Gauge groups and matter fields on some models of F-theory without section

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Abstract: We investigate F-theory on an elliptic Calabi-Yau 4-fold without a section to the fibration. To construct an elliptic Calabi-Yau 4-fold without a section, we introduce families of elliptic K3 surfaces which do not admit a section. A product K3 × K3, with one of the K3's chosen from these families of elliptic K3 surfaces without a section, realises an elliptic Calabi-Yau 4-fold without a section. We then compactify F-theory on such K3 × K3's.

We determine the gauge groups and matter fields which arise on 7-branes for these models of F-theory compactifications without a section. Since each K3 × K3 constructed does not have a section, gauge groups arising on 7-branes for F-theory models on constructed K3 × K3's do not have U(1)-part. Interestingly, exceptional gauge group E_6 appears for some cases.

Contents

1 Introduction

F-theory [1–3] is a non-perturbative extension of type IIB superstring theory, which is considered to encompass strong coupling regime. Description of high energy regime requires that a coupling constant vary over a base 3-fold. This is realised in F-theory, where the theory is compactified on an elliptic Calabi-Yau 4-fold over a base 3-fold, with fiber being an elliptic curve. A complex structure of a fiber as a parameter is identified with a coupling constant. So, in the framework of F-theory the coupling constant becomes a function over a base 3-fold. $SL(2, \mathbb{Z})$-symmetry present on an elliptic fiber can be seen as a geometrical realisation of S-duality.

Many aspects of gauge theory are described in the geometrical language in F-theory. *The discriminant locus* is the set of points in the base, over which an elliptic fiber becomes singular (i.e. not a smooth elliptic curve). The discriminant locus signals locations of 7-branes. 7-branes are wrapped on irreducible components of the discriminant locus. Not only the location of 7-branes, but the information on the structures of gauge groups can be read by studying the discriminant locus. Singular fibers lying over the discriminant locus encode the information on gauge groups in low energy theory. A type of singular fiber specifies the gauge groups which arise on 7-branes wrapped on an irreducible component of discriminant locus. So, by looking at a type of a singular fiber, the structure of gauge groups can be known. The types of singular fibers of an elliptic surface were classified by Kodaira [4, 5].

Many models of F-theory on elliptic 4-folds have been studied, mostly 4-folds with sections [6–16]. However, among all the elliptic Calabi-Yau 4-folds, ones which admit a

section to the fibration only form a special subset. F-theory compactifications on elliptic manifolds without a section was considered in [17]. Recently, Braun and Morrison [18] investigated F-theory on spaces which admit elliptic fibration without a section, and Morrison and Taylor [19] studied F-theory on elliptic Calabi-Yau manifolds which do not have a section. For recent models of F-theory without a section, see also [20–26].

In the present paper, we explore another model of F-theory on elliptic Calabi-Yau 4-folds which do not admit a section. To build a model without section, we consider an elliptic K3 surface times a K3 surface. We choose a direct product K3 × K3 as an elliptic Calabi-Yau 4-fold, for the reason that the techniques from algebraic geometry enable to perform a detailed analysis of F-theory compactifications on K3 × K3.

By using the method of algebraic geometry, we construct families of elliptic K3 surfaces without a section to the fibration. By considering a product of an elliptic K3 surface without a section times a K3 surface, we obtain an elliptic Calabi-Yau 4-fold without a section. We explicitly determine gauge groups which arise on 7-branes for F-theory compactified on these spaces. Non-Abelian part of the gauge groups are read from the types of singular fibers of an elliptic K3 surface. For F-theory on K3 × K3, the number of $U(1)$-factors in the gauge group equals the rank of the Mordell-Weil group of an elliptic K3 surface [3]. An elliptic K3 surface we consider in this paper does not have a section. So, gauge groups for these models without a section do not have $U(1)$-part. Interestingly, exceptional gauge group E_6 arises in some of our models.

We next investigate matter fields which arise on 7-branes for the models. Matter fields in the F-theory context were studied by Katz and Vafa [27]. As discussed in [27], a singularity of a compactifying space encodes information on charged matters on 7-branes. As 7-branes localised at the singularity move apart from one another, they generate light matter fields, and from geometric viewpoint this corresponds to a deformation of singularity. A singular fiber of ADE-type of an elliptic fibration corresponds to a singularity of ADE-type. When 7-branes move so a singularity of ADE-type deforms accordingly, the corresponding gauge group G breaks to H of one rank less. Resultantly, matters arise on 7-branes in irreducible representations of H. Therefore, matter fields on 7-branes can be read by deforming singularities of ADE-type. See for instance [28–30] for related issues. For recent advances on the correspondence between matter and geometry, see [31]. Since we limit the consideration to F-theory compactifications on K3 × K3, all the 7-branes are parallel. So the Yukawa-type interaction is absent for our models of F-theory on K3 × K3.

The paper is structured as follows: in section 2 we introduce families of elliptic K3 surfaces which do not admit a section. K3 × K3, with one of the K3's chosen from this introduced class of elliptic K3's, gives an elliptic Calabi-Yau 4-fold without a section. In section 3, we determine gauge groups arising on 7-branes for F-theory compactified on K3 × K3, with one of the K3's chosen from the class introduced in section 2. By construction, such K3 × K3 does not admit a section. So, the gauge groups which arise on 7-branes do not have $U(1)$ factor. We also see that for some cases, consideration on monodromy uniquely determines the gauge groups. F-theory on K3 × K3 gives 4d theory with enhanced $N = 2$ supersymmetry. Therefore, the theory is highly constrained by the anomaly cancellation conditions. We check if our solutions are consistent with these

anomaly cancellation conditions. In section 4, we calculate the matter spectra which arise on 7-branes, with and without flux. Since potential matter may actually vanish due to anomalies, we can fully determine the matter spectra for some limited cases. In section 5, we derive conclusions on F-theory compactifications on K3 × K3 without a section.

2 Class of elliptic K3 surfaces which do not admit section

In this section, we introduce a class of elliptic K3 surfaces which do not admit a section.

We can explicitly construct a class of elliptic K3 surfaces without a section as follows: consider a smooth hypersurface S of bidegree (2,3) in $\mathbb{P}^1 \times \mathbb{P}^2$. Then a hypersurface S is a K3 surface. By restricting projections from $\mathbb{P}^1 \times \mathbb{P}^2$ onto $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively to this hypersurface S, we obtain a map onto $\mathbb{P}^1$, and a degree 3 map into $\mathbb{P}^2$. These maps give an explicit description of the hypersurface S as an elliptic fibration over a base $\mathbb{P}^1$.

$$\mathbb{P}^1 \times \mathbb{P}^2 \supset S \xrightarrow{\text{degree 3}} \mathbb{E} \subset \mathbb{P}^2$$
$$\downarrow$$
$$\mathbb{P}^1$$

For a generic hypersurface S described above, pullbacks of the hyperplane classes in $\mathbb{P}^1$ and $\mathbb{P}^2$ generate Néron-Severi lattice of S, and Picard number $\rho(S) = 2$ [32, 33]. Let D_1 and D_2 denote divisors of hypersurface S corresponding to pullbacks of hyperplanes in $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively. Let p_1 and p_2 denote projections from $\mathbb{P}^1 \times \mathbb{P}^2$ onto $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively:

$$\mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{p_2} \mathbb{P}^2$$
$$p_1 \downarrow$$
$$\mathbb{P}^1$$

Then $p_1^* \mathcal{O}_{\mathbb{P}^1}(1) = \{pt\} \times \mathbb{P}^2$ and $p_2^* \mathcal{O}_{\mathbb{P}^2}(1) = \mathbb{P}^1 \times line$. Restrictions of these to a hypersurface S are D_1 and D_2, and $S \sim p_1^* \mathcal{O}_{\mathbb{P}^1}(2) + p_2^* \mathcal{O}_{\mathbb{P}^2}(3)$, so $D_1^2 = 0$, $D_2^2 = 2$ and $D_1 \cdot D_2 = 3$. Therefore, the generic Néron-Severi lattice of hypersurfaces of bidegree (2,3) in $\mathbb{P}^1 \times \mathbb{P}^2$ has intersection matrix

$$\begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}. \tag{2.1}$$

The divisor class D_1 represents a fiber class, and the divisor class D_2 is a 3-section.

When an elliptic surface has a section, let us denote a divisor class of section by O and fiber class by F respectively, then section O and fiber class F satisfy the relation $O \cdot F = 1$. The divisor class D_1 represents a fiber class. (So it deserves another notation F.) But the Néron-Severi lattice of a generic hypersurface S in $\mathbb{P}^1 \times \mathbb{P}^2$ constructed above has rank 2, generated by divisors D_1 and D_2. So any divisor D in S can be written as an integral sum of D_1 and D_2, $D = nD_1 + mD_2$. Then for any divisor D in S, $D \cdot D_1 (= D \cdot F) = 3m \neq 1$. Therefore, generic member of a bidegree (2,3) hypersurface S in $\mathbb{P}^1 \times \mathbb{P}^2$ does *not* have a section. (The above argument holds true only for a *generic* choice of an equation for S; there are exceptions. Special cases such as a hypersurface S is

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given by the equation $T_1^2 ZY^2 = T_1^2 X^3 + T_0^2 X Z^2 + T_0^2 Z^3$, where $[X : Y : Z]$ is a coordinate on $\mathbb{P}^2$ and $[T_0 : T_1]$ is a coordinate on $\mathbb{P}^1$, should be excluded. Such an equation locally becomes $y^2 = x^3 + t^2 x + t^2$, where $x = X/Z, y = Y/Z$ and $t = T_0/T_1$. An equation of Weierstrass form admits a constant section $[0 : 1 : 0]$ over the base. Therefore, to construct a K3 surface without section, we need to avoid a hypersurface given by an equation which admits a transformation into Weierstrass form.)

In subsequent sections, we perform a detailed analysis of F-theory on an elliptic Calabi-Yau 4-fold without section. We restrict our consideration to F-theory compactification on a product an elliptic K3 surface times a K3 surface, with an elliptic K3 surface chosen from hypersurfaces of bidegree (2,3) in $\mathbb{P}^1 \times \mathbb{P}^2$ constructed above. As shown above, such chosen elliptic K3 surface does not have a section. Therefore, K3 $\times$ K3, with one of the K3's chosen from hypersurfaces of bidegree (2,3) in $\mathbb{P}^1 \times \mathbb{P}^2$ constructed above, does not have section as an elliptic Calabi-Yau 4-fold.

To study F-theory compactifications on a class of K3 $\times$ K3 without a section in detail, we limit ourselves to consider two specific families of K3 surfaces,[1] among hypersurfaces of bidegree (2,3) in $\mathbb{P}^1 \times \mathbb{P}^2$. One family consists of K3 surfaces of *Fermat type* defined by equations of the form:

$$fX^3 + gY^3 + hZ^3 = 0. \tag{2.2}$$

$[X : Y : Z]$ is a coordinate on $\mathbb{P}^2$, and f, g, h are homogeneous polynomials of degree 2 in $[T_0 : T_1]$ where $[T_0 : T_1]$ is a coordinate on $\mathbb{P}^1$. For notational simplicity we use $t := T_0/T_1$. Then we may rewrite f, g, h as polynomials in t of degree at most 2. (Original homogeneous forms can be easily recovered.)

Notice that an elliptically fibered hypersurface having a section over the base $\mathbb{P}^1$ means that an equation for a hypersurface has a $K(\mathbb{P}^1)$−rational point. Here $K(\mathbb{P}^1)$ denotes a function field over $\mathbb{P}^1$. (So $K(\mathbb{P}^1) \cong \mathbb{C}(t)$.) For generic polynomials f, g, h of degree 2 in t, the K3 surface determined by (2.2) does not admit a section, i.e. the equation (2.2) with variables X, Y, Z does not have a solution in $\mathbb{C}(t)$.

We rewrite the equation (2.2) as

$$(t - \alpha_1)(t - \alpha_2)X^3 + (t - \alpha_3)(t - \alpha_4)Y^3 + (t - \alpha_5)(t - \alpha_6)Z^3 = 0 \tag{2.3}$$

where $\alpha_i \in \mathbb{C}$, α_1 and α_2 are the roots of f, α_3 and α_4 are the roots of g, and α_5 and α_6 are the roots of h.[2] Then the K3 surface given by the equation (2.3) has singular fibers at $t = \alpha_i$, $i = 1, \cdots, 6$. A special case in which some α_i, say α_1 is ∞ corresponds to the equation

$$(t - \alpha_2)X^3 + (t - \alpha_3)(t - \alpha_4)Y^3 + (t - \alpha_5)(t - \alpha_6)Z^3 = 0. \tag{2.4}$$

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[1]For a generic hypersurface S of bidegree (2,3) in $\mathbb{P}^1 \times \mathbb{P}^2$, Picard number $\rho(S) = 2$. For some specific surfaces, Picard number may enhance. This means divisors increase, but it is expected that divisors increase only in the fiber direction. Therefore, basic line of the above argument holds true for such specific K3 surfaces with enhanced Picard numbers, and they still do not have section.

[2]We suppressed top coefficients as they are irrelevant to locations and types of singular fibers, so are irrelevant to gauge groups.

This way, reduction of a factor in t from the coefficients of X, Y, Z in (2.3) results in singular fiber at $\infty \in \mathbb{P}^1$.

One needs to be careful about the choice of $\{\alpha_i\}_{i=1}^6$; a surface given by the equation (2.3) develops singularity worse than a rational double point with a wrong choice of $\{\alpha_i\}_{i=1}^6$. This bad singularity ruins the triviality of canonical bundle, and the equation (2.3) does not give a K3 surface then.

A simple condition on the multiplicity of α_i's in $\{\alpha_i\}_{i=1}^6$ ensures that the equation (2.3) correctly gives a K3 surface. The rule is that any two α_j and α_k among 6 α_i's may coincide, but any triplet among 6 α_i's are not allowed to coincide. In other words, the equation (2.3) describes a K3 surface if and only if the multiplicity of α_i is at most 2 for every $i = 1, \cdots, 6$. For instance, the equation (2.3) with $\alpha_1 = \alpha_2 = p$, $\alpha_3 = \alpha_4 = q$, $\alpha_5 = \alpha_6 = r$, where p, q, r are mutually distinct, correctly describes a K3 surface, but the one with $\alpha_1 = \alpha_2 = \alpha_3$ does not give a K3 surface.

In addition to the above K3 surfaces defined by Fermat type equation (2.3), we also would like to consider another family of K3 surfaces defined by equation in *Hesse form*:

$$aX^3 + bY^3 + cZ^3 - 3dXYZ = 0, \tag{2.5}$$

where a, b, c, d are homogeneous polynomials in $[T_0 : T_1]$ of degree 2. For generic polynomials a, b, c, d, the above equation of Hesse form (2.5) does not have a $K(\mathbb{P}^1)$−rational point, i.e. a K3 surface defined by the equation (2.5) does not have a section. Again for notational simplicity, we rewrite a, b, c, d as polynomials in t of degree at most 2.

In section 3, we determine gauge groups arising on 7-branes for F-theory compactified on K3 × K3, with the one of K3's being a K3 surface of Fermat type defined by the equation (2.3), and being a K3 surface in Hesse form defined by the equation (2.5), by computing their singular fibers. These K3 surfaces do not have a section, so the gauge groups which arise on 7-branes do not have U(1)-factor.

3 Gauge groups on the model

In this section, we compute gauge groups which arise on 7-branes for F-theory compactification on an elliptic K3 surface times a K3 surface. For an elliptic K3 surface in the product K3 × K3, we choose a K3 surface defined by the equation of Fermat type

$$(t - \alpha_1)(t - \alpha_2)X^3 + (t - \alpha_3)(t - \alpha_4)Y^3 + (t - \alpha_5)(t - \alpha_6)Z^3 = 0, \tag{3.1}$$

and a K3 surface defined by the equation in Hesse form

$$aX^3 + bY^3 + cZ^3 - 3dXYZ = 0, \tag{3.2}$$

as discussed in the previous section. With these settings, the product K3 × K3 does not have a section. So, the gauge groups arising on 7-branes for F-theory on these models do not have U(1)-factor.

Fiber of an elliptic manifold is generically a smooth elliptic curve, but along a locus of codimension 1 in the base a fiber becomes singular. This locus is called the discriminant locus.
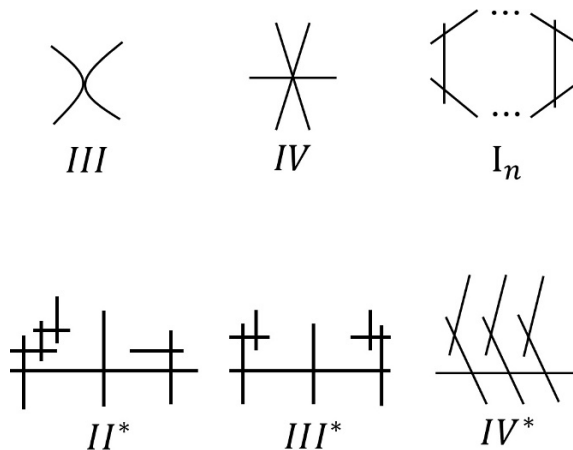
Figure 1. Singular fibers.

The singular fibers of an elliptic surface were classified by Kodaira [4, 5]. We use Kodaira's notation for types of singular fibers. Singular fiber of an elliptic surface is either the sum of smooth $\mathbb{P}^1$'s intersecting in specific ways, or a $\mathbb{P}^1$ with one singularity. The latter case is not a singularity of a manifold itself. When a singular fiber is $\mathbb{P}^1$ with a single singularity, it is either a nodal rational curve denoted by I_1, or a cuspidal rational curve denoted by type II, respectively in Kodaira's notation. When singular fiber is reducible into the sum of smooth $\mathbb{P}^1$'s, there are seven types of singular fibers: two infinite series I_n and I_n^*, and five types III, IV, II^*, III^* and IV^*.

To each singular fiber an extended Dynkin diagram is associated with a vertex for each $\mathbb{P}^1$ component of the fiber. Two vertices are joined by an edge if and only if the corresponding $\mathbb{P}^1$'s intersect. Extended Dynkin diagrams $\tilde{A}_{n-1}$, $\tilde{D}_{n+4}$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ describe the configurations of $\mathbb{P}^1$'s of singular fibers of types I_n, I_n^*, IV^*, III^* and II^* respectively. Type III is two $\mathbb{P}^1$'s tangent to each other at a point, and type IV is three $\mathbb{P}^1$'s meeting at a single point. Extended Dynkin diagrams associated to fiber types III and IV are $\tilde{A}_1$ and $\tilde{A}_2$ respectively. For pictures of singular fibers, see figure 1. Each line in a picture represents a $\mathbb{P}^1$ component. Pictures show the configurations of $\mathbb{P}^1$ components of each fiber type and how they intersect one another.

Extended Dynkin diagrams $\tilde{A}_n$, $\tilde{D}_n$ and $\tilde{E}_n$ correspond to SU($n+1$), SO($2n$) and E_n gauge groups respectively, on 7-branes. So, singular fibers of types I_n, I_n^* give rise to SU(n) and SO($2n+8$) gauge groups. Types IV^*, III^* and II^* generate E_6, E_7 and E_8 gauge groups respectively. Singular fibers of types I_1 and II do not generate non-Abelian gauge groups on 7-branes. Type III fiber gives SU(2) gauge group, and type IV generates SU(3) gauge group. The correspondence between the fiber types and the gauge groups are shown in the table 1 below.

In the following subsections, we determine gauge groups on 7-branes for F-theory on a K3 surface of Fermat type times K3, and a K3 surface in Hesse form times K3, respectively.

After we deduce gauge groups on 7-branes for F-theory compactifications on K3 × K3 without a section, we make a comment on possible monodromies. We find that for a

fiber type	gauge group
I_n	$SU(n)$
I_n^*	$SO(2n + 8)$
III	$SU(2)$
IV	$SU(3)$
IV^*	E_6
III^*	E_7
II^*	E_8

Table 1. The correspondence between the fiber types and the gauge groups.

Fermat type K3 × a K3, it is essential to consider monodromies to determine the gauge symmetries.

We also discuss anomaly cancellation conditions on the models. F-theory compactification on K3 × K3 gives 4d theory with enhanced $N = 2$ supersymmetry. Resultantly, theory is highly constrained by consistency conditions from anomalies. We check if the solutions we obtain are consistent with anomaly cancellation conditions. We also see that the form of the discriminant locus is determined by the consistency conditions from anomalies.

3.1 Gauge groups on K3 surfaces of Fermat type (times K3)

In this subsection we determine gauge groups arising on 7-branes for F-theory compactification on K3 × K3, with one of the K3's given by the equation of Fermat type (3.1).

As we saw in the previous section, K3 surface of Fermat type does not admit a section. This means that the equation (3.1) does not transform into Weierstrass form. So we cannot use Tate's algorithm [34] to determine singular fibers directly from the defining equation.

So we consider Jacobian fibration $J(S)$ of K3 surface of Fermat type S to compute singular fibers. Jacobian $J(S)$ of S is an elliptic surface constructed from S with exactly the same types of singular fibers at exactly the same locations over the base $\mathbb{P}^1$ as S. Even when S does not have a section, $J(S)$ has a section, so $J(S)$ does admit transformation into Weierstrass form.

Jacobian $J(S)$ of K3 surface S defined by (3.1) is given by the equation

$$X^3 + Y^3 + \Pi_{i=1}^6(t - \alpha_i)Z^3 = 0, \tag{3.3}$$

where t runs through $\mathbb{P}^1$. Converting the equation (3.3) by a linear change of variables to the Weierstrass form and applying Tate's algorithm, we find the locations of singular fibers over the base $\mathbb{P}^1$ and their types of Jacobian $J(S)$. Both Jacobian $J(S)$ and the original surface S have the same positions and types of singular fibers. So the computational results for the Jacobian $J(S)$ precisely tell us the locations and types of singular fibers of the original K3 surface S.

The Weierstrass form into which the equation (3.3) is transformed is locally given by the equation

$$y^2 = x^3 - 2^4 \cdot 3^3 \cdot \Pi_{i=1}^6 (t - \alpha_i)^2. \tag{3.4}$$

Therefore the discriminant of the Jacobian $J(S)$ is given by

$$\Delta = \Pi_{i=1}^6 (t - \alpha_i)^4. \tag{3.5}$$

We can determine the locations and types of the singular fibers of the Jacobian $J(S)$ from the discriminant Δ given by (3.5). Singular fibers are at $t = \alpha_i$, $i = 1, \cdots, 6$ and they are seen to be additive. Using Tate's algorithm, we see that when $\alpha_i \neq \alpha_j$ for all $j \neq i$, the singular fiber has type IV at $t = \alpha_i$. When there is a single $j \neq i$ such that $\alpha_i = \alpha_j$, the singular fiber at $t = \alpha_i$ has type IV^*. Intuitively speaking, when two points α_i and α_j on $\mathbb{P}^1$ approach each other and they eventually coincide (i.e. $\alpha_i = \alpha_j$), two singular fibers of type IV at $t = \alpha_i$ and $t = \alpha_j$ merge to give a singular fiber of type IV^*. As noted in section 2, more than two α_i's cannot be coincident. When more than two α_i's coincide (for instance $\alpha_i = \alpha_j = \alpha_k$ with i, j, k mutually distinct), the equation (3.1) does not give a K3 surface.

In this way we find that singular fibers of Fermat type K3 surface S defined by the equation (3.1) are located at $t = \alpha_i$, $i = 1, \cdots, 6$, and when α_i's are mutually distinct, their fiber types are all IV. So K3 surface S of Fermat type generically has 6 singular fibers of type IV. When two of α_i's coincide, two fibers of type IV collide and enhance to a singular fiber of type IV^*. Therefore, we have 4 cases of the configurations of the singular fibers in total: (i) six fibers of type IV, (ii) four fibers of type IV and one fiber of type IV^*, (iii) two fibers of type IV and two fibers of type IV^*, and (iv) three fibers of type IV^*.

Summarising the above, we have 4 cases of gauge groups which arise on 7-branes. They are:

- $SU(3)^{\oplus 6}$

- $E_6 \times SU(3)^{\oplus 4}$

- $E_6^{\oplus 2} \times SU(3)^{\oplus 2}$

- $E_6^{\oplus 3}$.

The gauge group which arises on 7-branes is generically $SU(3)^{\oplus 6}$ for F-theory on K3 × K3, with one of the K3's being a K3 surface of Fermat type S. But when two of α_i's coincide, two SU(3)'s collide and enhance to the exceptional gauge group E_6, and we have more E_6's as we have more pairs of coincident α_i's. As mentioned in the previous section, when three of α_i's coincide, the singularity at the point of triple coincidence becomes worse than a rational double point. This is not allowed for the equation (3.1) to describe a K3 surface.

The above 4 cases classify the gauge groups which arise on 7-branes for F-theory on an elliptic K3 surface of Fermat type times a K3 surface. The product K3 × K3, with one of the K3's being Fermat type, does not have a section as an elliptic Calabi-Yau 4-fold. Therefore the gauge groups on 7-branes do not have U(1)-factor.

3.2 Gauge groups on K3 surfaces of Hesse form (times K3)

In this subsection, we investigate gauge groups arising on 7-branes when F-theory is compactified on K3 $\times$ K3, with one of the K3's being a K3 surface of Hesse form defined by the equation (3.2). K3 surface S of Hesse form does not have a section for generic polynomials a, b, c and d with degrees 2 in t.

By inspection on the condition that the equation (3.2) has a singularity, we can directly compute the discriminant of the equation (3.2). We denote by Δ the discriminant of the equation (3.2). For notational convenience, we set

$$f_{Hesse} := aX^3 + bY^3 + cZ^3 - 3dXYZ. \tag{3.6}$$

Then the equation (3.2) has a singularity precisely when the equations

$$\partial_X f_{Hesse} = \partial_Y f_{Hesse} = \partial_Z f_{Hesse} = 0 \tag{3.7}$$

have a solution. From the equations in (3.7), we deduce that the discriminant Δ contains the factors abc and $(abc - d^3)$. Then by considering some special case of the equation (3.2), we see that the multiplicity of the factor $(abc - d^3)$ in the discriminant Δ is 3. Both Δ and $abc(abc - d^3)^3$ have degree 24 as polynomials in t, so we can conclude that they are equal. Therefore, the discriminant Δ is given by

$$\Delta = abc(abc - d^3)^3. \tag{3.8}$$

(We suppressed an irrelevant constant factor of the discriminant.)

$abc - d^3$ is a polynomial in t of degree 6, so we can rewrite the equation (3.8) as

$$\Delta = abc \cdot \Pi_{i=1}^6 (t - \gamma_i)^3, \tag{3.9}$$

where γ_i's are six roots of $abc - d^3$. From the expression of the discriminant (3.9), we see that singular fibers are at $t = \gamma_i$ ($i = 1, \cdots, 6$), and at zeros of a, b and c.

It can be seen that every fiber is multiplicative. So each singular fiber has fiber type I_n for some n, where n equals the number of irreducible components of a singular fiber. When there is a multiplicative singular fiber at $t = t_0$, the number of the components of the singular fiber at $t = t_0$ is equal to the multiplicity of the discriminant Δ at $t = t_0$. In this way, we can determine the types of singular fibers from the expression for the discriminant Δ (3.9).

When γ_i's are mutually distinct, the multiplicity of each γ_i in the discriminant Δ is 3, so all the six fibers at $t = \gamma_i$ ($i = 1, \cdots, 6$) are of type I_3. When two of γ_i's coincide, the multiplicities of the two add up to 6 so the fiber has type I_6 at the coincident γ_i. Intuitively speaking, two fibers of type I_3 collide and enhance to type I_6 fiber. When a does not have a multiple root, say $a = const.(t - \beta_1)(t - \beta_2)$, $\beta_1 \neq \beta_2$, then singular fibers at $t = \beta_i$, $i = 1, 2$ both have type I_1. These type I_1 fibers do not correspond to a surface singularity, and they do not generate non-Abelian gauge groups on 7-branes. When a has a multiple root, i.e. when $\beta_1 = \beta_2$, two singular fibers of type I_1 collide and enhance to a singular fiber of type I_2. The same argument applies to b and c. When the polynomial abc has a

multiple root of multiplicity 3, say abc has a triple root $t = \beta$ so $(t - \beta)^3$ divides abc, then there is a singular fiber of type I_3 at $t = \beta$.

Summarising the above, the gauge group on 7-branes is generically $\mathrm{SU}(3)^{\oplus 6}$ for F-theory on K3 × K3, with one of the K3's being a K3 surface of Hesse form. When abc has a multiple root, the gauge group enhances to $\mathrm{SU}(3)^{\oplus 6} \times \mathrm{SU}(2)$ when the multiplicity of the root is 2, and it enhances to $\mathrm{SU}(3)^{\oplus 7}$ when the multiplicity is 3. When two of γ_i's coincide in the discriminant Δ given by the equation (3.9), two $\mathrm{SU}(3)$'s collide and enhance to the group $\mathrm{SU}(6)$. From the expression for the discriminant Δ (3.9), all the singular fibers are seen to be multiplicative. So, for a K3 surface of Hesse form, each singular fiber has type I_n for some n. Therefore, only $\mathrm{SU}(N)$ gauge groups arise on 7-branes for this model. $\mathrm{SO}(2N)$ gauge groups and exceptional gauge groups do not arise on 7-branes for F-theory on a K3 surface of Hesse form times a K3 surface. This determines the gauge groups arising on 7-branes for F-theory compactified on an elliptic K3 surface in Hesse form given by the equation (3.2) times a K3 surface. The product K3 × K3, with one of the K3's being a K3 surface in Hesse form, does not have a section as an elliptic Calabi-Yau 4-fold. Therefore the gauge groups on 7-branes do not have $\mathrm{U}(1)$-part.

3.3 Consideration on monodromy and consistency conditions from anomalies

In this subsection, we comment on the monodromies in the gauge groups arising on 7-branes. We find that for F-theory on a Fermat type K3 surface times a K3 surface, considering monodromy uniquely determines the gauge groups which arise on 7-branes. We also check if the results we obtained above meet the consistency conditions from anomaly cancellation.

We first see that consideration on the j-invariant greatly constrains the possible types of singular fibers for a Fermat type K3 surface. Every smooth fiber of a Fermat type K3 surface over the base is the Fermat curve. The j-invariant of the Fermat curve is known to be 0. Therefore, a fiber of a Fermat type K3 surface has j-invariant 0 throughout the base. This forces j-invariants of singular fibers of a Fermat type K3 surface to be 0.

The j-invariant of each fiber type was computed by Kodaira in [4, 5]. We summarise his results in table 2. "Finite" for the j-invariant of fiber type I_0^* in the table means that the j-invariant takes a finite value for fiber type I_0^* (against the infinity). The value of j-invariant of fiber type I_0^* depends on the situations. So fiber type I_0^* may have the j-invariant 0. Only singular fibers of type II, II^*, IV and IV^* have j-invariant 0. Fiber type I_0^* can have j-invariant 0. Therefore, we deduce that these 5 are the only possible types of singular fibers for a Fermat type K3 surface. Equivalently, possible gauge groups on 7-branes for F-theory on a Fermat type K3 surface times a K3 surface are: $\mathrm{SU}(3)$, $\mathrm{SO}(8)$, E_6 and E_8. (Type II fiber does not generate a non-Abelian gauge group on 7-branes.) We saw the above that the gauge groups which arise on 7-branes for this model are $\mathrm{SU}(3)$ and E_6. So we confirm that our results obtained above agree with the requirement from a consideration on j-invariants.

Since j-invariant of elliptic curve given by the equation of Hesse form varies over the base $\mathbb{P}^1$, we cannot apply the above argument to F-theory on a K3 surface of Hesse form times a K3 surface.

fiber type	j-invariant	Monodromy	order of Monodromy	# of 7-branes (Euler number)
I_0^*	finite	$-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	6
I_b	∞	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	infinite	b
I_b^*	∞	$-\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	infinite	$b+6$
II	0	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	6	2
II^*	0	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	6	10
III	1728	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	4	3
III^*	1728	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	4	9
IV	0	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	3	4
IV^*	0	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	3	8

Table 2. Fiber types and their j-invariants, monodromies, and associated numbers of 7-branes.

$\mathrm{SL}_2(\mathbb{Z})$ has non-trivial finite subgroups of orders only 2, 3, 4 and 6. They are all cyclic: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$. It follows that there are only finite order elements of $\mathrm{SL}_2(\mathbb{Z})$ with orders 1,2,3,4 and 6. Singular fibers of fiber type I_0^*, II, III, IV, II^*, III^* and IV^* have finite order monodromies, represented by these elements. For monodromies of fiber types and their orders, see table 2. Results which appear in table 2 were derived by Kodaira [4, 5].[3] Singular fibers of type II and II^* have monodromies of finite order 6, while the fibers of type IV and IV^* have monodromies of order 3. Therefore, from a consideration on j-invariant we learn that possible monodromies for singular fibers of a Fermat type K3 surface have order 3 or 6. Combined with consistency conditions from anomalies, below we will see that only the monodromies of order 3 occur for singular fibers of a Fermat type K3 surface.

Now we consider the consistency conditions from anomalies on our models. We will see that the above consideration on j-invariants together with consistency conditions from anomalies precisely determines the gauge groups on 7-branes for F-theory on a Fermat type K3 surface times a K3 surface.

F-theory compactification on K3 $\times$ K3 gives 4d theory with $N = 2$ supersymmetry. The formula for the obstruction to cancelling the tadpole anomaly by turning on branes, for type IIA, M-theory and F-theory on a Calabi-Yau 4-fold, was derived in [35]. This

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[3]Kodaira [4, 5] computed Euler numbers of singular fibers. Euler number of a singular fiber can be interpreted as the number of 7-branes associated to the fiber type of a singular fiber.

anomaly cancellation is required to preserve supersymmetry of the theory. In particular, for F-theory on K3 × K3, the tadpole can be cancelled by turning on N 3-branes, where

$$N = \chi(K3 \times K3)/24 = 24. \tag{3.10}$$

(They also derived some topological constraints on the formula for an elliptic Calabi-Yau 4-fold, with assumptions that an elliptic Calabi-Yau 4-fold has a section and admits a smooth Weierstrass model. So the topological constraints they derived do not apply to our model.)

When 24 3-branes are turned on, F-theory has 3-brane charge 24. For a product K3 × K3 (with one of the K3's being elliptically fibered), the base is $\mathbb{P}^1 \times$ K3. Codimension-1 locus of $\mathbb{P}^1$ is set of discrete points, so each 7-brane is wrapped on {pt}× K3, i.e. a 7-brane is wrapped on a K3 surface. Each 7-brane wrapped on a K3 surface produces 3-brane charge −1. Since 3-brane charge from 7-branes wrapped on K3 surfaces cancel the charges from 3-branes and the net 3-brane charge is 0, we learn that there are 24 7-branes in total. Therefore, we see that the tadpole anomaly cancellation condition determines the form of the discriminant locus to be {24 points (counted with multiplicity)} × K3. Here the counting of points is done with weight of multiplicity assigned, so the actual number of points can be less than 24.

Now we combine the requirement from the consideration on j-invariants above with the consistency condition from anomalies to deduce gauge groups on 7-branes for F-theory on a Fermat type K3 surface times a K3 surface. We find that with these constraints, the gauge groups are precisely determined.

From the form of the equation (3.1), we see that a fiber degenerates precisely when $t = \alpha_i$, $i = 1, \cdots, 6$. So we expect that the six points $\{\alpha_i\}_{i=1}^{6}$ on $\mathbb{P}^1$ give all the locations of singular fibers. When α_i's are mutually distinct, there are 6 singular fibers. We have 24 7-branes in total, and by a symmetry argument there should be 4 7-branes at each singular fiber. From the consideration on j-invariant we learned that the possible types of singular fibers are I_0^*, II, II^*, IV and IV^*. From table 2, we see that only the fiber type IV corresponds to 4 7-branes. Therefore we can conclude that when α_i's are mutually distinct, singular fiber at each $t = \alpha_i$ has type IV, in agreement with what we obtained above.

To determine the fiber type at $t = \alpha_j$ when α_j and α_k coincide for some $k \neq j$, we consider the special case of the equation (3.1):

$$(t - \alpha_1)^2 X^3 + (t - \alpha_3)^2 Y^3 + (t - \alpha_5)^2 Z^3 = 0, \tag{3.11}$$

so we can apply a symmetry argument. From the form of the equation (3.11), we see that singular fibers are $t = \alpha_i$, $i = 1, 3, 5$, and they have the same fiber type. So there are 3 singular fibers of the same type, and there should be 8 7-branes at each fiber. As mentioned in the previous paragraph, since singular fibers have j-invariant 0, possible types of a singular fiber are type I_0^*, II, II^*, IV and IV^* for a Fermat type K3 surface. From table 2, we see that the only fiber type IV^* corresponds to 8 7-branes. Therefore we learn that when two of α_i's are coincident, say $\alpha_j = \alpha_k$, then singular fiber at $t = \alpha_j$ has type IV^*. So we recover the results that we obtained the above.

For F-theory compactification on a K3 in Hesse form $\times$ a K3, we see that the expression (3.8) for the discriminant Δ of a K3 in Hesse form has degree 24 in the variable t. This corresponds to the physical condition that there are in total 24 7-branes wrapped on the components of the discriminant locus. Therefore, we can conclude that the solutions for the gauge groups on 7-branes we obtained the above are in accord with the anomaly cancellation conditions for F-theory compactification on a K3 in Hesse form $\times$ a K3 as well.

For F-theory on K3 $\times$ K3, the above consistency conditions from anomaly cancellation corresponds to the mathematical fact that the sum of Euler numbers of singular fibers equals 24 for an elliptic K3 surface. Euler number of a singular fiber is equal to the number of 7-branes needed to generate a gauge group associated to the fiber type. This mathematical fact about elliptic K3 holds true, regardless of whether an elliptic K3 surface admits a section or not.

Fiber types IV and IV^* are the unique fiber types with monodromies of order 3. Fiber types II and II^* are the unique fiber types with monodromies of order 6. Consideration on j-invariants limited possible fiber types to the ones with monodromies of finite orders 3 and 6. The consideration on j-invariant, combined with tadpole anomaly cancellation condition precisely determined the types of singular fiber of a Fermat type K3 surface to be type IV and IV^*. Type IV and IV^* are the exact fiber types with monodromies of order 3. So the monodromies of order 3 characterise singular fibers for F-theory on a Fermat type K3 surface times a K3 surface. Therefore we see that for a Fermat type K3 surface, it is essential to consider a monodromy group in determining the gauge symmetries.

4 Matter fields on 7-branes

In this section, we investigate matter fields on 7-branes for F-theory compactified on K3 $\times$ K3, with one of the K3's chosen to be a Fermat type K3 surface or a K3 surface in Hesse form. With these choices, K3 $\times$ K3 does not have a section.

For F-theory on K3 $\times$ K3, all the 7-branes are parallel, so we do not have Yukawa-type interactions for our models. F-theory compactification on K3 $\times$ K3 gives a 4d theory with $N = 2$ supersymmetry in absence of flux. By including flux, half the supersymmetry is broken, and theory becomes 4d with $N = 1$ supersymmetry. Since all the 7-branes are parallel, without fluxes the only light matter for these models are the adjoint matters of gauge groups on 7-branes. When fluxes are turned on, vector-like pairs could also arise. Including flux, however, potential matter may vanish due to anomalies.

It turns out that, a Fermat type K3 surface with 3 singular fibers of type IV^* is an attractive[4] K3 (i.e. a K3 surface with highest possible Picard number $\rho = 20$), and whose complex structure can be precisely determined. Therefore, some detailed analysis on the matter spectrum with flux is possible for F-theory on K3 $\times$ K3, with one of the K3's being a Fermat type with $E_6 \times E_6 \times E_6$ gauge group on 7-branes. For other cases, we can only say that vector-like pairs *could* arise, but they may in fact vanish. The net chirality of light matter arising on 7-branes for F-theory compactification on K3 $\times$ K3 is 0 [37].

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[4]It is standard to call a K3 surface with the highest possible Picard number $\rho = 20$ a singular K3 surface in mathematics. We follow the convention of the term used in [36].

Now we see that for F-theory compactification on a Fermat type attractive K3 with gauge groups $E_6 \times E_6 \times E_6$ times a K3 (with K3 being appropriately chosen), tadpole anomaly cancels by turning on appropriate number of 3-branes. Consequently, we can calculate the full matter spectrum with flux for these cases.

Transcendental lattice T_S of a K3 surface S is the orthogonal complement of Néron-Severi lattice NS in K3 lattice $H^2(S, \mathbb{Z})$:

$$T_S := NS^\perp \subset H^2(S, \mathbb{Z}). \tag{4.1}$$

When a K3 surface S is attractive (i.e. when the Picard number $\rho(S) = 20$), its transcendental lattice T_S is a positive definite even lattice of rank 2.

Attractive K3 surfaces are classified by transcendental lattices, in the following sense. There is a bijective correspondence between complex structures of attractive K3 surfaces and the set of 2×2 even positive definite integral matrices modded out by the conjugacy action of $\mathrm{SL}_2(\mathbb{Z})$ [38]. Let $\mathcal{Q}$ denote the set of 2×2 even positive definite integral matrices. Then $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}$ by

$$Q \to g^t \cdot Q \cdot g, \tag{4.2}$$

where Q is in $\mathcal{Q}$ and g is in $\mathrm{SL}_2(\mathbb{Z})$. So Q and Q' are identified in $\mathcal{Q}/\mathrm{SL}_2(\mathbb{Z})$ exactly when there is some g in $\mathrm{SL}_2(\mathbb{Z})$ such that

$$Q' = g^t \cdot Q \cdot g. \tag{4.3}$$

Each attractive K3 surface S has the transcendental lattice T_S. Then with a basis $\{v_1, v_2\}$ of T_S, the transcendental lattice T_S has an associated Gram matrix Q_S,

$$Q_S := \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \tag{4.4}$$

for some integers a,b,c in $\mathbb{Z}$. Then we can define the map $S \to Q_S$, assigning the Gram matrix Q_S of the transcendental lattice to each attractive K3 S. Theorem 4 in [38] states that this map gives bijective correspondence between complex structures of attractive K3 surfaces and $\mathcal{Q}/\mathrm{SL}_2(\mathbb{Z})$. ($\mathrm{SL}_2(\mathbb{Z})$-action on $\mathcal{Q}$ corresponds to the change of basis of T_S.) In other words, the triplet of integers $[2a \quad b \quad 2c]$ (modded out by some $\mathrm{SL}_2(\mathbb{Z})$-action) parameterises the complex structure moduli of attractive K3 surfaces, and each fixed $[2a \quad b \quad 2c]$ can be identified with the transcendental lattice T_S of an attractive K3 S via

$$T_S = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}. \tag{4.5}$$

Since a triplet $[2a \quad b \quad 2c]$ parameterises the complex structure moduli of attractive K3 surfaces, we use the symbol $S_{[2a\,b\,2c]}$ to represent an attractive K3 surface S, whose transcendental lattice T_S has the Gram matrix $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$.

The transcendental lattice of an attractive Fermat type K3 surface with $E_6 \times E_6 \times E_6$ gauge groups (equivalently a Fermat type K3 surface with 3 singular fibers of type IV^*) has

the Gram matrix $\left(\begin{smallmatrix} 6 & 3 \\ 3 & 6 \end{smallmatrix}\right)$.[5] So, $S_{[6\ 3\ 6]}$ represents an attractive Fermat type K3 surface with $E_6 \times E_6 \times E_6$ gauge groups, with the above notational convention. From lattice theoretic argument, Néron-Severi lattice $NS(S_{[6\ 3\ 6]})$ of the attractive Fermat type K3 surface $S_{[6\ 3\ 6]}$ is determined to be

$$NS(S_{[6\ 3\ 6]}) = \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix} \oplus W, \tag{4.6}$$

where W represents a (negative definite) overlattice of $E_6 \oplus E_6 \oplus E_6$ with discriminant 3.[6] From the expression of Néron-Severi lattice (4.6), we can explicitly see that additional divisors W increase only in the fiber direction (i.e. they are orthogonal to the lattice $\left(\begin{smallmatrix} 0 & 3 \\ 3 & 2 \end{smallmatrix}\right)$, which is spanned by fiber class and 3-section), and the attractive Fermat type K3 surface $S_{[6\ 3\ 6]}$ with fibration specified by gauge groups $E_6 \times E_6 \times E_6$ does not have a section.

We will see below that, the full matter spectrum with flux can be calculated for F-theory compactification on $S_{[6\ 3\ 6]} \times$ K3, with some appropriate choices of K3 surfaces.

Flux compactification of M-theory on a product of attractive K3's $S_1 \times S_2$ was considered in [39]. 4-form flux G is subject to a quantisation condition [40]

$$G \in H^4(S_1 \times S_2, \mathbb{Z}), \tag{4.7}$$

and has decomposition

$$G = G_0 + G_1, \tag{4.8}$$

where

$$G_0 \in H^{1,1}(S_1, \mathbb{R}) \otimes H^{1,1}(S_2, \mathbb{R}) \tag{4.9}$$

$$G_1 \in H^{2,0}(S_1, \mathbb{C}) \otimes H^{0,2}(S_2, \mathbb{C}) + \text{h.c.} \tag{4.10}$$

In the presence of 4-form flux G, the tadpole cancellation condition [35] becomes

$$\frac{1}{2} \int_{\text{K3} \times \text{K3}} G \wedge G + N_3 = \frac{1}{24} \chi(\text{K3} \times \text{K3}) = 24, \tag{4.11}$$

where N_3 is the number of 3-branes turned on. With assumptions that

$$G_0 = 0 \tag{4.12}$$

and

$$N_3 = 0, \tag{4.13}$$

[39] obtained all the pairs $S_{[2a\ b\ 2c]} \times S_{[2d\ e\ 2f]}$ which satisfy the tadpole cancellation condition (4.11). Relaxing the condition (4.13) to

$$N_3 \geq 0, \tag{4.14}$$

the list of the pairs of attractive K3's was extended in [37].

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[5]We would like to thank Shigeru Mukai for pointing this out to us.

[6]We define ADE-lattice with negative sign in this paper.

The lists in [39] and [37] both contain only finitely many pairs of attractive K3's. So the complex structure moduli are the sets of finitely many discrete points. Therefore, the complex structure moduli in [37, 39] are stabilised. The attractive K3 surface $S_{[6\ 3\ 6]}$ does not appear in the list of [39], but it appears in the extended list of [37][7].

2 pairs $S_{[6\ 3\ 6]} \times S_{[4\ 2\ 4]}$ and $S_{[6\ 3\ 6]} \times S_{[2\ 1\ 2]}$ satisfy the tadpole cancellation condition (4.11), with $N_3 = 6$ and 15 respectively. So for F-theory compactifications on $S_{[6\ 3\ 6]} \times S_{[4\ 2\ 4]}$ and on $S_{[6\ 3\ 6]} \times S_{[2\ 1\ 2]}$, the tadpole anomalies are cancelled with appropriated number of 3-branes turned on. Therefore for these cases, we can say that the vector-like pairs *will* arise. Fermat type K3's with gauge groups $E_6 \times E_6 \times SU(3)^2$, $E_6 \times SU(3)^4$ and $SU(3)^6$ are not attractive. For F-theory compactifications on these non-attractive K3 × a K3 or a K3 surface of Hesse form × a K3, we can only say that vector-like pair could arise with flux.

As discussed in Katz and Vafa [27], singularity of a space (on which theory is compactified) encodes information on charged matter. As 7-branes move apart from one another, they generate matter fields on 7-branes. In geometrical language, deformation of singularity of ADE-type (ADE-type of a singularity corresponds to ADE-type of a singular fiber) describes this generation of matter. In subsections below, we describe light matter fields arising on 7-branes for F-theory compactified on a Fermat type K3 × a K3, and on a K3 surface of Hesse form × a K3, respectively.

4.1 Matter fields for Fermat type K3 surfaces (times K3)

Recall we saw in section 3 that singular fibers of a Fermat type K3 surface have only types IV or IV^*. (We classified all the 4 cases of configuration of singular fibers and gauge groups for F-theory compactifications on a Fermat type K3 × a K3.) Therefore, the corresponding surface singularities that a Fermat type K3 surface develops have types A_2 and E_6.

F-theory compactification on K3 × K3 gives 4d theory with $N = 2$ supersymmetry. All the 7-branes are parallel for F-theory on K3 × K3, so in absence of flux the only light matter are the adjoints of gauge groups on 7-branes. When fluxes are turned on, half the supersymmetry is broken, left with $N = 1$ supersymmetry. Vector-like pairs from hypermultiplets could also arise with flux, but the vector-like pairs may vanish due to anomalies. As argued above, for F-theory compactifications on $S_{[6\ 3\ 6]} \times S_{[4\ 2\ 4]}$ and on $S_{[6\ 3\ 6]} \times S_{[2\ 1\ 2]}$, turning on appropriated number of 3-branes cancel the tadpole anomalies. For these models, vector-like pairs will arise by including flux.

We compute the matter fields which can arise from E_6-singularity first. The only enhancement which generates the adjoint matters and hypermultiplets (an $N = 2$ hypermultiplet is split into an $N = 1$ vector-like pair with flux) is

$$A_5 \subset E_6. \tag{4.15}$$

Under this enhancement, the adjoint of E_6 decomposes into irreducible representations of A_5 as

$$\mathbf{78} = \mathbf{35} + \mathbf{20} + \overline{\mathbf{20}} + 3 \times \mathbf{1}, \tag{4.16}$$

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[7][37] uses different notation for attractive K3's. They denote by [a b c] the subscript for the attractive K3 surface whose transcendental lattice has the Gram matrix $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$. [3 3 3] in the list of [37] represents the attractive K3 surface denoted by $S_{[6\ 3\ 6]}$ in this paper.

where **35** is the adjoint of SU(6). The pair $\mathbf{20} + \overline{\mathbf{20}}$ forms an $N = 2$ hypermultiplet. By including flux, the $N = 2$ hypermultiplets are split into $N = 1$ vector-like pairs $\mathbf{20} + \overline{\mathbf{20}}$. This vector-like pair $\mathbf{20} + \overline{\mathbf{20}}$ is only potential candidate for matter spectrum with flux, and the vector-like pairs may actually vanish due to anomalies.

We next consider matters arising from A_2-singularity. For this case, the enhancement is

$$A_1 \subset A_2, \tag{4.17}$$

and the adjoint of A_2 decomposes under this enhancement as

$$\mathbf{8} = \mathbf{3} + \mathbf{2} + \overline{\mathbf{2}} + \mathbf{1}, \tag{4.18}$$

where **8** and **3** are the adjoints of SU(3) and SU(2), respectively. The pair $\mathbf{2} + \overline{\mathbf{2}}$ forms an $N = 2$ hypermultiplet. By including flux, the $N = 2$ hypermultiplets are split into the vector-like pairs $\mathbf{2} + \overline{\mathbf{2}}$. Again in general, we can only say that the vector-like pairs $\mathbf{2} + \overline{\mathbf{2}}$ could arise from A_2 type singularity.

Summarising the above, for F-theory compactification on a Fermat type K3 × a K3, matter arising from an E_6-singularity are only the adjoints **35** of SU(6) without flux. The vector-like pairs $\mathbf{20} + \overline{\mathbf{20}}$ could also arises on 7-branes when flux is turned on. Matter fields arising from an A_2-singularity are only the adjoints **3** of SU(3) without flux, and the vector-like pairs $\mathbf{2} + \overline{\mathbf{2}}$ could also arise on 7-branes by including flux.

For F-theory compactifications on $S_{[6\ 3\ 6]} \times S_{[4\ 2\ 4]}$ and on $S_{[6\ 3\ 6]} \times S_{[2\ 1\ 2]}$, the tadpole anomaly is cancelled by including sufficiently many 3-branes. Therefore, for these 2 models the vector-like pairs $\mathbf{20} + \overline{\mathbf{20}}$ *will* arise by including flux. Recall that $S_{[6\ 3\ 6]}$ represents a Fermat type K3 surface with $E_6 \times E_6 \times E_6$ gauge groups. This attractive K3 surface has singular fibers only of type IV^*, so all the singularities on $S_{[6\ 3\ 6]}$ have type E_6. It follows that only the pairs $\mathbf{20} + \overline{\mathbf{20}}$ arise from the singularities as vector-like pairs. So, the full matter spectrum with flux for F-theory compactifications on $S_{[6\ 3\ 6]} \times S_{[4\ 2\ 4]}$ and on $S_{[6\ 3\ 6]} \times S_{[2\ 1\ 2]}$ are the adjoints **35** of SU(6) and the vector-like pairs $\mathbf{20} + \overline{\mathbf{20}}$.

4.2 Matter fields for K3 surfaces of Hesse form (times K3)

As we saw in section 3, singular fibers of a K3 surface of Hesse form are generically of type I_3. For some special members among the whole family of K3 surfaces of Hesse form, two I_3 fibers collide and enhance to I_6 fiber. The singularity types corresponding to fiber types I_3 and I_6 are A_2 and A_5, respectively.

We saw the matter fields which can arise from A_2-singularity just the above: the adjoint **3** of SU(3) and the vector-like pair $\mathbf{2} + \overline{\mathbf{2}}$. (In general, we can only say that the vector-like pair $\mathbf{2} + \overline{\mathbf{2}}$ could arise by including flux.) It remains to compute the matter fields which can arise from A_5-singularity. The enhancement is

$$A_4 \subset A_5. \tag{4.19}$$

The adjoint of A_5 decomposes into irreducible representations of A_4 as

$$\mathbf{35} = \mathbf{24} + \mathbf{5} + \overline{\mathbf{5}} + \mathbf{1}, \tag{4.20}$$

where **35** and **24** are the adjoints of SU(6) and SU(5) respectively. The pair $\mathbf{5} + \overline{\mathbf{5}}$ forms an $N = 2$ hypermultiplet. By including flux, the hypermultiplets are split into $N = 1$ vector-like pairs $\mathbf{5} + \overline{\mathbf{5}}$.

Summarising the above, for F-theory compactification on a *generic* K3 surface of Hesse form × a K3, the only matter on 7-branes are the adjoints **3** of SU(3) without flux. The vector-like pairs $\mathbf{2} + \overline{\mathbf{2}}$ could also arise by including flux. For F-theory on some special K3 surface of Hesse form (with a singular fiber of enhanced type I_6) × a K3, the adjoints **24** of SU(5) also arise on 7-branes without flux. The vector-like pairs $\mathbf{5} + \overline{\mathbf{5}}$ could also arise with flux turned on.

5 Conclusion

We considered F-theory compactified on a product K3 × K3, with one of the K3's chosen to be of Fermat type or in Hesse form, as introduced in section 2. K3 surfaces of Fermat type and K3 surfaces in Hesse form do not have a section, so such constructed K3 × K3 does not admit a section to the fibration, as an elliptic Calabi-Yau 4-fold. Therefore, F-theory compactifications on these constructed K3 × K3 provide models without a section.

We determined gauge groups and matter fields which arise on 7-branes for these F-theory models without a section. For F-theory compactifications on both a Fermat type K3 × a K3 and a K3 surface in Hesse form × a K3, gauge groups arising on 7-branes are generically SU(3)$^{\oplus 6}$. When two singular fibers collide, SU(3) $\oplus$ SU(3) enhances to an exceptional gauge group E_6 for F-theory on a Fermat type K3 × a K3, and enhances to SU(6) for F-theory on a K3 surface in Hesse form × a K3. So exceptional gauge group E_6 appears for some of F-theory compactifications on a Fermat type K3 × a K3. We completely classified the configurations of gauge groups arising on 7-branes, for F-theory compactifications on a Fermat type K3 × a K3. There are 4 cases in total. K3 × K3 we consider in this paper does not have a section. So, the gauge groups arising on 7-branes do not have U(1)-part.

We saw that the gauge groups arising on 7-branes we computed are in agreement with the tadpole cancellation condition. The tadpole cancellation requires that there be 24 7-branes in total, and we confirmed that our solutions meet this consistency condition. We also saw that consideration on monodromy uniquely determines the gauge symmetries for F-theory compactifications on a Fermat type K3 × a K3.

Since all the 7-branes are parallel for F-theory compactified on K3 × K3, Yukawa-type interaction is absent for our F-theory models on K3 × K3. There are no matter curves. F-theory on K3 × K3 gives 4d theory with $N = 2$ supersymmetry. By including flux, half the supersymmetry is broken, left with $N = 1$. Only light matter are adjoint matters of gauge groups on 7-branes without flux. When flux is turned on, vector-like pairs could also arise on 7-branes. In general, we can only say that the vector-like pairs could arise, because potential candidate for matter spectrum may actually vanish due to anomalies. For some special cases of F-theory compactifications on a Fermat type K3 × a K3, we saw that the tadpole anomaly can be cancelled. For these special cases, vector-like pairs actually arise.

Acknowledgments

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