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Gauge Fields on Coset Spaces

by

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Classical gauge theories are constructed on associated fibre bundles. The connection coefficients are identified with gauge potentials. If the fibre is isomorphic to G/K, where G is the structural group, K its maximal subgroup, the number of dynamically independent gauge fields equals the dimension of the coset. The independent gauge fields support a nonlinear realization of G. An attempt is made to interpret the theory in terms of a spontaneously broken symmetry.

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ABSTRACT

Classical gauge theories are constructed on associated fibre bundles. The connection coefficients are identified with gauge potentials. If the fibre is isomorphic to G/K, where G is the structural group, K its maximal subgroup, the number of dynamically independent gauge fields equals the dimension of the coset. The independent gauge fields support a nonlinear realization of G. An attempt is made to interpret the theory in terms of a spontaneously broken symmetry.

1. INTRODUCTION

Gauge theories are widely believed to serve as useful models of elementary particle interactions. Nevertheless, their status - as far as the interpretation of experimental data is concerned- is still somewhat uncertain. In particular, gauge theories endowed with a generally accepted local symmetry group (SU(4), perhaps SU(5) or one of the exceptional Lie groups, like E(7), see Gürsey in particular[1]) predict the existence of a very large number of gauge bosons: in fact, the number of gauge bosons is equal to the dimension of the adjoint representation of the local symmetry group. Yet, experimentally only one gauge boson has been discovered so far, the photon. One may argue that some of the gauge bosons are not seen because they carry color quantum numbers and, therefore, they are "confined", just as quarks - supposedly - are. Other gauge mesons may become "superheavy" (with masses of the order of the Planck mass) as a consequence of some peculiar mechanism of spontaneous symmetry breaking and they are therefore, safely beyond the accessible energy range. In our opinion, however, none of the arguments referred to above is an entirely convincing one. Therefore, one feels justified to inquire whether it is possible to construct models which are as close in their structure to standard gauge theories as possible, yet, they contain fewer gauge fields than the usual gauge models do.

The purpose of this note is to point out one possible way towards the construction of such models.

Our approach is based upon a generalization of the geometrical structure underlying the usual gauge theories, as clarified, in particular, by the works of Trautman, De Witt, Kerner, Cho and

Freund and of Chang, Macrae and Mansouri[2].

As one understands it now, the mechanism of constructing a "conventional" gauge model involves the following essential steps.

i) Given a physically desirable symmetry group, G and a space-time manifold, M, one constructs a principal fibre bundle, P, such that P is locally isomorphic to the direct product $M \otimes G$.

ii) One induces a Cartan-Ehresmann connection[3] in P, essentially by lifting an arbitrary curve in M into one in P. The connection coefficients are identified - apart from an over-all scale factor, the charge - with the gauge potentials. The existence of a connection permits, in particular, the construction of a (horizontal) lift basis in the tangent space T_p , for any $p \in P$. The horizontal vectors in T_p are just the "gauge invariant differential operators". By the same token, one cán, of course, also define the dual to the lift basis in T_p^*, the space of one-forms at p. The latter construction serves to define the connection forms.

iii) The bundle P is endowed with a Riemannian structure. In particular, one prescribes a metric, $g(\cdot,\cdot)$,which is block-diagonal in the horizontal lift basis. In that basis T_p appears as a direct sum,

$$T_p \sim T_m \oplus T_\gamma$$

with $m \in M$ and $\gamma \in \mathfrak{g}$ (the Lie agebra of G). Further, if the vectors H, V lie in T_m and T_γ , respectively, such that under the Lie bracket

$$[H, H] \in T_m \quad , [V, V] \in T_\gamma \quad , [H, V] = 0,$$

then one demands

$$\left.\begin{array}{l} g(H,H) = g_H(H,H), \quad g(V,V) = g_\gamma(V,V), \\ g(H,V) = 0, \end{array}\right\} \qquad (1.1)$$

where g_M is the usual Riemann metric on M and $g_{\mathcal{G}}$ is the Cartan-Killing metric on $\mathcal{G}$.

Step iii) allows one to apply the standard machinery of Riemann geometry to the bundle P. It turns out that the usual Einstein-Yang-Mills action is obtained (apart from a trivial factor) as the integral of the Ricci scalar over the bundle.

We propose here to replace a principal bundle by bundles of other types. It is intuitively obvious that a physically acceptable bundle has to satisfy two criteria.

a) The base space should be a manifold which can serve as a model of space-time.

b) An acceptable internal symmetry group, G (usually assumed to be a simple, compact Lie group) should act effectively on the fibre.

Identification of the fibre with G itself gives a principal bundle and, hence, it leads to a gauge theory of the usual type.

Here we examine theories based on <u>associated bundles</u> instead, in which the fibre is isomorphic to a coset space of G. (For the precise definition of associated bundles, see e.g. ref.1.) It is obvious that both criteria listed above can be satisfied with associated bundles. We find that a geometrically acceptable gauge theory can be built on associated bundles. The number of gauge fields equals to the dimension of the coset space and, hence, it is generally smaller than in a usual gauge theory. However, the gauge fields transform non-linearly under the action of G.

In the next Section we review the construction of a connection on associated bundles. The reduction of the number of fields is carried out in Sec.3. We show that the connection coefficients belonging to the maximal subgroup are redundant

variables and they can be transformed out. As a result, however, the transformation law of the remaining variables in the connection form becomes nonlinear. The geometrical construction of a gauge theory based on associated bundles is completed by providing the bundle with a metric structure (Sec.4). The results are discussed in Sec.5.

<u>2.CARTAN-EHRESMANN CONNECTION ON ASSOCIATED BUNDLES.</u>

Let $\mathcal{A}$ be an associated bundle with base manifold M, and structure group G. Locally $\mathcal{A}$ is isomorphic to $M\otimes G/H$, where H is a subgroup of G. The local action of G on M is assumed to be trivial. (Physically: the internal symmetry group does not act on space-time.) The action of G on the fibre ($\sim G/H$) is defined in the usual way. We define a connection on $\mathcal{A}$ by lifting a curve in M into $\mathcal{A}$.

Assume that x^μ are local coordinates of M in some neighborhood of the point $m_o = m(x_o^\mu) \in M$ and y^a are local coordinates of the fibre.(Thus, y^a may be chosen as some set of group parameters parametrizing the coset G/H.) A point $a_o \in \mathcal{A}$ is given by the local coordinates (x_o^μ, y_o^a). Let $a(t) \doteq (x^\mu(t), y^a(t))$ be a curve in $\mathcal{A}$ passing through the point a_o and choose the parameter t such that $a(0) = a_o$. The part of the curve lying in the fibre is generated by the action of a one-parameter subgroup of G on G/H. If $y^{a'} = f^a(\alpha, y)$ gives the action of G on G/H, where α^A are the parameters of G, then we put

$$y^a(t) = f^a(\alpha(t), y_0). \tag{2.1}$$

The tangent vector of a(t) at $a(x^\mu, y^a)$ is given by

$$\frac{\partial}{\partial t} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{y}^a \frac{\partial}{\partial y^a} . \tag{2.2}$$

However, by (2.1) we have:

$$dy^a = \frac{\partial f^a(\alpha, y^0)}{\partial \alpha^A} d\alpha^A = \frac{\partial f^a(\beta, y_0)}{\partial \beta^A}\Big|_{\beta^A=0} \epsilon^A ,$$

where ϵ^A is infinitesimal.

By the group composition law we have further,

$$\epsilon^A = \psi^A{}_B(\alpha)\, d\alpha^B$$

where $\psi^A{}_B(\alpha)$ is a function of the parameters α ; its form is completely determined by the structure of G.

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t} &= \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{\alpha}^B \psi^A{}_B(\alpha) \frac{\partial f^a(\beta, y)}{\partial \beta^A}\Big|_{\beta=0} \frac{\partial}{\partial y^a} \\ &\equiv \dot{x}^\mu \frac{\partial}{\partial x^\mu} - \dot{\alpha}^B \psi^A{}_B(\alpha) X_A\end{aligned} \tag{2.3}$$

X_A being the generators of G realized on G/H. A (Cartan-Ehresmann) connection[3] expresses $\dot{\alpha}^A$ as a linear function of $\dot{x}^\mu$. We put

$$\dot{\alpha}^B = \Theta^B{}_C(\alpha)\, \Gamma^C_\mu(x,\alpha)\, \dot{x}^\mu$$

where Θ is the inverse of ψ ,

$$\Theta_A{}^B \psi^C_B = \delta^C_A ,$$

whereas the Γ^C_μ are the connection coefficients. We have finally,

$$\frac{\partial}{\partial t} = \dot{x}^\mu \left(\frac{\partial}{\partial x^\mu} - \Gamma^A_\mu(x,y) X_A \right) ; \tag{2.4}$$

in other words, if there is a connection given in $\mathcal{A}$, then any curve in $\mathcal{A}$ is completely determined by its projection on the base manifold and by the action of G on the fibre. (It is assumed, of course, that the parameters $\alpha(t)$ are eliminated from

Γ in terms of the coordinates y. This is always possible since (2.1) defines the action of a Lie group on the fibre.)

The horizontal lift basis of T_a is spanned by the vectors

$$F_\mu = \frac{\partial}{\partial x^\mu} - \Gamma^A_\mu X_A(y) \tag{2.5}$$

defined by (2.4) (the horizontal vectors) and by a basis of the linear complement of the space spanned by the vectors (2.5). The latter can be chosen to be the subset $Y_a \subset \{X_A\}$ generating the coset G/H.

The vector $\Gamma^A_\mu X_A$ in (2.5) is assumed to be an invariant vector, i.e.

$$[X_B(y), \Gamma^A_\mu(y) X_A(y)] = \left(X_B(y)\Gamma^A_\mu(y)\right) X_A(y) + \Gamma^A_\mu(y) C^C_{BA} X_C(y) = 0,$$

where C^C_{AB} stand for the structure constants of $\mathcal{G}$.

Equation (2.6) is completely integrable (as a consequence of Lie's second theorem), hence $\Gamma^A_\mu X_A$ can be always transported to a fixed point (say, the origin) of the fibre.

The freedom in the choice of the cross section in $\mathcal{A}$ corresponds to the invariance of F_μ under local transformations; in infinitesimal form,

$$[\epsilon^A(x) X_A , F_\mu] = 0 . \tag{2.7}$$

Physically, the coefficients Γ^A_μ correspond to gauge potentials, so that (2.7) generates gauge transformations.

The reader realizes now that up to this point, the construction parallels the procedure applicable to principal bundles, see ref.2.

3. ELIMINATION OF THE REDUNDANT FIELDS. THE TRANSFORMATION LAW OF GAUGE FIELDS ON COSETS.

One realizes that eq. (2.5) contains terms which, in a sense, are redundant. Indeed, the horizontal vector F_A contains a general vector in the Lie algebra $\mathcal{G}$, whereas the tangent space of the fibre at any point is spanned by only a subset of the X_A (corresponding to the generators of the coset G/H). It is desirable, therefore, to eliminate those gauge fields which multiply the generators of H. This is indeed possible at the cost of making the transformation law of the remaining fields more complicated.+ In order to be explicit, from now on we

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+The necessity of eliminating the redundant fields is obvious from a geometrical point of view. In particular, one should be able to obtain the vectors of the horizontal lift basis from e.g. a coordinate basis by means of a non-singular transformation.

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focus our attention on classical, off-diagonal coset spaces based on a Cartan decomposition, K being the maximal subgroup of G, (throughout this paper, G is assumed to be compact) although the procedure itself is applicable to more general spaces. The coset spaces in question are parametrised in a standard way as follows[5].

a) In some representation (say, in the adjoint representetion) the elements of K are exponentials of antihermitean, block-diagonal matrices:

$$\mathbb{A} = \exp\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad \text{for } \mathbb{A} \in K \tag{3.1}$$

with $A_1^+ = -A_1$, $A_2^+ = -A_2$.

b) The elements of G/K are exponentials of off-diagonal, antihermitean matrices, which can be parametrised by inhomogeneous

projective coordinates:

$$\mathbb{B} = \exp\begin{pmatrix} 0 & B \\ -B^+ & 0 \end{pmatrix} = \begin{pmatrix} (1+z z^+)^{-1/2} & z(1+z^+ z)^{-1/2} \\ -(1+z^+ z) z^+ & (1+z^+ z)^{-1/2} \end{pmatrix} \tag{3.2}$$

for $B \in G/K$, where

$$z = B(B^+B)^{-\frac{1}{2}} \tan(B^+B)^{1/2}. \tag{3.3}$$

Under the action of G on G/K the matrices z undergo a fractional linear transformation. If

$$\mathcal{G} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$$

then from $\mathcal{G}\mathbb{B} = \mathbb{B}'\mathbb{k}$ ($\mathbb{k} \in K$), one finds

$$z' = (Az+B)(Cz+D)^{-1}. \tag{3.4}$$

It follows that the horizontal vectors defined by a Cartan-Ehresmann connection, eq.(2.5), are of the form:

$$\mathbb{F}_\mu(x, z=0) = \partial_\mu - \begin{pmatrix} A_{1\mu}(x) & 0 \\ 0 & A_{2\mu}(x) \end{pmatrix} - \begin{pmatrix} 0 & B_\mu(x) \\ -B_\mu^+(x) & 0 \end{pmatrix} \equiv \partial_\mu - \mathbb{A}_\mu - \mathbb{B}_\mu, \tag{3.5}$$

whereas the vertical part of the tangent space of the bundle is spanned by the representatives of the coset elements at $z=0$, *viz.* a vector of the vertical space at $z=0$ is represented by

$$\mathbb{V} = \begin{pmatrix} 0 & v \\ -v^+ & 0 \end{pmatrix}.$$

In writing down eq.(3.5) we took advantage of the invariance property (2.6) of the vector $\mathbb{A}_\mu + \mathbb{B}_\mu$ entering the expression of the horizontal basis vector $\mathbb{F}_\mu$. In fact, due to that invariance property, it is always sufficient to perform the subsequent calculations at the origin ($z=0$) of the fibre, where the expression of $\mathbb{F}_\mu$ is simple. The resulting vectors can be transported afterwards to an arbitrary point of the fibre. The vector $\mathbb{A}_\mu$ can be removed from (3.5) with the help of a gauge transformation generated by (2.7). Indeed, choose a matrix,

$$\mathbb{K}(x) = \begin{pmatrix} K_1(x) & 0 \\ 0 & K_2(x) \end{pmatrix} \tag{3.6}$$

from $\mathbb{K}$, so that

$$\mathbb{E}_\mu \equiv \mathbb{K}(x)\mathbb{F}_\mu \mathbb{K}(x)^{-1} = \partial_\mu - (\partial_\mu \mathbb{K}(x) + \mathbb{K}(x)\mathbb{A}_\mu(x))\mathbb{K}(x)^{-1} - \mathbb{K}(x)\,\mathbb{B}_\mu \mathbb{K}(x)^{-1}.$$

Therefore, if $\mathbb{K}$ satisfies

$$\partial_\mu \mathbb{K}(x) + \mathbb{K}(x)\mathbb{A}_\mu(x) = 0, \tag{3.7}$$

then the transformed horizontal vector is of the form

$$\mathbb{E}_\mu = \partial_\mu - \mathbb{K}(x)\,\mathbb{B}_\mu\,\mathbb{K}(x)^{-1} \equiv \partial_\mu - \mathbb{C}_\mu(x), \tag{3.8}$$

with

$$\mathbb{C}_\mu(x) = \begin{pmatrix} 0 & K_1(x)B_\mu K_2^{-1}(x) \\ -K_2(x)B_\mu^+ K_1(x)^{-1} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & C_\mu(x) \\ -C_\mu^+(x) & 0 \end{pmatrix}. \tag{3.9}$$

The vector $\mathbb{C}_\mu$ lies entirely in G/K, as one can readily verify with the help of (3.1). One notices, of course, that the relation between the fields $B_\mu(x)$ and $C_\mu(x)$ is a non-local one, since from (3.7) one gets formally

$$\mathbb{K}(x) = T\exp - \int_{-\infty}^{x} dx^{\mu'} \mathbb{A}_{\mu'}(x'),$$

where Texp stands for an ordered exponential. In addition, if the covariant curl of $\mathbb{A}_\mu$ does not vanish, the matrix $\mathbb{K}(x)$ is path dependent. However, for the time being, we may proceed without having to consider this question in detail.

The transformation law of the fields $\mathbb{C}_\mu(x)$ under the local action of G is readily established. It is convenient to consider the action of K and of G/K separately.

Consider first the action of K, matrices of the form (3.1). We have:

$$\mathbb{E}_\mu + [\,\mathbb{A}(x), \mathbb{E}_\mu] = \partial_\mu - \begin{pmatrix} A_{1,\mu} & C_\mu + C_\mu A_2 - A_1 C_\mu \\ -C_\mu^+ + C_\mu^+ A_1 - A_2 C_\mu^+ & A_{2,\mu} \end{pmatrix}. \tag{3.10}$$

The block-diagonal part, diag($A_{1,\mu}$, $A_{2,\mu}$), can be removed by performing an infinitesimal gauge transformation of the form (3.6). However, one immediately verifies that the appropriate infinitesimal transformation is given by

$$1 - \int^{x} dx^{\mu'} \mathbb{A}_{,\mu'}(x') = 1 - \mathbb{A}(x).$$

The action of this, "corrective gauge transformation" exactly cancels the contribution of $\mathbb{A}(x)$ to (3.10). Hence, <u>the action of K is trivial on $\mathbb{E}_\mu$</u>.

Next, we turn to the action of the coset generators, (local matrices of the form (3.2)).

We have with

$$\mathbb{B}(x) \Longrightarrow \begin{pmatrix} 0 & u(x) \\ -u^+(x) & 0 \end{pmatrix},$$

$$\mathbb{E}_\mu + [\mathbb{B}(x), \mathbb{E}_\mu] = \partial_\mu - \begin{pmatrix} C_\mu u^+ - u C_\mu^+ & C_\mu + u_{,\mu} \\ -C^+_{,\mu} - u^+_{,\mu} & C_\mu^+ u - u^+ C_\mu \end{pmatrix}. \tag{3.11}$$

The off-diagonal structure of the horizontal vector is again reestablished by means of a gauge transformation of the form (3.6). One verifies that the appropriate infinitesimal gauge transformation is given by

$$1-\int^{x} dx^{\nu'} \begin{pmatrix} u(x)C_\nu^+(x) - C_\nu(x)\overset{+}{u}(x) & 0 \\ 0 & \overset{+}{u}(x)C_\nu(x) - C_\nu^+(x)u(x)\end{pmatrix}. \qquad (3.12)$$

Thus, the combined action of the transformations (3.11) and (3.12) results in the transformation law of the field $C_\mu(x)$:

$$\delta C_\mu(x) = u_{,\mu}(x) + \int^{x} dx^{\nu'}\Big[\big(u(x)C_\nu(x) - C_\nu(x)\overset{+}{u}(x)\big)C_\mu(x) - C_\mu(x)\big(\overset{+}{u}(x)C_\nu(x) - C_\nu(x)u(x)\big)\Big]. \qquad (3.13)$$

The field tensor is defined in the usual way, <u>viz.</u>

$$\mathbb{F}_{\mu\nu} = [\mathbb{E}_\mu, \mathbb{E}_\nu]. \qquad (3.14)$$

The transformation law of this oject is easily computed from eqs. (3.13) and (3.14), the transformation under elements of K being trivial. The explicit expression of the infinitesimal change of $\mathbb{F}_{\mu\nu}$ is, however, neither simple, nor is it particularly instructive. The geometrically important fact is that $\mathbb{F}_{\mu\nu}$ supports a nonlinear realization of the group G, just as the gauge potential C_μ does. Indeed, the combined effect of the infinitesimal transformations (3,11) and (3,12) can be represented by a single commutator, say,

$$\delta\mathbb{E}_\mu = [\mathbb{M}, \mathbb{E}_\mu], \qquad (3.15)$$

where $\mathbb{M}$ is a functional of the gauge potentials. Explicitly,

$$\mathbb{M} = \begin{pmatrix} \int^{x} dx^{\nu'}\big(C_\nu(x)\overset{+}{u}(x) - u(x)C_\nu^+(x)\big) & u(x) \\ -\overset{+}{u}(x) & \int^{x} dx^{\nu'}\big(C_\nu^+(x)u(x) - \overset{+}{u}(x)C_\nu(x)\big)\end{pmatrix}. \qquad (3.16)$$

As a consequence,

$$\begin{aligned}\delta\mathbb{F}_{\mu\nu} &= [[\mathbb{M},\mathbb{E}_\mu],\mathbb{E}_\nu] + [\mathbb{E}_\mu,[\mathbb{M},\mathbb{E}_\nu]] \\ &= [[\mathbb{M},[\mathbb{E}_\mu,\mathbb{E}_\nu]] = [\mathbb{M},\mathbb{F}_{\mu\nu}]\end{aligned} \qquad (3.17)$$

in virtue of the Jacobi identity. Hence, the usual quadratic expression, <u>viz.</u> $Tr(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu})$

is invariant under (3.17), just as in the case of a linear realization of the group G. This observation will be of use in the following Section.

We end the discussion of the transformation properties by constructing the vectors which are "perpendicular" (in the sense of the vanishing of the Lie bracket) to the vectors (3.8). The construction is elementary. Indeed, it follows from (2.5) and (2.7) that any vector $\mathbb{V}$, which is off-diagonal at Z=0, is perpendicular to $\mathbb{F}_\mu$ given by (2.5),

$$[\mathbb{F}_\mu, \mathbb{V}] = 0$$

Since the vector $\mathbb{E}_\mu$ -eq.(3.8)- is derived from $\mathbb{F}_\mu$ via a conjugation, it immediately follows that the conjugate of $\mathbb{V}$,

$$\mathbb{V}(x) = \mathbb{K}(x)\mathbb{V}\mathbb{K}^{-1}(x) \qquad (3.18)$$

is perpendicular to $\mathbb{E}_\mu$.

4. METRIC PROPERTIES OF THE ASSOCIATED BUNDLE.

The construction of a gauge theory on coset spaces is completed by imposing a metric structure on $\mathcal{A}$. We assume (as it is done in gauge theories based on principal bundles, see ref.1) that the metric is block-diagonal in the horizontal lift basis, cf. eq.(1.1).However, the metric tensor of the group is now replaced by the metric on the coset. The latter is induced by the Cartan-Killing metric of the structural group. In particular, if an infinitesimal displacement at the origin of the fibre is given by

$$d\mathbb{Z} = \begin{pmatrix} 0 & dz \\ -dz^+ & 0 \end{pmatrix}, \tag{4.1}$$

then the metric at the origin is

$$g_0(dZ, dZ) = \frac{1}{2}\,\mathrm{Tr}(d\mathbb{Z}\, d\mathbb{Z}) = -\frac{1}{2}\,\mathrm{Tr}(dz\,dz^+ + dz^+ dz). \tag{4.2}$$

Next, we transport this metric to an arbitrary point z of the coset. From eq.(3.4) we deduce that under a general group element $\mathcal{G}$, the one-forms dZ transform according to the formula:

$$dZ(z') = (A - z'C)\, dZ(z)\,(Cz + D)^{-1} \tag{4.3}$$

Z' being given by (3.4). If $\mathcal{G}$ is a coset element (3.2), then dZ(Z) is given in terms of dZ as follows:

$$dZ(z) = (1 + zz^+)^{\frac{1}{2}}\, dz\,(1 + z^+z)^{1/2} \tag{4.4}$$

and the metric at the point Z becomes:

$$g_z(dZ(z), dZ(z)) = -\frac{1}{2}\,\mathrm{Tr}\left((1+zz^+)^{-1} dZ(z)(1+z^+z)^{-1} dZ^+(z)\right) - \frac{1}{2}\,\mathrm{Tr}\left((1+z^+z)\, dZ^+(z)(1+zz^+)^{-1} dZ(z)\right). \tag{4.5}$$

We notice that in the basis spanned by the conjugate forms,

$$d\mathbb{Z}(x) = \mathbb{K}(x)\, d\mathbb{Z}\, \mathbb{K}^{-1}(x)$$

we have:

$$dZ(z,x) = (1 + z(x)z^+(x))^{1/2}\, dZ(x)\,(1 + z^+(x)z(x))^{1/2},$$

where, according to (3.4) ,

$$z(x) = K_1(x)\, z\, K_2(x)^{-1},$$

Hence, the metric (4.5) is form-invariant under conjugation, as expected.

By taking the metric of space-time to be a usual Riemann metric, one can compute the curvature tensor in the usual way. The calculation is somewhat laborious, but elementary. We merely quote the form of the Ricci scalar, which serves as the density of an invariant action. One finds at an arbitrary point of the fibre:

$$R = R_M + R_F - \frac{1}{4}\,\mathrm{Tr}(\mathbb{F}_{\mu\nu}\, \mathcal{G}_z\, \mathbb{F}^{\mu\nu}) \tag{4.6}$$

Here R_M and R_F stand for the Ricci scalars of the base manifold and fibre, respectively, while $\mathcal{G}_z$ is the metric tensor defined by (4.5). The essential feature of the expression (4.6) is that thedependence on the base point (introduced by the conjugation with $\mathbb{K}(x)$) disappears from the second and third terms. Thus the splitting of the Ricci scalar is complete, just as in the case of an underlying principal bundle. The gauge invariance of (4.6) is manifest. Indeed, by transporting the last term to the origin of the fibre, one gets

$$\mathrm{Tr}(\mathbb{F}_{\mu\nu}\, \mathcal{G}_z\, \mathbb{F}_{\mu\nu}) \longrightarrow \mathrm{Tr}(\mathbb{F}_{\mu\nu}\, \mathbb{F}^{\mu\nu}),$$

cf. eq.(4.2). However, $Tr(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu})$ is gauge invariant (despite of the nonlinear transformation law of the fields) as it was shown in the preceding Section.

There is a simple way of understanding the complete splitting of the Ricci scalar. Indeed, the calculation of the curvature could have been performed in a basis which contains redundant components, as in eq.(3.5). In such basis the transformation law of the connection coefficients under the action of G is linear and the fibre coordinates are independent of the base point. The form of R at the origin of the fibre is the same as in (4.6) except for the appearence of an extra term proportional to $Tr(\mathbb{A}_{\mu\nu}\mathbb{A}^{\mu\nu})$, where $\mathbb{A}_{\mu\nu}$ is the field tensor formed from $\mathbb{A}_\mu$ alone. However, the Ricci scalar is gauge invariant; therefore - as shown in Sec.3 - the extra term can be removed by a gauge transformation without changing the value of the Ricci scalar. (In other words, the transformation laws of the connection allow one to consider $\mathbb{A}_\mu$ to be a "pure gauge", viz. $\mathbb{A}_\mu = -\mathbb{K}^{-1}\partial_\mu \mathbb{K}$ with identically vanishing field tensor.)

The classical action principle derived from (4.6) is

$$W = \int dV R$$

where dV is the invariant volume element of the bundle. Due to the block-diagonal structure of the metric, this volume element is factorizable $dV = dV_M dV_F$, the first and second factors standing for the volume elements of the manifold and coset, respectively. In view of the invariance property of R with respect to translations in the fibre, one may integrate the first and the third terms trivially over the fibre. As a result, the third term in (4.6) is replaced by its value at Z=0. Finally, by dividing out with the volume, V_F, of the fibre, the action may be reduced to the form

$$W = \int dV_M R_M + V_F^{-1}\int dV_M dV_F R_F - \frac{1}{4}\int dV_M\, Tr(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}).$$

This action is of the conventional form of an Einstein-Yang-Mills theory with a cosmological constant,

$$\Lambda = V_F^{-1}\int dV_F R_F$$

except for the important fact that only the fields $\mathbb{C}_\mu$ enter the expression of $\mathbb{F}_{\mu\nu}$.

5.DISCUSSION.

We have demonstrated the possibility of constructing gauge theories based on associated, rather than principal bundles. The essential result emerging from the geometrical approach is that it is possible to remove some of the gauge fields from the theory provided the fibre is a coset space on a maximal subgroup. This result is obviously independent of the parametrization of the coset and, hence, it can be generalised to other groups. (Explicit calculations, however, may become rather complicated unless the coset is parametrized appropriately.) The geometrical argument leading to the elimination of the connection coefficients belonging to the subgroup K is quite compelling. The resulting nonlinear transformation law of the remaining connection coefficients suggests that the theory outlined here describes a "spontaneously broken" local symmetry. However, if this interpretation is indeed correct, the mechanism of symmetry breaking appears to be somewhat unusual. In particular, unlike in a standard Higgs-type model, here some gauge potentials can be transformed out of the theory altogether. Nevertheless, at least some connection can be found with a symmetry breaking mechanism involving scalar fields. Indeed, eq.(3.7) is not completely integrable unless the covariant curl of $\mathcal{A}_\mu$ vanishes. In that case, however, $\mathcal{A}_\mu$ is of the form $L^{-1}\partial_\mu L$, with L in K. Therefore, the parameters of an element of K play a geometrical role which is similar to the one played by pseudo-Goldstone bosons.

The theory outlined in this paper is a classical one. We have no results to present concerning the quantization of such a theory. One obvious obstacle standing in the way of quantization is the complicated transformation law of the gauge potentials. This problem may be circumvented by reintroducing the redundant fields, or perhaps the scalar fields just mentioned. Even so, it is not clear whether the quantized theory would reflect the attractive features of its classical counterpart.

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