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GATEAUX DIFFERENTIABLE POINTS WITH SPECIAL REPRESENTATION

SEUNG JAE OH

ABSTRACT. If (x_n) is a bounded sequence in a Banach space, is there an element $x = \sum_{n=1}^{\infty} a_n x_n$ such that $\sum_{n=1}^{\infty} \|a_n x_n\| < \infty$ and the directional derivative of the norm at x, $D(x, x_n)$, exists for every n? In fact, there are such x's dense in the closed span of $\{x_n\}$. An application of this fact is made to a proof of Rybakov's theorem on vector measures.

A real-valued function f defined on a linear topological space X is said to be *Gateaux differentiable at* $x \in X$ if, for every $y \in X$,

$$Df(x; y) = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}$$

exists and converges to $x^*(y)$ for a unique $x^* \in X^*$, where X^* denotes the linear space of all continuous linear functionals on X. If $Df(x; y)$ exists for a particular direction $y \in X$, then we call this limit the *Gateaux derivative of f at x in the direction of y*. The notation for the directional derivative of the norm is

$$D(x; y) = \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}.$$

A real-valued function f on a subset A of a linear space X is said to be *subdifferentiable* at $x \in A$ if there exists $x^* \in X^*$ such that

$$x^*(y - x) \leqslant f(y) - f(x) \quad \text{for all } y \text{ in } A.$$

We say that x^* is a *subgradient of f at x*. We denote by $\partial f(x)$ the set of all subgradients of f at x and call this set the *subdifferential of f at x*.

THEOREM 1. *Let f be a continuous convex function defined on a Banach space X, and (x_n) a bounded sequence in X; then there is an element $x = \sum_{n=1}^{\infty} a_n x_n$ such that $\sum_{n=1}^{\infty} \|a_n x_n\| < \infty$ and $Df(x; x_n)$ exists for every n. Further, such x's are dense in the closed span of x_n's.*

PROOF. Suppose (x_n) is a bounded sequence in X and f is a continuous convex function on X. Define $T: l_1 \to X$ by

$$T(a) = \sum_{n=1}^{\infty} a_n x_n \quad \text{where } a = (a_n) \in l_1.$$

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The norm of T is the surpemum of the norms of x_n's, so T is a continuous linear operator and $f \circ T$ is a continuous convex function on the separable space l_1. By Mazur's theorem [**4**] there is a dense G_δ subset $G \subseteqq l_1$ at each point of which $f \circ T$ is Gateaux differentiable. By the continuity of T the set $D = T(G)$ is a dense subset of $T(l_1)$, which is, in turn, dense in the closed span S of $\{x_n\}$. Thus, D is dense in the latter. If x_1^* and x_2^* are elements of the subdifferential $(\partial f)(Ta)$ for some $a \in G$, then

$$x_i^*(x - Ta) \leqq f(x) - f(Ta) \quad \text{for all } x \in X,\ i = 1,2.$$

In particular,

$$x_i^*(Tb - Ta) \leqq f(Tb) - f(Ta) \quad \text{for all } b \in l_1,\ i = 1,2,$$

which means $x_1^* \circ T$ and $x_2^* \circ T$ are in $\partial(f \circ T)(a)$. But $f \circ T$ is Gateaux differentiable on G, and, hence, $x_1^* \circ T = x_2^* \circ T$ on l_1. This implies that $x_1^* = x_2^*$ on $T(l_1)$ and also on S; that is, f restricted to S is Gateaux differentiable at each point of the dense set D in S since $\partial f(Ta)$ is a singleton set [**3**, p. 122]. D has the form $\sum_{n=1}^\infty a_n x_n$ with $\sum_{n=1}^\infty \|a_n x_n\| < \infty$, so the proof is complete.

This theorem applies to spaces of measures. Let Σ be a σ-field of subsets of the point set Ω, and let X be a Banach space. The space $\mathrm{ca}(\Sigma, X)$ consists of all countably additive X-valued measures on Σ normed by the semivariation. The following is an obvious corollary.

COROLLARY. *If* (μ_n) *is a bounded sequence in* $\mathrm{ca}(\Sigma, X)$, *then there is an element* $\mu = \sum_{n=1}^\infty a_n \mu_n$ *such that* $\sum_{n=1}^\infty \|a_n \mu_n\| < \infty$ *and* $D(\mu, \mu_n)$ *exists for every* n.

Let $F \in \mathrm{ca}(\Sigma, X)$, and let μ be a finite nonnegative real-valued measure on Σ. F is called μ-continuous ($F \ll \mu$) if $\lim_{\mu(E) \to 0} F(E) = 0$. We now state Rybakov's theorem. ($|\,|$ stands for the variation norm.)

THEOREM 2 (RYBAKOV [**5**]). *If* $F \in \mathrm{ca}(\Sigma, X)$, *then there is an element* $x^* \in X^*$ *such that* $F \ll |x^* F|$.

In order to prove the theorem we need to use the following Lemma. A proof may be found in [**2**, pp. 11–13]. We also use the key fact [**1**] that if $\lambda, \mu \in \mathrm{ca}(\Sigma, R)$ and $D(\lambda, \mu)$ exists, then $\mu \ll \lambda$.

LEMMA. *If* $F \in \mathrm{ca}(\Sigma, X)$, *then there exists a nonnegative real-valued countably additive measure* μ *on* Σ *such that* $F \ll \mu$. *Moreover*, μ *can be chosen so that* $\mu = \sum_{n=1}^\infty \beta_n |x_n^* F|$ *for some* (x_n^*), *with* $\|x_n^*\| = 1$ *for all* n, *where the* β_n*'s can be selected to be nonnegative and satisfy* $\sum_{n=1}^\infty \beta_n = 1$.

PROOF OF THEOREM 2. By using the Lemma let $F \ll \sum_{n=1}^\infty \beta_n |x_n^* F|$, where $(x_n^* F)$ is a bounded sequence in $\mathrm{ca}(\Sigma, R)$. Hence, by the Corollary, there is an element μ in $\mathrm{ca}(\Sigma, R)$ such that $\mu = \sum_{n=1}^\infty a_n (x_n^* F)$ and $D(\mu, x_n^* F)$ exists for every n. If we set $x^* = \sum_{n=1}^\infty a_n x_n^*$, then $x^* F = \sum_{n=1}^\infty a_n x_n^* F$. Now since $D(x^* F, x_n^* F)$ exists for all n, we have $|x_n^* F| \ll |x^* F|$ for all n [**1**]. Therefore,

$$F \ll \sum_{n=1}^\infty \beta_n |x_n^* F| \ll |x^* F|.$$

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DEPARTMENT OF MATHEMATICS, HANNAM UNIVERSITY, TAEJEON, CHOONGNAM, KOREA