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Gauge Condition in Helicity Formalism

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Through the example of boson Compton scattering, the gauge invariance requirement is formulated in the helicity formalism. The formalism assumes that the longitudinal helicity amplitudes vanish as powers of photon invariant "mass", and the virtual Compton scattering can be treated in a unified way with the real one. The formalism is used to discuss the problem of Born terms, Regge pole theory and other related topics. Since the formalism exploit only the knowledge of crossing matrix, it should be applicable to more complicated reactions.

§ 1. Introduction

The problem of constructing the kinematical singularities free (KSF) helicity amplitudes (HA's) has been studied extensively.[1] The KSF amplitudes thus constructed are used to study the dynamical properties of hadron physics. On the other hand, the structure of the helicity amplitudes involving the electromagnetic interaction is not free of problems. For Compton scattering, it is known that the prescription given by reference 1) is not sufficient to give the kinematical zeros free amplitude.[2] It is also known that for a photon process the poles coupled to photon(s) through the "minimal" interaction must be regarded as a "kinematical reflection" of the poles in the crossed channel, since a photon state can have only transversal polarization.[3),4] This problem becomes more than a matter of aesthetics, when we try to treat the processes involving virtual photons. Experiments like electron-proton scattering indicate that the virtual photon state is nothing but a massive (space-like or time-like) state of spin one. Furthermore, at least semi-quantitatively, it is known that a photon behaves similarly to those vector mesons ρ, ω and φ. These experimental facts make it undesirable and unlikely that the processes involving the electro-magnetic interaction demand different kinematics from those of purely strong interaction.

In a series of papers,[5)~7] it is shown that one can treat the real and virtual photon processes in a unified way provided "smoothness" on photon mass variables is required. Under the assumption of "smoothness" the helicity amplitudes involving the longitudinal photon helicity must vanish as a power of photon "mass", and these longitudinal helicity amplitudes are important to achieve the analyticity of the invariant amplitudes. It was shown, among others, that the Pomeranchuk trajectory with non-flat slope can contribute to the forward Compton scattering with regular residue function.[6]

On the other hand, the treatment presented in these papers is not satisfactory in the following respects. First, some "unknown" and seemingly arbitrary functions are introduced, and secondly in construction of the KSFHA's one must rely upon the knowledge on the analyticity of the invariant amplitudes. For a complicated process, the analytic properties of the invariant amplitudes themselves are not clear, because of the gauge invariance requirement.[8),9]

The purpose of the present paper is to show that the gauge condition can be formulated with the knowledge of helicity amplitudes alone.

In § 2, we present the formulation of gauge requirement in the helicity formalism. Our essential assumption is the "power law" on photon mass variable. After demanding the "power law", we proceed to construct the KSFHA's with the usual technique.[1] The formalism is presented here through the example of Compton scattering by a spinless boson target. It is clear, however, that the formalism is of wider applicability. In this section, it will be shown that the additional kinematical zeros associated with photon processes mentioned earlier can be explained easily.

In § 3, we compare the formalism presented in the previous section with the usual invariant amplitudes formalism. It can be seen that the formalism presented in § 2 is equivalent to the "power law" together with the analyticity of the invariant amplitudes.

In § 4, we present various consequences which follow from the formalism. These include the discussion of Born terms, the partial wave expansion, asymptotic behavior, the "vector dominance" model and the low energy theorem.

In § 5, we discuss the connection of crossing symmetry with gauge invariance formulated in § 2. We also discuss the problem of possible non-sense poles in the present formalism.

§ 2. Gauge condition in helicity formalism

The process we consider in the s-channel is

$$V_\mu(k_1)+K(p_1)\longrightarrow V_\nu(k_2)+K(p_2), \tag{2·1}$$

where $V_\mu(k_1)$ indicates a spin one boson of mass m, polarization μ and momentum k_1, and $K(p_1)$ designates a ps-meson of mass M and momentum p_1. With the crossing relation given by Trueman-Wick,[10] we can write down the s-channel helicity amplitudes G in terms of the t-channel helicity amplitudes F

$$G_{++}=-\tfrac{1}{2}\sin^2\chi F_{++}-\tfrac{1}{2}(1+\cos^2\chi)F_{+-}+\tfrac{1}{2}\sin^2\chi F_{00}+\sqrt{2}\sin\chi\cos\chi F_{0+}\,,$$

$$G_{+-}=-\tfrac{1}{2}(1+\cos^2\chi)F_{++}-\tfrac{1}{2}\sin^2\chi F_{+-}-\tfrac{1}{2}\sin^2\chi F_{00}-\sqrt{2}\sin\chi\cos\chi F_{0+}\,,$$

$$G_{+0}=\frac{1}{\sqrt{2}}\sin\chi\cos\chi(-F_{++}+F_{+-}+F_{00})+(\cos^2\chi-\sin^2\chi)F_{0+}\,,$$

$$G_{00}=\sin^2\chi(-F_{++}+F_{+-})-\cos^2\chi F_{00}+2\sqrt{2}\sin\chi\cos\chi F_{0+}\,. \qquad (2\cdot2)^{*)}$$

With the standard notation

$$S^2=[s-(M+m)^2][s-(M-m)^2], \qquad (2\cdot3)$$

the angle χ is given by

$$\sin\chi=2m[-(S^2+st)]^{1/2}/S(t-4m^2)^{1/2},$$
$$\cos\chi=\sqrt{t}\,(s-M^2+m^2)/S(t-4m^2)^{1/2}, \qquad (2\cdot4)$$

where s and t are the standard Mandelstam variables satisfying

$$s+t+u=2(M^2+m^2). \qquad (2\cdot5)$$

The s- and t-channel scattering angles θ_s and θ_t are given by

$$\sin\theta_s=2[-st(S^2+st)]^{1/2}/S^2,$$
$$\cos\theta_s=(S^2+2st)/S^2, \qquad (2\cdot6)$$

and

$$\sin\theta_t=2[-(S^2+st)]/[(t-4m^2)(t-4M^2)]^{1/2},$$
$$\cos\theta_t=[2s+t-2(m^2+M^2)]/[(t-4m^2)(t-4M^2)]^{1/2}$$
$$=(s-u)/[(t-4m^2)(t-4M^2)]^{1/2}. \qquad (2\cdot7)$$

We assume that each of the helicity amplitudes G and F is "smooth" in mass m. Namely, these helicity amplitudes have limit for $m\to0$ smoothly connected with the case $m\neq0$. For these amplitudes containing longitudinal helicity, this limit must be zero. We further demand that the longitudinal helicity amplitudes behave as power of m. Namely,

$$G_{+0},\ F_{0+}\propto m\,,$$
$$G_{00},\ F_{00}\propto m^2. \qquad (2\cdot8)$$

The assumption of power behavior (2·8) is understandable, if one notes that the longitudinal polarization vector of a state moving in z-direction with momentum k is given by

$$\epsilon_L=(k/m,0,0,k_0/m), \qquad (2\cdot9)$$

and that the definition of m is given by $k^2=m^2$ or, equivalently, by Eq. (2·5). The "power law" (2·8) is consistent with the result of perturbation theory and the invariant amplitudes formalism satisfying the divergenceless condition with

*) The phases are chosen such that the Tables I and III in reference 6) can be readily used, if the "u-channel" in reference 6) is read as "s-channel" in this paper.

respect to the photon polarization indices (cf. § 3). Furthermore, we notice that the power requirement (2·8) is the maximal possible one consistent with the structure of Eqs. (2·2) for general value of s and t.

We now examine the analytical property of the crossing relation (2·2). In accordance with the general prescription of reference 1), and with the assumption (2·8), we define $\overline{G}$'s and $\overline{F}$'s as follows:

$$\begin{aligned}
G_{++} &= [2(S^2+st)/S^2]\overline{G}_{++}\,,\\
G_{+-} &= -[2st/S^2]\overline{G}_{+-}\,,\\
G_{0+} &= 2m[-st(S^2+st)]^{1/2}\overline{G}_{0+}/S^2,\\
G_{00} &= m^2\overline{G}_{00}\,,
\end{aligned} \tag{2·10}$$

and

$$\begin{aligned}
F_{++} &= \overline{F}_{++}\,,\\
F_{+-} &= -[4(S^2+st)/(t-4m^2)(t-4M^2)]\overline{F}_{+-}\,,\\
F_{0+} &= 2m[-(S^2+st)/(t-4m^2)(t-4M^2)]\overline{F}_{0+}\,,\\
F_{00} &= m^2\overline{F}_{00}\,.
\end{aligned} \tag{2·11}$$

The s-channel helicity amplitudes G_{++} and G_{+-} have contribution from states with different parity. The separation can be achieved*) by considering

$$X_{1,2} = \overline{G}_{++} \pm \overline{G}_{+-}\,. \tag{2·12}$$

The amplitudes X_1, G_{0+} and G_{00} contain contribution from states with abnormal parity $J^P=0^-, 1^+$, etc. The amplitude X_2 contains states of normal parity 1^-, 2^+, etc.

Equations (2·2), then, can be rewritten as

$$\begin{aligned}
X_1 = \frac{1}{2st(t-4m^2)}&[\{t(s-M^2+m^2)^2-2m^2S^2\}\overline{F}_{++}\\
&+4\{2m^2S^2(S^2+st)+st^2(s-M^2+m^2)^2\}\overline{F}_{+-}/(t-4m^2)(t-4M^2)\\
&-2m^4(S^2+2st)\overline{F}_{00}\\
&-4\sqrt{2}m^2(s-M^2+m^2)\{t/(t-4m^2)(t-4M^2)\}^{1/2}(S^2+2st)\overline{F}_{0+}],\\
X_2 = \frac{S^2}{2st(t-4m^2)}&[-(t-2m^2)\overline{F}_{++}\\
&+4\{st(t-4m^2)-2m^2(S^2+st)\}\overline{F}_{+-}/(t-4m^2)(t-4M^2)\\
&+2m^4\overline{F}_{00}+4\sqrt{2}m^2(s-M^2+m^2)\{t/(t-4m^2)(t-4M^2)\}^{1/2}\overline{F}_{0+}],
\end{aligned}$$

*) The separation is possible only with respect to the leading power in $\cos\theta_s$ in partial wave expansion.

$$\sqrt{s}\overline{G}_{0+}=\frac{1}{\sqrt{2}}\frac{1}{t-4m^2}[-(s-M^2+m^2)\overline{F}_{++}$$
$$-4(s-M^2+m^2)(S^2+st)\overline{F}_{+-}/(t-4m^2)(t-4M^2)$$
$$+m^2(s-M^2+m^2)\overline{F}_{00}$$
$$+\sqrt{2}\{t(s-M^2+m^2)^2+4m^2(S^2+st)\}\{1/t(t-4m^2)(t-4M^2)\}^{1/2}\overline{F}_{0+}],$$

$$\overline{G}_{00}=\frac{1}{S^2(t-4m^2)}[4(S^2+st)\overline{F}_{++}+16(S^2+st)^2\overline{F}_{+-}/(t-4m^2)(t-4M^2)$$
$$-(s-M^2+m^2)^2t\overline{F}_{00}$$
$$-8\sqrt{2}(s-M^2+m^2)(S^2+st)\{t/(t-4m^2)(t-4M^2)\}^{1/2}\overline{F}_{0+}]. \quad (2\cdot13)$$

One can determine the kinematical singularities free t-channel HA's from the above equations as

$$\widetilde{F}_{++}=\overline{F}_{++},$$
$$\widetilde{F}_{+-}=\overline{F}_{+-}/[(t-4m^2)(t-4M^2)],$$
$$\widetilde{F}_{00}=\overline{F}_{00},$$
$$\widetilde{F}_{0+}=\sqrt{2}\overline{F}_{0+}/[t(t-4m^2)(t-4M^2)]^{1/2}. \quad (2\cdot14)^{*)}$$

These equations imply constraint equations at $t=0$ and $t=4m^2$,[11] to make the s-channel HA's finite at these points:

$$m^2(\widetilde{F}_{++}-4S^2\widetilde{F}_{+-}+m^2\widetilde{F}_{00})|_{t=0}=0 \quad (2\cdot15)$$

and

$$\widetilde{F}_{++}+4(s-M^2+m^2)^2\widetilde{F}_{+-}-8m^2(s-M^2+m^2)\widetilde{F}_{0+}-m^2\widetilde{F}_{00}|_{t=4m^2}=0. \quad (2\cdot16)$$

As for the s-channel helicity amplitudes, one sees that $\widetilde{X}_2$ and $\widetilde{G}_{00}$, defined by

$$X_2=\frac{1}{s}S^2\widetilde{X}_2,$$
$$G_{00}=\widetilde{G}_{00}/S^2, \quad (2\cdot17)$$

are KSF. As for the remaining two helicity amplitudes, one observes that the structure of Eqs. (2·13) is such that one can write

$$X_1=\frac{1}{s}\widetilde{X}_1=\frac{1}{s}[(s-M^2+m^2)^2\widetilde{X}_1^{(0)}+m^2\widetilde{X}_1^{(1)}],$$
$$\overline{G}_{0+}=\frac{1}{\sqrt{2s}}\widetilde{G}_{0+}=\frac{1}{\sqrt{2s}}[(s-M^2+m^2)\widetilde{G}_{0+}^{(0)}+m^2\widetilde{G}_{0+}^{(1)}]. \quad (2\cdot18)$$

Note that our "smoothness" assumption (2·8) implies that for $m\to0$, Eqs. (2·18)

*) There is a typographical error in the definition for F_{0+} in reference 6). Note that the $\widetilde{F}_{++}$ defined here is different from the corresponding one in references 2) and 9) by a factor of t.

reduce to the form

$$X_1 = \frac{1}{s}(s-M^2)^2 \widetilde{X}_1^{(0)},$$

$$X_2 = \frac{1}{s}(s-M^2)^2 \widetilde{X}_2 . \tag{2·19}$$

This extra factor $(s-M^2)^2$ agrees with the one obtained by Horn[2] and Abarbanel and Goldberger,[9] and one can see why this factor is peculiar to the photon process. In the formulation of reference 6), the terms $\widetilde{X}_1^{(1)}$ and $\widetilde{G}_{0+}^{(1)}$, which do not contribute to the real Compton scattering, $m=0$, are ignored.

Inversion of Eqs. (2·2) or (2·13) yields:

$$\begin{aligned}
\overline{F}_{++} = {} & [\{st^2(s-M^2+m^2)^2 + 2m^2S^2(S^2+st)\} X_1 \\
& + S^2\{-st(t-4m^2) + 2m^2(S^2+st)\} X_2 + 2m^4 S^2(S^2+st)\overline{G}_{00} \\
& + 4(S^2+st)(s-M^2+m^2)t\sqrt{2s}\,\overline{G}_{0+}]/[(S^2)^2(t-4m^2)], \\
\overline{F}_{+-} = {} & -(t-4M^2)[\{-t(s-M^2+m^2)^2 + 2m^2S^2\} X_1 - S^2(t-2m^2) X_2 \\
& -2m^4S^2\overline{G}_{00} - 4m^2(s-M^2+m^2)t\sqrt{2s}\,\overline{G}_{0+}]/[4(S^2)^2], \\
\overline{F}_{00} = {} & -[4(S^2+st)(S^2+2st)X_1 + 4S^2(S^2+st)X_2 \\
& + (s-M^2+m^2)^2 S^2 t\overline{G}_{00} + 8(S^2+st)(s-M^2+m^2)t\sqrt{2s}\,\overline{G}_{0+}] \\
& \times [(S^2)^2(t-4m^2)]^{-1}, \\
[t/(t-4m^2)(t-4M^2)]^{1/2}\overline{F}_{0+} = {} & t[(s-M^2+m^2)(S^2+2st)X_1 + (s-M^2+m^2)S^2X_2 \\
& + m^2(s-M^2+m^2)S^2\overline{G}_{00} \\
& + \{t(s-M^2+m^2)^2 + 4m^2(S^2+st)\}\sqrt{2s}\,\overline{G}_{0+}]/[\sqrt{2}(S^2)^2(t-4m^2)].
\end{aligned} \tag{2·20}$$

In analogy to the constraints (2·15), (2·16), the analyticity of $\widetilde{F}$'s demands that both

$$X_1 + [(s-M^2+m^2)/\sqrt{2s}]\overline{G}_{0+} \tag{2·21}$$

and

$$(s-M^2+m^2)S^2\overline{G}_{00} + 2(S^2+2st)\sqrt{2s}\,\overline{G}_{0+} \tag{2·22}$$

are proportional to $(S^2)^2$. In terms of the t-channel amplitudes $\widetilde{F}$, the above constraints imply

$$2m\widetilde{F}_{++} + 8m(M\pm m)^2 t\widetilde{F}_{+-} - m(t-2m^2)\widetilde{F}_{00} - 2(M\pm m)t^2\widetilde{F}_{0+} = 0 \tag{2·23}$$

for $s=(M\pm m)^2$. Another constraint worth mentioning is

$$\widetilde{X}_1 + (M^2-m^2)^2\widetilde{X}_2|_{s=0} = 0 . \tag{2·24}$$

Thus, with the assumption of the power law (2·8), we have been able to construct the KSFHA's with the formalism of reference 1).

§ 3. Invariant amplitudes

To see the content of the gauge conditionpr esented in the previous section, one can express these conditions in terms of the invariant amplitudes. Define

$$\epsilon_\nu' T_{\nu\mu}\epsilon_\mu = (\epsilon'\cdot\epsilon) A_1 + (P\cdot\epsilon')(P\cdot\epsilon) A_2 + [(P\cdot\epsilon')(k_2\cdot\epsilon) + (P\cdot\epsilon)(k_1\cdot\epsilon')] A_3 + (k_1\cdot\epsilon')(k_2\cdot\epsilon) A_4\,, \tag{3·1}$$

where $P=p_1+p_2$. Then, we have

$$\begin{aligned} A_1 &= \widetilde{F}_{++} - 4(S^2+st)\widetilde{F}_{+-}\,, \\ A_2 &= 2(t-4m^2)\widetilde{F}_{+-}\,, \\ A_3 &= 2(s-u)\widetilde{F}_{+-} + 2m^2\widetilde{F}_{0+}\,, \\ A_4 &= \frac{1}{t(t-4m^2)}[2(t-2m^2)\widetilde{F}_{++} \\ &\quad + 2\{t(t-4m^2)(t-4M^2) + 8m^2(S^2+st)\}\widetilde{F}_{+-} + 4m^2t(s-u)\widetilde{F}_{0+} - 4m^4\widetilde{F}_{00}]. \end{aligned} \tag{3·2}$$

One can easily see that Eqs. (3·2) imply

$$\begin{aligned} A_1 + (P\cdot k_1)A_3 + (k_1\cdot k_2)A_4 = m^2[&-\{4m^2/t(t-4m^2)\}\widetilde{F}_{++} \\ &-\{8(t-2m^2)(S^2+st) + t(t-4m^2)(t-4M^2)\}\widetilde{F}_{+-}/\{t(t-4m^2)\} \\ &+\{t(s-u)/(t-4m^2)\}\widetilde{F}_{0+} + \{2m^2(t-2m^2)/t(t-4m^2)\}\widetilde{F}_{00}] \end{aligned} \tag{3·3}$$

and

$$(P\cdot k_1)A_2 + (k_1\cdot k_2)A_3 = -2m^2[(s-u)\widetilde{F}_{+-} + (t-2m^2)\widetilde{F}_{0+}]. \tag{3·4}$$

Thus, the assumption that the F's remain finite as $m\to 0$, gives the gauge condition for $m\to 0$,

$$\begin{aligned} A_1 + (P\cdot k_1)A_3 + (k_1\cdot k_2)A_4 &= 0\,, \\ (P\cdot k_1)A_2 + (k_1\cdot k_2)A_3 &= 0\,. \end{aligned} \tag{3·5}$$

Thus, the formalism presented in the previous section can be said to be the requirement of the "power law" Eq. (2·8) plus the analyticity of amplitudes A_i.

Comparison of our treatment with the "usual" one may be useful. In constructing the invariant amplitudes, one usually *a priori* demands fulfillment of gauge invariance. For example,[12] one could write

$$\begin{aligned} \epsilon_\nu' T_{\nu\mu}\epsilon_\mu = {} & [(k_2\cdot k_1)(\epsilon'\cdot\epsilon) - (k_1\cdot\epsilon')(k_2\cdot\epsilon)]B_1 \\ & + [(P\cdot k_1)(P\cdot k_2)(\epsilon\cdot\epsilon') + (k_1\cdot k_2)(P\cdot\epsilon')(P\cdot\epsilon) \\ & \quad - (P\cdot k_1)(P\cdot\epsilon)(k_1\cdot\epsilon') - (P\cdot k_2)(P\cdot\epsilon')(k_2\cdot\epsilon)]B_2 \\ & + [m^4(\epsilon'\cdot\epsilon) + (k_1\cdot k_2)(k_1\cdot\epsilon)(k_2\cdot\epsilon') \end{aligned}$$

$$-m^2(k_1\cdot\epsilon)(k_2\cdot\epsilon')-m^2(k_2\cdot\epsilon)(k_1\cdot\epsilon')]B_3$$
$$+[2m^2(P\cdot k_1)(\epsilon'\cdot\epsilon)-(P\cdot k_1)(k_1\cdot\epsilon)(k_2\cdot\epsilon')$$
$$-(P\cdot k_1)(k_2\cdot\epsilon)(k_1\cdot\epsilon')$$
$$+(k_1\cdot k_2)(P\cdot\epsilon)(k_2\cdot\epsilon')+(k_1\cdot k_2)(P\cdot\epsilon')(k_1\cdot\epsilon)$$
$$-m^2(P\cdot\epsilon)(k_1\cdot\epsilon')-m^2(P\cdot\epsilon')(k_2\cdot\epsilon)]B_4\,. \qquad (3\cdot6)$$

The assumption that the B_i's have a finite limit as $m\to0$, implies the power law (2·8). On the other hand, the requirement that the A_i's are KSF, are not compatible with the requirement that the B_i's are KSF when $m\neq0$.

One notices that A_4 is regular at $t=0$ and $t=4m^2$ because of Eqs. (2·15) and (2·16), respectively. The situation is worthy of special attention. If we put $m=0$ in Eqs. (3·2), our expression implies a kinematical pole for A_4. This peculiar situation is due to the kinematical factor $t-4m^2$. In considering the crossing relation (2·2), one notices that for $t=0$, we have from Eqs. (2·4)

$$\sin\chi=1\,,$$
$$\cos\chi=0\,, \qquad (3\cdot7)$$

independent of $m\neq0$. On the other hand, if we start from $m\equiv0$, we would have $\sin\chi=0$, $\cos\chi=1$. A similar situation exists also in the gauge condition (3·3). For later convenience, we give $\widetilde{F}_{00}$ expressed in terms of A_i's:

$$m^4\widetilde{F}_{00}=\frac{1}{2}(t-2m^2)A_1$$
$$-\frac{1}{4}[t/(t-4m^2)](s-u)^2A_2+\frac{1}{2}t(s-u)A_3-\frac{t}{4}(t-4m^2)A_4\,. \qquad (3\cdot8)$$

§ 4. Application

4.1 *The Born terms*

Gauge invariant perturbation theory gives the Born terms

$$A_1=+2e^2,$$
$$A_2=A_4=e^2(t-2m^2)/(s-M^2)(u-M^2),$$
$$A_3=e^2(s-u)/(s-M^2)(u-M^2). \qquad (4\cdot1)$$

An interesting question to ask is whether the 0^- pole at $s=M^2$ is "dynamical" for the real Compton case, $m=0$, since for the transverse helicity amplitudes the 0^- state corresponds to the nonsense point.

From Eqs. (3·1), we have

$$\widetilde{G}_{00}^{\text{Born}}=e^2[M^2(4M^2-m^2)t$$
$$+(4M^2-m^2)(s-M^2)t+\tfrac{1}{4}S^2(t+8M^2-2m^2)]/(s-M^2)(u-M^2), \qquad (4\cdot2)$$

and there is no pole in any of $\widetilde{X}_1$, $\widetilde{X}_2$ and $\widetilde{G}_{0+}$ at $s=M^2$ when $m\neq 0$. Thus, it can be said from our formalism that the pole in Eqs. (4·1) at $s=M^2$ is indeed the dynamical one. One can also show that the Born amplitudes other than $\widetilde{G}_{00}$ (4·2), i.e. $\widetilde{G}_{0+}$, $\widetilde{F}_{0+}$ and $\widetilde{F}_{00}$, satisfy the "power law" (2·8).

4.2 *The partial wave amplitudes*

The threshold constraint equations (2·21) and (2·22) can be used to parametrize the partial wave amplitudes.[6] To achieve the constraints

$$\widetilde{X}_1+\tfrac{1}{2}(s-M^2+m^2)\widetilde{G}_{0+}\propto(S^2)^2,$$
$$(s-M^2+m^2)\widetilde{G}_{00}+2(S^2+2st)\widetilde{G}_{0+}\propto(S^2)^2, \tag{4·3}$$

one can use the recurrence formula for Legendre functions

$$zP_l'(z)-P_{l-1}'(z)=lP_l(z),$$
$$P_l'(z)+zP_l''(z)=lP_l'(z)+P_{l-1}''(z), \tag{4·4}$$

together with the identity

$$(s-M^2+m^2)^2-4m^2s=S^2, \tag{4·5}$$

since the partial wave expansion for those helicity amplitudes are given by

$$\widetilde{X}_1=\sum_{J=1}^{\infty}\widetilde{X}_1^J[P_J'+zP_J''+a_JP_J''],$$
$$\widetilde{G}_{0+}=\sum_{J=1}^{\infty}\widetilde{G}_{0+}^JP_J'(z),$$
$$\widetilde{G}_{00}=\sum_{J=1}^{\infty}\widetilde{G}_{00}^JP_J(z). \tag{4·6}$$

In a theory not considering the analyticity of the amplitudes in the complex J-plane, the constraints (4·3) imply that $\widetilde{G}_{00}^0$, the $J=0$ amplitude, must be proportional to $(S^2)^2$. The perturbation result (4·2) is seen to be consistent with this result:

$$\widetilde{G}_{00}^0=\frac{e^2}{2Ek_0-m^2}\frac{1}{s-M^2}\frac{1}{8s}(S^2)^2. \tag{4·7}$$

Thus, in perturbation, the dynamical pole in $\widetilde{G}_{00}^0$ appears as a second order zero in $\widetilde{G}_{00}$ when the $m=0$ limit is realized.

4.3 *Asymptotic behavior*[6]

It has been recognized widely[11] that Compton scattering can have a constant cross section due to Pomeron only when the residue has a $1/t$ singularity. It indeed is the case, when one constructs the kinematics with $m=0$. One can see readily from Eqs. (3·2) that the KSF amplitudes $\widetilde{F}_{++}$ and $\widetilde{F}_{+-}$ defined by Eqs. (2·12) are incompatible with the analyticity requirement for A_i's, since

$m \equiv 0$ means $F_{0+} \equiv 0$ and $F_{00} \equiv 0$, and there is no room to invoke the constraint equation $(2\cdot15)$.

On the other hand, for $m \neq 0$, the asymptotic behavior of A_1, which determines the asymptotic cross section, is given by

$$A_1 = [\beta_{++} - 4\beta_{+-}\alpha_P(t)\,(\alpha_P(t)-1)]\frac{1+e^{-i\pi\alpha_P}}{\sin \pi\alpha_P}\left(\frac{s}{s_0}\right)^{\alpha_P} \tag{4·8}$$

with

$$\tilde{F}_{++} = \beta_{++}\frac{1+e^{-i\pi\alpha_P}}{\sin \pi\alpha_P}\left(\frac{s}{s_0}\right)^{\alpha_P},$$

$$\tilde{F}_{+-} = \beta_{+-}\frac{1+e^{-i\pi\alpha_P}}{\sin \pi\alpha_P}\alpha_P(\alpha_P-1)\left(\frac{s}{s_0}\right)^{\alpha_P-2}, \tag{4·9}$$

where α_P is the trajectory of the non-flat Pomeron. One can preserve this situation and approach the limit $m=0$. In this approach, one can achieve the asymptotic constant cross section *without* introducing a fixed pole at $t=0$.

In terms of the helicity amplitude $\overline{G}_{++}$, which contributes to the forward process, the equivalent of Eq. $(4\cdot8)$ can be written as

$$\overline{G}_{++} = [m^2\tilde{F}_{++} + 2\{(t-2m^2)S^2 + 2m^2st\}\tilde{F}_{+-}$$
$$- m^4\tilde{F}_{00} - 2m^2t(s-M^2+m^2)\tilde{F}_{0+}]/(t-4m^2). \tag{4·10}$$

At $t=0$, we have

$$\overline{G}_{++}(t=0) = \tfrac{1}{2}[\tilde{F}_{++} - 4S^2\tilde{F}_{+-}], \tag{4·11}$$

because of the constraint $(2\cdot15)$, and this agrees with the result $(4\cdot8)$. One should note that the threshold factor $(t-4m^2)$ and the kinematical factor $(t-2m^2)$ play a specially important role in realizing the limit $m\to0$. For example, if we put $t=0$ in Eq. $(3\cdot8)$, we have

$$A_1 = -m^2\tilde{F}_{00}. \tag{4·12}$$

Thus, at $t=0$, $\tilde{F}_{00}$ is singular as $m\to0$. To see this point more clearly, we give the result of the Born terms $(4\cdot1)$:

$$\tilde{F}_{00}^{\text{Born}} = e^2[-2(s-u)^2 + (t-2m^2)$$
$$\times (t-4m^2)]/(t-4m^2)(s-M^2)(u-M^2). \tag{4·13}$$

Indeed, the Born term shows the peculiar behavior corresponding to Eq. $(4\cdot12)$. However, one notes that the partial wave amplitudes obtainable from Eq. $(4\cdot13)$ is regular at $t=4m^2$, despite the apparent kinematical pole in Eq. $(4\cdot13)$.

Actually, the situation for Eq. $(4\cdot8)$ is more complicated than stated above. For Regge-pole theory, where analyticity in the angular momentum plane is considered, the threshold condition $(4\cdot3)$ does not mean that the $J=0$ partial wave $\tilde{G}_{00}^0$ is proportional to S^4. One can see[6] that the Regge K-pole contribution

$$\widetilde{G}_{00}^{K}=\beta_K\frac{1+e^{-i\pi\alpha_K}}{\sin\pi\alpha_K}P_{\alpha_K}(z) \tag{4·14}$$

will imply

$$\widetilde{X}_1^{(0)K}=\frac{\beta_K}{4S^2}\frac{1}{\alpha_K{}^2}\frac{1+e^{-i\pi\alpha_K}}{\sin\pi\alpha_K}[P'_{\alpha_K}+zP''_{\alpha_K}+\alpha_K P''_{\alpha_K}]. \tag{4·14′}$$

The quantity $\widetilde{X}_1^{(0)}$ is defined in Eq. (2·16). The expression (4·14′) is necessary to have a dynamical K-meson pole in Regge-pole theory. Now, a similar situation exists also for the t-channel amplitudes. The threshold constraint equation can be rewritten as

$$\widetilde{F}_{++}+(s-u)^2\widetilde{F}_{+-}|_{t=4m^2}\propto(t-4m^2),$$
$$4(s-u)\widetilde{F}_{0+}+\widetilde{F}_{00}|_{t=4m^2}\propto(t-4m^2). \tag{4·15}$$

Remembering that

$$(s-u)=[(t-4m^2)(t-4M^2)]^{1/2}\cos\theta_t, \tag{4·16}$$

one notes that the threshold condition (4·15) can be achieved with the conditions

$$\beta_{+-}^{P}=-\{\beta_{++}^{P}/[\alpha_P(\alpha_P-1)]\}[(t-4m^2)(t-4M^2)]^{-1},$$
$$\beta_{0+}^{P}=-\frac{1}{4}\frac{1}{\alpha_P}\beta_{00}[(t-4m^2)(t-4M^2)]^{-1/2}, \tag{4·17}$$

because of the recurrence formula

$$z^2P_\alpha''(z)=\alpha(\alpha-1)P_\alpha(z)+(\alpha-1)P'_{\alpha-1}(z)+zP''_{\alpha-1}(z),$$
$$zP_\alpha'(z)=\alpha P_\alpha(z)+P_{\alpha-1}(z). \tag{4·18}$$

In the above, β_{00} and β_{0+} are defined similarly to β_{++} and β_{+-} given in Eqs. (4·9), and we discarded the part in β's which is explicitly proportional to m^2.

The conditions (4·17) imply that both $\widetilde{F}_{++}$ and $\widetilde{F}_{+-}$ can contribute to forward Compton scattering without invoking any singular residue.

4.4 *The "vector dominance model"*

The terminology "vector dominance model" is not well defined. One can say that our whole presentation given in § 2 is nothing but "vector dominance". The terminology is, however, used customarily to imply, for example,[13)]

$$\widetilde{X}_1(m^2)\propto\widetilde{X}_1(0). \tag{4·19}$$

One sees that our formalism does not imply Eq. (4·19). It is possible, however, to demand that the relation (4·19) hold approximately.[7)] From Eq. (2·18), Eq. (4·19) is seen to hold provided the relation

$$\widetilde{X}_1^{(1)}=-\{2(s-M^2)+m^2\}\widetilde{X}_1^{(0)} \tag{4·20}$$

is approximately satisfied. If we demand Eq. (4·20) hold rigorously, it implies that $\widetilde{X}_1$ defined by Eq. (2·16) has the additional factor of $(s-M^2)^2$, which may have some significance when one considers the problem of unwanted poles in the complex angular momentum plane.[14] (For this problem, see next section.)

4.5 *The low energy theorem*

Abarbanel and Goldberger[9] discussed the low energy theorem in terms of the helicity amplitude formalism for the case of real photon, $m=0$. One can see that the procedure can be followed closely in our formalism. It is perhaps worthwhile to note, however, that the s-channel and u-channel poles in the transverse helicity amplitudes discussed are only apparent, in the sense that they are the kinematical factor discussed in Eqs. (2·19), though these poles in the invariant amplitudes (3·1) as given by Eqs. (4·1) are truly dynamical.

§ 5. Remarks

5.1 *Crossing symmetry and gauge invariance*

Construction of the kinematical singularities free helicity amplitudes with the requirement of the "power law", as is shown in § 2, did not take into account the crossing property of the amplitudes under $s\leftrightarrow u$ crossing. As is most clearly seen in Eqs. (3·2), our formulation is consistent with the crossing symmetry of real Compton scattering.

One can easily see that the analyticity of the helicity amplitudes is *not* compatible with the "power law" (2·8), when one considers the case of *opposite* crossing. Physically, such a case would correspond to charge exchange scattering by virtual ρ-mesons,

$$\rho^- + K^+ \longrightarrow \rho^0 + K^0.$$

To show this incompatibility, one notes first that the Bose statistics requires $\widetilde{F}_{++}$, $\widetilde{F}_{+-}$, $\widetilde{F}_{00}$ to start from $(s-u)$ in power expansion in terms of $(s-u)$, and only $\widetilde{F}_{0+}$ can have a constant term. Thus, with the definition for $\widetilde{F}_{0+}$ given by Eq. (2·14), every s-channel helicity amplitude $\widetilde{X}_1$, etc., would have a pole at $t=4m^2$ when $s=u$. From the crossing relations which give $\widetilde{F}$'s in terms of $\widetilde{X}_1$, etc., one finds that there is no way to correct this shortcoming unless all of the $\widetilde{F}$'s defined by Eqs. (2·14) have a zero at $t=4m^2$. Such a circumstance is not compatible with the result of perturbation. One might wonder about the case $\widetilde{F}_{0+}\propto(s-u)^2$. This corresponds to neglecting the intermediate ρ-pole and $J=1^-$ states, and again is not compatible with perturbation theory.

Our analysis indicates that for a particle like the ρ-meson, one cannot demand the "power law" together with analyticity without breaking $SU(2)$ symmetry. This corresponds to the case of chiral Lagrangian theory with the coupling $\rho_\mu A_\mu$,[15] where the divergence condition for the ρ-field is given by

$$(\partial_\mu \pm ieA_\mu)\rho_\mu{}^\pm = 0\,,$$

$$\partial_\mu \rho_\mu{}^0 = 0\ . \qquad (5\cdot1)$$

5.2 *The fixed pole singularity in partial wave amplitudes*

Recently, Collins and Gault[14] have shown that for a photon process, $m=0$, the Reggeization should proceed analogous to the one presented in § 4.3. They have shown that for $m=0$, the pole representation like Eqs. (4·15) and (4·17) are allowed since a kinematical factor can be invoked to cancel the unwanted fixed pole in the physical partial wave amplitudes.

The argument of Collins and Gault may seem somewhat unfavorable to our situation, since their argument is strongly dependent on the kinematics of $m=0$. To accomodate our result with that of Collins and Gault, we present here two possibilities. For the s-channel K-meson Regge pole, one could appeal to the "vector dominance model" relation (4·21); then, the argument can be made parallel to those in reference 14). Another possibility is to ask to the contribution from the back-ground integral. One can see that the presentation like Eq. (4·15) can only be made compatible with intuition by admitting a contribution from the "background integral".*) For example, the $J=0$ state in $\widetilde{G}^0_{00}$ is made of a ρK system with orbital angular momentum one, and the physically observable amplitude $\widetilde{G}^0_{00}$ defined by Eq. (4·6), should be proportional to $(S^2)^2$. Thus, for s near the threshold $S^2=0$, the contribution from the Regge pole, Eq. (4·15), should be cancelled by the "background" contribution. One can also appeal to similar argument in the case of t-channel amplitudes.

5.3 *The $t=am^2$ singularity*

As seen from Eqs. (3·3) and (3·4), one cannot get the gauge condition (3·5) when we put $t=am^2$, with a an arbitrary constant not equal to 0 or 4, and then take the limit $m\to0$. Such a singular behavior is not surprising. We notice that if we take $t=am^2$ as above, then for $m\to0$, we have

$$\sin\chi = 2/(4-a)^{1/2},$$
$$\cos\chi = [-a/(4-a)]^{1/2} \qquad (5\cdot2)$$

from Eq. (2·4). This limit is quite different from the value $\cos\chi=1$ for $m\equiv0$, and $\cos\chi=0$ for $t=0$, $m\neq0$. Thus, it should be interesting to analyze the virtual photon process under the constraint $t=am^2$ for small values of $|m^2|$, if possible.

In our formalism, we preferred to start from $m\neq0$. That the procedure is consistent with the kinematics at $m=0$ can be seen from the papers of references 14) and 16).

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*) This could include a cut contribution. What we claim is that there could be terms important in low s or t but irrelevent to the asymptotic behavior.

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Note added in proof:

(1) The paper F. Arbab and R. C. Brower, Phys. Rev. **181** (1969), 2124, should be added to the references 5)~7) as the one pursuiting the similar idea.

(2) Recently, A. H. Mueller and T. L. Trueman published a paper (to be published in Phys. Rev. **D**) very relevant to the approach presented in our paper. Though our article covers the topics they discussed, a more explicit discussion can be found in a preprint by two of us (T.A., T.E.) together with M. Sakuraoka (TU/70/62).

(3) Our kinematical factor for F as given by Eq. (2·14) is different from the usual one by an over all factor of $(t-4m^2)$. This difference disappears when $m\to 0$ limit is taken.