Gauge Invariant Effective Action in Abelian Chiral Gauge Theory on the Lattice

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Lüscher's recent formulation of Abelian chiral gauge theories on the lattice, in the vacuum (or perturbative) sector in infinite volume, is reinterpreted in terms of the lattice covariant regularization. The gauge invariance of the effective action and the integrability of the gauge current in anomaly-free cases become transparent. The real part of the effective action is simply one-half that of the Dirac fermion and, when the Dirac operator behaves properly in the continuum limit, the imaginary part in this limit reproduces the η-invariant.

We have gained a new perspective on chiral symmetries on the lattice, after the recent discovery [1]‑[3] of gauge covariant Dirac operators which satisfy the Ginsparg-Wilson relation. [4] When the Dirac operator is applied to lattice QCD, in which the chiral symmetries are global, the action can be made invariant under all the flavor $U(N_f)$ axial symmetries. Quantum mechanically, on the other hand, the flavor-singlet $U(1)$ symmetry suffers from the axial anomaly, due to the non-trivial Jacobian factor. [5],[6] The desired breaking pattern of the Ward-Takahashi identities in vectorial gauge theories is thus restored. In this context, one can even formulate the (analytic) index theorem with finite lattice spacing. [2],[5]

Quite recently, Lüscher [7] formulated (Abelian) chiral gauge theories on the lattice, on the basis of the chirality separation with respect to the Ginsparg-Wilson chiral matrix. [8]‑[10] He proved there exists a gauge invariant effective action of Weyl fermions, when (and only when) the anomaly cancellation condition *in the continuum field theory* is fulfilled. (Neuberger [11] made a similar observation in the overlap formalism [12],[13] for a particular kind of gauge field configuration.) The crucial ingredient in Lüscher's proof is the complete clarification of the structure of the axial anomaly with finite lattice spacing. [14]

In the present note, we give a reinterpretation of Lüscher's formulation in Ref. 7) in terms of the "lattice covariant regularization" proposed in the past by the present author and a collaborator. [15],[16] This reinterpretation is possible at least in the vacuum sector (implying that the Dirac operator has no zero modes and the inverse of the Dirac operator exists) in infinite lattice volume (for which the results of Ref. 14) can be used straightforwardly). Although the most interesting part of Ref. 7) is the analysis of the topological sector in a finite lattice volume, we will clarify some of properties of the formulation in this simpler situation, by giving a one parameter integral representation of Lüscher's effective action.

Now, in the scheme of Refs. 15) and 16), the primarily-defined quantity is the variation of the effective action with respect to the gauge field, "the gauge current".

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The effective action is regarded as the secondary object, which is deduced from the gauge current. The basic idea is that using the *vectorial* gauge covariant Dirac propagator (without species doublers) with an appropriate chirality projection, one may always preserve the gauge covariance of the gauge current of the Weyl fermion. In this way, the gauge symmetry is maximally preserved even in anomalous cases, and the gauge invariance is automatically restored (in the continuum limit) when the gauge representation is anomaly-free.

According to this scheme, we temporarily identify a variation of the effective action W with (see, Eq. (9) of Ref. 16))

$$\delta W \sim \operatorname{Tr} \delta D \mathcal{P}_{\mathrm{H}} D^{-1}. \tag{1}$$

We use a gauge covariant Dirac operator D that is assumed to satisfy the Ginsparg-Wilson relation,[4]

$$\gamma_5 D + D \gamma_5 = a D \gamma_5 D, \tag{2}$$

with the lattice spacing a. We will also assume the Hermitian conjugate of the Dirac operator satisfies $D^\dagger = \gamma_5 D \gamma_5$. In Eq. (1), H $= \pm$ denotes the chirality of the Weyl fermion, and the projection operator in Eq. (1) is defined with respect to the modified chiral matrix,[8]-[10]

$$\Gamma_5 = \gamma_5(1 - aD) = \gamma_5 - a\mathcal{D}, \qquad \mathcal{P}_\pm = \frac{1}{2}(1 \pm \Gamma_5). \tag{3}$$

Here $\mathcal{P}_\pm$ is in fact the projection operator, because $\Gamma_5^2 = 1$ due to the relation (2). (Γ_5 is Hermitian: $\Gamma_5^\dagger = \Gamma_5$.) For convenience, we introduce the Hermitian operator $\mathcal{D} = \gamma_5 D$ and $\mathcal{D}^{-1} = D^{-1}\gamma_5$. The following relations, which are the consequence of Eq. (2), are also useful:

$$\Gamma_5 \mathcal{D}^{-1} = -\mathcal{D}^{-1}\Gamma_5 - a = -\mathcal{D}^{-1}\gamma_5, \qquad \{\Gamma_5, \delta\mathcal{D}\} = 0. \tag{4}$$

We restrict ourselves to the case of the Abelian gauge group. In this case, and if the lattice volume is infinite, one may always associate[14] the gauge potential A_μ with the link variable through

$$U_\mu(x, t) = \exp[iatTA_\mu(x)], \tag{5}$$

where T denotes the generator of the Abelian gauge group, $T_{\alpha\beta} = e_\alpha \delta_{\alpha,\beta}$. Here e_α is the $U(1)$ charge of the flavor α. In Eq. (5), we have introduced the "gauge coupling parameter" t. We denote the dependence on t in five-dimensional notation as $A_\mu(x, t) = t A_\mu(x)$ (where the original link variable and the gauge potential are given by the value at $t = 1$), but the argument t will often be omitted when there is no danger of confusion.

In the Abelian case, we may differentiate the effective action with respect to the parameter t and then integrate it over this parameter. The identification (1) then motivates the following definition of the effective action W':

$$\begin{aligned}
W' &= \int_0^1 dt\, \partial_t W' \\
&\equiv \int_0^1 dt\, \operatorname{Tr} \partial_t D \mathcal{P}_{\mathrm{H}} D^{-1} = \int_0^1 dt\, \operatorname{Tr} \partial_t \mathcal{D} \mathcal{P}_{\mathrm{H}} \mathcal{D}^{-1}.
\end{aligned} \tag{6}$$

In the continuum field theory, the corresponding definition of the effective action of the Weyl fermion is known to have interesting properties. [17]

The property of the would-be effective action (6) under the gauge transformation is the central point of our discussion. (The following analysis is analogous to that in Ref. 16) with the Wilson propagator.) We first split the functional (6) into real and imaginary parts. By noting that the operators $\partial_t \mathcal{D}$, $\mathcal{P}_{\mathrm{H}}$ and $\mathcal{D}^{-1}$ are all Hermitian, we find from Eq. (4),

$$W'^* = \int_0^1 dt \, \mathrm{Tr}\, \partial_t \mathcal{D} \widetilde{\mathcal{P}}_{\mathrm{H}} \mathcal{D}^{-1} = \int_0^1 dt \, \mathrm{Tr}\, \partial_t D \widetilde{\mathcal{P}}_{\mathrm{H}} D^{-1}, \tag{7}$$

with $\widetilde{\mathcal{P}}_{\pm} = \mathcal{P}_{\mp}$. The real part is therefore given by

$$\mathrm{Re}\, W' = \frac{1}{2} \int_0^1 dt \, \mathrm{Tr}\, \partial_t D D^{-1} = \frac{1}{2} \ln \mathrm{Det}\, D. \tag{8}$$

This is simply one-half the effective action of the Dirac fermion in vectorial gauge theories. Equation (8) is manifestly gauge invariant, because the Dirac operator is gauge covariant.

The imaginary part, on the other hand, is given by

$$i\, \mathrm{Im}\, W' = \frac{\epsilon_{\mathrm{H}}}{2} \int_0^1 dt \, \mathrm{Tr}\, \partial_t \mathcal{D} \varGamma_5 \mathcal{D}^{-1} = -\frac{\epsilon_{\mathrm{H}}}{2} \int_0^1 dt \, \mathrm{Tr}\, \gamma_5 \partial_t \mathcal{D} \mathcal{D}^{-1}, \tag{9}$$

where $\epsilon_{\pm} = \pm 1$. Let us quickly verify that the imaginary part vanishes identically when the representation is "vector-like", i.e., when there exists a unitary matrix u such that[*] $uTu^{\dagger} = -T$. We assume the Dirac operator D transforms in the same way as the conventional lattice covariant derivative under charge conjugation. [7] This implies, in the functional notation, $CuDu^{\dagger}C^{-1} = D^T$, with the charge conjugation matrix C, $C\gamma^{\mu}C^{-1} = -\gamma^{\mu T}$ and $C\gamma_5 C^{-1} = \gamma_5^T$. Consequently, we have $Cu\partial_t \mathcal{D} \varGamma_5 \mathcal{D}^{-1} u^{\dagger} C^{-1} = \gamma_5^T \partial_t \mathcal{D}^T \varGamma_5^T \mathcal{D}^{-1T} \gamma_5^T = (\gamma_5 \mathcal{D}^{-1} \varGamma_5 \partial_t \mathcal{D} \gamma_5)^T$ and

$$\begin{aligned}
\mathrm{Tr}\, \partial_t \mathcal{D} \varGamma_5 \mathcal{D}^{-1} &= \mathrm{Tr}\, Cu\partial_t \mathcal{D} \varGamma_5 \mathcal{D}^{-1} u^{\dagger} C^{-1} \\
&= \mathrm{Tr}\, \mathcal{D}^{-1} \varGamma_5 \partial_t \mathcal{D} \\
&= -\, \mathrm{Tr}\, \partial_t \mathcal{D} \varGamma_5 \mathcal{D}^{-1} = 0. \tag{10}
\end{aligned}$$

It is thus seen that $\mathrm{Im}\, W' = 0$ for vector-like cases. This is certainly the desired property, because in this case it is possible to arrange the fermions so that the theory is left-right symmetric.

In general, the imaginary part (9) neither vanishes nor is gauge invariant. Its variation under the gauge transformation can be determined by noting that the infinitesimal gauge transformation on a gauge covariant object is represented by the commutator as $\delta_{\lambda} \mathcal{D} = -it[\lambda T, \mathcal{D}]$. We have

$$\begin{aligned}
i\delta_{\lambda} \mathrm{Im}\, W' &= -i\frac{\epsilon_{\mathrm{H}}}{2} \int_0^1 dt \, \mathrm{Tr}\Big\{ \partial_t(t[\lambda T, \mathcal{D}]) \varGamma_5 \mathcal{D}^{-1} + t\partial_t \mathcal{D}[\lambda T, \varGamma_5 \mathcal{D}^{-1}] \Big\} \\
&= i\epsilon_{\mathrm{H}} \int_0^1 dt \, \mathrm{Tr}\, \lambda T \gamma_5 \Big(1 - \frac{1}{2} aD \Big) \equiv i\epsilon_{\mathrm{H}} \int_0^1 dt\, a^4 \sum_x \lambda(x) \mathcal{A}(x, t). \tag{11}
\end{aligned}$$

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[*] For example, when the $U(1)$ charges come in pairs of opposite sign, the unitary matrix $u_{\alpha\beta} = \delta_{\alpha,\beta-1} - \delta_{\alpha,\beta+1}$ is sufficient.

From the first line to the second line, use of relation (4) has been made. In the last expression, we have defined the covariant gauge anomaly as

$$
\begin{aligned}
\mathcal{A}(x) &= \operatorname{tr} T \gamma_5 \Big[1 - \frac{1}{2} a D(x) \Big] \delta(x, x) \\
&\stackrel{a \to 0}{\to} \frac{1}{32\pi^2} \sum_\alpha e_\alpha^3 \, \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(x),
\end{aligned}
\tag{12}
$$

where the field strength has been defined by $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$. The anomaly in the continuum limit was perturbatively computed[6] by using the overlap-Dirac operator introduced in Ref. 3) (see also Refs. 18) and 19)) and we have used the result in Eq. (12). One can even prove[20] that the expression in Eq. (12) is insensitive to the choice of the Dirac operator, as long as this operator behaves properly in the continuum limit. Substituting the covariant anomaly (12) into Eq. (11), we have a consistent (Abelian) gauge anomaly (because $\int_0^1 dt \, t^2 = 1/3$), as should be the case. In continuum field theory, the prescription (6) provides a general recipe to produce the consistent anomaly from the covariant anomaly.[17]

Not only about its non-invariance under the gauge transformation, we can see more directly the validity of the prescription (9) in the continuum limit. The argument proceeds as follows:

$$
\begin{aligned}
i \operatorname{Im} W' &= -\frac{\epsilon_{\mathrm{H}}}{2} \int_0^1 dt \, \operatorname{Tr} \gamma_5 \partial_t D D^{-1} \\
&= -\lim_{\Lambda \to \infty} \frac{\epsilon_{\mathrm{H}}}{2} \int_0^1 dt \, \operatorname{Tr} \gamma_5 \partial_t D D^{-1} f(\mathcal{D}^2 / \Lambda^2) \\
&\stackrel{a \to 0}{\to} -\lim_{\Lambda \to \infty} \frac{\epsilon_{\mathrm{H}}}{2} \int_0^1 dt \, \operatorname{Tr} \gamma_5 \partial_t \slashed{D} \slashed{D}^{-1} f(\slashed{D}^2 / \Lambda^2) \\
&= -i \epsilon_{\mathrm{H}} \pi \eta(0) + \frac{i \epsilon_{\mathrm{H}}}{3 \cdot 8\pi^2} \sum_\alpha e_\alpha^3 \int_{M^4 \times R} A \, dA \, dA.
\end{aligned}
\tag{13}
$$

In the first step, we have introduced the regulator $f(x)$, which rapidly goes to zero as x increases and satisfies $f(0) = 1$. In the second step, we have exchanged the two limits $a \to 0$ and $\Lambda \to \infty$ by assuming the lattice integrals without the regulator $f(x)$ are finite in the $a \to 0$ limit. Since the corresponding expression in the continuum field theory is UV finite, once the gauge covariance is imposed, this is a reasonable assumption. Next, we have assumed the Dirac operator D is free of species doubling and thus that it diverges rapidly $\sim 1/a$ in the momentum region corresponding to species doublers. These momentum regions do not contribute in the $a \to 0$ limit, because of the existence of $f(x)$: $f(1/a^2 \Lambda^2) \stackrel{a \to 0}{\to} 0$. In the physical momentum region, we have assumed

$$
D(x) \stackrel{a \to 0}{\to} i c \slashed{D}(x), \qquad c: \text{real constant,}
\tag{14}
$$

where $\slashed{D}(x)$ is the covariant derivative in continuum field theory, $\slashed{D}(x) = \gamma^\mu (\partial_\mu + i T A_\mu)$. (Equation (14) is consistent with the assumed Hermiticity.) Our manipulation is quite similar to that in the general analysis of the continuum limit of the chiral Jacobian.[20] A detailed account of its justification can be found in Ref. 20).

In the final step of Eq. (13), we have appealed to a well-known result in continuum field theory: [21] - [23] The imaginary part of the effective action is given by the Atiyah-Patodi-Singer η-invariant (the spectral asymmetry) $\eta(0)$ plus the Chern-Simons 5-form. (See, for example, the first reference in Ref. 23) for a heuristic proof with $f(x) = (1 + x)^{-1/2}$. Note that the imaginary part is independent of the choice of $f(x)$.) Equation (13) in fact reproduces the consistent gauge anomaly in the continuum limit, Eq. (11) with Eq. (12).[*]

We have seen that the prescription (6) is satisfactory in the sense that the real part is always gauge invariant and, in the continuum limit, the imaginary part reproduces the correct result of the continuum field theory. In particular, the gauge invariance of the imaginary part is automatically restored in the continuum limit when the anomaly is canceled, i.e., when $\sum_\alpha e_\alpha^3 = 0$. However, at this stage, we cannot say anything regarding the gauge invariant property of the imaginary part (9) with *finite* lattice spacing: This was the main reason that the formulation of Ref. 16) could not be pursued.

Now, Lüscher has given a remarkable proof [14] that the gauge anomaly (12) with *finite* lattice spacing has the following structure:[**]

$$\mathcal{A}(x) = \frac{1}{32\pi^2} \sum_\alpha e_\alpha^3 \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x + a_\mu + a_\nu) + \partial_\mu^* \overline{k}_\mu(x). \tag{15}$$

(We have used the coefficient in Eq. (12).) In this expression, ∂_μ^* is the backward difference operator (see Ref. 14)) and $\overline{k}_\mu(x)$ is a *gauge invariant* current that depends locally on the gauge potential. The proof in Ref. 14) also gives an explicit method to construct $\overline{k}_\mu(x)$. Although the current $\overline{k}_\mu(x)$ is not unique, we can fix (partially) its form by requiring it to transform like the axial vector current under the lattice transformation. [14],[7]

Once having observed Eq. (15), we may improve the effective action (6) as $W = W' + \overline{K}$ where

$$\overline{K} = i\epsilon_H \int_0^1 dt \, a^4 \sum_x A_\mu(x) \overline{k}_\mu(x, t). \tag{16}$$

Since the gauge transformation of the gauge potential is given by $\delta_\lambda A_\mu(x) = \partial_\mu \lambda(x)$, the gauge variation of Eq. (16) is given by

$$\delta_\lambda \overline{K} = -i\epsilon_H \int_0^1 dt \, a^4 \sum_x \lambda(x) \partial_\mu^* \overline{k}_\mu(x, t), \tag{17}$$

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[*] The η-invariant is defined by $\eta(0) \equiv \sum_n \text{sign}\,\lambda_n$ from the eigenvalue of the five-dimensional Hermitian operator $H = i\gamma_5 \partial/\partial x^5 + \not{D}$. The boundary condition for the gauge field is specified as $A_\mu(x, x^5 = \infty) = A_\mu(x)$ and $A_\mu(x, x^5 = -\infty) = 0$, and $A_5(x, x^5) = 0$. The four-dimensional gauge transformation at the plane $x^5 = \infty$ can be regarded as a five-dimensional gauge transformation that is independent of x^5. Under such a gauge transformation, the eigenvalue λ_n is clearly gauge invariant, and thus is the η-invariant. On the other hand, the Chern-Simons 5-form in Eq. (13) is not invariant, and it reproduces the consistent gauge anomaly (11) with Eq. (12) on the boundary $x^5 = \infty$. The gauge-invariant information of the imaginary part is therefore carried by the η-invariant. [23]

[**] The author is grateful to Professor T. Inami for an informative discussion and for pointing out the importance of this proof in our approach, prior to the publication of Ref. 7).

because $\overline{k}_\mu$ is gauge invariant, i.e., $\delta_\lambda \overline{k}_\mu = 0$. Since $\partial_\mu^* \overline{k}_\mu(x) = \mathcal{A}(x)$ from Eq. (15) for anomaly-free cases, the gauge variation of $\overline{K}$ (17) in fact cancels the gauge variation of W', Eq. (11). Namely, when the anomaly cancellation condition *in the continuum field theory* is fulfilled, the improved effective action $W = W' + \overline{K}$ is gauge invariant even with *finite* lattice spacing. The improvement term $\overline{K}$ does not spoil the desired properties of W', because $\overline{K}$ contributes only the imaginary part of W ($\overline{k}_\mu(x)$ transforms like the axial current), and because the current $\overline{k}_\mu(x)$ is higher order in a, i.e., $\overline{K} \overset{a \to 0}{\to} 0$.

In the remainder of this paper, we show that the above expression of the improved effective action $W = W' + \overline{K}$ corresponds to the formulation of Ref. 7), in the vacuum sector for an infinite lattice volume. To see this, we consider a variation of the gauge potential $A_\mu(x)$, $\delta_\eta A_\mu(x) = \eta_\mu(x)$ and $\delta_\eta A_\mu(x,t) = t\eta_\mu(x)$. The variation of the functional (6) is then given by

$$\delta_\eta W' = \int_0^1 dt \, \text{Tr}\left(\partial_t \delta_\eta \mathcal{D} \mathcal{P}_\text{H} \mathcal{D}^{-1} - \frac{\epsilon_\text{H}}{2} a \partial_t \mathcal{D} \delta_\eta \mathcal{D} \mathcal{D}^{-1} - \partial_t \mathcal{D} \mathcal{P}_\text{H} \mathcal{D}^{-1} \delta_\eta \mathcal{D} \mathcal{D}^{-1} \right). \quad (18)$$

The relation (4) follows $\mathcal{P}_\text{H} \mathcal{D}^{-1} \delta_\eta \mathcal{D} = \mathcal{D}^{-1} \delta_\eta \mathcal{D} \mathcal{P}_\text{H} - \epsilon_\text{H} a \delta_\eta \mathcal{D}/2$. Therefore we have

$$\delta_\eta W' = \int_0^1 dt \, (\partial_t \, \text{Tr} \, \delta_\eta \mathcal{D} \mathcal{P}_\text{H} \mathcal{D}^{-1} - \text{Tr} \, \delta_\eta \mathcal{D} \partial_t \mathcal{P}_\text{H} \mathcal{D}^{-1})$$

$$= \text{Tr} \, \delta_\eta \mathcal{D} \mathcal{P}_\text{H} \mathcal{D}^{-1} + \epsilon_\text{H} \int_0^1 dt \, \frac{1}{2} a \, \text{Tr} \, \delta_\eta \mathcal{D} \partial_t \mathcal{D} \mathcal{D}^{-1}. \quad (19)$$

The second term here can be written from Eq. (4) as

$$\frac{1}{2} a \, \text{Tr} \, \delta_\eta \mathcal{D} \partial_t \mathcal{D} \mathcal{D}^{-1} = -\frac{1}{2} a \, \text{Tr} \, \delta_\eta \mathcal{D} \partial_t \mathcal{D} \Gamma_5 \mathcal{D}^{-1} \Gamma_5 - \frac{1}{2} a^2 \, \text{Tr} \, \delta_\eta \mathcal{D} \partial_t \mathcal{D} \Gamma_5$$

$$= -\frac{1}{4} a^2 \, \text{Tr} \, \Gamma_5 \delta_\eta \mathcal{D} \partial_t \mathcal{D}$$

$$= \epsilon_\text{H} \, \text{Tr} \, \mathcal{P}_\text{H} [\partial_t \mathcal{P}_\text{H}, \delta_\eta \mathcal{P}_\text{H}]. \quad (20)$$

Therefore the variation of W' is given by

$$\delta_\eta W' = \text{Tr} \, \delta_\eta \mathcal{D} \mathcal{P}_\text{H} \mathcal{D}^{-1} + \int_0^1 dt \, \text{Tr} \, \mathcal{P}_\text{H} [\partial_t \mathcal{P}_\text{H}, \delta_\eta \mathcal{P}_\text{H}]. \quad (21)$$

Combined with the variation of $\overline{K}$ (16), the variation of the total effective action may be expressed as

$$\delta_\eta W = \delta_\eta W' + \delta_\eta \overline{K}$$
$$\equiv \text{Tr} \, \delta_\eta D \mathcal{P}_\text{H} D^{-1} + i\epsilon_\text{H} \mathcal{L}_\eta^\star, \quad (22)$$

where

$$\mathcal{L}_\eta^\star \equiv a^4 \sum_x \eta_\mu(x) j_\mu^\star(x)$$

$$= -i\epsilon_\text{H} \int_0^1 dt \, \text{Tr} \, \mathcal{P}_\text{H} [\partial_t \mathcal{P}_\text{H}, \delta_\eta \mathcal{P}_\text{H}] + \int_0^1 dt \, a^4 \sum_x \Big[\eta_\mu(x) \overline{k}_\mu(x,t) + A_\mu(x) \delta_\eta \overline{k}_\mu(x,t) \Big]$$

$$= a^4 \sum_x \eta_\mu(x) \overline{k}_\mu(x) - i\epsilon_{\mathrm{H}} \int_0^1 dt \, \mathrm{Tr} \, \mathcal{P}_{\mathrm{H}}[\partial_t \mathcal{P}_{\mathrm{H}}, \delta_\eta \mathcal{P}_{\mathrm{H}}]$$

$$+ \int_0^1 dt \, a^4 \sum_x \Big[A_\mu(x) \delta_\eta \overline{k}_\mu(x,t) - \eta_\mu(x) t \partial_t \overline{k}_\mu(x,t) \Big]. \tag{23}$$

In deriving the last expression, we have performed a partial integration by inserting $1 = \partial t / \partial t$. It is also easy to see from Eq. (8) that $\mathcal{L}_\eta^\star$ arises entirely[*] from $\overline{K}$ and the imaginary part of W'.

Equation (23) is identically the linear functional $\mathcal{L}_\eta^\star$ in Eq. (5.8) of Ref. 7). When the lattice volume is infinite and the Dirac operator has no zero modes, the variation of the effective action in Ref. 7) is given by Eq. (22). (See Eq. (3.8) of Ref. 7).) Therefore, under the above conditions, the effective action formulated by Lüscher can be represented as $W = W' + \overline{K}$, i.e., Eq. (6) plus Eq. (16). The content of Theorem 5.3 of Ref. 7) also immediately follows in view of Eq. (22): (a) When the gauge group is Abelian, the variation δ_η and the gauge variation δ_λ commute if η does not depend on the gauge potential. From this, we see that $\delta_\eta W$ is gauge invariant because W is gauge invariant. The quantity $\mathrm{Tr} \, \delta_\eta D \mathcal{P}_{\mathrm{H}} D^{-1}$ is gauge covariant by construction. This is equivalent to the gauge invariance in the Abelian case. Therefore $\mathcal{L}_\eta^\star$ is gauge invariant. (b) $\mathcal{L}_\eta^\star$ arises from the imaginary part of W. Thus it is consistent to assume $j_\mu^\star(x)$ (and $\overline{k}_\mu(x)$) transforms like the axial vector current. (c) $(\delta_\eta \delta_\zeta - \delta_\zeta \delta_\eta) W = 0$ for the A_μ-independent parameters η and ζ. Using a calculation that is almost the same as that producing Eq. (20) from Eq. (18) immediately shows

$$\delta_\eta \, \mathrm{Tr} \, \delta_\zeta D \mathcal{P}_{\mathrm{H}} D^{-1} - \delta_\zeta \, \mathrm{Tr} \, \delta_\eta D \mathcal{P}_{\mathrm{H}} D^{-1} = - \, \mathrm{Tr} \, \mathcal{P}_{\mathrm{H}}[\delta_\eta \mathcal{P}_{\mathrm{H}}, \delta_\zeta \mathcal{P}_{\mathrm{H}}]. \tag{24}$$

Therefore $\mathcal{L}_\eta^\star$ satisfies the integrability condition Eq. (5.9) of Ref. 7). (d) The anomalous conservation law $\partial_\mu^* j_\mu^\star(x) = \mathcal{A}(x)$ holds (when $\sum_\alpha e_\alpha^3 = 0$) because W is gauge invariant, and the first term of Eq. (22) produces the gauge anomaly under the gauge variation $\delta_\lambda D = -i[\lambda T, D]$, as in Eq. (11).

The last line of Eq. (23) implies a difference between the "covariant gauge current", $\mathrm{Tr} \, \delta_\eta D \mathcal{P}_{\mathrm{H}} D^{-1} + i\epsilon_{\mathrm{H}} a^4 \sum_x \eta_\mu \overline{k}_\mu$, and the "consistent gauge current", $\mathrm{Tr} \, \delta_\eta D \mathcal{P}_{\mathrm{H}} D^{-1} + i\epsilon_{\mathrm{H}} a^4 \sum_x \eta_\mu j_\mu^\star$. This quantity is analogous to the quantity in the continuum field theory that relates the covariant anomaly and the consistent anomaly.[24]-[26],[17] Interestingly, the difference does not contribute to the integral (16), as can easily be verified.[17] Thus we may use $j_\mu^\star(x)$ instead of $\overline{k}_\mu(x)$ in Eq. (16). Of course, the structure of $\overline{k}_\mu(x)$ is simpler and the expression of $j_\mu^\star(x)$ (23) is not needed in our representation $W = W' + \overline{K}$. The existence of the "integrable current" $j_\mu^\star(x)$ is important in ensuring[7] the existence of the path integral expression that corresponds to the effective action W.

It is certainly desirable to perform the present analysis in finite volume.[**] The representation (9) might be useful in identifying a possible lattice counterpart of

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[*] For vector-like cases, the imaginary part of the effective action W' vanishes identically, and thus W' is gauge invariant. Therefore $\mathcal{A} = \overline{K} = 0$, and consequently $\mathcal{L}_\eta^\star = 0$ in these cases.[7]

[**] Some steps of our discussion may be formal due to possible IR divergences. This also prompts us to pursue an analysis on a finite volume lattice.

the η-invariant. Obtaining a lattice implementation of the t-integrals in Eqs. (6) and (16) is also an interesting problem. We postpone these for future projects.

Acknowledgements

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