GATEAUX DIFFERENTIABLE LIPSCHITZ FUNCTIONS NEED NOT BE FRÉCHET DIFFERENTIABLE ON A RESIDUAL SUBSET

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Although a Lipschitz function on a separable Hilbert space is necessarily Gateaux differentiable on a large set (see, for example, [1], [2], [5], [6]), it is not known whether it is Fréchet differentiable at least at one point. (See [4].) This problem cannot be solved with the help of the Baire category method, since even on the real line there are Lipschitz functions which are not differentiable on a residual subset. (See [7], where it is proved that for a set $E \subset R$ there is a Lipschitz function f such that E equals to the set of points where f' does not exist if and only if E is a $G_{\delta\sigma}$-set of measure zero.) Nevertheless, one may hope that the Baire category method can be used if f is an everywhere Gateaux differentiable Lipschitz function. Here we intend to show that even this is false; we shall construct an everywhere Gateaux differentiable Lipschitz function on a separable Hilbert space which is not Fréchet differentiable on any residual set.

Let H be a real Hilbert space and f a real-valued function on H. Recall that f is said to be Gateaux differentiable at a point x of H if there is an element df(x) of H such that for all $y \in H$

$$\lim_{t \to 0} t^{-1}(f(x+ty)-f(x)) = \langle df(x), y \rangle ,$$

and we call df(x) the Gateaux derivative of f at x.
The function f is said to be Fréchet differentiable at a point x of H if there is an element $f'(x)$ of H such that

$$\lim_{y \to 0} \|y\|^{-1}(f(x+y)-f(x)- \langle f'(x), y \rangle) = 0,$$

$f'(x)$ is called the Fréchet derivative of f at x. Clearly, if $f'(x)$ exists, then f is also Gateaux differentiable at x and $df(x) = f'(x)$.

Let R denotes the real line and R^n the n - dimensional Euclidean space.

We shall first construct Lipschitz functions on R, which are everywhere differentiable with the derivative equal zero on a dense open subset of R, but which are badly approximated by their derivatives. To do this, we use the following consequence of Lemma 7 from [8].

<u>Lemma 1</u>. There is a function $\varphi: R \to R$ and a constant $C \in R$ such that

(i) φ is everywhere differentiable and $0 \leq \varphi' \leq C$,

(ii) $\varphi = 0$ on $(-\infty, 0]$, $0 < \varphi < 1$ on $(0,1)$ and $\varphi = 1$ on $[1, +\infty)$, and

(iii) $\varphi' = 0$ on a dense open subset of R.

A simple application of this Lemma gives the following technical result.

<u>Lemma 2</u>. There is a constant $c \in (1, +\infty)$ such that, whenever ε, ε_n are positive numbers ($n = 1, 2, \dots$), there is a sequence of functions $h_n: R \to R$ such that

(i) $|h_n| \leq \varepsilon_n$,

(ii) h_n is everywhere differentiable and $|h_n'| \leq c$,

(iii) the derivative of h_n equals zero on a dense open set, and

(iv) whenever g is a convex combination of two functions h_n and h_m and $t \in R$, there is $s \in R$ such that $0 < |t-s| < \varepsilon$ and

$$|g(t)-g(s)| \geq c^{-1}|t-s|.$$

<u>Proof</u>. Let α_n be a sequence of positive numbers such that $\alpha_n < \varepsilon$, $\alpha_n < \varepsilon_n$ and $\alpha_{n+1} < (4C+4)^{-1}\alpha_n$. Let d(x) denote the distance from x to the nearest even integer.

Put $h(x) = \varphi(d(x))$, where φ is the function from Lemma 1. Then $0 \leq h \leq 1$, h' exists everywhere, $|h'| \leq C$, $h' = 0$ on a dense open subset of R, $h(x) = 0$ if x is an even integer and $h(x) = 1$ if x is an odd integer.

Let $h_n(t) = \alpha_n h_n(\alpha_n^{-1}t)$. Clearly $0 \leq h_n \leq \alpha_n$, h_n' exists on R, $|h_n'| \leq C$ and $h_n' = 0$ on a dense open subset of R. Whenever $t \in R$, we find $u_n \leq t \leq v_n$ such that $|v_n-u_n| = \alpha_n$ and $|h_n(v_n)-h_n(u_n)| = |v_n-u_n|$. If $g = ah_n+(1-a)h_m$ ($a \in [0,1]$, $n < m$), then
$|g(v_n)-g(u_n)| \geq a|v_n-u_n| - \alpha_m \geq (a-(4C+4)^{-1})|v_n-u_n|$ and
$|g(v_m)-g(u_m)| \geq (1-a)|v_m-u_m| - aC|v_m-u_m| = (1-a(C+1))|v_m-u_m|$.
Hence, if $a(C+1) \geq 1/2$, then $|g(v_n)-g(u_n)| \geq 1/(4C+4)|v_n-u_n|$, and if $a(C+1) \leq 1/2$, then $|g(v_m)-g(u_m)| \geq 1/2|v_m-u_m| \geq 1/(4C+4)|v_m-u_m|$. Consequently, among the points u_n, v_n, u_m, v_m there is at least one point $s \neq t$ such that $|g(t)-g(s)| \geq 1/(4C+4)|t-s|$. Since $|s-t| < \varepsilon$, this proves that the Lemma holds with $c = 4C+4$.

We shall also need a special partition of unity in R^p.

<u>Lemma 3</u>. Let $G \subset R^p$ be a nonempty open set. Then there is a

sequence of functions $\varphi_n : R^p \to [0,1]$ such that

(i) each φ_n is everywhere Fréchet differentiable, φ_n' is bounded and $\varphi_n' = 0$ on a dense open subset of R^p,

(ii) supp φ_n is a compact subset of G and supp $\varphi_n \cap$ supp $\varphi_m = \emptyset$ whenever $|n-m| > 1$, and

(iii) the sum of φ_n eguals to the charasteristic function of G.

Proof. Let η_n be a sequence of continuously differentiable functions with compact supports in G which forms a locally finite partition of unity on G (see, e.g., [3],pp.224-225). Put $\psi_0 = 0$ and, by induction, $\psi_{k+1} = \sum \{\eta_i;\ i \le k+1 \text{ or supp}\, \eta_i \cap \text{supp}\, \psi_k \ne \emptyset\}$. Then the sequence $\varphi_n = \varphi(\psi_n - \psi_{n-1}) / \sum \varphi(\psi_k - \psi_{k-1})$ (where φ is the function from Lemma 1) has the desired properties.

We shall construct our example by induction, the induction step being the following lemma.

Lemma 4. Let $G \subset R^p$ be an open dense set and let $\varepsilon > 0$. Then there is a function $f : R^{p+1} \to R$ such that

(i) $|f| \le \varepsilon$,

(ii) f' exists on R^{p+1},

(iii) $\|f'\| \le c+1$,

(iv) if $x,y \in R^p$ and $t \in R$, then $|f(x,t)-f(y,t)| \le \varepsilon \|x-y\|$,

(v) $f' = 0$ on a dense open subset of R^{p+1},

(vi) if $x \in R^p - G$ and $t \in R$, then $f'(x,t) = 0$, and

(vii) if $x \in G$ and $t \in R$ then there is $s \in R$ such that $0<|t-s|<\varepsilon$ and $|f(x,t)-f(x,s)| \ge c^{-1}|s-t|$.

Proof. We may assume $\varepsilon < 1$ and $R^p - G \ne \emptyset$. Let φ_n be a partition of unity on G with the properties from Lemma 3. Let $d_n > 0$ such that $|\varphi_n| \le d_n^{-1}$ and $\|\varphi_n'\| \le d_n^{-1}$. For the given $\varepsilon > 0$ and the sequence

$$\varepsilon_n = \min(\varepsilon\, d_n 2^{-n},\ d_n 2^{-n} \operatorname{dist}^2(R^p - G, \operatorname{supp} \varphi_n))$$

we construct a sequence h_n according to the Lemma 2.

Put $f(x,t) = \sum \varphi_n(x) h_n(t)$ for $(x,t) \in R^p \times R = R^{p+1}$. Then

(i) $|f(x,t)| \le \sum d_n^{-1} \varepsilon_n \le \varepsilon$,

(ii) is clear for $(x,t) \in G \times R$ and for other (x,t) it follows from (vi).

(iii) $\|f'(x,t)\| \le \sum |h_n(t)|\ \|\varphi_n'(x)\| + \sum |\varphi_n(x)|\ |h_n'(t)| \le 1+c$,

(iv) for each $t \in R$ the function $f_t(x) = f(x,t)$ is Fréchet differentiable and $\|f_t'(x)\| \le \sum |h_n(t)|\ \|\varphi_n'(x)\| \le \varepsilon$,

(v) if D_n is a dense open subset of $\{x;\ \varphi_n(x) > 0\}$ such that $\varphi_n' = 0$ on D_n, H_n is a dense open subset of R such that $h_n' = 0$ on H_n and $G_n = H_{n-1} \cap H_n \cap H_{n+1}$, then $f' = 0$ on $\bigcup D_n \times G_n$,

(vi) for each $(x,t) \in R^p \times R$ we have

$$|f(x,t)| \le \sum_{n,\ x\in \operatorname{supp}\varphi_n} |\varphi_n(x)|\ |h_n(t)|$$
$$\le \sum_{n,\ x\in \operatorname{supp}\varphi_n} 2^{-n}\operatorname{dist}^2(R^p-G, \operatorname{supp}\varphi_n) \le \operatorname{dist}^2(x, R^p-G).$$

Hence, if $z \in (R^p-G) \times R$ and $y \in R^{p+1}$ then $|f(y)-f(z)| \le \|y-z\|^2$.
(vii) Whenever $x \in G$, the function $g: t \to f(x,t)$ is a convex combination of two functions from the sequence h_n, hence (vii) follows from Lemma 2,(iv).

The rest of the construction is straightforward. Let E denote the Hilbert space of all sequences $x = (x_n;\ n=1,2,\dots)$ of real numbers such that $\|x\|^2 = \sum x_n^2 < \infty$.

Theorem. There is a Lipschitz function f on E which is Gateaux differentiable at each point of E and which is Fréchet differentiable at no point of some residual subset of E.

Proof. By induction we shall construct a sequence of functions $f_p: R^p \to R$ and a sequence of open dense subsets G_p of R^p such that
(i) $|f_p| \le 2^{-p}$,
(ii) f_p is Fréchet differentiable at each point of R^p,
(iii) $\|f_p'\| \le c+1$,
(iv) if (x,t), $(y,t) \in R^p \times R$ then $|f_{p+1}(x,t)-f_{p+1}(y,t)|$ $\le 2^{-p}c^{-1}$ x-y ,
(v) $f_p' = 0$ on G_p,
(vi) if $(x,t) \in (R^p-G_p) \times R$ then $f_{p+1}'(x,t) = 0$,
(vii) if $(x,t) \in G_p \times R$ then there is $s \in R$ such that $0 < |s-t| < 2^{-p}$ and $|f_{p+1}(x,s)-f_{p+1}(x,t)| \ge c^{-1}|s-t|$, and
(viii) $G_{p+1} \subset G_p \times R$.

(We put $f_1 = 0$, $G_1 = R$ and, whenever $f_1,\dots,f_p$, $G_1,\dots,G_p$ have been defined, we use Lemma 4 with $G = G_p$ and $\varepsilon = 2^{-p-1}c^{-1}$ to construct the function f_{p+1}. The set G_{p+1} we define as the intersection of G_p R with a dense open subset of R^{p+1} at each point of which $f_{p+1}' = 0$.)

For $x \in E$ we put $f(x) = \sum f_p(x_1,\dots,x_p)$.

Since $\sum \|f_p'\| \le c+1$ according to (iii), (v), (vi) and (viii), each of the functions $\sum_{p<q} f_p(x_1,\dots,x_p)$ has Lipschitz constant $\le c+1$. Consequently, the Lipschitz constant of f is $\le c+1$.

For each $x \in E$ and each natural k the function
$g_{k,x}(t_1,\dots,t_k) = f(t_1\dots,t_k,x_{k+1},\dots) = \sum_{p\le k} f_p(t_1,\dots,t_p) +$

$$+ \sum_{p>k} f_p(t_1,\dots,t_k,x_{k+1},\dots,x_p)$$

is Fréchet differentiable on R^k since the sum of Fréchet derivatives converges uniformly according to (iv). Since f is Lipschitz, this implies that f is Gateaux differentiable at each point of E.

Let $H_p = \{x \in E;\ (x_1,\dots x_p) \in G_p\}$ and let H be the intersection of the sequence H_p. Then H is a dense G_δ subset of E and $df(x) = 0$ at each $x \in H$. On the other hand, for each $x \in H$ and each natural k we may find $s \in R$ such that

$|f_{k+1}(x_1,\dots,x_k,s)-f_{k+1}(x_1,\dots,x_{k+1})| \geq c^{-1}|s-x_{k+1}|$ and

$0 < |s-x_{k+1}| < 2^{-k-1}$ (property (vii)). Hence

$$|f(x_1,\dots,x_k,s,x_{k+2},\dots)-f(x)| \geq c^{-1}|s-x_{k+1}| - \sum_{n>k} 2^{-n}c^{-1}|s-x_{k+1}|$$
$$\geq (2c)^{-1}|s-x_{k+1}|.$$

(The first inequality follows from (iv).) This shows that f is not Fréchet differentiable at x.

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