -\begin{pmatrix} U_{aa} & U_{ab} & U_{a\beta} & U_{a\zeta} \\ U_{ba} & U_{bb} & U_{b\beta} & U_{b\zeta} \\ U_{\beta a} & U_{\beta b} & U_{\beta\beta} & U_{\beta\zeta} \\ U_{\zeta a} & U_{\zeta b} & U_{\zeta\beta} & U_{\zeta\zeta} \end{pmatrix}$$

$$U_{aa} = \frac{-n}{a^2} - (b-1)\sum_{i=1}^{n}\frac{A_i^a B_i^2 (C_i + A_i^a)}{C_i^2} + b(\beta-1)\sum_{i=1}^{n}\frac{B_i^2 C_i^{b-2} A_i^a[-(b-1)D_i A_i^a + C_i D_i - bA_i^a C_i^b]}{D_i^2},$$

$$U_{ab} = -\sum_{i=1}^{n}\frac{A_i^a B_i}{C_i} + (\beta-1)\sum_{i=1}^{n}\frac{C_i^{b-1}A_i^a B_i}{D_i} + b(D_i + C_i^b)\sum_{i=1}^{n}\frac{C_i^{b-1}E_i A_i^a B_i (D_i + C_i^b)}{D_i^2},$$

$$U_{a\beta} = b\sum_{i=1}^{n}\frac{B_i C_i^{b-1} A_i^a}{D_i}, \qquad U_{b\beta} = -\sum_{i=1}^{n}\frac{C_i^b E_i}{D_i}, \qquad U_{\beta\beta} = \frac{-n}{\beta^2},$$

$$U_{bb} = \frac{-n}{b^2} - (\beta-1)\sum_{i=1}^{n}\frac{E_i^2 C_i^b (D_i + C_i^b)}{D_i^2}, U_{\beta\varepsilon_k} = ab\sum_{i=1}^{n}\frac{C_i^{b-1}A_i^{a-1}F_i}{D_i},$$

$$U_{b\varepsilon_k} = -a\sum_{i=1}^{n}\frac{A_i^{a-1}F_i}{C_i} + a(\beta-1)\sum_{i=1}^{n}\frac{C_i^{b-1}A_i^{a-1}F_i}{D_i} + ab(\beta-1)\sum_{i=1}^{n}\frac{C_i^{b-1}E_i A_i^{a-1}F_i (D_i + C_i^b)}{D_i^2},$$

$$U_{a\varepsilon_k} = \sum_{i=1}^{n}\frac{F_i}{A_i} - (b-1)\sum_{i=1}^{n}\frac{A_i^{a-1}F_i}{C_i} - a(b-1)\sum_{i=1}^{n}\frac{A_i^{a-1}B_i F_i (C_i + A_i^a)}{C_i^2} + b(\beta-1)\sum_{i=1}^{n}\frac{C_i^{b-1}A_i^{a-1}F_i}{D_i} + ab(\beta-1)\sum\frac{C_i^{b-2}A_i^{a-1}B_i F_i [C_i D_i - (b-1)A_i^a D_i - bC_i^b A_i^a]}{D_i^2},$$

$$U_{\varepsilon_k\varepsilon_l} = \sum_{i=1}^{n}\frac{G_i N_i - k_i}{N_i^2} + (a-1)\sum_{i=1}^{n}\frac{L_i A_i - F_i M_i}{A_i^2} - a(b-1)\sum_{i=1}^{n}\{\frac{A_i^{a-2}F_i M_i\left[(a-1)C_i + aA_i^a\right]}{C_i^2} + \frac{A_i^{a-1}L_i}{C_i}\} + ab(\beta-1)\sum_{i=1}^{n}\frac{C_i^{b-2}A_i^{a-2}}{D_i^2} \times\{F_i M_i\left[(a-1)D_i C_i - a(b-1)D_i A_i^a - abC_i^b A_i^a\right] + C_i A_i D_i L_i\}.$$

Where

$$A_i = H(x_i,\zeta) \;,\; B_i = \ln H(x_i,\zeta) \;,\; C_i = 1 - H(x_i,\zeta)^a,\; D_i = 1 - \left[1 - H(x_i,\zeta)^a\right]^b,$$

$$E_i = \ln\left(1 - H(x_i,\zeta)^a\right) \;,\; F_i = H_k^{\text{创}}(x_i,\zeta) \;, G_i = h_{k,l}^{?}(x_i,\zeta) \;,\; N_i = h(x_i,\zeta),$$

$$K_i = h_k^{\text{创}}(x_i,\zeta)\, h_l(x_i,\zeta) \;, L_i = H_{k,l}^{\text{创}}(x_i,\zeta) \;,\; M_i = H_l^{?}(x_i,\zeta).$$

where

$$H_k'(x_i,\zeta) = \tfrac{\partial}{\partial \zeta_k} H(x_i,\zeta) \;,\; H_l'(x_i,\zeta) = \tfrac{\partial}{\partial \zeta_l} H(x_i,\zeta)$$

and $H_{k,l}^{\text{创}}(x_i,\zeta) = \tfrac{\partial^2}{\partial \zeta_k\, \partial \zeta_l} H(x_i,\zeta).$

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