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# Gauge-invariant magnetic perturbations in perfect-fluid cosmologies

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**Abstract.** We develop further our extension of the Ellis–Bruni covariant and gauge-invariant formalism to the general relativistic treatment of density perturbations in the presence of cosmological magnetic fields. We present a detailed analysis of the kinematical and dynamical behaviour of perturbed magnetized FRW cosmologies containing fluid with non-zero pressure. We study the magnetohydrodynamical effects on the growth of density irregularities during the radiation era. Solutions are found for the evolution of density inhomogeneities on small and large scales in the presence of pressure, and some new physical effects are identified.

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## 1. Introduction

In a recent article [1] we examined the behaviour of cosmological density perturbations in a universe containing a large-scale primordial magnetic field, by means of the Ellis and Bruni covariant and gauge-invariant approach [2]. Our assumptions were: first, that the conductivity of the medium is infinite; and second, that the background universe, though permeated by a coherent magnetic field, remains spatially isotropic to leading order. The first approximation is a standard simplification of Maxwell's equations, ignoring any large-scale electric field, while preserving the desired coupling between matter and the magnetic field. The second approximation was introduced at a later stage of our analysis, to allow the direct comparison between our results and those from previous Newtonian treatments. Starting from a general, inhomogeneous and anisotropic, cosmological model we provided the exact, fully nonlinear, evolution formulae for all the basic gauge-invariant variables. These equations are valid irrespective of the field's strength and can be linearized about a 'variety' of smooth background universes. In [1], the main objective was to establish a fully relativistic treatment of magnetized density perturbations. To achieve this, we focused upon the dust era and compared our results to those obtained earlier by the Newtonian treatments of Ruzmaikina and Ruzmaikin [3] and Wasserman [4]. In addition to the desired agreement with the non-relativistic analysis, our method suggested weak corrections to the evolution of density disturbances. These are generated by both the isotropic and the anisotropic pressure that the magnetic field introduces into the cosmological model. Our results confirmed the relative unimportance of the field for the evolution of superhorizon-sized density disturbances.

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This paper extends our study to the radiation era by including the isotropic pressure of a perfect fluid in the calculations. During this period the kinematic evolution of the universe follows more complicated patterns than in the dust era. The sources of the extra complexities are changes in the fluid motion due to its non-vanishing pressure, which manifest themselves in a number of ways. For instance, the acceleration of the fluid depends now on pressure gradients as well as on the spatial variations of the magnetic field, with the two of them not necessarily acting in the same sense. So, unlike the pressure-free case, the field vector and the fluid acceleration are generally not orthogonal. Moreover, the time derivative of the magnetic field is no longer a spacelike 3-vector. All these factors leave their trace on the kinematics of the universe and further complicate its dynamic evolution. It should be emphasized that we do not consider plasma-physics complexities induced by dissipative stresses in the energy–momentum tensor of the fluid. A separate study of this magnetohydrodynamical problem, both at linear and nonlinear level, is possible due to the conformal invariance of the stress tensor and can be found in Subramanian and Barrow [5]. These studies provide a mathematical and physical basis for the evaluation of the observational effects of cosmological magnetohydrodynamics. The existence of magnetic fluctuations can have observable effects on the structure of the microwave background on small angular scales and these can be seen by future microwave background satellite missions if the magnetic field is strong enough to influence the formation of large-scale structure. Subramanian and Barrow [6] have shown that in general for a tangled magnetic field with a strength $B_0 \sim 3 \times 10^{-9}$ G, one can expect a RMS microwave background anisotropy signal of the order of 5 $\mu$K or larger, depending on the angular scale. The anisotropy in hot or cold spots could be several times larger. The formalism we have developed can also be used to trace the effects of the damping of magnetic field fluctuations on the photon and neutrino spectra emerging from the radiation era. On superhorizon scales the field evolves as though quasi-homogeneous and will create small expansion anisotropies which will produce anisotropies in the microwave background temperature distribution. The existing observational data allows us to place upper limits of $6.8 \times 10^{-9}(\Omega_0 h^2)^{1/2}$ G on the present strength of any *uniform* (spatially homogeneous) component of the magnetic field [7, 8] ($\Omega_0$ is the present density parameter and $h$ the Hubble constant in units of 100 km s$^{-1}$ Mpc$^{-1}$).

In this report we examine the case of a perturbed spatially flat Friedmann–Robertson–Walker (FFRW) magnetized universe filled with a single barotropic perfect fluid and derive the linear equations that determine its evolution. We consider the evolution of the basic kinematic and dynamic quantities and the magnetohydrodynamical effects upon them. We show that, in regions of subhorizon size, gradients in the energy density of the field, together with those in the fluid, decelerate the universal expansion and act as sources of positive spatial curvature. It also appears that the field tends to smooth out the curvature of the underlying 3-surfaces and, depending on the spatial curvature, it can act as a source of slightly accelerated expansion.

Given the current interest in structure formation, we provide a set of four linear first-order differential equations that governs the linear evolution of $\Delta$, the scalar variable that determines the gravitational clumping of matter and describes the formation of structure directly. During the radiation epoch the energy density of the fluid and that of the magnetic field fall as the inverse fourth power of the scale factor. As a result, the Alfvén velocity is time independent and our system of four differential equations becomes easier to handle. We provide analytic solutions at both limits of the wavelength spectrum and compare them to those of a non-magnetic universe. We show that although the density inhomogeneities retain their basic evolutionary patterns, the role of the field as an agent opposing their growth is clear. In particular, large-scale perturbations undergo a power-law evolution, similar to that of the non-magnetized case, but their growth rate is reduced by an amount proportional to the field's relative strength. At the opposite end of the spectrum the density contrast continues to oscillate. Here, the extra magnetic pressure has simply reduced the oscillation period. The conclusion is that during the radiation era the magnetic effects are just supplementary to those induced by the pressure of the relativistic fluid. After the radiation era ends, the field becomes the sole source of pressure in regions exceeding the Jeans length associated at the time. We find, in agreement with our earlier conclusions (see [1]), that any large-scale magnetic influence ceases completely by the later stages of the dust era, although at earlier times it could have forced the density contrast to enter a brief oscillatory phase. The negative role of the field is also confirmed in the subhorizon regions. On scales between the Jeans length and the horizon, the magnetic field slows the power-law growth of the inhomogeneities by an amount depending on its relative strength.

In [1] we considered a general FRW universe, allowing for spatially open and closed unperturbed backgrounds. However, it is important to recognize that the gauge invariance of the magnetic field perturbations holds if and only if the underlying spatial sections are flat. Accordingly, all equations in [1] must be linearized about an FFRW universe and every variable representing spatial curvature should be treated as a perturbation. As a result, terms in the linearized formulae of [1] that contain the background 3-curvature constant are nonlinear and can be dropped at first order. This does not affect the results presented there but restricts their validity to almost-FFRW models. Notice that the gauge invariance of the magnetic field gradients still holds within a perturbed Bianchi-I universe due to the latter's spatial flatness. In appendix A we give a full account of this question.

## 2. Preliminaries

### *2.1. Kinematic variables*

Following [9, 10], we assume that the average motion of matter in the universe defines a future-directed velocity 4-vector, $u_i$, corresponding to a *fundamental observer* ($u_iu^i = -c^2$), and generates a unique splitting of spacetime into 'time' and 'space' ($1+3$ decomposition). For any tensorial quantity $T$, the directional derivative $\dot{T} = T_{;i}u^i = u^i\nabla_i T$ denotes differentiation along the fluid-flow lines. The second-order symmetric tensor $h_{ij} = g_{ij} + u_iu_j/c^2$ projects orthogonal to $u_i$ onto what is known as the observer's *instantaneous 3D rest space* $\Sigma_\perp$†. We also introduce ${}^{(3)}\nabla_i$, the covariant derivative operator orthogonal to $u_i$ (${}^{(3)}\nabla_i h_{jk} = 0$), by totally projecting the corresponding 4D operator. This is not, however, a derivative on a hypersurface unless the fluid flow is irrotational. Nevertheless, we will call the 3-gradient ${}^{(3)}\nabla_i$ 'spatial' for simplicity.

The kinematic variables are established by decomposing the covariant derivative of $u_i$ into its spatial and temporal parts. In particular, we have

$$\nabla_j u_i = \sigma_{ij} + \omega_{ij} + \frac{\Theta}{3}h_{ij} - \frac{1}{c^2}a_iu_j, \tag{1}$$

where $\sigma_{ij} = {}^{(3)}\nabla_{(j}u_{i)} - \Theta h_{ij}/3$ is the shear ($\sigma_{ij}u^i = \sigma_{ij}u^j = 0$, $\sigma_i{}^i = 0$), $\omega_{ij} = {}^{(3)}\nabla_{[j}u_{i]}$ is the vorticity ($\omega_{ij}u^i = \omega_{ij}u^j = 0$), $\Theta = \nabla_i u^i$ is the volume expansion and $a_i = \dot{u}_i = u^j\nabla_j u_i$ is the acceleration ($a_iu^i = 0$). The magnitudes of the shear and the vorticity are $\sigma^2 = \sigma_{ij}\sigma^{ij}/2$ and $\omega^2 = \omega_{ij}\omega^{ij}/2$, respectively. The expansion scalar, $\Theta$, defines a

† At every event along the worldline of a fundamental observer, $\Sigma_\perp$ is the normal to $u_i$ 3D subspace of the 4D space tangent to that event.

representative length scale ($S$) along the fluid flow by means of $\dot{S}/S = \Theta/3$. In a non-rotating universe (i.e. when $\omega_{ij} = 0$), $u_i$ is a hypersurface orthogonal field and $\Sigma_\perp$ becomes a 3-surface, namely the instantaneous rest space of all the fundamental observers.

### 2.2. Spacetime geometry

The global geometry of the spacetime is determined by the Riemann curvature tensor, conveniently expressed by the decomposition

$$R_{ijkq} = C_{ijkq} + \tfrac{1}{2}(g_{ik}R_{jq} + g_{jq}R_{ik} - g_{jk}R_{iq} - g_{iq}R_{jk}) - \tfrac{1}{6}(g_{ik}g_{jq} - g_{iq}g_{jk})R, \tag{2}$$

where $C_{ijkq}$ is the Weyl conformal curvature tensor, $R_{ij} \equiv R_{i\ jk}^{\ k}$ is the Ricci tensor and $R \equiv R^i{}_i$ is the Ricci scalar. Both the Ricci tensor and the Ricci scalar are determined locally by matter through the Einstein field equations. Conversely, the Weyl tensor describes long-range gravitational effects, such as those of tidal forces and gravitational waves. By definition $C_{ijkq}$ satisfies all the symmetries of the Riemann tensor and is also trace-free. It decomposes into an electric and a magnetic part according to [11]

$$C_{ijkq} = \frac{1}{c^2}(g_{ijsr}g_{kqpt} - \eta_{ijsr}\eta_{kqpt})u^s u^p E^{rt} - \frac{1}{c^2}(\eta_{ijsr}g_{kqpt} + g_{ijsr}\eta_{kqpt})u^s u^p H^{rt}, \tag{3}$$

where

$$g_{ijkq} \equiv g_{ik}g_{jq} - g_{iq}g_{jk}, \tag{4}$$

$\eta_{ijkq}$ is the totally antisymmetric spacetime permutation tensor and $E_{ij}$, $H_{ij}$ are, respectively, the electric and the magnetic components of the Weyl tensor. The latter have nothing to do with actual electric or magnetic fields but derive their name from the Maxwell-like equations they comply with [12]. Also notice that

$$C_{ijkq} = 0 \quad \Leftrightarrow \quad \begin{cases} E_{ij} = 0, \\ H_{ij} = 0. \end{cases} \tag{5}$$

### 2.3. The electromagnetic field

The electromagnetic field is represented by the antisymmetric Maxwell tensor $F_{ij}$. This splits into an electric and a magnetic 4-vector, respectively defined by [10]

$$E_i = F_{ij}u^j, \qquad \text{and} \qquad H_i = \frac{1}{2c}\eta_{ijkq}u^j F^{kq}. \tag{6}$$

The above confirm that $E_i u^i = H_i u^i = 0$, which in turn mean that both fields lie on $\Sigma_\perp$; $E^2 \equiv E_i E^i$ and $H^2 \equiv H_i H^i$, respectively, denote the magnitudes of the electric and the magnetic fields.

### 2.4. The material component

As in [1], we consider a universe filled with a single perfect fluid of infinite conductivity (i.e. $\overline{\sigma} = cE_i J^i/E^2 \to \infty$, with $J_i$ representing the current density). We can now drop the electric field from Maxwell's equations, which reduce to†

$$2\omega^i H_i = \epsilon c, \tag{7}$$

$$\eta^{ijkq}u_j(a_k H_q - c^2\nabla_q H_k) = c^2 h^i{}_j J^j, \tag{8}$$

† The merit of the infinite conductivity assumption is that, based on Ohm's law, it can accommodate a zero electric field with non-vanishing spatial currents (i.e. $h_i{}^j J_j \neq 0$). The latter condition is essential if one wishes to preserve the coupling between matter and magnetic field (see appendix B in [1]).

$$\nabla^i H_i - \frac{1}{c^2} a^i H_i = 0, \tag{9}$$

$$\left(\sigma^i{}_j + \omega^i{}_j - \tfrac{2}{3}\Theta h^i{}_j\right) H^j = h^i{}_j \dot{H}^j, \tag{10}$$

where $\omega_i \equiv \eta_{ijkq} u^j \omega^{kq}/2c$ is the vorticity vector and $\epsilon \equiv -J_i u^i/c^2$ is the charge density. For our purposes the last two equations are the important ones. More specifically, (9) provides the familiar vanishing 3-divergence law for the magnetic field (i.e. ${}^{(3)}\nabla_i H^i = 0$), whereas (10), when contracted with the magnetic field vector, gives $\sigma_{ij} H^i H^j = 2\Theta H^2/3 + (H^2)\dot{}/2$ and simplifies the expression for the energy density conservation law [1].

The energy–momentum tensor for a magnetized single perfect fluid of infinite conductivity has the form [1]

$$T_{ij} = \left(\mu + \frac{H^2}{2c^2}\right) u_i u_j + \left(p + \tfrac{1}{6} H^2\right) h_{ij} + \Pi_{ij} \tag{11}$$

with the pressure ($p$) and the mass density ($\mu$), the latter including contributions from the internal thermal energy, related by a suitable equation of state. The symmetric, traceless and completely spacelike tensor $\Pi_{ij} = H^2 h_{ij}/3 - H_i H_j$ describes the anisotropic pressure induced by the magnetic field†.

### 2.5. *Inhomogeneity variables*

In an FFRW universe all physical quantities are functions of cosmic-time only, while the shear, the vorticity, the acceleration, all the anisotropic stresses, the curvature of the spatial sections and the electric and magnetic components of the Weyl tensor vanish. Within a nearly FFRW universe, spatial perturbations in the energy density and the pressure of the fluid, in the expansion and in the magnetic field are described by four key variables defined covariantly in [1, 2]. These are: the comoving fractional orthogonal spatial gradient of the energy density, $D_i \equiv S^{(3)}\nabla_i \mu/\mu$; the orthogonal spatial gradient of the pressure, $Y_i \equiv \kappa^{(3)}\nabla_i p$, where $\kappa = 8\pi G/c^4$ is the Einstein gravitational constant; the comoving orthogonal spatial gradient of the expansion, $\mathcal{Z} \equiv S^{(3)}\nabla_i \Theta$; and the comoving orthogonal spatial gradient of the magnetic field, $\mathcal{M}_{ij} \equiv \kappa S^{(3)}\nabla_j H_i$, with $\mathcal{M}_i{}^i = 0$, as (9) requires. Each one of these 3-gradients vanishes in a perfect FFRW model (see also appendix A), thus satisfying the criterion for gauge invariance [13]. Three additional gauge-invariant variables, which play a crucial role in our analysis, are the divergence of the acceleration (i.e. $A \equiv \nabla_i a^i$), its spatial gradient (i.e. $A_i \equiv {}^{(3)}\nabla_i A$), and the spatial gradient of the curvature scalar $K$ associated with $\Sigma_\perp$ (i.e. $K_i \equiv {}^{(3)}\nabla_i K$).

It is convenient to introduce the following local decomposition for the spatial gradient of $D_i$ [14]:

$$\Delta_{ij} \equiv S^{(3)}\nabla_j D_i = W_{ij} + \Sigma_{ij} + \tfrac{1}{3}\Delta h_{ij}, \tag{12}$$

where $W_{ij} \equiv \Delta_{[ij]}$ contains information about the rotational behaviour of $D_i$, $\Sigma_{ij} \equiv \Delta_{(ij)} - \Delta_i{}^i h_{ij}/3$ describes the formation of anisotropies (e.g. pancakes or cigar-like structures) and $\Delta \equiv \Delta_i{}^i$ is related to the spherically symmetric gravitational clumping of matter. Although in a general perturbation pattern we expect turbulence (i.e. $W_{ij} \neq 0$) and anisotropic structures (i.e. $\Sigma_{ij} \neq 0$), as well as material aggregation (i.e. $\Delta > 0$), it is the latter scalar which is crucial for the structure formation purposes.

† In [1] we represented the anisotropic magnetic stresses by $M_{ij}$ instead of $\Pi_{ij}$. Other changes relative to that article are; $a_i$ has replaced $\dot{u}_i$ as the acceleration vector; the 3-Ricci scalar has changed from $\mathcal{K}$ into $K$, while now $\mathcal{K} \equiv S^2 K$; the gradient ${}^{(3)}\nabla_i H^2$ is represented by $\mathcal{H}_i$ and not by $\mathcal{B}_i$ as in [1] and the scalar $\mathcal{B}$ is no longer the Laplacian ${}^{(3)}\nabla^2 H^2$ but equals the dimensionless ratio $S^{2(3)}\nabla^2 H^2/H^2$.

## 3. The linear regime

In reality $\mu$, $p$, $\Theta$ and $H_i$ must have a spatial dependence as well as a temporal one. Moreover, $K$, $a_i$, $\sigma_{ij}$, $\omega_{ij}$, $\Pi_{ij}$, $E_{ij}$ and $H_{ij}$ will generally take non-zero values. Assuming that the observed universe is close to an FFRW spacetime, we can linearize the evolution equations by treating all the gauge-invariant quantities, along with their derivatives, as first-order variables. The exact, fully nonlinear formulae have already been derived in [1]. Here we give their linearized versions only.

### 3.1. Evolution equations

The linear regime is monitored through the following combination of propagation formulae and constraint equations.

(i) The conservation laws of the energy and the momentum densities of the fluid, respectively, expressed by

$$\frac{\dot{\mu}}{\mu} + (1+w)\Theta = 0, \tag{13}$$

and

$$\kappa\mu(1+w)a_i + Y_i - \frac{2}{S}\mathcal{M}_{[ij]}H^j = 0, \tag{14}$$

where the ratio $w \equiv p/\mu c^2$ evolves according to

$$\dot{w} = -(1+w)\left(\frac{c_s^2}{c^2} - w\right)\Theta, \tag{15}$$

with $c_s^2 \equiv \dot{p}/\dot{\mu}$ representing the adiabatic sound speed. Although generally $w$ is allowed to vary, when it remains constant along the fluid-flow lines (i.e. when $\dot{w} = 0$) equation (15) suggests that $w = c_s^2/c^2 =$ constant, provided of course that $\Theta \neq 0$.

(ii) The propagation equations that determine the kinematics of the universe. These are Raychaudhuri's formula,

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \tfrac{1}{2}\kappa\mu c^4(1+3w) - A - \Lambda c^2 = 0, \tag{16}$$

where $\Lambda$ is the cosmological constant, and the propagation formulae of the vorticity† and the shear tensors, respectively, given by

$$\dot{\omega}_{ij} + \tfrac{2}{3}\Theta\omega_{ij} = {}^{(3)}\nabla_{[j}a_{i]}, \tag{17}$$

and

$$\dot{\sigma}_{ij} + \tfrac{2}{3}\Theta\sigma_{ij} = {}^{(3)}\nabla_{(j}a_{i)} - \tfrac{1}{3}Ah_{ij} + \tfrac{1}{2}\kappa c^2\Pi_{ij} - c^2E_{ij}, \tag{18}$$

where $E_{ij} \equiv C_{ikjq}u^k u^q/c^2$ is the 'electric' part of the Weyl tensor. To first order, the scalar $A$ is given by the 3-divergence of the fluid acceleration (i.e. $A = {}^{(3)}\nabla^i a_i$).

(iii) The linear propagation equations of $D_i$, $\mathcal{Z}_i$ and $\mathcal{M}_{ij}$, respectively, governing the growth of spatial inhomogeneities in the energy density of the fluid,

$$\dot{D}_i = w\Theta D_i - (1+w)\mathcal{Z}_i - \frac{2\Theta}{\kappa\mu c^2}\mathcal{M}_{[ij]}H^j + \frac{2S\Theta H^2}{3\mu c^4}a_i, \tag{19}$$

† Equation (17) monitors the model's rotational behaviour through the vorticity tensor, as opposed to the vorticity vector used in [1] (see equation (89) therein). Recalling that $\omega_i = \eta_{ijkq}u^j\omega^{kq}/2c$, the equivalence of the two formulae becomes evident.

in the expansion scalar†,

$$\dot{\mathcal{Z}}_i = -\tfrac{2}{3}\Theta\mathcal{Z}_i - \tfrac{1}{2}\kappa\mu c^4 D_i - 3c^2\mathcal{M}_{[ij]}H^j - c^2\mathcal{M}_{ji}H^j + SA_i, \tag{20}$$

and in the magnetic field vector,

$$\dot{\mathcal{M}}_{ij} = -\frac{2\Theta}{3}\mathcal{M}_{ij} - \frac{2\kappa}{3}H_i\mathcal{Z}_j + \kappa SH^{k\,(3)}\nabla_j(\sigma_{ik}+\omega_{ik}) - \frac{\kappa\Theta S}{3c^2}(2H_ia_j + a_iH_j) + \frac{\kappa\Theta S}{3c^2}a_kH^kh_{ij} + \kappa h_i{}^kR_{kqjs}H^qu^s, \tag{21}$$

recalling that $A_i = {}^{(3)}\nabla_i A$ by definition.

(iv) The evolution of the magnetic field is governed by the four decomposed Maxwell's equations (see equations (7)–(10)), of which only

$$\nabla_iH^i = \frac{1}{c^2}a_iH^i, \tag{22}$$

and

$$h_i{}^j\dot{H}_j = \left(\sigma^i{}_j + \omega^i{}_j - \tfrac{2}{3}\Theta h^i{}_j\right)H^j, \tag{23}$$

are crucial for our analysis. The former verifies that the magnetic field is a 'solenoidal' (i.e. ${}^{(3)}\nabla_iH^i = 0$), and the latter, when contracted with the field vector, provides a radiation-like linear evolution law for the magnetic energy density

$$H^2 = \frac{\mathbb{H}}{S^4}, \tag{24}$$

where $\dot{\mathbb{H}} = 0$.

(v) We close this section with a brief discussion on the geometry of $\Sigma_\perp$, the observer's instantaneous rest space. Its curvature is characterized by the scalar

$$K = 2\left(\kappa\mu c^2 - \frac{\Theta^2}{3c^2} + \Lambda\right), \tag{25}$$

so that $\Theta^2/3 = \kappa\mu c^4 + \Lambda c^2$ to zero order. When there is no vorticity and only then, $K$ coincides with the 3-Ricci scalar of the spacelike hypersurfaces that define the instantaneous rest space of all the fundamental observers. Its propagation formula,

$$\dot{K} = -\frac{2\Theta}{3}\left(K + \frac{2}{c^2}A\right), \tag{26}$$

suggests that in the linear regime the fluid acceleration acts as the sole source of spatial curvature through its 3-divergence. Following [14], we describe the spatial variations of the 3-curvature by the gauge-invariant vector $C_i \equiv S^3K_i$ and provide a supplementary relation between $D_i$, $Z_i$ and $\mathcal{M}_{ij}$,

$$C_i = 2\kappa\mu c^2S^2D_i + 2S^2\mathcal{M}_{ji}H^j - \frac{4\Theta S^2}{3c^2}\mathcal{Z}_i, \tag{27}$$

† In [1], based on the weakness of the magnetic field, we ignored the linear effects on the evolution of the expansion and the 3-curvature gradients resulting from the field's contribution to the active gravitational mass of the universe. Here we fully incorporate these effects via the second to last terms in the right-hand side of (20) and (27) (compare them to equations (91) and (99) in [1]). Notice that these quantities provide all the coupling between the magnetic and the matter inhomogeneities that is left, once the infinite conductivity approximation is abandoned in favour of a pure source-free magnetic field (see appendix B in [1]). Although they make no qualitative difference and introduce negligible quantitative changes, both terms are included here for completeness.

which, by means of (19)–(21), leads to

$$\dot{C}_i = -\frac{4\Theta S^3}{3c^2} A_i. \qquad (28)$$

The above propagation formula is consistent with equation (26) and, together with (42), confirms the interdependence between the spatial curvature and the acceleration of the fluid flow. The advantages of choosing $C_i$, instead of $K_i$, to describe the spatial variations in the 3-curvature, will become clear later.

## 4. The case of a barotropic perfect fluid

Among the propagation formulae given above, which refer to a general perfect fluid with pressure, there is no equation for the evolution of 3-gradients in the pressure. The reason is that the propagation of $Y_i$ will be determined directly from (19), once the material content of the universe has been specified.

### *4.1. Equation of state*

Here, we extend the analysis presented in [1] by considering a universe filled with a single barotropic perfect fluid. Its equation of state is†

$$p = p(\mu), \qquad (29)$$

suggesting that $\nabla_{[i} p \nabla_{j]} \mu = 0$. Consequently, the relation between pressure and energy density gradients becomes

$$SY_i = \kappa \mu c_s^2 D_i, \qquad (30)$$

since $c_s^2 = \mathrm{d}p/\mathrm{d}\mu$ relative to the observer's rest frame.

### *4.2. Kinematic evolution*

*4.2.1. The acceleration.* The energy density conservation law of the barotropic fluid is still expressed by (13). However, the momentum density conservation law (14), together with (30), gives

$$a_i = \frac{1}{(1+w)S}\left(\frac{2}{\kappa\mu}\mathcal{M}_{[ij]}H^j - c_s^2 D_i\right), \qquad (31)$$

for the acceleration of a fundamental observer. It depends both on gradients in the energy density of the fluid and on gradients in the magnetic field. Thus, the geodesic flow can still be preserved provided that the field gradients counterbalance those of the material component. The necessary and sufficient condition for this to occur is‡ $D_i = 2\mathcal{M}_{[ij]}H^j/\kappa\mu c_s^2$.

Unlike the pressure-free case (see [1]), the acceleration of the barotropic fluid is not always normal to the magnetic-field vector. Alternatively, one might say that, when $p \neq 0$, the time derivative $\dot{H}_i$ no longer lies on $\Sigma_\perp$. We verify these statements by simply contracting (31) with $H_i$. We find that

$$a_i H^i = u_i \dot{H}^i = -\frac{c_s^2}{(1+w)S} H^i D_i, \qquad (32)$$

† From now on, all our results will refer to a barotropic fluid unless otherwise stated. We will also ignore the entropy contribution to the fluid pressure.

‡ The geodesic flow condition can simplify the evolutionary relations of section 3.1 considerably. However, it does not appear to be consistent and we will not pursue the matter any further here.

where generally $H_iD^i \neq 0$. Obviously, $a_i$ and $H_i$ remain orthogonal if $H_iD^i = 0$. We can modify this condition by setting $E \equiv H^2/\mu c^2$, taking its 3-gradient and then contracting with the magnetic field vector. The result,

$$H^iD_i = -\frac{S}{E}H^{i\,(3)}\nabla_i E, \tag{33}$$

suggests that the acceleration of the fluid flow remains normal to the magnetic field if and only if the directional derivative $H^{i\,(3)}\nabla_i E$ vanishes (i.e. when the energy density ratio $E$ does not change along the magnetic field lines). In this case, the time derivative of the magnetic field lies on the observer's instantaneous rest space. This fact can simplify, among other, calculations involving commutations between the spatial gradients of $\dot{H}_i$.

Spatial gradients in the fluid acceleration affect the expansion dynamics directly (see equations (16)–(18)), as well as the spatial geometry (see equation (26)). Consequently the following new decomposition of the acceleration's 3-gradient is of major importance. It is obtained directly from equation (31) via the commutation laws for the 3-gradients of scalars and spacelike vectors (see equations (B1) and (B2) in appendix B), relations (12), (23) and the relativistic expression $\Pi_{ij} = H^2h_{ij}/3 - H_iH_j$ for the magnetic anisotropic stresses (see section 7.4.1 in [15] for more details),

$$\begin{aligned}{}^{(3)}\nabla_j a_i = &-\frac{c_s^2}{(1+w)S^2}\left(\Sigma_{ij} + W_{ij} + \tfrac{1}{3}\Delta h_{ij}\right) - \frac{4\Theta H^2}{9\mu c^2(1+w)}\omega_{ij} + \frac{H^2}{3\mu(1+w)}{}^{(3)}R_{ij} \\ &-\frac{1}{2\mu(1+w)}{}^{(3)}\nabla_j{}^{(3)}\nabla_i H^2 + \frac{1}{\kappa\mu(1+w)S}H^{k\,(3)}\nabla_k\mathcal{M}_{ij},\end{aligned} \tag{34}$$

where ${}^{(3)}R_{ij}$ is the 3-Ricci tensor of the spacelike regions (given by equation (83) in [1]). In what follows, the trace, the skew part and the symmetric part of the above will be employed to analyse the magnetohydrodynamical effects upon the kinematics and the spatial geometry of our cosmological model.

*4.2.2. The deceleration parameter.* To begin with, the trace of (34),

$$A = {}^{(3)}\nabla^i a_i = -\frac{c_s^2}{(1+w)S^2}\Delta + \frac{H^2}{3\mu(1+w)}K - \frac{H^2}{2\mu(1+w)S^2}\mathcal{B}, \tag{35}$$

where $\mathcal{B} \equiv S^{2\,(3)}\nabla^2 H^2/H^2$, is substituted into (16) to produce Raychaudhuri's equation for a magnetized universe filled with a single barotropic perfect fluid of infinite conductivity. This formula is recast into the following alternative expression:

$$\frac{\Theta^2}{3c^2}q = \frac{\kappa\mu c^2}{2}(1+3w) - \frac{H^2}{3\mu c^2(1+w)}K + \frac{1}{(1+w)S^2}\left(\frac{c_s^2}{c^2}\Delta + \frac{H^2}{2\mu c^2}\mathcal{B}\right) - \Lambda, \tag{36}$$

where $q \equiv -\ddot{S}S/\dot{S}^2$ is the 'deceleration parameter' (note that $\Delta = S^{2\,(3)}\nabla^2\mu/\mu$ to first order). Clearly, the sign of the quantity on the right-hand side of (36) determines whether the expansion slows down or continues unimpeded. Not surprisingly, a spherically symmetric increase in the energy density of the field (i.e. $\mathcal{B} > 0$), together with any material aggregation (i.e. $\Delta > 0$), slows the expansion down. Their combined effect is of first order and, as the Laplacians verify, it is confined to regions well within the horizon. However, while the energy density of ordinary matter (i.e. $w > -\frac{1}{3}$) always adds a positive value to the deceleration parameter, the contribution of the magnetic energy density depends on the geometry of $\Sigma_\perp$. According to (36), the coupling between the field and the 3-curvature slows down the expansion of spatially open almost-FRW universe (i.e. when $K < 0$) but accelerates perturbed Friedmannian cosmologies with positive spatial curvature (i.e. $K > 0$).

This rather unconventional magnetic effect is global, though still first order in magnitude since $K = 0$ in the background. It vanishes when the perturbed universe retains its spatial flatness.

*4.2.3. The vorticity tensor.* According to (17), only the antisymmetric part of (34) affects the vorticity propagation. Since $W_{ij} = -(1+w)\Theta S^2\omega_{ij}/c^2$ to first order (see [14]) and ${}^{(3)}\nabla_{[i}{}^{(3)}\nabla_{j]}H^2 = 4\Theta H^2\omega_{ij}/3c^2$, as the commutator of the 3-gradients of scalars (see equation (B1) in appendix B) and the last of Maxwell's equations (see equation (23)) imply, we obtain

$$\dot{\omega}_{ij} + \frac{2\Theta}{3}\left(1 - \frac{3c_s^2}{2c^2}\right)\omega_{ij} = \frac{1}{\kappa\mu(1+w)S}H^{k\,(3)}\nabla_k\mathcal{M}_{[ij]}. \tag{37}$$

Notice that a cosmic magnetic field influences the vorticity of the universe solely through the antisymmetric part of the gradient field $\mathcal{M}_{ij}$, which itself describes the rotational behaviour of the magnetic field vector (see appendix C.1 in [1]). Also, according to (37), the field has no effect at all when the directional derivative $H^{k\,(3)}\nabla_k\mathcal{M}_{[ij]}$ vanishes, that is when curl $H_i$ does not change along the magnetic field lines.

*4.2.4. The shear tensor.* The symmetric part of (34) together with its trace allows us to recast equation (18), for the linear evolution of the shear tensor, into the following form:

$$\begin{aligned}\dot{\sigma}_{ij} + \frac{2\Theta}{3}\sigma_{ij} = &-\frac{c_s^2}{(1+w)S^2}\Sigma_{ij} - \frac{1}{2\mu(1+w)}\left({}^{(3)}\nabla_{(i}{}^{(3)}\nabla_{j)} - \tfrac{1}{3}h_{ij}{}^{(3)}\nabla^2\right)H^2 \\ &+\frac{H^2}{3\mu(1+w)}\left({}^{(3)}R_{(ij)} - \tfrac{1}{3}\mathcal{K}h_{ij}\right) + \frac{\kappa c^2}{2}\Pi_{ij} \\ &+\frac{1}{\kappa\mu(1+w)S}H^{k\,(3)}\nabla_k\mathcal{M}_{(ij)} - c^2E_{ij}.\end{aligned} \tag{38}$$

Clearly, the shear anisotropies evolve in a rather complicated way under the simultaneous influence of a number of sources. According to (38) such sources are: the fluid; the magnetic field; the geometry of the observer's 3D rest space and the long-range source-free gravitational field. The magnetic influence is multi-faceted. In particular, anisotropic spatial variations in the energy density of the field have a similar effect to those in the energy density of the fluid (the latter represented by $\Sigma_{ij}$). Also, the magnetic energy density couples with anisotropies in the spatial curvature to create an additional effect. Notice that any anisotropic patterns in the distribution of the magnetic field vector (described by $\mathcal{M}_{(ij)}$ since $\mathcal{M}_i{}^i = 0$) exert no influence at all if the directional derivative $H^{k\,(3)}\nabla_k\mathcal{M}_{(ij)}$ vanishes.

*4.2.5. The 3-curvature scalar.* In the linear regime, the curvature scalar of the observer's instantaneous 3D rest space evolves according to equation (26). Substituting the trace of (34) into the latter we obtain

$$\dot{K} + \frac{2\Theta}{3}\left(1 + \frac{2H^2}{3\mu c^2(1+w)}\right)K = \frac{4\Theta}{3(1+w)S^2}\left(\frac{c_s^2}{c^2}\Delta + \frac{H^2}{2\mu c^2}\mathcal{B}\right), \tag{39}$$

or, since $H^2/\mu c^2(1+w) \ll 1$,

$$\dot{K} + \frac{2\Theta}{3}K = \frac{4\Theta}{3(1+w)S^2}\left(\frac{c_s^2}{c^2}\Delta + \frac{H^2}{2\mu c^2}\mathcal{B}\right). \tag{40}$$

Therefore, on regions of subhorizon size, any spherically symmetric spatial increase in the energy density of the fluid (i.e. $\Delta > 0$), or of the magnetic field (i.e. $\mathcal{B} > 0$), acts as a source of positive curvature. This is a first-order effect similar to the small-scale magnetohydrodynamical impact on the expansion of the universe (see the last term in the right-hand side of equation (36)). The magnetic influence also results in a global effect of the opposite type. As (39) reveals, the field tends to smooth out the curvature of the spacelike regions through its coupling with the background expansion. This 'magnetic smoothing', which here is of negligible magnitude, is analogous to the field's global magneto-geometrical impact upon the universal expansion (compare to the second term in the right-hand side of (36)) both qualitatively and quantitatively.

We close this section with a comment on the non-local magnetic effects illustrated in equations (36) and (39). We attribute such behaviour to the vectorial nature of the field, as opposed to the scalar nature of quantities such as the energy density of the fluid or its isotropic pressure. Being a vector, the field interacts with the curvature of the spacelike regions (e.g. through the 3-Ricci identity) and this dependence creates the aforementioned effects. However, we do not suggest that any perturbed spacelike vector would have a similar impact. The magnetic influence outlined above depends crucially upon the specific properties of the field, as these are reflected in Maxwell's equations, and also in the unique way general relativity describes the magnetic anisotropic stresses (see comments on the derivation of equation (34)).

### *4.3. Dynamic evolution*

*4.3.1. The growth of the inhomogeneities.* The introduction of a barotropic fluid modifies the key linear propagation equations (19)–(21) and (28). More specifically, by means of (31), the former becomes

$$\dot{D}_i = w\Theta D_i - (1+w)\mathcal{Z}_i - \frac{2\Theta}{\kappa\mu c^2}\mathcal{M}_{[ij]}H^j. \tag{41}$$

The direct barotropic influence on the evolution of the expansion gradients comes through the spatial gradient of $A$†,

$$A_i = -\frac{c_s^2}{(1+w)S}{}^{(3)}\nabla^2 D_i + \frac{H^2}{3\mu(1+w)S^3}C_i - \frac{1}{2\mu(1+w)}{}^{(3)}\nabla^2\mathcal{H}_i - \frac{2c_s^2\Theta}{c^2}{}^{(3)}\nabla_j\omega_j{}^j, \tag{42}$$

where $\mathcal{H}_i \equiv {}^{(3)}\nabla_i H^2$. Substituting the above into (20) and using (27) we obtain

$$\dot{\mathcal{Z}}_i = -\frac{2\Theta}{3}\mathcal{Z}_i - \frac{\kappa\mu c^4}{2}D_i - 3c^2\mathcal{M}_{[ij]}H^j - c^2\mathcal{M}_{ji}H^j - \frac{c_s^2}{1+w}{}^{(3)}\nabla^2 D_i$$
$$-\frac{S}{2\mu(1+w)}{}^{(3)}\nabla^2\mathcal{H}_i - \frac{2c_s^2\Theta S}{c^2}{}^{(3)}\nabla_j\omega_i{}^j. \tag{43}$$

By means of (2) and the fact that $h_i{}^j R_{jk}u^k = 0$ (see equation (63) in [1]), the last term in (21), which describes the effects of spacetime curvature upon the evolution of magnetic inhomogeneities, becomes

$$\kappa h_i{}^k R_{kqjs}H^q u^s = \kappa h_i{}^k C_{kqjs}H^q u^s. \tag{44}$$

† In deriving (42) we have treated ${}^{(3)}\nabla_i w$ and ${}^{(3)}\nabla_i c_s^2$ as first-order gauge-invariant quantities. Though the former is straightforward to prove, the latter requires the gauge independence of ${}^{(3)}\nabla_i \dot{p}$ and ${}^{(3)}\nabla_i \dot{\mu}$ to be shown first.

Moreover, using decomposition (3) of the Weyl tensor we may recast the above into

$$\kappa h_i{}^k C_{kqjs} H^q u^s = -\kappa \eta_{ik}{}^{qs} H^k u_q H_{sj}, \tag{45}$$

showing that only spacetime ripples caused by the long-range gravitational forces, here represented by the magnetic part $H_{ij} \equiv \eta_{ip}{}^{kq} C_{kqjs} u^p u^s / 2c^2$ of the Weyl tensor, affect the propagation of spatial inhomogeneities in the cosmic magnetic field. The above result together with equations (31) and (32) allows us to transform (21) into

$$\dot{\mathcal{M}}_{ij} = -\frac{2\Theta}{3}\mathcal{M}_{ij} - \frac{2\kappa}{3} H_i \mathcal{Z}_j + \kappa S H^{k\,(3)}\nabla_j(\sigma_{ik} + \omega_{ik}) + \frac{2\Theta H^2}{9\mu c^2(1+w)}\mathcal{M}_{[ij]} + \frac{\kappa c_s^2 \Theta}{3c^2(1+w)}(2H_i D_j + H_j D_i - H^k D_k h_{ij}) + \kappa S \eta_{ik}{}^{qs} H^k u_q H_{sj}. \tag{46}$$

According to (41) it is only the contraction $\mathcal{M}_{ij} H^j$ that contributes to the linear growth of spatial inhomogeneities in the energy density of the medium. Therefore, taking the skew part of (46) and then contracting with the magnetic field vector, we obtain†

$$\dot{\mathcal{M}}_{[ij]} H^j = -\tfrac{2}{3}\Theta \mathcal{M}_{[ij]} H^j - \tfrac{2}{3}\kappa H_{[i}\mathcal{Z}_{j]} H^j + \kappa S h_{[i}{}^k h_{j]}{}^q (\sigma_{ks} + \omega_{ks})_{;q} H^s H^j + \frac{\kappa c_s^2 \Theta}{3c^2(1+w)} H_{[i} D_{j]} H^j, \tag{47}$$

which will prove useful later. Notice the absence of the spacetime curvature term from the right-hand side of equation (47). As a result, long-range gravitational forces have no linear effect on the evolution of magnetized density perturbations.

As far as spatial inhomogeneities in the curvature scalar are concerned, equations (30), (31) and (42) reshape their propagation formula (28), into the following:

$$\dot{C}_i = \frac{4c_s^2 \Theta S^2}{3c^2(1+w)}{}^{(3)}\nabla^2 D_i + \frac{2\Theta S^3}{3\mu c^2(1+w)}{}^{(3)}\nabla^2 \mathcal{H}_i + \frac{8c_s^2 \Theta^2 S^3}{3c^4}{}^{(3)}\nabla_j \omega_i{}^j, \tag{48}$$

implying that the gradient field $C_i$ is invariant on large scales (i.e. when the Laplacian terms are negligible) if ${}^{(3)}\nabla_j \omega_i{}^j = 0$. Finally, under the barotropic fluid assumption, reflected in (32), Maxwell's equations become

$$\nabla_i H^i = -\frac{c_s^2}{c^2(1+w)S} H^i D_i \tag{49}$$

and

$$\dot{H}_i = \left(\sigma_{ij} + \omega_{ij} - \tfrac{2}{3}\Theta h_{ij}\right) H^j - \frac{c_s^2}{c^2(1+w)S} H^j D_j u_i, \tag{50}$$

with the non-zero right-hand side of (49) and the last term of (50) being direct results of the changes in the fluid motion relative to the pressureless case.

*4.3.2. The growth of the density gradient.* The dynamics of the inhomogeneity variable $D_i$ is governed by (41), together with equations (43)‡ and (47), or by the linear second-order

† See sections 5.4 and 7.5 in [15] for a detailed derivation of the complete set of the exact and linear propagation equations.

‡ Alternatively, one can use (48) instead of (43), on substituting $\mathcal{Z}_i$ by $C_i$ in (41) from (27).

differential equation, which follows from (41), by means of (13), (15), (16), (43), (47) and (50). This equation is

$$\ddot{D}_i = -\left(\frac{2}{3} + \frac{c_s^2}{c^2} - 2w\right)\Theta \dot{D}_i + \left(\left(\frac{1}{2} - \frac{3c_s^2}{c^2} + 4w - \frac{3w^2}{2}\right)\kappa\mu c^4 - \left(\frac{3c_s^2}{c^2} - 5w\right)\Lambda c^2\right)D_i$$
$$+c_s^{2\,(3)}\nabla^2 D_i + \frac{2c_s^2\Theta S(1+w)}{c^2}{}^{(3)}\nabla_j \omega_i{}^j$$
$$-\left(\left(\frac{c_s^2}{c^2} - w\right)6c^2 + \left(1 + \frac{c_s^2}{c^2}\right)\frac{6\Lambda}{\kappa\mu}\right)\mathcal{M}_{[ij]}H^j$$
$$+\frac{S}{2\mu}{}^{(3)}\nabla^2\mathcal{H}_i - \frac{2\Theta S}{\mu c^2}{}^{(3)}\nabla_{[j}\dot{H}_{i]}H^j. \tag{51}$$

This is the generalization of formula (26) in [14] for a magnetized almost-FFRW universe. It has the form of a wave equation with extra terms due to the universal expansion, gravity, the cosmological constant, the magnetic field and the vorticity. The difference in the vorticity terms between equation (51) above, and its corresponding formula (115) in [1], is due to the residual coupling between the divergence of the vorticity tensor and the energy density of the field that remains when $p = 0$.

## 5. The scalar variables

So far we have considered the evolution of gauge-invariant vector variables and in particular the propagation of $D_i$, the gradient field that describes orthogonal to the fluid flow variations of the energy density. However, regarding the growth (or decay) of density inhomogeneities, the vector field $D_i$ contains more information than actually required. We can extract the information we need by adopting the local decomposition (12). Of the three additional variables mentioned there, the scalar $\Delta \equiv S^{(3)}\nabla^i D_i$ (alternatively $\Delta = (S^2/\mu)^{(3)}\nabla^2\mu$ to first order) is the most important one when addressing the problem of structure formation.

### *5.1. Definitions*

Focusing upon $\Delta$, which describes spherically symmetric spatial variations in the energy density of the matter, we also consider the following complementary scalar variables:

$$\mathcal{Z} \equiv S^{(3)}\nabla^i \mathcal{Z}_i, \qquad \mathcal{B} \equiv \frac{S^2}{H^2}{}^{(3)}\nabla^2 H^2, \tag{52}$$

respectively related to spatial gradients in the expansion and the energy density of the magnetic field, and

$$\mathcal{K} = S^2 K, \tag{53}$$

representing perturbations in the spatial curvature. Notice that all but $\mathcal{Z}$ are dimensionless variables. Also, $\mathcal{B}$ describes spherically symmetric spatial variations in the energy density of the magnetic field and it will be treated as the magnetic analogue of $\Delta$.

*5.2. Evolutionary equations*

The propagation equations associated with the above-defined scalars (see section 7.6 in [15] for details on their derivation) are

$$\dot{\Delta} = w\Theta\Delta - (1+w)\mathcal{Z} - \frac{\Theta H^2}{3\mu c^2}\mathcal{K} + \frac{\Theta H^2}{2\mu c^2}\mathcal{B}, \tag{54}$$

$$\dot{\mathcal{Z}} = -\frac{2\Theta}{3}\mathcal{Z} - \frac{\kappa\mu c^4}{2}\Delta - \frac{c_s^2}{1+w}{}^{(3)}\nabla^2\Delta - \frac{\kappa c^2 H^2}{2}\mathcal{K} + \frac{\kappa c^2 H^2}{4}\mathcal{B} - \frac{H^2}{2\mu(1+w)}{}^{(3)}\nabla^2\mathcal{B}, \tag{55}$$

$$\dot{\mathcal{B}} = \frac{4c_s^2\Theta}{3c^2(1+w)}\Delta - \frac{4}{3}\mathcal{Z} - \frac{4\Theta H^2}{9\mu c^2(1+w)}\mathcal{K}, \tag{56}$$

and

$$\dot{\mathcal{K}} = \frac{4c_s^2\Theta}{3c^2(1+w)}\Delta + \frac{2\Theta H^2}{3\mu c^2(1+w)}\mathcal{B}. \tag{57}$$

The first two are obtained by linearizing the 3-divergence of their corresponding vector equations (41) and (43). The third results directly from definition (52) via the laws governing commutations between time derivatives and spatial gradients of scalars and spacelike vectors (see equations (B3) and (B4) or equation (B5) in appendix B). Finally, the last is a simple rearrangement of (40)†.

By combining equations (54)–(56), or by linearizing the 3-divergence of (51), we obtain the following second-order differential equation for the evolution of the spatial matter aggregations

$$\begin{aligned}\ddot{\Delta} = {} & -\left(\frac{2}{3} + \frac{c_s^2}{c^2} - 2w\right)\Theta\dot{\Delta} + \left(\left(\frac{1}{2} - \frac{3c_s^2}{c^2} + 4w - \frac{3w^2}{2}\right)\kappa\mu c^4 - \left(\frac{3c_s^2}{c^2} - 5w\right)\Lambda c^2\right)\Delta \\ & + c_s^{2\,(3)}\nabla^2\Delta + \left(\left(\frac{2}{3} - \frac{c_s^2}{c^2} + w\right)\kappa\mu c^2 - \left(\frac{1}{3} + \frac{c_s^2}{c^2}\right)\Lambda\right)\frac{H^2}{\mu}\mathcal{K} \\ & - \left(\left(\frac{1}{2} - \frac{3c_s^2}{2c^2} + w\right)\kappa\mu c^2 - \left(\frac{1}{2} + \frac{3c_s^2}{2c^2}\right)\Lambda\right)\frac{H^2}{\mu}\mathcal{B} + \frac{H^2}{2\mu}{}^{(3)}\nabla^2\mathcal{B}.\end{aligned} \tag{59}$$

The rest of the variables evolve in accordance with the propagation formulae,

$$\dot{\mathcal{K}} = \frac{4c_s^2\Theta}{3c^2(1+w)}\Delta + \frac{2\Theta H^2}{3\mu c^2(1+w)}\mathcal{B}, \tag{60}$$

and

$$\dot{\mathcal{B}} = \frac{4}{3(1+w)}\dot{\Delta} + \frac{4\Theta}{3(1+w)}\left(\frac{c_s^2}{c^2} - w\right)\Delta, \tag{61}$$

where the latter is obtained by substituting $\mathcal{Z}$ in (56) from (54)‡. This is the system that governs the evolution of spatial matter aggregations in a perturbed FFRW universe that

† The 3-divergence of (48) provides the evolution formula of $C \equiv S^{(3)}\nabla^i C_i$, the scalar associated with spatial inhomogeneities in the curvature of the spacelike regions. Its form,

$$\dot{C} = \frac{4c_s^2\Theta S^2}{3c^2(1+w)}{}^{(3)}\nabla^2\Delta + \frac{2\Theta S^2 H^2}{3\mu c^2(1+w)}{}^{(3)}\nabla^2\mathcal{B}, \tag{58}$$

verifies that $C$ is time invariant on large scales irrespective of the model's rotational behaviour. Notice that one immediately recovers (57) from (58) on using definition (53) and the commutation law (B5) in appendix B.

‡ Equations (15) and (61) suggest that when $\dot{w} = 0$, as is the case in the dust era for example, then $\dot{\mathcal{B}} = 4\dot{\Delta}/3(1+w)$. So, during these periods spherically symmetric spatial variations in the energy density of the magnetic field grow (or decay) proportionally to those in the energy density of the matter.

contains a single barotropic perfect fluid of infinite conductivity and is permeated by a weak cosmological magnetic field.

The equations obtained here are significantly simpler and more transparent than their vector counterparts of section 4.3, especially as far as the role of the magnetic field is concerned. The field no longer exerts its influence through some complicated combinations of curl's and vector products, but simply via the spatial gradients of its energy density. Moreover, these are exactly the quantities that matter for structure formation purposes.

## 6. Particular solutions

In [1] we considered the evolution of density inhomogeneities during the post-equilibrium era, when the universe is filled with a non-relativistic perfect fluid (i.e. $p = 0 \Rightarrow w, c_s^2 = 0$). There, based on the nature of the evolution equation for $\Delta$, we argued that the magnetic effects on the growth of large-scale material aggregations are relatively unimportant. Here, the existence of formulae (60) and (61) will enable us to confirm, refine and extend these conclusions as well as to study the behaviour of the density contrast in the radiation era.

### *6.1. Harmonic analysis*

Following [16–18], we harmonically decompose the inhomogeneity variable $\Delta$ by writing it in the form of the sum

$$\Delta = \sum_n \Delta^{(n)} Q^{(n)}, \tag{62}$$

with ${}^{(3)}\nabla_i \Delta^{(n)} = 0$, $\dot{Q}^{(n)} = 0$ and ${}^{(3)}\nabla^2 Q^{(n)} = -n^2 Q^{(n)}/S^2$. Similarly, $\mathcal{K}$ and $\mathcal{B}$ may be written as

$$\mathcal{K} = \sum_n \mathcal{K}^{(n)} Q^{(n)}, \qquad \text{and} \qquad \mathcal{B} = \sum_n \mathcal{B}^{(n)} Q^{(n)}, \tag{63}$$

where ${}^{(3)}\nabla_i \mathcal{K} = {}^{(3)}\nabla_i \mathcal{B} = 0$. Notice that the harmonic eigenvalue ($n$) coincides with the comoving wavenumber ($\nu$) because of the spatial flatness of the background universe†.

Substituting results (62) and (63) into equations (59)–(61), the harmonics decouple to provide the following autonomous system:

$$\begin{aligned}\ddot{\Delta}^{(\nu)} = &-\left(\frac{2}{3} + \frac{c_s^2}{c^2} - 2w\right)\Theta\dot{\Delta}^{(\nu)} + \left(\left(\frac{1}{2} - \frac{3c_s^2}{c^2} + 4w - \frac{3w^2}{2}\right)\kappa\mu c^4 - \frac{\nu^2 c_s^2}{S^2}\right. \\ &\left. -\left(\frac{3c_s^2}{c^2} - 5w\right)\Lambda c^2\right)\Delta^{(\nu)} + \left(\left(\frac{2}{3} - \frac{c_s^2}{c^2} + w\right)\kappa\mu c^2 - \left(\frac{1}{3} + \frac{c_s^2}{c^2}\right)\Lambda\right)c_A^2\mathcal{K}^{(\nu)} \\ &-\left(\left(\frac{1}{2} - \frac{3c_s^2}{2c^2} + w\right)\kappa\mu c^2 + \frac{\nu^2}{2S^2} - \left(\frac{1}{2} + \frac{3c_s^2}{c^2}\right)\Lambda\right)c_A^2\mathcal{B}^{(\nu)},\end{aligned} \tag{64}$$

$$\dot{\mathcal{K}}^{(\nu)} = \frac{4c_s^2\Theta}{3c^2(1+w)}\Delta^{(\nu)} + \frac{2\Theta c_A^2}{3c^2(1+w)}\mathcal{B}^{(\nu)}, \tag{65}$$

and

$$\dot{\mathcal{B}}^{(\nu)} = \frac{4}{3(1+w)}\dot{\Delta}^{(\nu)} + \frac{4\Theta}{3(1+w)}\left(\frac{c_s^2}{c^2} - w\right)\Delta^{(\nu)}, \tag{66}$$

† If the unperturbed universe has open spatial sections (i.e. $k = -1$) then $n^2 = \nu^2 + 1$, with $\nu^2 \geqslant 0$. Conversely, when the background model is spatially closed (i.e. $k = +1$) the associated relation is $n^2 = \nu(\nu + 2)$, where now $\nu = 1, 2, 3, \ldots,$ and the fundamental mode corresponds to $\nu = 1$ [19, 20].

where $c_A^2 \equiv H^2/\mu$ is the Alfvén speed characterizing the propagation of hydromagnetic waves.

### 6.2. The radiation era

When radiation dominates $w = c_s^2/c^2 = \frac{1}{3}$ and the energy density of the matter falls as $\mu = \mathbb{M}_R/S^4$ (see equation (13)), suggesting, together with equation (24), that the Alfvén velocity remains constant along the fluid-flow lines (i.e. $\dot{c}_A^2 = 0$). Ignoring the cosmological constant (i.e. $\Lambda = 0$), it is preferable to express equations (64)–(66) with respect to the scale factor, $S(t)$,

$$S^2\frac{\mathrm{d}^2\Delta^{(\nu)}}{\mathrm{d}S^2} = 2\left(1 - \frac{\nu^2 S^2}{2\kappa\mathbb{M}_R c^2}\right)\Delta^{(\nu)} + \frac{2c_A^2}{c^2}\mathcal{K}^{(\nu)} - \frac{c_A^2}{c^2}\left(1 + \frac{3\nu^2 S^2}{2\kappa\mathbb{M}_R c^2}\right)\mathcal{B}^{(\nu)}, \tag{67}$$

$$S\frac{\mathrm{d}\mathcal{K}^{(\nu)}}{\mathrm{d}S} = \Delta^{(\nu)} + \frac{3c_A^2}{2c^2}\mathcal{B}^{(\nu)}, \tag{68}$$

$$\frac{\mathrm{d}\mathcal{B}^{(\nu)}}{\mathrm{d}S} = \frac{\mathrm{d}\Delta^{(\nu)}}{\mathrm{d}S}. \tag{69}$$

In the long-wavelength limit (i.e. $\nu \to 0$, or equivalently $\nu^2 S^2/\kappa\mathbb{M}_R c^2 \ll 1$)† the above system reduces to

$$S^2\frac{\mathrm{d}^2\Delta^{(\nu)}}{\mathrm{d}S^2} = 2\Delta^{(\nu)} + \frac{2c_A^2}{c^2}\mathcal{K}^{(\nu)} - \frac{c_A^2}{c^2}\mathcal{B}^{(\nu)}, \tag{70}$$

$$S\frac{\mathrm{d}\mathcal{K}^{(\nu)}}{\mathrm{d}S} = \Delta^{(\nu)} + \frac{3c_A^2}{2c^2}\mathcal{B}^{(\nu)}, \tag{71}$$

$$\frac{\mathrm{d}\mathcal{B}^{(\nu)}}{\mathrm{d}S} = \frac{\mathrm{d}\Delta^{(\nu)}}{\mathrm{d}S}, \tag{72}$$

and accepts a power-law solution of the form

$$\Delta^{(\nu)}(S) = \sum_z \Delta_z^{(\nu)} z S^z, \tag{73}$$

where $\Delta_z^{(\nu)}$ are arbitrary positive constants. The parameter $z$ satisfies the cubic equation

$$z^3 - z^2 + \left(\frac{c_A^2}{c^2} - 2\right)z - \frac{2c_A^2}{c^2}\left(1 + \frac{3c_A^2}{2c^2}\right) = 0, \tag{74}$$

which has three real roots provided that $c_A^2/c^2 < \frac{3}{11}$ [21] given in trigonometric form by [22]

$$z \simeq \frac{1}{3}\left[1 + 2\sqrt{7}\left(1 - \frac{3c_A^2}{14c^2}\right)\cos\left(\frac{\theta + 2k\pi}{3}\right)\right], \tag{75}$$

with $k = 0, 1, 2$, and

$$\cos\theta \simeq \frac{10}{7\sqrt{7}}\frac{1 + 9c_A^2/4c^2}{1 - 9c_A^2/4c^2}. \tag{76}$$

In the absence of a magnetic field (i.e. $c_A^2 = 0$), expressions (75) and (76) provide the standard solutions $z = 0, 2, -1$ associated with a magnetic-free universe (see, for example,

† During the radiation epoch the scale factor evolves as $S \equiv \beta t^{1/2}$, with $\beta = (4\kappa\mathbb{M}_R c^4/3)^{1/4}$. Considering a physical scale much larger than the horizon (i.e. $\lambda_{phys} \gg d_H$), and taking into account that $\lambda_{phys} \sim S\lambda_{com}$, $\lambda_{com} \sim 1/\nu$ and $d_H \sim ct$, we find that $\nu^2 S^2/\kappa\mathbb{M}_R c^2 \ll 1$ on large scales. Clearly, subhorizon scales are characterized by the reverse inequality.

[23] or [17]). However, the coupling between the field and the 3-curvature obscures the overall magnetic effect upon the growth of the density contrast. As (76) reveals the field increases the cosine term in (75) but at the same time decreases this term's coefficient, with the net effect depending on the field's relative strength. To clarify the magnetic impact we consider the case of a spatially flat (i.e. $\mathcal{K}^{(\nu)} = 0$) perturbed universe. Then the system (70)–(72) reduces to

$$S^2\frac{\mathrm{d}^2\Delta^{(\nu)}}{\mathrm{d}S^2} = 2\Delta^{(\nu)} - \frac{c_A^2}{c^2}B^{(\nu)}, \tag{77}$$

$$\frac{\mathrm{d}B^{(\nu)}}{\mathrm{d}S} = \frac{\mathrm{d}\Delta^{(\nu)}}{\mathrm{d}S}, \tag{78}$$

while the density contrast evolves as

$$\Delta^{(\nu)}(S) = \Delta_1^{(\nu)}S^{z_1} + \Delta_2^{(\nu)}S^{z_2}, \tag{79}$$

where $\Delta_1^{(\nu)}$, $\Delta_2^{(\nu)}$ are constants and

$$z_{1,2} = \frac{1}{2}\left(1 \pm 3\sqrt{1 - \frac{4c_A^2}{9c^2}}\right). \tag{80}$$

When the field is absent we recover the familiar evolution law (i.e. $z_1 = 2$, $z_2 = -1$) of a magnetic-free universe. Generally, however, the large-scale magnetic effect is to reduce the growth rate of the density contrast in proportion to its relative strength.

Conversely, the evolution of short-wavelength (i.e. $\nu \to \infty$, or equivalently $\nu^2S^2/\kappa\mathbb{M}_Rc^2 \gg 1$) density aggregations is governed by the following set of equations:

$$S^2\frac{\mathrm{d}^2\Delta^{(\nu)}}{\mathrm{d}S^2} = -\frac{\nu^2S^2}{\kappa\mathbb{M}_Rc^2}\Delta^{(\nu)} + \frac{2c_A^2}{c^2}\mathcal{K}^{(\nu)} - \frac{3\nu^2c_A^2S^2}{2\kappa\mathbb{M}_Rc^4}\mathcal{B}^{(\nu)}, \tag{81}$$

$$S\frac{\mathrm{d}\mathcal{K}^{(\nu)}}{\mathrm{d}S} = \Delta^{(\nu)} + \frac{3c_A^2}{2c^2}\mathcal{B}^{(\nu)}, \tag{82}$$

$$\frac{\mathrm{d}\mathcal{B}^{(\nu)}}{\mathrm{d}S} = \frac{\mathrm{d}\Delta^{(\nu)}}{\mathrm{d}S}. \tag{83}$$

The lack of a general analytic solution forces us to ignore the effects of the spatial curvature. In this case the remaining equations accept the solution

$$\Delta^{(\nu)}(S) = \Delta_1^{(\nu)}\sin\left(\frac{\nu S}{c\sqrt{\kappa\mathbb{M}_R}}\sqrt{1+\frac{3c_A^2}{2c^2}}\right) + \Delta_2^{(\nu)}\cos\left(\frac{\nu S}{c\sqrt{\kappa\mathbb{M}_R}}\sqrt{1+\frac{3c_A^2}{2c^2}}\right). \tag{84}$$

So, small-scale matter aggregations oscillate with period $2\pi c\sqrt{\kappa\mathbb{M}_R}/\nu\sqrt{1+3c_A^2/2c^2}$. Relative to the non-magnetized case (see [23] for example), the excess pressure supplied by the field has simply increased the oscillation frequency of the density contrast.

Conclusively, the presence of a cosmological magnetic field during the radiation era does not cause significant changes in the evolutionary patterns of the density gradients. However, as far as their actual growth is concerned, the field impact is evidently negative, although still secondary to the effects induced by the pressure of the relativistic matter, which still dominates their evolution.

### *6.3. The dust era*

After the radiation era ends, dust dominates and the energy density evolves as $\mu = \mathbb{M}_D/S^3$ with $\dot{\mathbb{M}}_D = 0$. Thus, for vanishing cosmological constant the scale factor changes as $S = \alpha t^{2/3}$, where $t$ measures the observer's time and $\alpha \equiv (3\kappa\mathbb{M}_D c^4/4)^{1/3}$. Also, $\Theta = 2/t$ and $\mu = 4/3\kappa c^4 t^2$. During this period the Alfvén velocity falls as $c_A^2 = \mathbb{E}/\alpha t^{2/3}$, with $\dot{\mathbb{E}} = 0$, reflecting the fact that the magnetic energy density drops faster than that of the matter. So, relative to a reference frame comoving with the expanding fluid, equations (64)–(66) become

$$\frac{d^2\Delta^{(\nu)}}{dt^2} = -\frac{4}{3t}\frac{d\Delta^{(\nu)}}{dt} + \frac{2}{3t^2}\Delta^{(\nu)} + \frac{8\mathbb{E}}{9c^2\alpha t^{8/3}}\mathcal{K}^{(\nu)} - \frac{2\mathbb{E}}{3c^2\alpha t^{8/3}}\left(1 + \frac{3\nu^2c^2t^{2/3}}{4\alpha^2}\right)\mathcal{B}^{(\nu)}, \tag{85}$$

$$\frac{d\mathcal{K}^{(\nu)}}{dt} = \frac{4\mathbb{E}}{3c^2\alpha t^{5/3}}\mathcal{B}^{(\nu)}, \tag{86}$$

$$\frac{d\mathcal{B}^{(\nu)}}{dt} = \frac{4}{3}\frac{d\Delta^{(\nu)}}{dt}. \tag{87}$$

On superhorizon scales (i.e. $\nu \to 0$, or equivalently $\nu^2c^2t^{2/3}/\alpha^2 \ll 1$)† the above system simplifies into

$$\frac{d^2\Delta^{(\nu)}}{dt^2} = -\frac{4}{3t}\frac{d\Delta^{(\nu)}}{dt} + \frac{2}{3t^2}\Delta^{(\nu)} + \frac{8\mathbb{E}}{9c^2\alpha t^{8/3}}\mathcal{K}^{(\nu)} - \frac{2\mathbb{E}}{3c^2\alpha t^{8/3}}\mathcal{B}^{(\nu)}, \tag{88}$$

$$\frac{d\mathcal{K}^{(\nu)}}{dt} = \frac{4\mathbb{E}}{3c^2\alpha t^{5/3}}\mathcal{B}^{(\nu)}, \tag{89}$$

$$\frac{d\mathcal{B}^{(\nu)}}{dt} = \frac{4}{3}\frac{d\Delta^{(\nu)}}{dt}. \tag{90}$$

To obtain an analytic solution, we assume that the perturbed universe has flat spatial sections (i.e. $\mathcal{K} = 0$) and also consider the special case where $\mathcal{B}^{(\nu)} = 4\Delta^{(\nu)}/3$. Then, we are left with the following differential equation‡:

$$\frac{d^2\Delta^{(\nu)}}{dt^2} = -\frac{4}{3t}\frac{d\Delta^{(\nu)}}{dt} + \frac{2}{3t^2}\left(1 - \frac{4\mathbb{E}}{3c^2\alpha t^{2/3}}\right)\Delta^{(\nu)}. \tag{91}$$

Notice that at later times (i.e. $t \to \infty$) the magnetic term in the parenthesis becomes completely irrelevant. So, in agreement with [1], we recover the power-law evolution

$$\Delta^{(\nu)} = \Delta_-^{(\nu)}t^{-1} + \Delta_+^{(\nu)}t^{2/3}, \tag{92}$$

also familiar from the study of a non-magnetized cosmological model. The alternative early time solution

$$\Delta^{(\nu)}(t) = \left[\Delta_1^{(\nu)}\sin\left(\frac{\epsilon}{t^{1/3}}\right) + \Delta_2^{(\nu)}\cos\left(\frac{\epsilon}{t^{1/3}}\right)\right]\epsilon t^{1/3} + \left[\Delta_1^{(\nu)}\cos\left(\frac{\epsilon}{t^{1/3}}\right) - \Delta_2^{(\nu)}\sin\left(\frac{\epsilon}{t^{1/3}}\right)\right]\left(t^{2/3} - \tfrac{1}{3}\epsilon^2\right), \tag{93}$$

where $\epsilon \equiv 2\sqrt{2\mathbb{E}}/c\sqrt{\alpha}$, suggests that under the magnetic influence the long-wavelength aggregations of the material component oscillate with an amplitude that increases as $t^{2/3}$.

† The post-equilibrium evolution of the scale factor implies that the long-wavelength condition $\lambda_{phys} \gg d_H$ translates into $\nu^2c^2t^{2/3}/\alpha^2 \ll 1$.

‡ According to equation (90), the condition $\mathcal{B}^{(\nu)}/\Delta^{(\nu)} = \frac{4}{3}$ requires that the same ratio holds at the initial moment too. In other words, solutions (92) and (93) presume that, as the large-scale spatial variations in the magnetic and the fluid energy densities enter the post-equilibrium era, their ratio equals 4/3. Such a simplifying step is not unreasonable at all since, as equation (69) suggests, $B^{(\nu)} \sim \Delta^{(\nu)}$ by the end of the radiation era.

On scales well below the horizon (i.e. $\nu \to \infty$, or equivalently $\nu^2 c^2 t^{2/3}/\alpha^2 \gg 1$), equations (85)–(87) become

$$\frac{\mathrm{d}^2\Delta^{(\nu)}}{\mathrm{d}t^2} = -\frac{4}{3t}\frac{\mathrm{d}\Delta^{(\nu)}}{\mathrm{d}t} + \frac{2}{3t^2}\Delta^{(\nu)} + \frac{8\mathbb{E}}{9c^2\alpha t^{8/3}}\mathcal{K}^{(\nu)} - \frac{\nu^2\mathbb{E}}{2\alpha^3 t^2}\mathcal{B}^{(\nu)}, \tag{94}$$

$$\frac{\mathrm{d}\mathcal{K}^{(\nu)}}{\mathrm{d}t} = \frac{4\mathbb{E}}{3c^2\alpha t^{5/3}}\mathcal{B}^{(\nu)}, \tag{95}$$

$$\frac{\mathrm{d}\mathcal{B}^{(\nu)}}{\mathrm{d}t} = \frac{4}{3}\frac{\mathrm{d}\Delta^{(\nu)}}{\mathrm{d}t}. \tag{96}$$

Again, by ignoring any effects from the spatial curvature we obtain the following power-law evolution for the density contrast

$$\Delta^{(\nu)}(t) = \Delta_1^{(\nu)} t^{z_1} + \Delta_2^{(\nu)} t^{z_2}, \tag{97}$$

with

$$z_{1,2} = -\frac{1}{6}\left(1 \pm 5\sqrt{1 - \frac{24\nu^2\mathbb{E}}{25\alpha^3}}\right). \tag{98}$$

Notice that in the absence of the magnetic field (i.e. when $\mathbb{E} = 0$) we are left with the well known solution (i.e. $z_{1,2} = \frac{2}{3}, -1$) of the magnetic-free case. In quantitative agreement with Ruzmaikina and Ruzmaikin we find that the field presence reduces the growth rate of the inhomogeneities proportionally to the ratio $t^{2/3}H^2/\mu c^2$.

We conclude by arguing that the presence of a cosmological magnetic field always opposes the growth of matter aggregations, either by forcing them to oscillate or by reducing their growth rate. The magnetic influence ceases only at the later stages of the dust era, when the relative strength of the field becomes negligibly small. It should be emphasized that result (97) refers to wavelengths that lie within the horizon but are much larger than the Jeans length at the time. Otherwise the pressure effects of the ordinary non-relativistic matter become important, preventing the density gradients from growing. This fact, together with the oscillatory nature of solution (93), suggests that earlier in the dust era any actual growth is confined to scales comparable to the horizon size at the time.

## 7. Conclusions

We have explored the influence of a primordial magnetic field upon the kinematical and the dynamical evolution of perturbed cosmological models containing perfect fluids with non-vanishing pressure. We employed the Ellis–Bruni covariant and gauge-invariant formalism, first applied to the analysis of magnetized cosmologies in [1], to derive the full set of equations determining the linear evolution of an almost-FFRW universe containing a perfectly conducting medium. Relative to the dust era examined in [1], the principal new complexities are due to changes in the observer's motion under the simultaneous action of the perturbed medium and the magnetic field. These changes are best seen in the different form of the momentum density conservation law (see equation (14)), which in turn implies a modified acceleration for the fluid. In fact, this is the reason for essentially all the extra complications in the evolutionary patterns of the pre-equilibrium era. We have quantified the magnetohydrodynamical effects upon the kinematics and the dynamics of a universe dominated by a barotropic perfect fluid. We found an acceleration that depends on density gradients as well as on the gradients of the field. It is no longer normal to the field vector and can have subtle effects upon the evolution of fundamental cosmological parameters. Of particular interest is the first-order magneto-geometrical contribution to the deceleration parameter. We show that, unlike ordinary matter which always slows the expansion down, the magnetic field can act as a driving force through its interaction with the geometry of the spatial sections. An analogous effect is found upon the Ricci scalar of the observer's instantaneous rest space. On large scales, the magnetic field tends to smooth out the curvature of the spacelike surfaces and restore their initial flatness.

As in [1], we were primarily interested in studying the growth of density inhomogeneities in a magnetized environment. Here, we have defined four scalar variables that measure spatial variations in the energy densities of the medium ($\Delta$) and the magnetic field ($\mathcal{B}$), spatial inhomogeneities in the expansion ($\mathcal{Z}$) and deviations from the spatial flatness of the background universe ($\mathcal{K}$). We provide a system of four linear first-order differential equations that describes the evolution of these disturbances and ultimately dictates the behaviour of spatial matter aggregations. We have obtained analytic solutions both at the long- and at the short-wavelength limit during the radiation and the dust eras. In [1] we argued for the relative unimportance of the field during the dust era and on scales that exceed the horizon at the time. Here, we were able to confirm and also refine those results. More specifically, we have found that any magnetic effects upon long-wavelength matter aggregations cease completely as the dust era enters its later stages. During this period the inhomogeneities grow exactly as those in a non-magnetized universe. Soon after equilibrium however, the extra pressure of the field could have forced the density gradients to oscillate, thus preventing them from growing. Nevertheless, the weakness of the field means that such large-scale oscillations are short-lived and that the epoch of unimpeded growth begins almost immediately after equilibrium. On scales smaller than the horizon, but larger than the Jeans length associated at the time, the disturbances undergo a power-law growth but at a slower pace relative to the magnetic-free case. The field pressure also affects the evolution of the density contrast during the radiation era. Here, it adds to the pressure of the relativistic matter and impedes any further gravitational clumping of the medium. On large scales we have found that the field inhibits the growth of the inhomogeneities by an amount proportional to its relative strength, whereas on subhorizon regions it increases the frequency of their oscillations. In the radiation era the magnetic effect supplements that from the pressure of the relativistic matter. During this period, the fate of small-scale inhomogeneities is affected by plasma processes [5].

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## Appendix A. Gauge invariance for the 3-gradients of spatial vectors

In [1] (see appendix A therein) it was stated that the metric of an exact FRW or Bianchi-I spacetime can always be brought into the diagonal form

$$g_{ij} = \mathrm{diag}(g_{00}, g_{11}, g_{22}, g_{33}), \tag{A1}$$

with its components being functions of proper-time only (i.e. $g_{ii} = g_{ii}(t)$, no summation over $i$). Based on this we then proceeded to prove the gauge invariance of $\mathcal{M}_{ij}$, the spatial tensor that describes the variations of the magnetic field vector as seen by two neighbouring fundamental observers. However, though our initial statement is correct within a Bianchi-I and a spatially flat FRW cosmology (see, for example, [24] for verification), it cannot be extended to spatially curved FRW spacetimes. So, the gauge independence of the magnetic 3-gradients has been established simply within perturbed FFRW and Bianchi-I universes. What we actually showed in [1] was that the spatial flatness of the aforementioned spacetimes (together with their zero rotation) is sufficient for the 3-gradients of any homogeneous spacelike vector field to be gauge invariant. Next we argue that within the limits of an FRW model such a requirement is also necessary. Indeed, in the observer's rest space the Ricci identity takes the form (see appendix B)

$$ {}^{(3)}\nabla_{[i}{}^{(3)}\nabla_{j]}v_k = -\frac{1}{c^2}\omega_{ij}h_k{}^q\dot{v}_q + \frac{1}{2}{}^{(3)}R_{qkji}v^q, \tag{A2} $$

where $v_iu^i = 0$. The above equation, which is presented here in its exact form, clearly states that in a non-rotating spacetime (i.e. $\omega_{ij} = 0$) the 3-gradients of any spacelike vector vanish (i.e. ${}^{(3)}\nabla_i v_j = 0$) only when ${}^{(3)}R_{ijkq}v^i = 0$. Contracting the latter over the indices $j$ and $q$ we obtain the new restriction ${}^{(3)}R_{ij}v^j = 0$. In an FRW spacetime ${}^{(3)}R_{ij} = {}^{(3)}Rh_{ij}/3$, which means that the gauge invariance of ${}^{(3)}\nabla_i v_j$ requires ${}^{(3)}R$ to vanish. This in turn ensures that ${}^{(3)}R_{ijkq} = 0$ and therefore the spatial flatness of the model. Clearly, the introduction of 3-vectors into the spatially isotropic Friedmannian cosmologies is only an approximation. Nevertheless, the 3-gradients of such a homogeneous spacelike vector, cannot be treated as gauge-independent variables unless the FRW cosmology is spatially flat.

## Appendix B. Auxiliary relations

Following [14] we point out that generally the operator ${}^{(3)}\nabla_i$ cannot be treated as the standard covariant derivative of a three-dimensional hypersurface because in a rotating spacetime the *defect tensor* does not vanish. Thus one cannot assume the usual commutation relations but should use expressions that include possible rotational terms. A selection of such formulae can be found in [14] (see appendix A therein). Here we present only those essential to our analysis.

Commutations between the spatial gradients of scalars and spacelike vectors are given by, respectively,

$$ {}^{(3)}\nabla_{[i}{}^{(3)}\nabla_{j]}f = -\frac{1}{c^2}\omega_{ij}\dot{f}, \tag{B1} $$

and

$$ {}^{(3)}\nabla_{[i}{}^{(3)}\nabla_{j]}v_k = -\frac{1}{c^2}\omega_{ij}h_k{}^q\dot{v}_q + \frac{1}{2}{}^{(3)}R_{qkji}v^q, \tag{B2} $$

where $f$ can be any scalar and $v_i$ is a spatial vector (i.e. $v_iu^i = 0$). Equation (B2) is also regarded as the general expression of the 3-Ricci identity. Commutations between the spatial gradients and the time derivatives of these quantities are governed by

$$ {}^{(3)}\nabla_i\dot{f} - h_i{}^j\left({}^{(3)}\nabla_j f\right)^{\cdot} = -\frac{1}{c^2}\dot{f}a_i + \frac{1}{3}\Theta{}^{(3)}\nabla_i f + {}^{(3)}\nabla_j f\left(\sigma^j{}_i + \omega^j{}_i\right), \tag{B3} $$

and

$$ {}^{(3)}\nabla_i\dot{v}_j - h_i{}^k h_j{}^q\left({}^{(3)}\nabla_k v_q\right)^{\cdot} = \tfrac{1}{3}\Theta{}^{(3)}\nabla_i v_j, \tag{B4} $$

where the latter appears here in its linearized form and applies only to first order (i.e. $v_i \equiv 0$ in the background) spacelike vectors. Commutator (B4) provides an additional first-order relation, which plays an important role in our analysis. In particular, assuming that ${}^{(3)}\nabla_i f$ vanishes in the background, we may linearize the 3-divergence of (B3) to obtain

$$ \left({}^{(3)}\nabla^2 f\right)^{\cdot} - {}^{(3)}\nabla^2\dot{f} = \frac{1}{c^2}\dot{f}A - \frac{2\Theta}{3}{}^{(3)}\nabla^2 f, \tag{B5} $$

recalling that $A = {}^{(3)}\nabla_i a^i$ to first order. The above is used to derive the evolution formula of $\mathcal{B}$, the scalar that describes spherically symmetric changes in the energy density of the magnetic field.

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