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Gauge Field Theory of the Quantum Group $SU_q(2)$

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The gauge field theory of the quantum group $SU_q(2)$ is formulated. The parallelism to the case of $SU(2)$ is maintained with the help of the definition of the $SU_q(2)$ gauge transformations preserving some group-like properties, the differential calculus on $SU_q(2)$ developed by Woronowicz, and the q-deformed trace. The Lagrangian invariant under local $SU_q(2)$ transformations is obtained.

§ 1. Introduction

Nearly forty years have passed since Yang and Mills[1] constructed a field theory which is invariant under local $SU(2)$ transformations. Their scheme and its generalizations have found the most brilliant applications in particle physics. They are realized in the Glashow-Weinberg-Salam model[2] for the electroweak interaction and QCD for the strong interaction. Utiyama[3] showed that even the grativational interaction can be described along the line of Yang-Mills.

On the other hand, the notion of the Lie group has been generalized recently by Drinfel'd,[4] Jimbo[5] and Woronowicz.[6] Their generalized Lie group, i.e., a noncommutative and non-cocommutative Hopf algebra, is now called the quantum group and is under an enthusiastic study by a lot of mathematicians and physicists. In some integrable models of quantum field theory, the quantum group manifests itself as the group of hidden symmetry.[7] It should be mentioned that several authors attempted to quantize or q-deform the Lorentz group.[8]

The purpose of this article is to generalize Yang and Mills' idea to the case of quantum group. We shall be concerned only with the simplest example of quantum group, $SU_q(2)$, which was called $S_\nu U(2)$ in Ref. 6). Our scheme developed below is a generalization of $SU(2)$ Yang-Mills theory since $SU_q(2)$ contains a parameter ν and reduces to $SU(2)$ for $\nu=1$. Although the gauge field theory of quantum group has been investigated by Bernard[9] and Aref'eva and Volovich,[10] our scheme is quite different from theirs. Bernard[9] discussed the case that the gauge potentials are R-commuting. For the R-commutativity to be consistent, however, the R-matrix must satisfy a very severe condition, excluding the most fundamental case of $SU_q(2)$. Aref'eva and Volovich[10] considered the quantum enveloping algebra as the basic algebra for gauge fields. They concluded that an infinite number of component fields are needed to describe the gauge field in general representations. We discuss in this paper that the product of gauge transformations should be defined differently from the manner of Refs. 9) and 10) since the quantum group is not a group in the usual sense. We shall find that, with the help of the q-deformed trace and the differential calculus on $SU_q(2)$ invented by Woronowicz,[6),11),12)] the gauge field theory of $SU_q(2)$ can be constructed in a parallel way to the case of $SU(2)$.

This paper is organized as follows. In § 2, we discuss how the product of gauge transformations should be defined. In § 3, we investigate how the fields containing noncommutative group coordinates should be differentiated. The group-transformation law of fields is formulated in § 4. We discuss in § 5 how the invariants of the local $SU_q(2)$-transformations are constructed. The final section will be devoted to some concluding remarks. Some appendices are attached to explain the details of calculations.

§ 2. Product of $SU_q(2)$ transformations

2.1. *Repeated $SU_q(2)$ transformations*

According to Woronowicz,[6] the fundamental representation of $SU_q(2)$ is given by

$$w=\begin{pmatrix} \alpha & -\nu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}, \tag{2·1}$$

where α, γ, α^* and γ^* are operators satisfying the relations

$$\begin{aligned} &\alpha^*\alpha+\gamma^*\gamma=I\,, \quad \alpha\alpha^*+\nu^2\gamma^*\gamma=I\,, \\ &\gamma\gamma^*=\gamma^*\gamma\,, \quad \alpha\gamma=\nu\gamma\alpha\,, \quad \alpha\gamma^*=\nu\gamma^*\alpha\,, \\ &\gamma^*\alpha^*=\nu\alpha^*\gamma^*\,, \quad \gamma\alpha^*=\nu\alpha^*\gamma\,, \end{aligned} \tag{2·2}$$

ν a real parameter and I is the unit operator such that $\alpha I=I\alpha=\alpha$, etc. We denote by A the polynomial ring generated by I, α, γ, α^* and γ^* and by $M_N(B)$ the set of $N\times N$ matrices whose entries belong to set B. One can think of set A' of representations of A as operators acting on a Hilbert space H.[6] If

$$w_i=\begin{pmatrix} \alpha_i & -\nu\gamma_i^* \\ \gamma_i & \alpha_i^* \end{pmatrix}, \quad i=1,2\,, \tag{2·3}$$

are two such representations of w, the members of each set of $(\alpha_i, \gamma_i, \alpha_i^*, \gamma_i^*)$, $i=1, 2$, satisfy, by definition, the same algebra as (2·2). We define two kinds of product of w_1 and w_2. One is the conventional product,

$$w_2w_1=\begin{pmatrix} \alpha_2\alpha_1-\nu\gamma_2^*\gamma_1 & -\nu(\gamma_2^*\alpha_1^*+\alpha_2\gamma_1^*) \\ \gamma_2\alpha_1+\alpha_2^*\gamma_1 & \alpha_2^*\alpha_1^*-\nu\gamma_2\gamma_1^* \end{pmatrix}\in M_2(A') \tag{2·4}$$

which acts on H. The other is the product introduced by Woronowicz,[6]

$$w_2\oplus w_1=\begin{pmatrix} \alpha_2\otimes\alpha_1-\nu\gamma_2^*\otimes\gamma_1 & -\nu(\gamma_2^*\otimes\alpha_1^*+\alpha_2\otimes\gamma_1^*) \\ \gamma_2\otimes\alpha_1+\alpha_2^*\otimes\gamma_1 & \alpha_2^*\otimes\alpha_1^*-\nu\gamma_2\otimes\gamma_1^* \end{pmatrix}\in M_2(A'\otimes A') \tag{2·5}$$

which acts on $H\otimes H$ and is intimately related to the coproduct defined on A. The $*$-operation is the complex-conjugation for complex numbers and is defined by

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n)^* = a_1^* \otimes a_2^* \otimes \cdots \otimes a_n^* ,$$
$$(a_1 a_2 \cdots a_n)^* = a_n^* \cdots a_2^* a_1^* ,$$
$$(a_i^*)^* = a_i \tag{2·6}$$

for operators belonging to A or A'. The product of $\otimes$-products of operators of A or A' is defined by

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n)(b_1 \otimes b_2 \otimes \cdots \otimes b_n) = a_1 b_1 \otimes a_2 b_2 \otimes \cdots \otimes a_n b_n . \tag{2·7}$$

Although the entries of $w_2 w_1$ do not satisfy the algebra (2·2), those of $w_2 \oplus w_1$ inherit the properties of w:

$$(w_2 \oplus w_1)_{22} = (w_2 \oplus w_1)_{11}^* \equiv \alpha'^* ,$$
$$(w_2 \oplus w_1)_{12} = -\nu (w_2 \oplus w_1)_{21}^* \equiv -\nu \gamma'^* , \tag{2·8}$$
$$\alpha'^* \alpha' + \gamma'^* \gamma' = I \otimes I , \quad \alpha' \alpha'^* + \nu^2 \gamma'^* \gamma' = I \otimes I ,$$
$$\gamma' \gamma'^* = \gamma'^* \gamma' , \quad \alpha' \gamma' = \nu \gamma' \alpha' , \quad \alpha' \gamma'^* = \nu \gamma'^* \alpha' ,$$
$$\gamma'^* \alpha'^* = \nu \alpha'^* \gamma'^* , \quad \gamma' \alpha'^* = \nu \alpha'^* \gamma' , \tag{2·9}$$

where we denoted the unit operator on H also by I. It can be readily understood that the product $w_m \oplus w_{m-1} \oplus \cdots \oplus w_1$, with the entries of w_i, $i=1, 2, \cdots, m$, satisfying (2·2), preserves properties analogous to (2·8) and (2·9). The only difference is that, for $w_m \oplus w_{m-1} \oplus \cdots \oplus w_1$, the operator

$$I_m = I \otimes I \otimes \cdots \otimes I \quad (m \text{ times}) , \tag{2·10}$$

instead of I or $I \otimes I$, plays the role of the unit operator. We have thus observed that the quantum group $SU_q(2)$ is a group-like object with respect to the $\oplus$-product. It is then natural to regard $w_m \oplus w_{m-1} \oplus \cdots \oplus w_1$, rather than $w_m w_{m-1} \cdots w_1$, as the matrix corresponding to the repeated application of $SU_q(2)$ transformations. We denote the set of $w_m \oplus w_{m-1} \oplus \cdots \oplus w_1$ by C_m. We hereafter consider the transformations caused by the elements of C_m and the group-theoretic representations thereof. This is the main difference of our approach from those of Refs. 9) and 10).

2.2. *Inverse of* w

The matrix w given in (2·1) has the inverse w^{-1}:

$$w^{-1} = \begin{pmatrix} \alpha^* & \gamma^* \\ -\nu\gamma & \alpha \end{pmatrix} , \tag{2·11}$$

$$w^{-1} w = w w^{-1} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} , \tag{2·12}$$

$$(w^{-1})_{ij} = w_{ji}^* , \quad i, j = 1, 2 . \tag{2·13}$$

It turns out that $w_m \oplus w_{m-1} \oplus \cdots \oplus w_1 \in C_m$ has the inverse in the sense that

$$(w_m \oplus w_{m-1} \oplus \cdots \oplus w_1)^{-1} (w_m \oplus w_{m-1} \oplus \cdots \oplus w_1)$$

$$=(w_m\oplus w_{m-1}\oplus\cdots\oplus w_1)(w_m\oplus w_{m-1}\oplus\cdots\oplus w_1)^{-1}$$

$$=\begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix}. \tag{2·14}$$

Its entries are given by

$$(w_m\oplus w_{m-1}\oplus\cdots\oplus w_1)^{-1}_{ij}=(w_m\oplus w_{m-1}\oplus\cdots\oplus w_1)^*_{ji}\,, \quad i,j=1,2\,. \tag{2·15}$$

Note, however, that $(w_m\oplus w_{m-1}\oplus\cdots\oplus w_1)^{-1}$ does not belong to C_m. From the algebra (2·2), we obtain

$$\sum_{k,l=1}^{2}\sigma_{kl}w_{ki}w^*_{lj}=\sigma_{ij}I\,, \quad i,j=1,2\,, \tag{2·16}$$

$$\sum_{k,l=1}^{2}(\sigma^{-1})_{kl}w^*_{ik}w_{jl}=(\sigma^{-1})_{ij}I\,, \quad i,j=1,2\,, \tag{2·17}$$

$$\sigma=\begin{pmatrix} 1 & 0 \\ 0 & \nu^2 \end{pmatrix}, \tag{2·18}$$

which should be compared with the unitarity of w, i.e.,

$$\sum_{k=1}^{2}w_{ik}w^*_{jk}=\sum_{k=1}^{2}w^*_{ki}w_{kj}=\delta_{ij}I\,, \quad i,j=1,2\,. \tag{2·19}$$

2.3. *Group-theoretic representation of w*

The group-theoretic representation theory of $SU_q(2)$ was discussed by many authors.[6),13),14)] It turned out that the representation theory of $SU_q(2)$ is quite similar to that of $SU(2)$. The matrix $W\in M_N(A)$ is said to be a representation of w if it satisfies[6)]

$$\Phi(W_{ij})=(W\oplus W)_{ij}\equiv\sum_{k=1}^{N}W_{ik}\otimes W_{kj}\,, \quad i,j=1,2,\cdots,N\,, \tag{2·20}$$

where Φ is the coproduct defined by

$$\Phi(w)\equiv\begin{pmatrix} \Phi(\alpha) & \Phi(-\nu\gamma^*) \\ \Phi(\gamma) & \Phi(\alpha^*) \end{pmatrix}=w\oplus w\,, \tag{2·21}$$

$$\Phi(ab)=\Phi(a)\Phi(b)\,, \quad \Phi(c_1a+c_2b)=c_1\Phi(a)+c_2\Phi(b)\,,$$

$$a,b\in A\,, \quad c_1,c_2\in \mathcal{C}\,. \tag{2·22}$$

We shall say that the irreducible unitary representation $W\in M_N(A)$ is canonical if it satisfies the N-dimensional version of (2·16) and (2·17):

$$\sum_{k,l=1}^{N}\sigma^N_{kl}W_{ki}W^*_{lj}=\sigma^N_{ij}I\,,$$

$$\sum_{k,l=1}^{N}(\sigma^N)^{-1}_{kl}W^*_{ik}W_{jl}=(\sigma^N)^{-1}_{ij}I\,, \tag{2·23}$$

$$\sigma^N_{kl}=\nu^{2(k-1)}\delta_{kl}\,, \quad k,l=1,2,\cdots,N\,. \tag{2·24}$$

The fundamental representation w is canonical as is seen from (2·16) and (2·17). The canonical representations of general dimensions were obtained in Ref. 14) with the aid of the little-q Jacobi polynomials. One can consider more general representations of w than canonical ones since $W_t = tWt^{-1}$, $t \in U(N) \subset M_N(\mathbb{C})$, is a representation of w if W is. Although the discussions below can be applied, with slight modifications, to W_t, we hereafter consider only the canonical representations and the $\oplus$-product thereof for the sake of definiteness. We define the set $C_m{}^N$ by

$$C_m{}^N = \{W_m \oplus W_{m-1} \oplus \cdots \oplus W_1\}, \tag{2·25}$$

where $W_i \in M_N(A')$, $i=1, 2, \cdots, m$, are the canonical representations of $w_i \in M_2(A')$, $i=1, 2, \cdots, m$, respectively. It is easy to see that the element of $C_m{}^N$ satisfies the analogue of (2·23):

$$\sum_{k,l=1}^{N} \sigma_{kl}^{N} W_{ki} W_{lj}^{*} = \sigma_{ij}^{N} I_m, \quad W \in C_m{}^N. \tag{2·26}$$

§ 3. Differential calculus on $SU_q(2)$

3.1. *3D calculus on $SU_q(2)$*

Woronowicz discussed the differential calculus on $SU_q(2)$, the coordinates of which being non-commutative operators.[6),11),12)] The differential calculus on $SU_q(2)$ is not unique. The $4D_\pm$ scheme developed in Ref. 12) is bicovariant, while the $3D$ scheme studied in Ref. 6) is only left-covariant but simple and works mysteriously well even for the higher order differential calculi. For simplicity, we make use of the latter scheme. We here briefly recapitulate the 3D calculus of Woronowicz.[6)]

The linear functionals χ_0, χ_1, χ_2, f_0, f_1, f_2, e on A are defined by

$$\chi_0(w) \equiv \begin{pmatrix} \chi_0(\alpha) & \chi_0(-\nu\gamma^*) \\ \chi_0(\gamma) & \chi_0(\alpha^*) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\chi_1(w) = \begin{pmatrix} 1 & 0 \\ 0 & -\nu^2 \end{pmatrix}, \quad \chi_2(w) = \begin{pmatrix} 0 & 0 \\ -\nu & 0 \end{pmatrix}, \quad \chi_k(I) = 0, \tag{3·1}$$

$$f_0(w) = f_2(w) = \begin{pmatrix} \nu^{-1} & 0 \\ 0 & \nu \end{pmatrix}, \quad f_1(w) = \begin{pmatrix} \nu^{-2} & 0 \\ 0 & \nu^2 \end{pmatrix}, \quad f_k(I) = 1, \tag{3·2}$$

$$e(w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e(I) = 1, \ k=0, 1, 2 \tag{3·3}$$

and

$$e(ab) = e(a)e(b), \tag{3·4}$$

$$\chi_k(ab) = \chi_k(a) f_k(b) + e(a)\chi_k(b), \tag{3·5}$$

$$f_k(ab) = f_k(a) f_k(b), \quad a, b \in A, \quad k=0, 1, 2. \tag{3·6}$$

The convolution product of a linear functional X on A and $a\in A$ is defined by

$$X*a=\sum_i X(a_i')a_i''\in A, \tag{3·7}$$

where a_i' and a_i'' are given by $\Phi(a)=\sum_i a_i'\otimes a_i''$. The differential operator d is defined by

$$da=\sum_{k=0}^{2}(\chi_k*a)\omega_k, \quad a\in A, \tag{3·8}$$

where ω_k, $k=0, 1, 2$, are the bases of the space of differential one-forms. The multiplication of the one-form

$$\omega=\sum_{k=0}^{2}a_k\omega_k, \quad a_k\in A, \quad k=0, 1, 2, \tag{3·9}$$

by $a\in A$ is defined by

$$a\omega=\sum_{k=0}^{2}(aa_k)\omega_k, \tag{3·10}$$

$$\omega a=\sum_{k=0}^{2}a_k(f_k*a)\omega_k, \tag{3·11}$$

assuring the associativity of the multiplication laws

$$(b\omega)a=b(\omega a), \quad \omega(ab)=(\omega a)b, \quad a(b\omega)=(ab)\omega, \tag{3·12}$$

and the desired property

$$d(ab)=(da)b+adb \tag{3·13}$$

for any $a, b\in A$. The higher order differential calculus can be defined so as to maintain the property

$$d^2=0. \tag{3·14}$$

Above definitions are sufficient to derive the following results:

$$W^{-1}dW=\sum_{k=0}^{2}\chi_k(W)\omega_k, \tag{3·15}$$

$$\chi_k(W)\chi_l(W)-c_{lk}\chi_l(W)\chi_k(W)=\sum_{m=0}^{2}d_{mkl}\chi_m(W), \quad k, l=0, 1, 2 \tag{3·16}$$

for any $W\in C_1{}^N$, where c_{lk} and d_{mkl} are given by

$$c_{kk}=1, \quad k=0, 1, 2,$$
$$c_{20}=(c_{02})^{-1}=\nu^2, \quad c_{21}=(c_{12})^{-1}=\nu^4, \quad c_{10}=(c_{01})^{-1}=\nu^4 \tag{3·17}$$

and

$$d_{120}=\nu^{-1}, \quad d_{102}=-\nu, \quad d_{212}=d_{001}=-\nu^2(1+\nu^2)$$
$$d_{221}=d_{010}=\nu^{-2}(1+\nu^2), \quad d_{mkl}=0;\ \text{otherwise}. \tag{3·18}$$

The hermitian conjugate $\chi_k(W)^\dagger$ of $\chi_k(W)\in M_N(C)$ is given by

$$-\nu\chi_0(W)^\dagger=\chi_2(W)\,,\quad \chi_1(W)^\dagger=\chi_1(W)\,,\quad W\in C_1^N\,. \tag{3·19}$$

From (3·7), (3·8), (3·11) and (2·20), we obtain

$$\omega_k W=Wf_k(W)\omega_k\,,\quad k=0,1,2\,,\quad W\in C_1^N\,. \tag{3·20}$$

Other properties of ω_k obtained in Ref. 6) are summarized in Appendix A. The formulae (A·5) and (A·6) exhibit what ω_0, ω_1 and ω_2 should be identified with.

3.2. *Local $SU_q(2)$*

Let $x=(x^0, x^1, x^2, x^3)$ be the coordinate of the four dimensional Minkowski spacetime and $\alpha(x)$, $\gamma(x)$, $\alpha^*(x)$, $\gamma^*(x)\in A'$ be the x-dependent representations of α, γ, α^*, $\gamma^*\in A$, respectively, as operators acting on the Hilbert space H introduced in 2.1. To discuss the gauge theory of $SU_q(2)$, it is inevitable to consider functions of x, $\alpha(x)$, $\gamma(x)$, $\alpha^*(x)$, $\gamma^*(x)$ and their derivatives with respect to x^μ. We denote the set of functions of the form $g[x]\equiv g(x, \alpha(x), \gamma(x), \alpha^*(x), \gamma^*(x))$ by A'^x. The functional X^x on A'^x should be so introduced that $X^x(w(x))=X(w)$, e.g.,

$$\chi_k{}^x(w(x))=\chi_k(w)\,,\quad f_k{}^x(w(x))=f_k(w)\,,\quad e^x(w(x))=e(w)\,,\quad k=0,1,2\,, \tag{3·21}$$

where $w(x)$ is defined by

$$w(x)=\begin{pmatrix}\alpha(x) & -\nu\gamma^*(x)\\ \gamma(x) & \alpha^*(x)\end{pmatrix} \tag{3·22}$$

and w, χ_k, f_k and e are those defined hitherto. Recalling (3·8), the differential operator d^x should be defined to act on $g[x]$ as

$$d^x g[x]=\sum_{k=0}^{2}(\chi_k{}^x * g[x])\omega_k{}^x+(\partial_\mu g[x])dx^\mu\,, \tag{3·23}$$

where $\omega_k{}^x$, $k=0, 1, 2$, are the analogue of previous ω_k and $\partial_\mu g[x]$ is the conventional partial derivative of $g[x]$ with respect to the explicit x-dependence of $g[x]$. A consistent set of rules are derived from the result of Woronowicz[6] by supposing that $\omega_k{}^x$ and $d^x g[x]$ are decomposed as $\omega_{k,\mu}^x dx^\mu$ and $(D_\mu g[x])dx^\mu$, respectively, and assuming $\{dx^\mu, dx^\nu\}=[dx^\mu, \omega_{k,\nu}^x]=[dx^\mu, a]=0$, $a\in A'^x$, μ, ν,$=0, 1, 2, 3$. We call the above procedure the Z-procedure. We notice that the results of the Z-procedure can be reproduced more systematically in a manner mimicking (3·10) and (3·11). The Z-procedure leads us to the following definition of the partial derivative $D_\mu g[x]$ of $g[x]\in A'^x$:

$$D_\mu g[x]=\sum_{k=0}^{2}(\chi_k{}^x * g[x])\omega_{k,\mu}^x+\partial_\mu g[x]\,. \tag{3·24}$$

It can be seen that the relation

$$\omega_{k,\mu}^x w(x)=w(x)f_k{}^x(w(x))\omega_{k,\mu}^x=w(x)f_k(w)\omega_{k,\mu}^x \tag{3·25}$$

induced by (3·20) and the Z-procedure ensure

$$D_\mu(g[x]h[x])=(D_\mu g[x])h[x]+g[x](D_\mu h[x]) \tag{3·26}$$

for any $g[x], h[x]\in A'^x$. We also obtain from (A·1)~(A·4) and the Z-procedure that

$$(\omega^x_{\bar{0},\mu})^*=\nu\omega^x_{\bar{2},\mu}\,,\quad (\omega^x_{\bar{1},\mu})^*=-\omega^x_{\bar{1},\mu}\,,\quad (\omega^x_{\bar{2},\mu})^*=\nu^{-1}\omega^x_{\bar{0},\mu}\,, \tag{3·27}$$

$$\omega^x_{\bar{k},\mu}\omega^x_{\bar{l},\nu}=c_{kl}\omega^x_{\bar{l},\nu}\omega^x_{\bar{k},\mu}\,,\quad k,\ l=0,1,2\,, \tag{3·28}$$

$$D_\mu\omega^x_{\bar{k},\mu}=-d_{klm}\omega^x_{\bar{l},\mu}\omega^x_{\bar{m},\nu}\,,\quad (k,\ l,\ m)=(0,0,1),(1,0,2),(2,1,2)\,, \tag{3·29}$$

where the c_{kl}'s are given by (3·17). From (3·14), (3·15), (3·21) and the Z-procedure, we obtain

$$D_\mu D_\nu g[x]=D_\nu D_\mu g[x]\,,\quad g[x]\in A'^x\,, \tag{3·30}$$

$$W(x)^{-1}D_\mu W(x)=\sum_{k=0}^{2}\chi_k(W)\omega^x_{\bar{k},\mu}+W(x)^{-1}\partial_\mu W(x)\,, \tag{3·31}$$

where $W(x)\in M_N(A'^x)$ is the canonical representation of $w(x)\in M_2(A'^x)$ just as $W\in C_1{}^N\subset M_N(A)$ is that of $w\in M_2(A)$. From the above definition of $W(x)$, we readily understand that (3·25) can be generalized to

$$\omega^x_{\bar{k},\mu}W(x)=W(x)f_k(W)\omega^x_{\bar{k},\mu}\,. \tag{3·32}$$

We note that $W(x)$ satisfies the following relations:

$$\mathrm{tr}(\sigma^N(D_\mu W(x))W(x)^{-1})=\mathrm{tr}((\sigma^N)^{-1}W(x)^{-1}D_\mu W(x))=0\,, \tag{3·33}$$

the proof of which being given in Appendix B.

We have thus far introduced $w\in M_2(A)$, $w_i\in C_1{}^2\subset M_2(A')$, $w_m\oplus w_{m-1}\oplus\cdots\oplus w_1\in C_m{}^2\subset M_2(A'^{\otimes m})$ where $A'^{\otimes m}=A'\otimes A'\otimes\cdots\oplus A'$($m$ times), $w(x)\in M_2(A'^x)$, $W\in C_1{}^N$ and $W(x)\in M_N(A'^x)$ which are respectively the N-dimensional canonical representations of w and $w(x)$. According to the viewpoint stated in § 2.1, we have to consider $\oplus$-products of the $W(x)$'s. Let us say that $W_m(x)\oplus W_{m-1}(x)\oplus\cdots\oplus W_1(x)$ belongs to $C_m{}^N(x)$ if each $W_i(x)$, $i=1,2,\cdots,m$, is the N-dimensional canonical representation of $w(x)$. In the next section, we seek the field theory which is invariant under transformations belonging to $C_m{}^N(x)$, or more exactly $\{\otimes^x C_m{}^N(x);\ x\in\text{spacetime}\}$. As was mentioned below (2·24), it is not difficult to discuss the more general class of representations intertwined with $W_m(x)\oplus W_{m-1}(x)\oplus\cdots\oplus W_1(x)$ by $t\in U(N)$.

§ 4. Gauge field

4.1. *Transformation law of gauge field*

We now introduce the gauge potential and the gauge field. We suppose that the components of the gauge potential, $a_{k,\mu}(x)$, $k=0,1,2$, $\mu=0,1,2,3$, obey the same multiplication- and $*$-rules as those of $i\omega^x_{\bar{k},\mu}$. Namely, corresponding to (3·25), (3·28) and (3·27), we postulate that

$$a_{k,\mu}(x)w(x)=w(x)f_k(w)a_{k,\mu}(x)\,, \tag{4·1}$$

$$a_{k,\mu}(x)a_{l,\nu}(x)=c_{kl}a_{l,\nu}(x)a_{k,\mu}(x)\,, \tag{4·2}$$

$$a_{0,\mu}(x)^* = -\nu a_{2,\mu}(x)\,,\quad a_{1,\mu}(x)^* = a_{1,\mu}(x)\,,\quad a_{2,\mu}(x)^* = -\nu^{-1} a_{0,\mu}(x)\,. \tag{4·3}$$

In the same way as the previous discussion, the derivative D_μ can be introduced so as to respect the Leibniz rule, the $*$-property $(D_\mu a_{k,\nu}(x))^* = D_\mu(a_{k,\nu}(x)^*)$, and the commutativity $D_\mu D_\nu = D_\nu D_\mu$. They correspond to (A·7), (A·8) and (A·9), respectively.

Throughout the present subsection, we denote $W_m(x) \oplus W_{m-1}(x) \oplus \cdots \oplus W_1(x) \in C_m{}^N(x)$ and $W'_n(x) \oplus W'_{n-1}(x) \oplus \cdots \oplus W'_1(x) \in C_n{}^N(x)$ by $W(x)$ and $W'(x)$, respectively. We call the field $A_\mu{}^W(x)$ defined by

$$A_\mu{}^W(x) = W(x)\{I_{m-1} \otimes a_\mu{}^W(x)\} W(x)^{-1} - \frac{1}{ig}(D_\mu W(x)) W(x)^{-1}\,, \tag{4·4}$$

$$a_\mu{}^W(x) = \sum_{k=0}^{2} \chi_k(W_1) a_{k,\mu}(x) \tag{4·5}$$

the gauge potential in the gauge $W(x)$. In (4·4), g is the gauge coupling constant, $D_\mu W(x)$ is defined by

$$D_\mu W(x) = \sum_{l=1}^{m} W_m(x) \oplus \cdots \oplus W_{l+1}(x) \oplus D_\mu W_l(x) \oplus W_{l-1}(x) \oplus \cdots \oplus W_1(x)\,, \tag{4·6}$$

and $\chi_k(W_1)$ is equal to $\chi_k{}^x(W_1(x))$. The gauge transform $(A_\mu{}^W(x))^{W'}$ of $A_\mu{}^W(x)$ by $W'(x)$ is defined by

$$\begin{aligned}(A_\mu{}^W(x))^{W'} &= (W'(x) \otimes I_m)(I_n \otimes A_\mu{}^W(x))(W'(x)^{-1} \otimes I_m)\\ &\quad - \frac{1}{ig}(D_\mu W'(x)) W'(x)^{-1} \otimes I_m\,.\end{aligned} \tag{4·7}$$

Then we have

$$(A_\mu{}^W(x))^{W'} = A_\mu{}^{W' \oplus W}(x)\,, \tag{4·8}$$

showing that the above definitions correspond to the standpoint stated in § 2. Written more explicitly, (4·7) reads

$$\begin{aligned}(A_\mu{}^W(x))^{W'}_{ij} &= \sum_{s,t=1}^{N} W'(x)_{is} W'(x)^{-1}_{tj} \otimes A_\mu{}^W(x)_{st}\\ &\quad - \frac{1}{ig} \sum_{s=1}^{N} (D_\mu W'(x)_{is}) W'(x)^{-1}_{sj} \otimes I_m\,,\end{aligned} \tag{4·9}$$

which reduces to the usual Yang-Mills transformation law when the entries of $W(x)$ and $W'(x)$ as well as $a_{k,\mu}(x)$ become commutative numbers and the $\otimes$-symbol is omitted by some equivalence relation, e.g., $1 \otimes 1 \sim 1$. Denoting by B'^x the set of linear combinations of $a_{k,\mu}(x)$ and $\omega^x_{k,\mu}$ with coefficients belonging to A'^x and defining $\Gamma^x = A'^x \oplus B'^x$, we see $A_\mu{}^W(x) \in M_N((\Gamma^x)^{\otimes m})$, $(A_\mu{}^W(x))^{W'} \in M_N((\Gamma^x)^{\otimes(m+n)})$. In general, we are working in the space $(\Gamma^x)^{\otimes} \equiv \oplus_{m=1}^{\infty} (\Gamma^x)^{\otimes m}$.

We define the gauge field $F^W_{\mu\nu}(x)$ in the gauge $W(x)$ by

$$F^W_{\mu\nu}(x) = [\nabla_\mu{}^W, \nabla_\nu{}^W]\,,\quad \nabla_\mu{}^W = D_\mu + igA_\mu{}^W(x)\,. \tag{4·10}$$

Then we find that

$$F_{\mu\nu}^{W}(x)=W(x)\{I_{m-1}\otimes f_{\mu\nu}^{W}(x)\}W(x)^{-1}, \tag{4·11}$$

$$f_{\mu\nu}^{W}(x)=\sum_{k=0}^{2}\chi_k(W_1)f_{k,\mu\nu}(x), \tag{4·12}$$

$$f_{k,\mu\nu}(x)=D_\mu a_{k,\nu}(x)-D_\nu a_{k,\mu}(x)+ig\sum_{p,q=0}^{2}d_{kpq}a_{p,\mu}(x)a_{q,\nu}(x), \tag{4·13}$$

where we have made use of the algebra (3·16). The transformation law of $F_{\mu\nu}^{W}(x)$ is given by

$$\begin{aligned}(F_{\mu\nu}^{W}(x))^{W'}&=(W'(x)\otimes I_m)(I_n\otimes F_{\mu\nu}^{W}(x))(W'(x)^{-1}\otimes I_m)\\ &=F_{\mu\nu}^{W'\oplus W}(x).\end{aligned} \tag{4·14}$$

In contrast to the complicated structure of $A_\mu{}^W(x)$, we see $F_{\mu\nu}^{W}(x)\in M_N((A'^x)^{\otimes(m-1)}\otimes G'^x)$ where G'^x is the set of linear combinations of $f_{k,\mu\nu}(x)$ with coefficients belonging to A'^x.

Combining (4·3) with (3·19), we have

$$A_\mu{}^W(x)^\dagger==A_\mu{}^W(x), \tag{4·15}$$

where † means the $*$-operation associated with the transposition. From (3·17), (3·18), (4·2) and (4·3), we obtain

$$f_{0,\mu\nu}(x)^*=-\nu f_{2,\mu\nu}(x), \quad f_{1,\mu\nu}(x)^*=f_{1,\mu\nu}(x), \quad f_{2,\mu\nu}(x)^*=-\nu^{-1}f_{0,\mu\nu}(x), \tag{4·16}$$

which yield

$$F_{\mu\nu}^{W}(x)^\dagger=F_{\mu\nu}^{W}(x). \tag{4·17}$$

It is evident that both of (4·15) and (4·17) are independent of the choice of the gauge.

4.2. *Trace property of gauge field*

We know that each $W_i(x)$, $i=1,\cdots,m$, consisting the $W(x)$ of the previous subsection satisfies (2·23) and (3·33). The generalized and inverted version of (4·1) can be obtained from (3·6) and the definition of $W_i(x)$:

$$a_{k,\mu}(x)W_i(x)^{-1}=f_k(W_i^{-1})W_i(x)^{-1}a_{k,\mu}(x), \quad W_i(x)\in C_1{}^N(x). \tag{4·18}$$

With the help of (3·32), (3·33), (4·4) and (4·18), we obtain

$$\mathrm{tr}(\sigma^N A_\mu{}^W(x))=I_{m-1}\otimes\sum_{k=0}^{2}\mathrm{tr}(\sigma^N\chi_k(W_1)f_k(W_1{}^{-1}))a_{k,\mu}(x), \tag{4·19}$$

where we have made use of the formula $\mathrm{tr}(\sigma^N W_i T W_i{}^{-1})=\mathrm{tr}(\sigma^N T)$, $T\in M_N(C)$, $W_i\in C_1{}^N$. As is shown in Appendix C, we have

$$\mathrm{tr}(\sigma^N\chi_k(W_1)f_k(W_1{}^{-1}))=0, \quad W_1\in C_1{}^N, \quad k=0,1,2, \tag{4·20}$$

yielding

$$\mathrm{tr}(\sigma^N A_\mu{}^W(x))=0, \quad W(x)\in C_m{}^N(x). \tag{4·21}$$

Similarly, (4·1), (4·10), (4·13), (4·21) and the formula

$$d_{kpq}\mathrm{tr}(\sigma^N\chi_k(W_1)f_p(W_1^{-1})f_q(W_1^{-1}))=0\,,\quad k,p,q=0,1,2\,,\quad W_1\in C_1{}^N\,, \tag{4·22}$$

lead us to

$$\mathrm{tr}(\sigma^N F^W_{\mu\nu}(x))=0\,,\quad W(x)\in C_m{}^N(x)\,. \tag{4·23}$$

The proof of (4·22) is similar to that of (4·20). Both of (4·21) and (4·23) are gauge independent.

4.3. *Tensor product representation of gauge field*

For two representations $u\in C_1{}^L$ and $v\in C_1{}^N$ given by

$$u=\sum_i a_i m_i\,,\quad a_i\in A\,,\quad m_i\in M_L(\mathcal{C})\,,$$

$$v=\sum_j b_j n_j\,,\quad b_j\in A\,,\quad n_j\in M_N(\mathcal{C})\,, \tag{4·24}$$

the tensor product representation $u \textcircled{\scriptsize T} v$ is defined by

$$u \textcircled{\scriptsize T} v=\sum_{i,j}a_i b_j(m_i \textcircled{\scriptsize T} n_j)\in M_{LN}(A)\,, \tag{4·25}$$

where $m_i \textcircled{\scriptsize T} n_j$ on the right-hand side means the conventional Kronecker product of the matrices m_i and n_j. It can be seen that $u \textcircled{\scriptsize T} v$ indeed satisfies (2·20) by making use of the identity[6),11)]

$$(u\oplus u')\textcircled{\scriptsize T}(v\oplus v')=(u\textcircled{\scriptsize T}v)\oplus(u'\textcircled{\scriptsize T}v')\,,$$

$$u,\,u'\in C_1{}^L,\,v,\,v'\in C_1{}^N\,. \tag{4·26}$$

The intriguing formula[6)]

$$\chi_k(u\textcircled{\scriptsize T}v)=\chi_k(u)\textcircled{\scriptsize T}f_k(v)+e(u)\textcircled{\scriptsize T}\chi_k(v) \tag{4·27}$$

is the direct result of (3·3)~(3·5) where $e(u)$ is the $L\times L$ unit matrix. Putting $W(x)=u(x)\textcircled{\scriptsize T}v(x)\in C_1{}^{LN}(x)\subset M_{LN}(A'^x)$ in (4·4) and making use of (4·27), we obtain

$$A_\mu{}^{u\textcircled{\scriptsize T}v}(x)=A_\mu{}^u(x)\textcircled{\scriptsize T}e(v)+(u\textcircled{\scriptsize T}e(v))(e(u)\textcircled{\scriptsize T}A_\mu{}^v(x))(u^{-1}\textcircled{\scriptsize T}e(v))\,, \tag{4·28}$$

which should be compared with $A_\mu{}^{u\textcircled{\scriptsize T}v}=A_\mu{}^u\textcircled{\scriptsize T}e(v)+e(u)\textcircled{\scriptsize T}A_\mu{}^v$ valid in the Yang-Mills theory. The identity (4·26) indicates

$$(A_\mu{}^{u\textcircled{\scriptsize T}v}(x))^{u'\textcircled{\scriptsize T}v'}=A_\mu{}^{(u'\oplus u)\textcircled{\scriptsize T}(v'\oplus v)}(x)\,. \tag{4·29}$$

Similarly, we have

$$\begin{aligned}F^{u\textcircled{\scriptsize T}v}_{\mu\nu}(x)=&(u\textcircled{\scriptsize T}v)\{\sum_{k=0}^{2}\chi_k(u)f_{k,\mu\nu}(x)\textcircled{\scriptsize T}f_k(v)\}(u\textcircled{\scriptsize T}v)^{-1}\\&+(u\textcircled{\scriptsize T}e(v))(e(u)\textcircled{\scriptsize T}F^v_{\mu\nu}(x))(u^{-1}\textcircled{\scriptsize T}e(v))\end{aligned} \tag{4·30}$$

and

$$(F^{u\textcircled{\scriptsize T}v}_{\mu\nu}(x))^{u'\textcircled{\scriptsize T}v'}=F^{(u'\oplus u)\textcircled{\scriptsize T}(v'\oplus v)}_{\mu\nu}(x)\,. \tag{4·31}$$

§ 5. Lagrangian of gauge field

To keep the parallelism to the Yang-Mills case, the Lagrangian density of the local $SU_q(2)$ invariant field theory should be a bilinear combination of $F^W_{\mu\nu}(x)$, $W(x) \in C_m{}^N(x)$, introduced in the previous section. It should be independent of the choice of $W(x)$, the dimensionality N and the integer m.

We begin with defining S^W_{kl} by

$$S^W_{kl} = \mathrm{tr}(\rho^N \chi_k(W)(\rho^N)^2 \chi_l(W)), \quad W \in C_1{}^N, \quad k, l = 0, 1, 2, \tag{5·1}$$

where ρ^N is given by

$$\rho^N = (\sigma^N)^{-1}. \tag{5·2}$$

Since ρ^N and $\chi_1(W)$ are diagonal and $\chi_0(W)$ and $\chi_2(W)$ are off-diagonal, we have

$$S^W_{kl} = 0, \quad (k, l) \neq (0, 2), (2, 0), (1, 1). \tag{5·3}$$

The important property of S^W_{kl} is that the ratios of its non-vanishing members depend on neither W nor N (see Appendix D):

$$S^W_{02} : S^W_{20} : S^W_{11} = -1 : -\nu^2 : \nu(1+\nu^2). \tag{5·4}$$

We note that the ratios of non-vanishing members of simpler expressions, e.g., tr $(\chi_k(W)\chi_l(W))$ and $\mathrm{tr}(\sigma^N \chi_k(W)\chi_l(W))$ depend on the representations adopted. If we define K_N by

$$K_N = -\frac{\nu}{8}(S^W_{20})^{-1}, \quad W \in C_1{}^N, \tag{5·5}$$

e.g.,

$$K_2 = \frac{\nu^2}{8}, \quad K_3 = \frac{\nu^8}{8(1+\nu^2)(1+\nu^4)}, \tag{5·6}$$

the product $K_N S^W_{kl}$ is independent of W.

We now define $L^W(x)$, $W(x) = W_m(x) \oplus \cdots \oplus W_1(x)$, by

$$L^W(x) = K_N \mathrm{tr}(\sigma^N G^W_{\mu\nu}(x) G^{W,\mu\nu}(x)), \tag{5·7}$$

$$G^W_{\mu\nu}(x) = \tau(W) F^W_{\mu\nu}(x) \tau(W)^\dagger, \tag{5·8}$$

$$\tau(W) = (W_m(x) \oplus \cdots \oplus W_2(x) \oplus \rho^N I) W(x)^{-1}. \tag{5·9}$$

Then we are led to

$$L^W(x) = I_{m-1} \otimes L(x) \tag{5·10}$$

with $L(x)$ given by

$$L(x) = \sum_{k,l=0}^{2} K_N S^{W_1}_{kl} f_{k,\mu\nu}(x) f_l^{\mu\nu}(x). \tag{5·11}$$

More explicitly, $L(x)$ can be written as

$$L(x)=\frac{1}{8}\{f_{0,\mu\nu}(x)^*f_0{}^{\mu\nu}(x)+(1+\nu^2)f_{1,\mu\nu}(x)^*f_1{}^{\mu\nu}(x)+f_{2,\mu\nu}(x)^*f_2{}^{\mu\nu}(x)\}\,, \qquad (5\cdot12)$$

where explicit expressions of $f_{k,\mu\nu}(x)$ are given by

$$f_{0,\mu\nu}(x)=D_\mu a_{0,\nu}(x)-D_\nu a_{0,\mu}(x)+ig(1+\nu^2)\{\nu^{-2}a_{1,\mu}(x)a_{0,\nu}(x)-\nu^2 a_{0,\mu}(x)a_{1,\nu}(x)\}\,,$$
$$f_{1,\mu\nu}(x)=D_\mu a_{1,\nu}(x)-D_\nu a_{1,\mu}(x)+ig\{\nu^{-1}a_{2,\mu}(x)a_{0,\nu}(x)-\nu a_{0,\mu}(x)a_{2,\nu}(x)\}\,,$$
$$f_{2,\mu\nu}(x)=D_\mu a_{2,\nu}(x)-D_\nu a_{2,\mu}(x)+ig(1+\nu^2)\{\nu^{-2}a_{2,\mu}(x)a_{1,\nu}(x)-\nu^2 a_{1,\mu}(x)a_{2,\nu}(x)\}\,, \qquad (5\cdot13)$$

through (4·13) and (3·18). The result (5·10) shows that $L^W(x)$ still depends on m. If we define the equivalence relation $\sim$ by

$$I\otimes a\sim a\,, \qquad (5\cdot14)$$

$L^W(x)$ is equivalent to $L(x)$:

$$L^W(x)\sim L(x)\,. \qquad (5\cdot15)$$

We interpret (5·15) as the gauge invariance of $L^W(x)$. This interpretation stems from the definition of the $SU_q(2)$ transformations through $\oplus$-product. Thus the Lagrangian has been fixed by (5·12) or, noticing (4·16),

$$L(x)=\frac{1}{8}\{(1+\nu^2)f_{1,\mu\nu}(x)f_1{}^{\mu\nu}(x)-\nu f_{2,\mu\nu}(x)f_0{}^{\mu\nu}(x)-\nu^{-1}f_{0,\mu\nu}(x)f_2{}^{\mu\nu}(x)\}\,. \qquad (5\cdot16)$$

The derivation of field equations from the above Lagrangian might require a rather careful definition of the variation procedure since $f_{k,\mu\nu}(x)$ is non-commutative. We leave this problem to the forthcoming investigations.

§ 6. Concluding remarks

We have investigated how the symmetry group of the Yang-Mills theory can be extended to the quantum group $SU_q(2)$. The gauge transformation that we have adopted preserves some group-like properties and differs from those considered by previous authors.[9),10)] Our theory is described in any representations by three component fields $a_{k,\mu}(x)$, $k=0,1,2$, in contrast to Ref. 10). The component fields are non-commutative objects obeying (4·1)$\sim$(4·3), which should be contrasted with the R-commutativity assumed in Ref. 9).

The Lagrangian of the gauge field has been given by (5·12) or (5·16). In the $\nu=1$ and commutative limit, $L(x)$ reduces to the Lagrangian of the conventional $SU(2)$ Yang-Mills theory. The gauge invariance of the theory has been maintained with the help of the equivalence relation (5·14). It turned out that the q-deformed trace $\mathrm{tr}(\sigma^N T)$, $T\in M_N$ and the differential calculus of Woronowicz play important roles in formulating the $SU_q(2)$ gauge theory. Although we have not discussed the interaction between the gauge and the matter fields, it can be readily understood that the building block of the interaction Lagrangian should be $\nabla_\mu{}^W\phi^W(x)$, $\nabla_\mu{}^W$ being given in (4·10), where $\phi^W(x)$ defined by

$$\phi^W(x)=W(x)(I_{m-1}\otimes\phi(x)),$$

$$\phi(x)\in M_{N\times 1}(A'^x),\quad W(x)\in C_m{}^N(x), \tag{6·1}$$

is the matter field in the gauge $W(x)$. In (6·1), $M_{N\times 1}$ means the set of $N\times 1$ matrices. The gauge invariant matter Lagrangian can be constructed in a parallel way to the case of Yang-Mills.

There remain many problems to be resolved. One of the most tractable problems would be to extend the above discussion to the case of $SU_q(3)$. If some discrepancies from the standard model are revealed by future experiments, the quantum group version of the $SU(3)$ QCD and the $SU(2)\times U(1)$ electroweak model should be seriously taken into account. If not, we are left with a different problem to explain why Nature selects the case $\nu=1$, resembling the problem of the cosmological constant.[15]

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Appendix A
——*Properties of ω_k*——

$$\omega_0^*=\nu\omega_2,\quad \omega_1^*=-\omega_1,\quad \omega_2^*=\nu^{-1}\omega_0. \tag{A·1}$$

$$\omega_k\omega_k=0,\quad k=0,1,2. \tag{A·2}$$

$$\omega_2\omega_0=-\nu^2\omega_0\omega_2,\quad \omega_1\omega_0=-\nu^4\omega_0\omega_1,\quad \omega_2\omega_1=-\nu^4\omega_1\omega_2. \tag{A·3}$$

$$d\omega_0=\nu^2(1+\nu^2)\omega_0\omega_1,\quad d\omega_1=\nu\omega_0\omega_2,\quad d\omega_2=\nu^2(1+\nu^2)\omega_1\omega_2. \tag{A·4}$$

$$dI=0,\quad d\alpha=\alpha\omega_1+\nu^2\gamma^*\omega_2,\quad d\gamma^*=-\nu^{-1}\alpha\omega_0-\nu^2\gamma^*\omega_1,$$

$$d\gamma=\gamma\omega_1-\nu\alpha^*\omega_2,\quad d\alpha^*=\gamma\omega_0-\nu^2\alpha^*\omega_1. \tag{A·5}$$

$$\omega_0=\gamma^*d\alpha^*-\nu\alpha^*d\gamma^*,\quad \omega_2=\gamma d\alpha-\nu^{-1}\alpha d\gamma,$$

$$\omega_1=\alpha^*d\alpha+\gamma^*d\gamma=-\gamma d\gamma^*-\nu^{-2}\alpha d\alpha^*. \tag{A·6}$$

$$d(\zeta\zeta')=(d\zeta)\zeta'+(-1)^{\partial\zeta}\zeta(d\zeta'), \tag{A·7}$$

$$(d\zeta)^*=d\zeta^*, \tag{A·8}$$

$$d^2\zeta=0, \tag{A·9}$$

where ζ and ζ' are differential forms on A, $\partial\zeta$ the grade of ζ, and we have abbreviated the symbol of the edge product.

Appendix B
——*Proof of (3·33)*——

Any higher dimensional representations of w are obtained by block-diagonalizing the tensor product of lower dimensional ones. An example can be found in Appendix E. It can be seen that the formula (3·33) is valid for any $W(x)\in C_1{}^N(x)$ if it is the case for $w(x)\in C_1^2(x)$. The general element of $C_1^2(x)$ is given by

$$w(x)=\begin{pmatrix} \alpha(x) & -\nu\gamma^*(x)e^{-i\varphi(x)} \\ \gamma(x)e^{i\varphi(x)} & \alpha^*(x) \end{pmatrix}, \tag{B$\cdot$1}$$

where $\varphi(x)$ is an arbitrary real function. From (A$\cdot$5) and the Z-procedure, we have $D_\mu\alpha(x)=\alpha(x)\omega_{1,\mu}^x+\nu^2\gamma^*(x)\omega_{0,\mu}^x$, etc. Then, the direct calculation, with the help of (2$\cdot$2), (2$\cdot$18), (3$\cdot$2) and (3$\cdot$25), yields $\mathrm{tr}(\sigma(D_\mu w(x))w(x)^{-1})=0$. Similarly, the latter equality of (3$\cdot$33) can be proved.

Appendix C
——*Proofs of (4$\cdot$20) and (4$\cdot$22)*——

For the tensor product representation $u\textcircled{T}v$, $u\in C_1^L$, $v\in C_1^N$, we have (4$\cdot$27) and $f_k(u\textcircled{T}v)=f_k(u)\textcircled{T}f_k(v)$. The σ matrix for $u\textcircled{T}v$ is defined by $\sigma^{u\textcircled{T}v}=\sigma^u\textcircled{T}\sigma^v$. Then, we have

$$\begin{aligned} T_k^{u\textcircled{T}v}&\equiv\mathrm{tr}(\sigma^{u\textcircled{T}v}\chi_k(u\textcircled{T}v)f_k(u\textcircled{T}v)) \\ &=\mathrm{tr}(\sigma^L\chi_k(u)f_k(u))\mathrm{tr}(\sigma^N f_k(v)f_k(v))+\mathrm{tr}(\sigma^L f_k(u))\mathrm{tr}(\sigma^N\chi_k(v)f_k(v)) \\ &\equiv T_k^u\mathrm{tr}(\sigma^N f_k(v)f_k(v))+\mathrm{tr}(\sigma^L f_k(u))T_k^v\,. \end{aligned} \tag{C$\cdot$1}$$

We see that $T_k^{u\textcircled{T}v}$ vanishes if T_k^u and T_k^v do. Recalling the comment at the top of Appendix B, we realize that (4$\cdot$20) holds for general $W\in C_1^N$ if it does for the fundamental representation. It is easy to see that the above is the case. Thus (4$\cdot$20) has been proved.

Quite a similar discussion to the above leads us to (4$\cdot$22).

Appendix D
——*Proof of (5$\cdot$4)*——

Similarly to Appendix C, we obtain

$$S_{kl}^{u\textcircled{T}v}=\mathrm{tr}(\sigma^N(\sigma^N)^{-2}f_k(v)(\sigma^N)^{-2}f_l(v))S_{kl}^u+\mathrm{tr}(\sigma^L(\sigma^L)^{-2}(\sigma^L)^{-2})S_{kl}^v\,, \tag{D$\cdot$1}$$

where the relation

$$f_1(v^{-1})=(f_0(v^{-1}))^2=(f_2(v^{-1})^2=\nu^{2(N-1)}(\sigma^N)^{-2} \tag{D$\cdot$2}$$

and (4$\cdot$22) have been made use of. Noticing

$$f_k(v)=(f_k(v^{-1}))^{-1}\,, \tag{D$\cdot$3}$$

$$\mathrm{tr}(\sigma^N)=\nu^{2(N-1)}\mathrm{tr}((\sigma^N)^{-1})\,, \tag{D$\cdot$4}$$

we have

$$S_{kl}^{u\textcircled{T}v}=\lambda_N S_{kl}^u+\mu_L S_{kl}^v\,, \tag{D$\cdot$5}$$

where λ_N and μ_L are constants independent of k and l:

$$\lambda_N=\nu^{-4(N-1)}(1+\nu^2+\nu^4+\cdots+\nu^{2(N-1)})\,, \tag{D$\cdot$6}$$

$$\mu_L = \nu^{-6(L-1)}(1+\nu^6+\nu^{12}+\cdots+\nu^{6(L-1)})\,. \tag{D·7}$$

Discussions similar to the top of Appendix B yield the conclusion that the ratio S^W_{02}: S^W_{20}: S^W_{11} is independent of W. Through a simple calculation in the fundamental representation, we obtain (5·4).

Appendix E
——*Example of Irreducible Decomposition*——

We consider the irreducible decomposition of the tensor product of the fundamental representation w given by (2·1). It can be readily seen that

$$B(w \textcircled{T} w)B^{-1} = \begin{pmatrix} & & & 0 \\ & V & & 0 \\ & & & 0 \\ 0 & 0 & 0 & v \end{pmatrix}, \tag{E·1}$$

where B, V and v are given by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \nu c & c & 0 \\ 0 & 0 & 0 & 1 \\ 0 & c & -\nu c & 0 \end{pmatrix}, \quad c = \frac{1}{\sqrt{1+\nu^2}}, \quad B^\dagger = B^{-1}, \tag{E·2}$$

$$V = \begin{pmatrix} \alpha^2 & -c^{-1}\alpha\gamma^* & \nu^2\gamma^{*2} \\ \nu^{-1}c^{-1}\alpha\gamma & \alpha\alpha^* - \gamma\gamma^* & -c^{-1}\gamma^*\alpha^* \\ \gamma^2 & \nu^{-1}c^{-1}\gamma\alpha^* & \alpha^{*2} \end{pmatrix}, \quad v = I\,. \tag{E·3}$$

V and v are the 3- and 1-dimensional representations of w, respectively. We also have

$$B(\sigma \textcircled{T} \sigma)B^{-1} = \begin{pmatrix} & & & 0 \\ & \sigma^3 & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \sigma^1 \end{pmatrix}, \tag{E·4}$$

$$\sigma \textcircled{T} \sigma = \mathrm{diag}\,(1, \nu^2, \nu^2, \nu^4)\,, \tag{E·5}$$

$$\sigma^3 = \mathrm{diag}\,(1, \nu^2, \nu^4)\,, \quad \sigma^1 = 1\,. \tag{E·6}$$

From (3·1)~(3·6), we obtain

$$\chi_0(V) = c^{-1}\begin{pmatrix} 0 & \nu^{-1} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \chi_2(V) = -c^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \nu & 0 \end{pmatrix},$$

$$\chi_1(V) = c^{-2}\mathrm{diag}\,(\nu^{-2}, 0, -\nu^2)\,. \tag{E·7}$$

After some manipulations, we obtain

$$S^w_{02} = -\nu^{-3}, \quad S^w_{20} = -\nu^{-1}, \quad S^w_{11} = \nu^{-2}(1+\nu^2)\,, \tag{E·8}$$

$$S_{02}^{V}=-\nu^{-9}(1+\nu^2)(1+\nu^4)\,,\quad S_{20}^{V}=-\nu^{-7}(1+\nu^2)(1+\nu^4)\,,$$
$$S_{11}^{V}=\nu^{-8}(1+\nu^2)^2(1+\nu^4)\,. \tag{E·9}$$

The formulae (5·4)~(5·6) can now be understood.

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