Gauge Invariant Perturbations of General Spherically Symmetric Spacetimes

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In this paper, the gauge choices in general spherically symmetric spacetimes are explored. In particular, we construct the gauge invariant variables and the master equations for both the Detweiler easy gauge and the Regge-Wheeler gauge, respectively. The particular cases for $l = 0, 1$ are also investigated. Our results provide analytical calculations of metric perturbations in general spherically symmetric spacetimes, which can be applied to various cases, including the effective-one-body problem. A simple example is presented to show how the metric perturbation components are related to the source perturbation terms.

I. INTRODUCTION

Metric perturbations of spacetimes are an important issue. The solution of the Einstein field equations (EFEs) for static, vacuum and spherically symmetric spacetime is the Schwarzschild spacetime. And the metric perturbations of the Schwarzschild black hole have been studied for a long time. To begin with, Regge, Wheeler and Vishveshwara et.al studied the odd-parity perturbation [1, 2], while Zerilli and Moncrief investigated the even-parity perturbation [3–5]. The perturbation theory of the Schwarzschild spacetime has been well summarized in Chandrasekhar's monograph [6]. After decades of research and development, this theory can be applied to a variety of different physical problems. A useful application is the quasi-normal modes of the perturbed black holes, which was initiated by Vishveshwara [7], Chandrasekhar [8] and Mashhoon [9] et.al., and the review articles of this topic can be found in [10–15]. Another application is studying a particle moving around the Schwarzschild black hole. One can treat this point-particle as a perturbation of the Schwarzschild spacetime [16, 17]. In addition, studying the metric perturbation can promote the analysis of the stability of the Schwarzschild spacetime [18–20].

In perturbation theory in general relativity, the redundant coordinate freedom can be eliminated by choosing specific gauges. The most familiar gauge in the Schwarzschild spacetime is the Regge-Wheeler (RW) gauge, which was first presented by Regge and Wheeler [1]. And where they also analysed the spherical harmonics and decomposed the general perturbation in the Schwarzschild spacetime into odd-parity and even-parity sectors. The RW gauge has the obvious advantage of algebraic simplicity, and it is widely used in the literature. Since then, the construction and the physical meaning of the gauge-invariant properties have attracted lots of attention. Using Lagrangian and Hamiltonian variational principles for the perturbation, Moncrief considered that the metric perturbations can be decomposed into the gauge invariant part and the gauge dependent part [5, 21]. Gerlach and Sengupta discussed the construction of gauge invariant properties in general spherically symmetric spacetimes [22, 23]. Thorne reviewed and summarized various scalar, vector and tensor spherical harmonics with a uniform notation [24]. Martel and Poisson presented a gauge-invariant and covariant formalism, and also showed that the energy or angular-momentum radiation can be expressed in terms of gauge-invariant scalar functions [25]. Recently, Sopuerta considered that the master functions are linear combinations of the metric perturbations and their first-order derivatives, and discussed about the master equation for vacuum spherically symmetric spacetimes [26]. Besides the RW gauge, there exists a variety of gauge choices. For example, the light-cone gauge, which presented by Preston and Poisson [27], can provide geometrical meaning to the coordinates in perturbed spacetimes. Another gauge choice is named as easy (EZ) gauge [28], which was devised by Detweiler when he considered the gravitational self-force problem in the perturbed Schwarzschild spacetime [29]. In the EZ gauge, the metric perturbation is singular on the black-hole horizon [30].

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Generally speaking, for metric perturbation of a spherically symmetric spacetime, the standard process is to decouple the even-parity and the odd-parity EFEs, obtain the wave equations and then solve the one dimensional Schrödinger-like equation with an effective potential. One of the most important step is to construct the gauge-invariant variable and obtain the master equation. When the background metric takes the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

the problems have been thoroughly studied for the theories of Einstein [31, 32], Einstein-Maxwell [33, 34], and Lovelock [34–36] in higher dimensions. However, if one considers a non-vacuum spherical static black hole with hairs [37–39], the metric in general cannot be cast in the above form. Instead, the most general spherically symmetric static spacetimes should be described by the metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2}$$

where $A \cdot B \neq 1$, for which perturbations have not been studied in detail so far. In particular, based on the Post-Newtonian (PN) approximation, Darmour et.al investigated the gravitational radiation generated by inspiralling compact binary systems and presented a novel approach to map the two-body problem onto an effective-one-body (EOB) system [40, 41]. Recently, the discussions of self-consistent of radiation-reaction force in the EOB system shows that one should first solve the gravitational perturbation in the most general shyerically symmetric spacetimes [42]. Therefore, a natural question is how to construct gauge-invariant perturbation variables in the most general spherically symmetric spacetimes (2), and then study the even-parity and odd-parity perturbations.

With the above considerations as our main motivations, in this paper we consider the most general spherically symmetric background spacetimes with metric perturbations, and the construction of gauge invariant variables. For even-parity perturbations, we find that there exist several gauge choices, including the EZ gauge and the RW gauge. Under the EZ gauge, we construct the gauge-invariant variables and obtain a third-order master equations. However, the third-order equation can be written as a second-order equation after the separation of radial and time variables. Under the RW gauge, a similar situation also occurs. For the odd-parity perturbations, the master equation remains a second-order wave equation as usual. It should be noted that such developed formulas are not only applicable to the most general EOB system, as pointed above [42], but also to other modified theories of gravity, in which the background is described by the most general metric (2). These include theories with high-order derivative terms [43–45]. In such theories, the field equations can be always written as $G_{\mu\mu} = \kappa T_{\mu\nu}^{\text{eff.}}$, where $T_{\mu\nu}^{\text{eff.}}$ represent the modifications to general relativity (GR). Certainly, in such theories extra fields are often introduced. In the latter, we need to consider not only the effective Einstein field equations, but also the equations for matter fields. In this paper we shall mainly focus on the effective Einstein field equations [cf. Eq.(20) to be given below and other components given in Appendix B], that is, the perturbations of the most general spherically symmetric metric, and leave the studies of perturbations for matter field equations to another occasion, as the latter will be involved with specific modified theories.

Considering the general metric perturbations in static spherically symmetric spacetimes, Throne showed how to construct a ten-spherical-harmonic basis [24]. Through this paper, we use the A-K notation [28], which dealt only with the Schwarzschild spacetime as the background, and was first presented by Detweiler when he considered the self-force problem. The advantage of using this notation is that one can find the relation between the metric perturbation components and the gauge invariants. In this paper, through the gauge invariants representing different combinations of the metric components, we show that the gauge-invariant variables have the similar structure under the EZ and RW gauges.

The rest of this paper is organized as follows. In Sect. II, we discuss the basic framework. First, the ten orthogonal harmonics basis are introduced. Then the decomposition of non-vacuum Einstein equations and the A-K notation are reviewed. After the investigation of gauge freedom, we consider the EZ and RW gauges. In Sect. III, we first consider the gauge invariant properties. Then we focus on constructing the master equations for both even-parity and odd-parity perturbations. We also study the cases for $l = 0, 1$. In Sect. IV, an example that a small particle goes around a circular orbit in spherically symmetric spacetimes is investigated. Finally, we summarize our main results with some discussions.

Throughout this paper, we use the A-K notation similarly to [28]. Units will be chosen in which $c = G = 1$. In Sect. III, the subscript, e.g., ψ_0 and ψ_1, always represents quantities for $l = 0$ and $l = 1$ cases, respectively. And the superscript with Roman letters, i.e. χ^{I} and χ^{II}, represent the quantities under the EZ gauge or under the RW gauge, respectively.

II. BASIC FRAMEWORK

A. Orthogonal Harmonics Basis

Let us start with the most general spherically symmetric spacetimes

$$ds^2 = g_{ab}^{(0)} dx^a dx^b = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3}$$

To decompose tensor fields on the above background, we choose the orthogonal basis composed of scalar spherical harmonics, vector harmonics and tensor harmonics. First, we define two unnormalized and orthogonal co-vectors v and n

$$v_a = (-1, 0, 0, 0), \qquad n_a = (0, 1, 0, 0), \tag{4}$$

the projection operator onto the sphere surface

$$\Omega_{ab} = g_{ab}^{(0)} + e^{2\Phi} v_a v_b - e^{2\Lambda} n_a n_b = r^2 \text{diag}(0, 0, 1, \sin^2\theta), \tag{5}$$

and the spatial Levi-Civita tensor, $\epsilon_{abc} \equiv v^d \epsilon_{dabc}$, where $\epsilon_{tr\theta\phi} = e^{\Phi+\Lambda} r^2 \sin\theta$.

In general, the complete basis on the 2-sphere are constructed by 1-scalar spherical harmonic, $Y^{lm} = Y^{lm}(\theta, \varphi)$, 3 pure-spin vector harmonics and 6 tensor harmonics [24]. The pure-spin vector harmonics are given by

$$Y_a^{E,lm} = r\nabla_a Y^{lm}, \quad Y_a^{B,lm} = r\epsilon_{ab}{}^c n^b \nabla_c Y^{lm}, \quad Y_a^{R,lm} = n_a Y^{lm}. \tag{6}$$

And the pure-spin tensor harmonics are given by

$$T_{ab}^{T0,lm} = \Omega_{ab} Y^{lm}, \qquad T_{ab}^{L0,lm} = n_a n_b Y^{lm}, \tag{7}$$

$$T_{ab}^{E1,lm} = rn_{(a}\nabla_{b)} Y^{lm}, \qquad T_{ab}^{B1,lm} = rn_{(a}\epsilon_{b)c}{}^d n^c \nabla_d Y^{lm}, \tag{8}$$

$$T_{ab}^{E2,lm} = r^2\left(\Omega_a{}^c\Omega_b{}^d - \frac{1}{2}\Omega_{ab}\Omega^{cd}\right)\nabla_c\nabla_d Y^{lm}, \qquad T_{ab}^{B2,lm} = r^2\Omega_{(a}{}^c\epsilon_{b)e}{}^d n^e \nabla_c\nabla_d Y^{lm}. \tag{9}$$

Note that the vector harmonics are orthogonal to each other

$$\oint Y_a^{A,lm}(Y_{A',l'm'}^a)^* d\Omega = N^{(vec)}(A, r, l)\delta_{AA'}\delta_{ll'}\delta_{mm'}, \tag{10}$$

with $\{A, A'\} = \{E, B, R\}$ and $N^{(vec)}(A, r, l)$ is the specific normalization factor for vector harmonics. The tensor harmonics are also orthogonal to each other

$$\oint T_{ab}^{A,lm}(T_{A',l'm'}^{ab})^* d\Omega = N^{(ten)}(A, r, l)\delta_{AA'}\delta_{ll'}\delta_{mm'}, \tag{11}$$

with $\{A, A'\} = \{T0, L0, E1, E2, B1, B2\}$ and $N^{(ten)}(A, r, l)$ is the specific normalization factor for tensor harmonics. The expression for these normalization functions $N^{(vec)}$ and $N^{(ten)}$ are given in Appendix A.

B. Decomposition of Linearized Einstein Equations

For perturbed spacetimes, we use h_{ab} to represent the linear perturbation of the background spacetime $g_{ab}^{(0)}$, i.e., the metric of the perturbed spacetime can be written as

$$g_{ab} = g_{ab}^{(0)} + h_{ab}. \tag{12}$$

The background metric and the perturbed metric satisfied the Einstein Field Equations (EFEs)

$$G_{ab}(g^{(0)}) = 8\pi T_{ab}, \tag{13}$$

$$G_{ab}(g^{(0)} + h) = 8\pi(T_{ab} + \mathcal{T}_{ab}), \tag{14}$$

where T_{ab} and $\mathcal{T}_{ab}$ denote the non-vacuum background and the perturbed energy-momentum tensor, respectively. Expanding the EFEs in terms of h_{ab}, we get

$$G_{ab}(g^{(0)} + h) = G_{ab}(g^{(0)}) - \frac{1}{2}E_{ab}(h), \tag{15}$$

where E_{ab} is the linearized Einstein operator

$$
\begin{aligned}
E_{ab}(h) =& \Box h_{ab} + \nabla_a \nabla_b h^c{}_c - 2\nabla_{(a}\nabla^c h_{b)c} + 2R_a{}^c{}_b{}^d h_{cd} - (R_a{}^c h_{bc} + R_b{}^c h_{ac}) \\
& + g_{ab}(\nabla^c \nabla^d h_{cd} - \Box h^d{}_d) - g_{ab}R^{cd}h_{cd} + Rh_{ab} \\
=& -16\pi\mathcal{T}_{ab}.
\end{aligned} \tag{16}
$$

Note that now the Ricci curvature R_{ab} and the scalar curvature R of the background do not vanish in general. If R_{ab} and R vanish, then the background metric would reduce to the Schwarzschild metric, which is the situation discussed in [28].

Detweiler decomposed the harmonic modes of the perturbed metric h_{ab} as

$$
\begin{aligned}
h^{lm}_{ab} =& \mathrm{A}\ v_a v_b Y^{lm} + 2\mathrm{B}\ v_{(a}Y^{E,lm}_{b)} + 2\mathrm{C}\ v_{(a}Y^{B,lm}_{b)} + 2\mathrm{D}\ v_{(a}Y^{R,lm}_{b)} + \mathrm{E}\ T^{T0,lm}_{ab} \\
& + \mathrm{F}\ T^{E2,lm}_{ab} + \mathrm{G}\ T^{B2,lm}_{ab} + 2\mathrm{H}\ T^{E1,lm}_{ab} + 2\mathrm{J}\ T^{B1,lm}_{ab} + \mathrm{K}\ T^{L0,lm}_{ab},
\end{aligned} \tag{17}
$$

where all coefficients A through K are scalar functions of (t, r), which were referred to as the A-K coefficients in [28].

For even-parity (polar part) perturbations with $l \geq 2$, the perturbed metric can be decomposed as

$$
h^{\mathrm{even}}_{ab} = \begin{pmatrix}
\mathrm{A}\,Y_{lm} & -\mathrm{D}\,Y_{lm} & -r\mathrm{B}\,\partial_\theta Y_{lm} & -r\mathrm{B}\,\partial_\phi Y_{lm} \\
\mathrm{Sym} & \mathrm{K}\,Y_{lm} & r\mathrm{H}\,\partial_\theta Y_{lm} & r\mathrm{H}\,\partial_\phi Y_{lm} \\
\mathrm{Sym} & \mathrm{Sym} & r^2\left[\mathrm{E} + \mathrm{F}\left(\partial_\theta^2 + \frac{1}{2}l(l+1)\right)\right]Y_{lm} & r^2\mathrm{F}\left[\partial_\theta\partial_\phi - \cot\theta\partial_\phi\right]Y_{lm} \\
\mathrm{Sym} & \mathrm{Sym} & \mathrm{Sym} & r^2\sin^2\theta\left[\mathrm{E} - \mathrm{F}\left(\partial_\theta^2 + \frac{1}{2}l(l+1)\right)\right]Y_{lm}
\end{pmatrix}. \tag{18}
$$

And for the odd-parity (axial part) perturbations with $l \geq 2$, the perturbed metric can be decomposed as

$$
h^{\mathrm{odd}}_{ab} = e^{-\Phi-\Lambda}\begin{pmatrix}
0 & 0 & r\csc\theta\mathrm{C}\,\partial_\phi Y_{lm} & -r\sin\theta\ \mathrm{C}\,\partial_\theta Y_{lm} \\
0 & 0 & -r\csc\theta\mathrm{J}\,\partial_\phi Y_{lm} & r\sin\theta\ \mathrm{J}\,\partial_\theta Y_{lm} \\
\mathrm{Sym} & \mathrm{Sym} & -r^2\csc\theta\mathrm{G}\left[\partial_\theta\partial_\phi - \cot\theta\partial_\phi\right]Y_{lm} & -\frac{r^2}{2}\mathrm{G}\left[\csc\theta\partial_\phi^2 + \cos\theta\partial_\theta - \sin\theta\partial_\theta^2\right]Y_{lm} \\
\mathrm{Sym} & \mathrm{Sym} & \mathrm{Sym} & r^2\mathrm{G}\left[\sin\theta\partial_\theta\partial_\phi - \cos\theta\partial_\phi\right]Y_{lm}
\end{pmatrix}. \tag{19}
$$

These A-K notations could be linearly transformed into the notation taken by Regge and Wheeler [28]. To get the explicit expression of coefficients A-K, one can project the metric perturbation onto the tensor harmonic basis. We present these coefficients in Appendix A.

Now, we can get the A-K components of any rank-2 tensor in the spherically symmetric background. For example, from Eq.(16), one can write

$$-16\pi\mathcal{T}_{\mathrm{A}} = E_{\mathrm{A}} = e^{4\Phi}\oint E_{ab}(h)(v^a v^b Y^*_{lm})d\Omega, \tag{20}$$

to represent the A-term of $\mathcal{T}_{ab}$ or $E_{ab}(h)$, which along the direction of $v^a v^b Y^*_{lm}$. The expression for E_{A}-E_{K} are given in Appendix B.

C. The Gauge Transformation

Under a gauge transformation, $\tilde{x}^a = x^a + \xi^a$, the first-order metric perturbation h_{ab} would be transformed as

$$\tilde{h}_{ab} = h_{ab} - 2\nabla_{(a}\xi_{b)}. \tag{21}$$

As a vector, ξ^a can be projected onto the pure-spin harmonic basis as

$$\xi_a = \mathrm{P}\ v_a Y_{lm} + \mathrm{R}\ Y^{R,lm}_a + \mathrm{S}\ Y^{E,lm}_a + \mathrm{Q}\ Y^{B,lm}_a, \tag{22}$$

where P, R, S and Q are scalar functions of (t, r). The functions P, R and S describe three degrees of gauge freedom for even-parity perturbations, while the function Q describes one degree of the gauge freedom for odd-parity perturbations. Next we use Δ to represent the A-K projections of $2\nabla_{(a}\xi_{b)}$. For example,

$$\Delta A = A - \tilde{A} = 2e^{4\Phi} \oint \nabla_{(a}\xi_{b)}(v^a v^b Y_{lm}^*)d\Omega, \tag{23}$$

here A and $\tilde{A}$ correspond to the projections of h_{ab} and $\tilde{h}_{ab}$, respectively. Projecting $2\nabla_{(a}\xi_{b)}$ onto the tensor harmonic basis, we obtain the A-K components of the term $2\nabla_{(a}\xi_{b)}$,

$$\Delta A = -2\frac{\partial P}{\partial t} - 2e^{-2\Lambda + 2\Phi}\frac{\partial \Phi}{\partial r}R, \qquad \Delta B = -\frac{\partial S}{\partial t} + \frac{1}{r}P,$$

$$\Delta C = -\frac{\partial}{\partial t}Q, \qquad \Delta D = \left(\frac{\partial}{\partial r} - 2\frac{\partial \Phi}{\partial r}\right)P - \frac{\partial R}{\partial t},$$

$$\Delta E = 2\frac{e^{-2\Lambda}}{r}R - \frac{l(l+1)}{r}S, \qquad \Delta F = \frac{2}{r}S,$$

$$\Delta G = \frac{2}{r}Q, \qquad \Delta H = \frac{1}{r}R + \left(\frac{\partial}{\partial r} - \frac{1}{r}\right)S,$$

$$\Delta J = \left(\frac{\partial}{\partial r} - \frac{1}{r} - \frac{\partial \Phi}{\partial r} - \frac{\partial \Lambda}{\partial r}\right)Q, \qquad \Delta K = \left(2\frac{\partial}{\partial r} - 2\frac{\partial \Lambda}{\partial r}\right)R. \tag{24}$$

D.   Gauge Choices

Generally speaking, ξ^a has four independent functions, representing four degrees of freedom in spherically symmetric spacetimes. Hence by properly choosing these four functions we can work with different gauges. For example, under the gauge transformation, the scalar function $F(t, r)$ would transform as

$$\tilde{F} = F - \Delta F = F - \frac{2}{r}S. \tag{25}$$

By setting $S = rF/2$, one degree of the gauge freedom is fixed, and $\tilde{F} = 0$. Then substituting this S back into Eq.(24), one can move on to eliminate the next degrees of freedom. In even-parity perturbations, properly choosing the functions P, R and S would fix three variables of the metric perturbation. In the odd-parity properly choosing the function Q, one can fix one variable of the metric perturbation.

Note that in the odd-parity sector, ΔG is proportional to Q, but there exists some derivative relation between ΔC, ΔJ and Q. If we want to eliminate $\tilde{G}$ under the gauge transformation, just set $Q = rG/2$ then $\tilde{G} = 0$ and ΔC and ΔJ are uniquely determined. If we eliminate $\tilde{C}$ under the gauge transformation rather than $\tilde{G}$, then Q could be an arbitrary function of r with some integration constant, which could not determine ΔG and ΔJ completely. Such a choice cannot completely fix the gauge freedom. Similarly, in the even-parity sector, in order to fix the gauge completely, one may first fix S from ΔF. Afterward, there are still several choices to fix P and R via ΔA, ΔB, ΔE or ΔH. Below, we would discuss two useful gauge choices.

Regge-Wheeler Gauge. Regge and Wheeler first presented the RW gauge by setting certain RW variables to zero [1]. In the even-parity sector of the Schwarzschild spacetime, to eliminate three gauge freedom, RW set

$$h_0^{RW,\text{even}} = h_1^{RW,\text{even}} = G^{RW} = 0, \tag{26}$$

which corresponds to setting $\tilde{B} = \tilde{F} = \tilde{H} = 0$ in the A-K notation. And in the odd-parity sector, RW set $h_2^{RW} = 0$, which corresponds to setting $\tilde{G} = 0$ [28]. For general spherically symmetric spacetimes, we set

$$S^{RW} = \frac{r}{2}F, \qquad P^{RW} = rB + \frac{r^2}{2}\frac{\partial F}{\partial t},$$

$$R^{RW} = rH - \frac{r}{2}\frac{\partial F}{\partial r}, \qquad Q^{RW} = \frac{r}{2}G, \tag{27}$$

which means

$$\xi_a^{RW} = \left(rB + \frac{r^2}{2}\frac{\partial F}{\partial t}\right)v_a Y_{lm} + \left(rH - \frac{r}{2}\frac{\partial F}{\partial r}\right)Y_a^{R,lm} + \frac{r}{2}FY_a^{E,lm} + \frac{r}{2}GY_a^{B,lm}. \tag{28}$$

It should note that this gauge choice is exactly the same as the gauge choice with the A-K notation in the Schwarzschild spacetime, see, for example, Eq.(6.7) in [28].

Easy gauge. The EZ gauge was first introduced by Detweiler when he considered the self-force problem, in which the following metric components are set to zero,

$$\tilde{B} = \tilde{E} = \tilde{F} = \tilde{G} = 0. \tag{29}$$

To eliminate these metric components, the components of the gauge vector are chosen as

$$
\begin{aligned}
S^{EZ} &= \frac{r}{2}F, & P^{EZ} &= rB + \frac{r^2}{2}\frac{\partial F}{\partial t}, \\
R^{EZ} &= \frac{r}{4}l(l+1)e^{2\Lambda}F + \frac{r}{2}e^{2\Lambda}E, & Q^{EZ} &= \frac{r}{2}G,
\end{aligned}
\tag{30}
$$

which means

$$\xi_a^{EZ} = \left(rB + \frac{r^2}{2}\frac{\partial F}{\partial t}\right)v_a Y_{lm} + \left(rl(l+1)e^{2\Lambda}F + \frac{r}{2}e^{2\Lambda}E\right)Y_a^{R,lm} + \frac{r}{2}FY_a^{E,lm} + \frac{r}{2}GY_a^{B,lm}. \tag{31}$$

III.   GAUGE INVARIANTS AND MASTER EQUATIONS

In this section, we would first introduce a general set of gauge invariants in the spherically symmetric backgrounds. With these gauge invariants, we shall investigate how to construct the single master equation under certain gauge choices.

A.   Gauge Invariants

Generally speaking, under any arbitrary gauge transformation, the gauge invariants can be constructed from Eq.(24). For example,

$$\Delta G = \frac{2}{r}Q, \tag{32}$$

$$\Delta J = \left(\frac{\partial}{\partial r} - \frac{1}{r} - \frac{\partial \Phi}{\partial r} - \frac{\partial \Lambda}{\partial r}\right)Q, \tag{33}$$

from which, we obtain

$$\Delta J + \frac{r}{2}\left(\frac{\partial \Phi}{\partial r} + \frac{\partial \Lambda}{\partial r} - \frac{\partial}{\partial r}\right)\Delta G = 0. \tag{34}$$

The above equation indicates that one can define

$$\alpha = J + \frac{r}{2}\left(\frac{\partial \Phi}{\partial r} + \frac{\partial \Lambda}{\partial r} - \frac{\partial}{\partial r}\right)G, \tag{35}$$

and α is a gauge invariant quantity. Note that there are seven even-parity metric components and three odd-parity metric components in the metric perturbation, while there are three even-parity components and one odd-parity component in ξ^a, which tell us that we can construct four independent even-parity gauge invariants and two independent odd-parity gauge invariants for $l \geq 2$ cases. Following the construction of α, we find that they can be

constructed as

$$\alpha = J + \frac{r}{2}\left(\frac{\partial\Phi}{\partial r} + \frac{\partial\Lambda}{\partial r} - \frac{\partial}{\partial r}\right)G,$$

$$\beta = -C - \frac{r}{2}\frac{\partial}{\partial t}G,$$

$$\chi = H - \frac{1}{2}e^{2\Lambda}E - \frac{l(l+1)}{4}e^{2\Lambda}F - \frac{r}{2}\frac{\partial}{\partial r}F,$$

$$\psi = \frac{1}{2}K - \frac{r}{2}e^{2\Lambda}\frac{\partial\Lambda}{\partial r}E - \frac{1}{2}e^{2\Lambda}E - \frac{r}{2}e^{2\Lambda}\frac{\partial}{\partial r}E - \frac{r}{4}l(l+1)e^{2\Lambda}\frac{\partial\Lambda}{\partial r}F - \frac{1}{4}l(l+1)e^{2\Lambda}F - \frac{r}{4}l(l+1)e^{2\Lambda}\frac{\partial}{\partial r}F,$$

$$\delta = D + \frac{r}{2}e^{2\Lambda}\frac{\partial}{\partial t}E + \left(2r\frac{\partial\Phi}{\partial r} - 1\right)B - r\frac{\partial}{\partial r}B - \frac{r^2}{2}\frac{\partial^2}{\partial t\partial r}F - \left[r - r^2\frac{\partial\Phi}{\partial r} - \frac{r}{4}l(l+1)e^{2\Lambda}\right]\frac{\partial}{\partial t}F,$$

$$\epsilon = -\frac{1}{2}A - \frac{r}{2}e^{2\Phi}\frac{\partial\Phi}{\partial r}E - r\frac{\partial}{\partial t}B - \frac{r}{4}l(l+1)e^{2\Phi}\frac{\partial\Phi}{\partial r}F - \frac{r^2}{2}\frac{\partial^2}{\partial t^2}F. \tag{36}$$

These relations are the same as Eq. (7.5) of [28] when the background is vacuum, which degenerates to the Schwarzschild spacetime.

From now on, we shall work in the coordinates $\tilde{x}^a$. And for the save of simplicity, all the tildes will be dropped from now on. Note that we use the superscript I or II to denote the quantities or parameters under the EZ gauge or the RW gauge, respectively. Adopting the specific EZ gauge, we have

$$B = E = F = G = 0. \tag{37}$$

Then, the gauge invariants become

$$\alpha = J, \qquad\qquad\qquad \beta = -C,$$

$$\chi^{\mathrm{I}} = H, \qquad\qquad\qquad \psi^{\mathrm{I}} = \frac{1}{2}K,$$

$$\delta^{\mathrm{I}} = D, \qquad\qquad\qquad \epsilon^{\mathrm{I}} = -\frac{1}{2}A. \tag{38}$$

Here α and β are not superscripted because they are the same under the EZ and RW gauges. Similarly, adopting the specific RW gauge, we have

$$B = F = H = G = 0, \tag{39}$$

and the even-parity gauge invariants become

$$\chi^{\mathrm{II}} = -\frac{1}{2}e^{2\Lambda}E, \qquad\qquad \psi^{\mathrm{II}} = \frac{1}{2}K - \frac{1}{2}\left(r\Lambda' + 1\right)e^{2\Lambda}E - \frac{r}{2}e^{2\Lambda}\frac{\partial}{\partial r}E,$$

$$\delta^{\mathrm{II}} = D + \frac{r}{2}e^{2\Lambda}\frac{\partial}{\partial t}E, \qquad\qquad \epsilon^{\mathrm{II}} = -\frac{1}{2}A - \frac{1}{2}r\Phi'e^{2\Phi}E. \tag{40}$$

Using these gauges, each A-K projection of the linearized Einstein equations, i.e. Eqs.(B1)-(B10), can be rewritten as a combination of the gauge invariants listed in Eq.(36). The results can be found in Appendix C. It is obvious that under the EZ gauge, the relationship between the gauge invariants and the perturbed metric components seems simpler, hence we first study the master equation under the EZ gauge.

B. Master Equations for $l \geq 2$

Assuming that the perturbation of the stress-energy $\mathcal{T}_{ab}$ is known, i.e. E_{A}-E_{K} are known quantities, now we look for the master equations in terms of gauge invariants.

1. Even-parity Perturbations and the EZ gauge

The Bianchi identities indicate that not all seven even-parity projection equations in Appendix C are independent. It has been shown that there are four independent gauge invariants. Noting the specific structure of the expressions of

E_A^I, E_D^I, E_F^I, E_H^I and E_K^I, we find that one can obtain $\partial \delta^I / \partial t$ from $\partial E_D^I / \partial t$, and ϵ^I from E_A^I or E_F^I. Substituting these relations into $2E_H^I + E_K^I$, after a large but tedious calculation, we find that the coupled partial differential equations for χ^I and ψ^I can be constructed as

$$\frac{\partial}{\partial r}\chi^I = \frac{2e^{2\Lambda-2\Phi}r}{\tau^I\eta^I}\left(\sigma^I\frac{\partial^2}{\partial t^2}\chi^I - 2\frac{\partial^2}{\partial t^2}\psi^I + \frac{1}{2}e^{2\Lambda}r\frac{\partial}{\partial t}E_D^I\right) + \frac{\gamma^I}{r\tau^I}\chi^I + \frac{\rho^I}{r\tau^I}\psi^I$$
$$- \frac{e^{2\Lambda}r}{2\tau^I}\left(e^{2\Lambda}\lambda + 2r\Phi' - 2\right)E_F^I + \frac{e^{2\Lambda}r}{2\tau^I}\left(E_K^I + 2E_H^I\right), \tag{41}$$

$$\frac{\partial}{\partial r}\psi^I = \frac{e^{2\Lambda-2\Phi}r}{\tau^I}\left(\sigma^I\frac{\partial^2}{\partial t^2}\chi^I - 2\frac{\partial^2}{\partial t^2}\psi^I + \frac{1}{2}e^{2\Lambda}r\frac{\partial}{\partial t}E_D^I\right) + \frac{\mu^I}{r\tau^I}\chi^I + \frac{\nu^I}{r\tau^I}\psi^I$$
$$+ \frac{e^{2\Lambda}r}{4\tau^I}\left[\kappa^I E_F^I + \eta^I\left(E_K^I + 2E_H^I\right)\right] - \frac{1}{4}e^{4\Lambda-2\Phi}rE_A^I, \tag{42}$$

where $\lambda = l(l+1)$, and σ^I, τ^I, η^I, γ^I, ρ^I, μ^I, ν^I, κ^I are functions determined by the background spacetime, which can be find in Appendix D. Note that in these two equations, there are no spatial derivatives of the source terms. Now the goal becomes to decouple the gauge invariants χ^I and ψ^I, e.g. Eqs.(41) and (42). Introducing the gauge invariant

$$Z^{I(+)} = \sigma^I\chi^I - 2\psi^I, \tag{43}$$

we find that $Z^{I(+)}$ satisfies the master equation

$$\left[\frac{2e^{2\Lambda-2\Phi}r}{N^I\eta^I}\left(2r(\eta^I - \sigma^I)\sigma^I\frac{\partial^3}{\partial t^2\partial r} + \frac{N_T^I}{N^I\eta^I\tau^I}\frac{\partial^2}{\partial t^2}\right) + \frac{2r\sigma^I\tau^I}{N^I}\frac{\partial^2}{\partial r^2} + \frac{N_R^I}{(N^I)^2}\frac{\partial}{\partial r} + \frac{N_Z^I}{r(N^I)^2\tau^I}\right]Z^{I(+)} = S_{\text{even}}^I, \tag{44}$$

with the source term given by

$$S_{\text{even}}^I = \frac{e^{4\Lambda-2\Phi}r}{N^I}\left[\frac{N_A^I}{2N^I}E_A^I + r\sigma^I\tau^I\frac{\partial}{\partial r}E_A^I + \frac{rN_D^I}{N^I(\eta^I)^2\tau^I}\frac{\partial}{\partial t}E_D^I - \frac{2r^2(\eta^I - \sigma^I)\sigma^I}{\eta^I}\frac{\partial^2}{\partial t\partial r}E_D^I\right] - \frac{2e^{2\Lambda}r^2 M_F^I\sigma^I}{N^I}\frac{\partial}{\partial r}E_F^I$$
$$+ \frac{e^{2\Lambda}r}{(N^I)^2\tau^I}\left[\frac{1}{2}N_F^I E_F^I + N_{HK}^I\left(E_H^I + \frac{1}{2}E_K^I\right)\right] - \frac{e^{2\Lambda}r^2(\eta^I - \sigma^I)\sigma^I}{N^I}\left(2\frac{\partial}{\partial r}E_H^I + \frac{\partial}{\partial r}E_K^I\right). \tag{45}$$

In the above equation, N^I, N_T^I, N_R^I, N_Z^I and N_A^I, N_D^I, M_F^I, N_F^I, N_{HK}^I are functions depending only on the background. The explicit expressions of them can be found in Appendix D. Unlike the well-known Zerilli equation, the master equation under the EZ gauge is a third-order equation. However, rewriting Eq.(44) in the form

$$\left(a\frac{\partial^3}{\partial t^2\partial r} + b\frac{\partial^2}{\partial t^2} + c\frac{\partial^2}{\partial r^2} + d\frac{\partial}{\partial r} + e\right)Z^{I(+)} = S_{\text{even}}^I, \tag{46}$$

setting $Z^{I(+)} = Z^{I(+)}(r)e^{i\omega t}$ and $S_{\text{even}}^I = S_{\text{even}}^I(r)e^{i\omega t}$, we obtain a second-order differential equation

$$\left[c\frac{\partial^2}{\partial r^2} + (d - a\omega^2)\frac{\partial}{\partial r} + (e - b\omega^2)\right]Z^{I(+)}(r) = S_{\text{even}}^I(r). \tag{47}$$

If the background becomes the Schwarzschild spacetime, the master equation (44) will reduce to the result given in [28], in other words, the result of Zerilli [3]. Further discussions about the degeneration of our result could be found in Appendix E.

Once Eq.(44) is solved, together with the definition of the master variable given by Eq.(43), the gauge invariants χ^I and ψ^I can be solved from

$$\chi^I = \frac{1}{N^I}\left[\frac{4e^{2\Lambda-2\Phi}r^2(\eta^I - \sigma^I)}{\eta^I}\frac{\partial^2}{\partial t^2}Z^{I(+)} + 2r\tau^I\frac{\partial}{\partial r}Z^{I(+)} + M_1^I Z^{I(+)} - e^{4\Lambda-2\Phi}r^2\tau^I E_A^I\right.$$
$$\left.+ \frac{2e^{4\Lambda-2\Phi}r^3(\eta^I - \sigma^I)}{\eta^I}\frac{\partial}{\partial t}E_D^I + 2e^{2\Lambda}r^2 M_F^I E_F^I + e^{2\Lambda}r^2(\eta^I - \sigma^I)(E_K^I + 2E_H^I)\right], \tag{48}$$

$$\psi^{\mathrm{I}} = \frac{\sigma^{\mathrm{I}}}{2N^{\mathrm{I}}} \left[\frac{4e^{2\Lambda - 2\Phi} r^2 \left(\eta^{\mathrm{I}} - \sigma^{\mathrm{I}} \right)}{\eta^{\mathrm{I}}} \frac{\partial^2}{\partial t^2} Z^{\mathrm{I}(+)} + 2r\tau^{\mathrm{I}} \frac{\partial}{\partial r} Z^{\mathrm{I}(+)} - M_2^{\mathrm{I}} Z^{\mathrm{I}(+)} - e^{4\Lambda - 2\Phi} r^2 \tau^{\mathrm{I}} E_{\mathrm{A}}^{\mathrm{I}} \right.$$
$$\left. + \frac{2e^{4\Lambda - 2\Phi} r^3 \left(\eta^{\mathrm{I}} - \sigma^{\mathrm{I}} \right)}{\eta^{\mathrm{I}}} \frac{\partial}{\partial t} E_{\mathrm{D}}^{\mathrm{I}} + 2e^{2\Lambda} r^2 M_{\mathrm{F}}^{\mathrm{I}} E_{\mathrm{F}}^{\mathrm{I}} + e^{2\Lambda} r^2 \left(\eta^{\mathrm{I}} - \sigma^{\mathrm{I}} \right) \left(E_{\mathrm{K}}^{\mathrm{I}} + 2E_{\mathrm{H}}^{\mathrm{I}} \right) \right], \tag{49}$$

where M_1^{I} and M_2^{I} are given in Appendix D. Then, the remaining two even-parity gauge invariants δ^{I} and ϵ^{I} can be obtained from Eqs.(C3) and (C5), given respectively by

$$\delta^{\mathrm{I}} = \frac{1}{-2 - e^{2\Lambda} \left(-2 + \lambda \right) + 4r\Lambda'} \left(e^{2\Lambda} r^2 E_{\mathrm{D}}^{\mathrm{I}} + e^{2\Lambda} \lambda r \frac{\partial}{\partial t} \chi^{\mathrm{I}} - 4r \frac{\partial}{\partial t} \psi^{\mathrm{I}} \right), \tag{50}$$

$$\epsilon^{\mathrm{I}} = e^{-2\Lambda + 2\Phi} \left[\left(1 - r\Lambda' + r\Phi' \right) \chi^{\mathrm{I}} - \psi^{\mathrm{I}} + r \frac{\partial}{\partial r} \chi^{\mathrm{I}} + \frac{1}{2} e^{2\Lambda} r^2 E_{\mathrm{F}}^{\mathrm{I}} \right]. \tag{51}$$

From Eq.(38), the even-parity metric perturbation components A, D, H, K could be read out directly.

2. *Even-parity Perturbations and the RW gauge*

Similar to the development provided in the previous subsection, using the field equations $E_{\mathrm{A}}^{\mathrm{II}}$ to $E_{\mathrm{K}}^{\mathrm{II}}$ in Appendix C, one can eliminate the gauge invariants δ^{II} and ϵ^{II}, and then obtain the following coupled equations

$$\frac{\partial}{\partial r} \chi^{\mathrm{II}} = \frac{2e^{2\Lambda - 2\Phi} r}{\tau^{\mathrm{II}} \eta^{\mathrm{II}}} \left(\sigma^{\mathrm{II}} \frac{\partial^2}{\partial t^2} \chi^{\mathrm{II}} - 2 \frac{\partial^2}{\partial t^2} \psi^{\mathrm{II}} + \frac{1}{2} e^{2\Lambda} r \frac{\partial}{\partial t} E_{\mathrm{D}}^{\mathrm{II}} \right) + \frac{\gamma^{\mathrm{II}}}{r\tau^{\mathrm{II}}} \chi^{\mathrm{II}} + \frac{\rho^{\mathrm{II}}}{r\tau^{\mathrm{II}}} \psi^{\mathrm{II}}$$
$$- \frac{e^{2\Lambda} r}{2\tau^{\mathrm{II}}} (e^{2\Lambda} \lambda + 2r\Phi' - 2) E_{\mathrm{F}}^{\mathrm{II}} + \frac{e^{2\Lambda} r}{2\tau^{\mathrm{II}}} \left(2E_{\mathrm{H}}^{\mathrm{II}} + E_{\mathrm{K}}^{\mathrm{II}} \right), \tag{52}$$

$$\frac{\partial}{\partial r} \psi^{\mathrm{II}} = \frac{e^{2\Lambda - 2\Phi} r}{\tau^{\mathrm{II}}} \left(\sigma^{\mathrm{II}} \frac{\partial^2}{\partial t^2} \chi^{\mathrm{II}} - 2 \frac{\partial^2}{\partial t^2} \psi^{\mathrm{II}} + \frac{1}{2} e^{2\Lambda} r \frac{\partial}{\partial t} E_{\mathrm{D}}^{\mathrm{II}} \right) + \frac{\mu^{\mathrm{II}}}{r\tau^{\mathrm{II}}} \chi^{\mathrm{II}} + \frac{\nu^{\mathrm{II}}}{r\tau^{\mathrm{II}}} \psi$$
$$+ \frac{e^{2\Lambda} r}{4\tau^{\mathrm{II}}} \left(\kappa^{\mathrm{II}} E_{\mathrm{F}}^{\mathrm{II}} + 2\eta^{\mathrm{II}} E_{\mathrm{H}}^{\mathrm{II}} + \eta^{\mathrm{II}} E_{\mathrm{K}}^{\mathrm{II}} \right) - \frac{1}{4} e^{4\Lambda - 2\Phi} r E_{\mathrm{A}}^{\mathrm{II}}, \tag{53}$$

where the parameters σ^{II}, τ^{II}, η^{II}, κ^{II}, ρ^{II}, μ^{II}, ν^{II} depend only on the background, and the explicit expressions of them can be found in Appendix D.

We find that the master variable can be similarly constructed as

$$Z^{\mathrm{II}(+)} = \sigma^{\mathrm{II}} \chi^{\mathrm{II}} - 2\psi^{\mathrm{II}}, \tag{54}$$

and then the master equation under the RW gauge is given by

$$\left[\frac{2e^{2\Lambda - 2\Phi} r}{N^{\mathrm{II}} \eta^{\mathrm{II}}} \left(4r^2 (\Lambda' + \Phi') \sigma^{\mathrm{II}} \frac{\partial^3}{\partial t^2 \partial r} + \frac{N_{\mathrm{T}}^{\mathrm{II}}}{N^{\mathrm{II}} \eta^{\mathrm{II}} \tau^{\mathrm{II}}} \frac{\partial^2}{\partial t^2} \right) - \frac{2r\sigma^{\mathrm{II}} \tau^{\mathrm{II}}}{N^{\mathrm{II}}} \frac{\partial^2}{\partial r^2} + \frac{N_{\mathrm{R}}^{\mathrm{II}}}{(N^{\mathrm{II}})^2} \frac{\partial}{\partial r} + \frac{N_{\mathrm{Z}}^{\mathrm{II}}}{r \left(N^{\mathrm{II}} \right)^2 \tau^{\mathrm{II}}} \right] Z^{\mathrm{II}(+)} = S_{\mathrm{even}}^{\mathrm{II}}, \tag{55}$$

with the source term given by

$$S_{\mathrm{even}}^{\mathrm{II}} = \frac{e^{4\Lambda - 2\Phi} r}{N^{\mathrm{II}}} \left[\frac{N_{\mathrm{A}}^{\mathrm{II}}}{2N^{\mathrm{II}}} E_{\mathrm{A}}^{\mathrm{II}} - r\sigma^{\mathrm{II}} \tau^{\mathrm{II}} \frac{\partial}{\partial r} E_{\mathrm{A}}^{\mathrm{II}} + \frac{r N_{\mathrm{D}}^{\mathrm{II}}}{N^{\mathrm{II}} \left(\eta^{\mathrm{II}} \right)^2 \tau^{\mathrm{II}}} \frac{\partial}{\partial t} E_{\mathrm{D}}^{\mathrm{II}} - \frac{4r^3 (\Lambda' + \Phi') \sigma^{\mathrm{II}}}{\eta^{\mathrm{II}}} \frac{\partial^2}{\partial t \partial r} E_{\mathrm{D}}^{\mathrm{II}} \right] - \frac{2e^{2\Lambda} r^2 M_{\mathrm{F}}^{\mathrm{II}} \sigma^{\mathrm{II}}}{N^{\mathrm{II}}} \frac{\partial}{\partial r} E_{\mathrm{F}}^{\mathrm{II}}$$
$$+ \frac{e^{2\Lambda} r}{\left(N^{\mathrm{II}} \right)^2 \tau^{\mathrm{II}}} \left[\frac{1}{2} N_{\mathrm{F}}^{\mathrm{II}} E_{\mathrm{F}}^{\mathrm{II}} + N_{\mathrm{HK}}^{\mathrm{II}} \left(E_{\mathrm{H}}^{\mathrm{II}} + \frac{1}{2} E_{\mathrm{K}}^{\mathrm{II}} \right) \right] - \frac{2e^{2\Lambda} r^3 \left(\Lambda' + \Phi' \right) \sigma^{\mathrm{II}}}{N^{\mathrm{II}}} \left(2 \frac{\partial}{\partial r} E_{\mathrm{H}}^{\mathrm{II}} + \frac{\partial}{\partial r} E_{\mathrm{K}}^{\mathrm{II}} \right), \tag{56}$$

where N^{II}, $N_{\mathrm{T}}^{\mathrm{II}}$, $N_{\mathrm{R}}^{\mathrm{II}}$, $N_{\mathrm{Z}}^{\mathrm{II}}$, $N_{\mathrm{A}}^{\mathrm{II}}$, $N_{\mathrm{D}}^{\mathrm{II}}$, $M_{\mathrm{F}}^{\mathrm{II}}$, $N_{\mathrm{F}}^{\mathrm{II}}$ all depend on the background, which can be found in Appendix D. Similarly to the case under the EZ gauge, the master equation is a third-order equation, and which can be transformed as a second-order differential equation. Note that when the background metric takes the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \tag{57}$$

The relation $\Lambda' + \Phi' = 0$ makes the third-order terms in Eq.(55) vanishes. In the Schwarzschild case, the master equation Eq.(55) will also reduce to the result of Zerilli, see Appendix E. Once Eq.(55) is solved, one can get χ^{II} and ψ^{II} from

$$
\begin{aligned}
\chi^{\mathrm{II}} = \frac{1}{N^{\mathrm{II}}} & \left[\frac{8e^{2\Lambda-2\Phi}r^3\left(\Lambda'+\Phi'\right)}{\eta^{\mathrm{II}}} \frac{\partial^2}{\partial t^2} Z^{\mathrm{II}(+)} - 2r\tau^{\mathrm{II}} \frac{\partial}{\partial r} Z^{\mathrm{II}(+)} + M_1^{\mathrm{II}} Z^{\mathrm{II}(+)} + e^{4\Lambda-2\Phi}r^2\tau^{\mathrm{II}} E_{\mathrm{A}}^{\mathrm{II}} \right. \\
& \left. + \frac{4e^{4\Lambda-2\Phi}r^4\left(\Lambda'+\Phi'\right)}{\eta^{\mathrm{II}}} \frac{\partial}{\partial t} E_{\mathrm{D}}^{\mathrm{II}} + 2e^{2\Lambda}r^2 M_{\mathrm{F}}^{\mathrm{II}} E_{\mathrm{F}}^{\mathrm{II}} + 2e^{2\Lambda}r^3\left(\Lambda'+\Phi'\right)\left(E_{\mathrm{K}}^{\mathrm{II}}+2E_{\mathrm{H}}^{\mathrm{II}}\right) \right],
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
\psi^{\mathrm{II}} = \frac{\sigma^{\mathrm{II}}}{2N^{\mathrm{II}}} & \left[\frac{8e^{2\Lambda-2\Phi}r^3\left(\Lambda'+\Phi'\right)}{\eta^{\mathrm{II}}} \frac{\partial^2}{\partial t^2} Z^{\mathrm{II}(+)} - 2r\tau^{\mathrm{II}} \frac{\partial}{\partial r} Z^{\mathrm{II}(+)} + \frac{M_2^{\mathrm{II}}}{\sigma^{\mathrm{II}}} Z^{\mathrm{II}(+)} + e^{4\Lambda-2\Phi}r^2\tau^{\mathrm{II}} E_{\mathrm{A}}^{\mathrm{II}} \right. \\
& \left. + \frac{4e^{4\Lambda-2\Phi}r^4\left(\Lambda'+\Phi'\right)}{\eta^{\mathrm{II}}} \frac{\partial}{\partial t} E_{\mathrm{D}}^{\mathrm{II}} + 2e^{2\Lambda}r^2 M_{\mathrm{F}}^{\mathrm{II}} E_{\mathrm{F}}^{\mathrm{II}} + 2e^{2\Lambda}r^3\left(\Lambda'+\Phi'\right)\left(E_{\mathrm{K}}^{\mathrm{II}}+2E_{\mathrm{H}}^{\mathrm{II}}\right) \right],
\end{aligned}
\tag{59}
$$

and the remaining two gauge invariants δ^{II} and ϵ^{II} can be obtained from $E_{\mathrm{D}}^{\mathrm{II}}$ and $E_{\mathrm{F}}^{\mathrm{II}}$, which are given respectively by

$$
\delta^{\mathrm{II}} = \frac{1}{-2 - e^{2\Lambda}\left(\lambda-2\right) + 4r\Lambda'} \left[e^{2\Lambda}r^2 E_{\mathrm{D}}^{\mathrm{II}} + r\left(2 + e^{2\Lambda}\left(\lambda-2\right) + 4r\Phi'\right) \frac{\partial}{\partial t}\chi^{\mathrm{II}} - 4r\frac{\partial}{\partial t}\psi^{\mathrm{II}} \right],
\tag{60}
$$

$$
\epsilon^{\mathrm{II}} = e^{-2\Lambda+2\Phi} \left[\left(1 - r\Lambda' + r\Phi'\right)\chi^{\mathrm{II}} - \psi^{\mathrm{II}} + r\frac{\partial}{\partial r}\chi^{\mathrm{II}} + \frac{1}{2}e^{2\Lambda}r^2 E_{\mathrm{F}}^{\mathrm{II}} \right].
\tag{61}
$$

When χ^{II}, ψ^{II}, δ^{II} and ϵ^{II} are solved, the even-parity metric perturbation components A, D, E, K are given by

$$
\mathrm{A} = 2e^{-2\Lambda+2\Phi}r\Phi'\chi^{\mathrm{II}} - 2\epsilon^{\mathrm{II}}, \qquad\qquad \mathrm{D} = \delta^{\mathrm{II}} + r\frac{\partial}{\partial t}\chi^{\mathrm{II}},
$$

$$
\mathrm{E} = -2e^{-2\Lambda}\chi^{\mathrm{II}}, \qquad\qquad \mathrm{K} = 2\psi^{\mathrm{II}} + 2(r\Lambda'-1)\chi^{\mathrm{II}} - 2r\frac{\partial}{\partial r}\chi^{\mathrm{II}}.
\tag{62}
$$

Under the RW gauge, the master variable can be written as the combination of the gauge invariants χ^{II} and ψ^{II}. Note that Zerilli constructed the master variable as the combination of K_{LM} and R_{LM} [3], which correspond to a combination of χ^{II} and δ^{II} in our paper.

3. Odd-parity Perturbations

For the odd-parity perturbations, the EZ gauge and the RW gauge are identical. So, in the following we shall not distinguish them. We construct the odd-parity master variable as

$$
Z^{(-)} = \left(r + r^2\Lambda' + r^2\Phi'\right)\beta - r^2\frac{\partial}{\partial r}\beta + r^2\frac{\partial}{\partial t}\alpha,
\tag{63}
$$

then the master equation is given by

$$
\left\{ -\frac{\partial^2}{\partial t^2} + e^{-2\Lambda+2\Phi}\left[\frac{\partial^2}{\partial r^2} - \left(\frac{X'}{X} + 3\Lambda' + 3\Phi'\right)\frac{\partial}{\partial r} + \frac{N_{\mathrm{odd}}}{r^2} \right] \right\} Z^{(-)} = S_{\mathrm{odd}},
\tag{64}
$$

with the source term given by

$$
S_{\mathrm{odd}} = e^{2\Phi}r^2 \left[\left(\frac{1}{r} + \Lambda' - \Phi' - \frac{X'}{X}\right)E_{\mathrm{C}} + \frac{\partial}{\partial r}E_{\mathrm{C}} + \frac{\partial}{\partial t}E_{\mathrm{J}} \right],
\tag{65}
$$

where X and N_{odd} are functions only depending on the background, and the explicit expressions of which can be found in Appendix D. Once Eq.(64) is solved, together with the structure of the odd-parity master variable Eq.(63) and the perturbed field equations Eqs.(C15) and (C17), we have

$$
\alpha = -\frac{e^{2\Lambda-2\Phi}}{X}\left(e^{2\Phi}r^2 E_{\mathrm{J}} + \frac{\partial}{\partial t}Z^{(-)} \right),
\tag{66}
$$

and

$$
\beta = \frac{1}{X}\left[e^{2\Lambda}r^2 E_{\mathrm{C}} - \left(\frac{1}{r} - 2\Lambda' - 2\Phi'\right)Z^{(-)} - \frac{\partial}{\partial r}Z^{(-)} \right].
\tag{67}
$$

Then, the odd-parity metric perturbation components C and J can be read off from these expressions.

C. Specific Cases for $l = 0, 1$

In the previous subsection, we have discussed the construction of master variables and master equations for even-parity and odd-parity perturbations for the $l \geq 2$ cases. However, the decomposition of the metric perturbation would take some other forms for the specific cases of $l = 0, 1$. In this section, we investigate how to solve the metric perturbation for $l = 0, 1$. Note that in this subsection, we use, e.g. A_0 and A_1, to represent the scalar functions in the metric perturbation and the gauge vector for $l = 0$ and $l = 1$ cases, respectively. And we also use, e.g. $E_A^{(l=0)}$ and $E_A^{(l=1)}$ to represent the projection function E_A for $l = 0$ and $l = 1$ cases, respectively.

1. $l = 0$ case

For the special case $l = 0$, there remains one scalar spherical harmonic function $Y^{00} = \frac{1}{2\sqrt{\pi}}$, which leads the metric perturbation to

$$h_{ab}^{(l=0)} = \frac{1}{2\sqrt{\pi}} (A_0 v_a v_b + 2D_0 v_{(a} n_{b)} + E_0 \sigma_{ab} + K_0 n_a b_b), \tag{68}$$

and the gauge vector takes the form

$$\xi_a^{(l=0)} = \frac{1}{2\sqrt{\pi}} (P_0 v_a + R_0 n_a). \tag{69}$$

In the $l = 0$ case, we only have four metric perturbation components, and all of them are even-parity. Under the gauge transformation, the metric perturbation will be transformed as

$$\Delta A_0 = -2\frac{\partial}{\partial t} P_0 - 2e^{-2\Lambda+2\Phi} \Phi' R_0, \qquad \Delta D_0 = \left(\frac{\partial}{\partial r} - 2\Phi'\right) P_0 - \frac{\partial}{\partial t} R_0,$$

$$\Delta E_0 = 2\frac{e^{-2\Lambda}}{r} R_0, \qquad \Delta K_0 = \left(2\frac{\partial}{\partial r} - 2\Lambda'\right) R_0. \tag{70}$$

The structure of ΔA_0, ΔD_0 and ΔK_0 are the same as the $l \geq 2$ cases, but ΔE_0 is different from the expression given in Eq.(24) since S does not exist for $l = 0$. From Eq.(70), we can construct two gauge invariants

$$\psi_0 = \frac{1}{2} K_0 - \frac{r}{2} e^{2\Lambda} \Lambda' E_0 - \frac{1}{2} e^{2\Lambda} E_0 - \frac{r}{2} e^{2\Lambda} \frac{\partial}{\partial r} E_0,$$

$$o_0 = \frac{1}{2} \frac{\partial}{\partial r} A_0 - \Phi' A_0 + \frac{\partial}{\partial t} D_0 + \frac{1}{2} e^{2\Phi} (\Phi' + r\Phi'') E_0 + \frac{1}{2} e^{2\Phi} r\Phi' \frac{\partial}{\partial r} E_0 + \frac{1}{2} e^{2\Lambda} r \frac{\partial^2}{\partial t^2} E_0. \tag{71}$$

Choosing the gauge $D_0 = E_0 = 0$, we find that $E_A^{(l=0)}$, $E_D^{(l=0)}$, $E_E^{(l=0)}$ and $E_K^{(l=0)}$ are given by

$$E_A^{(l=0)} = 2r^{-2} e^{-2\Lambda} (-1 + e^{2\Lambda} + 2r\Lambda') A_0 + 2r^{-2} e^{-4\Lambda+2\Phi} (4r\Lambda' - 1) K_0 - 2r^{-1} e^{-4\Lambda+2\Phi} \frac{\partial}{\partial r} K_0,$$

$$E_D^{(l=0)} = 2r^{-1} e^{-2\Lambda} \frac{\partial}{\partial t} K_0,$$

$$E_E^{(l=0)} = 2r^{-1} e^{-2\Lambda-2\Phi} (r\Lambda'\Phi' - r\Phi'' - \Phi') A_0 - r^{-1} e^{-2\Lambda-2\Phi} (r\Lambda' + 2r\Phi' - 1) \frac{\partial}{\partial r} A_0 + e^{-2\Lambda-2\Phi} \frac{\partial^2}{\partial r^2} A_0$$

$$+ 2r^{-1} e^{-4\Lambda} (\Phi' + r\Phi'^2 + r\Phi'' - 2r\Lambda'\Phi' - 2\Lambda') K_0 + r^{-1} e^{-4\Lambda} (r\Phi' + 1) \frac{\partial}{\partial r} K_0 + e^{-2\Lambda-2\Phi} \frac{\partial^2}{\partial t^2} K_0,$$

$$E_K^{(l=0)} = -4r^{-1} e^{-2\Phi} \Phi' A_0 + 2r^{-1} e^{-2\Phi} \frac{\partial}{\partial r} A_0 + 2r^{-2} K_0. \tag{72}$$

The gauge invariants ψ_0 and o_0 can be constructed as

$$\psi_0 = \frac{1}{2} K_0, \qquad o_0 = \frac{1}{2} \frac{\partial}{\partial r} A_0 - \Phi' A_0. \tag{73}$$

And then $E_{\mathrm{D}}^{(l=0)}$, $E_{\mathrm{E}}^{(l=0)}$ and $E_{\mathrm{K}}^{(l=0)}$ can be written as

$$E_{\mathrm{D}}^{(l=0)} = 4r^{-1}e^{-2\Lambda}\frac{\partial}{\partial t}\psi_0, \tag{74}$$

$$E_{\mathrm{E}}^{(l=0)} = -2r^{-1}e^{-2\Lambda-2\Phi}(r\Lambda'-1)o_0 + 2e^{-2\Lambda-2\Phi}\frac{\partial}{\partial r}o_0 + 4r^{-1}e^{-4\Lambda}(\Phi'-2\Lambda'+r\Phi'^2+r\Phi''-2r\Lambda'\Phi')\psi_0$$
$$+ 2r^{-1}e^{-4\Lambda}(r\Phi'+1)\frac{\partial}{\partial r}\psi_0 + 2e^{-2\Lambda-2\Phi}\frac{\partial^2}{\partial t^2}\psi_0, \tag{75}$$

$$E_{\mathrm{K}}^{(l=0)} = 4r^{-1}e^{-2\Phi}o_0 + 4r^{-2}\psi_0. \tag{76}$$

From Eq.(76), we have

$$o_0 = \frac{1}{4}e^{2\Phi}\left(rE_{\mathrm{K}}^{(l=0)} - \frac{4}{r}\psi_0\right). \tag{77}$$

Substituting Eqs.(74) and (77) into Eq.(75), we find that ψ_0 satisfied

$$\left(\iota_0\frac{\partial}{\partial r} - \sigma_0\right)\psi_0 = \frac{r}{2}e^{4\Lambda}S^{(l=0)}, \tag{78}$$

where ι_0 and σ_0 are functions depend on background, which can be found in Appendix D, and $S^{(l=0)}$ is the source term,

$$S^{(l=0)} = \frac{1}{2}e^{-2\Phi}r\frac{\partial}{\partial t}E_{\mathrm{D}}^{(l=0)} - E_{\mathrm{E}}^{(l=0)} + \frac{1}{2}e^{-2\Lambda}\left[r\frac{\partial}{\partial r}E_{\mathrm{K}}^{(l=0)} + (2 - r\Lambda' + 2r\Phi')E_{\mathrm{K}}^{(l=0)}\right]. \tag{79}$$

2. $l = 1$ *case*

For the case $l = 1$, the tensor harmonic basis $T_{ab}^{E2,1m}$ and $T_{ab}^{B2,1m}$ vanish, hence the scalar functions F_1 and G_1 would no longer exist. The four components of the gauge vector ξ_a imply that there are only three gauge invariants for even-parity and one gauge invariant for odd-parity. The remaining eight projections of $2\nabla_{(a}\xi_{b)}$ are the same as for $l \geq 2$ cases.

First, we investigate the even-parity sector. As we explain for $l \geq 2$ cases, one should first determine the function S of the gauge vector ξ_a to make F vanish under the gauge transformation. However, when $l = 1$, the lack of F_1 prevents us from constructing the gauge invariants as the cases for $l \geq 2$. The even-parity $l = 1$ metric perturbations are related to the linear momentum of the system [4]. Note that if we take the gauge choice $B_1 = E_1 = H_1 = 0$, which was introduced by Zerilli in the Schwarzschild spacetime, the relation between E_A and metric components would no longer be a simple relationship. However, taking the gauge

$$A_1 = E_1 = H_1 = 0, \tag{80}$$

we find that $E_{\mathrm{A}}^{(l=1)}$ and the metric perturbed component K_1 have a simple relation. In particular, we find

$$E_{\mathrm{A}}^{(l=1)} = -2r^{-1}e^{-4\Lambda+2\Phi}\left(\frac{\partial}{\partial r}K_1 + r^{-1}(1 + e^{2\Lambda} - 4r\Lambda')K_1\right), \tag{81}$$

$$E_{\mathrm{D}}^{(l=1)} = 2r^{-2}(1 - 2r\Phi')B_1 + 2r^{-1}\frac{\partial}{\partial r}B_1 + 2r^{-1}e^{-2\Lambda}\frac{\partial}{\partial t}K_1 - 2r^{-2}e^{-2\Lambda}(1 - 2r\Lambda')D_1, \tag{82}$$

$$E_{\mathrm{K}}^{(l=1)} + 4E_{\mathrm{H}}^{(l=1)} = 2r^{-2}e^{-2\Lambda}(-2 + e^{2\Lambda} - 2r\Phi')K_1 + 4r^{-1}e^{-2\Phi}(1 - e^{2\Lambda})\frac{\partial}{\partial t}B_1 - 4e^{-2\Phi}\frac{\partial^2}{\partial t\partial r}B_1. \tag{83}$$

From Eq.(81), the metric perturbation function K_1 can be found. This solution can be used in Eq.(83) to solve B_1, and then from Eq.(82), D_1 could also be solved. The remaining quantities, such as E_{B} or E_{E}, can be used to check the consistency of the solutions.

Then we investigate the odd-parity sector. For $l \geq 2$ cases, one should determine the function Q of the gauge vector ξ_a to make G vanish under the gauge transformation. For $l = 1$ case, G_1 no longer exists, which means that we can

not construct the gauge invariants α and β as in the $l \geq 2$ cases. However, from the components of the projection of $2\nabla_{(a}\xi_{b)}$

$$\Delta C_1 = -\frac{\partial}{\partial t}Q, \tag{84}$$

$$\Delta J_1 = \left(\frac{\partial}{\partial r} - \frac{1}{r} - \Phi' - \Lambda'\right)Q, \tag{85}$$

we can construct a gauge invariant property α_1 as

$$\alpha_1 = r^2\frac{\partial}{\partial t}J_1 + r^2\frac{\partial}{\partial r}C_1 - r\left(1 + r\Phi' + r\Lambda'\right)C_1. \tag{86}$$

Then, for $l = 1$, the odd-parity of the projection of EFEs are

$$E_C^{(l=1)} = -r^{-1}e^{-2\Lambda}(-2 + 3r\Lambda' + 3r\Phi')\frac{\partial}{\partial r}C_1 + e^{-2\Lambda}\frac{\partial^2}{\partial r^2}C_1 - r^{-1}e^{-2\Lambda}(-3 + 2r\Lambda' + 2r\Phi')\frac{\partial}{\partial t}J_1 + e^{-2\Lambda}\frac{\partial^2}{\partial t\partial r}J_1,$$

$$- r^{-2}e^{-2\Lambda}\left(-2 + 2r^2\Lambda'^2 - 3r\Phi' + r\Lambda\left(1 + 6r\Phi'\right) - r^2\Lambda'' - 3r^2\Phi''\right)C_1 \tag{87}$$

$$E_J^{(l=1)} = -r^{-1}e^{-2\Lambda}\left(2\Lambda'\left(1 + r\Phi'\right) - 2(\Phi' + r\Phi'^2 + r\Phi'')\right)J_1 - e^{-2\Phi}\frac{\partial^2}{\partial t^2}J_1$$

$$+ r^{-1}e^{-2\Phi}(1 + r\Lambda' + r\Phi')\frac{\partial}{\partial t}C_1 - e^{-2\Phi}\frac{\partial^2}{\partial t\partial r}C_1. \tag{88}$$

Together with Eq.(86), Eq.(87) and Eq.(88) can be decoupled, and the master equation for the odd-parity perturbation is given by

$$\left\{-\frac{\partial^2}{\partial t^2} + e^{-2\Lambda+2\Phi}\left[\frac{\partial^2}{\partial r^2} - \left(\frac{X_1'}{X_1} + 3\Lambda' + 3\Phi'\right)\frac{\partial}{\partial r} + \frac{N_{\text{odd}}^{(l=1)}}{r^2}\right]\right\}\alpha_1 = S_{\text{odd}}^{(l=1)}, \tag{89}$$

where $S_{\text{odd}}^{(l=1)}$ is the source term,

$$S_{\text{odd}}^{(l=1)} = e^{2\Phi}r^2\left[\left(\frac{1}{r} + \Lambda' - \Phi' - \frac{X_1'}{X_1}\right)E_C^{(l=1)} + \frac{\partial}{\partial r}E_C^{(l=1)} + \frac{\partial}{\partial t}E_J^{(l=1)}\right]. \tag{90}$$

In the above equations, X_1 and $N_{\text{odd}}^{(l=1)}$ are all depend only on the background, which can be found in Appendix D. Once Eq.(89) is solved, the metric perturbation components C_1 and J_1 can be determined by

$$C_1 = -\frac{1}{X_1}\left[e^{2\Lambda}r^2 E_C^{(l=1)} - \left(\frac{1}{r} - 2\Lambda' - 2\Phi'\right)\alpha_1 - \frac{\partial}{\partial r}\alpha_1\right], \tag{91}$$

$$J_1 = -\frac{e^{2\Lambda-2\Phi}}{X_1}\left(e^{2\Phi}r^2 E_J^{(l=1)} + \frac{\partial}{\partial t}\alpha_1\right). \tag{92}$$

IV. A POINT PARTICLE AS THE SOURCE

In this section, we present a simple example that a small object moves along a circular orbit around the center of a spherically symmetric spacetime. We analyse the solutions for $l = 0, 1$ in this section. For $l \geq 2$ cases, we only provide a general outline.

Assuming that the small object moves along the worldline $z(\tau)$ with mass μ and four velocity u_a, the stress-energy tensor of this point particle takes the form [46]

$$T_{ab} = \int\frac{\mu u_a u_b}{\sqrt{-g^{(0)}}}\delta^4[x - z(\tau)]d\tau, \tag{93}$$

where τ is the proper time, $g^{(0)}$ and δ^4 are the determinant of the background and the four-dimensional Dirac delta function, respectively. Generalizing the standard analysis from the textbook [47], the time-like geodesics in general spherically symmetric spacetime is

$$-1 = g_{ab}u^a u^b = -e^{2\Phi}\dot{t}^2 + e^{2\Lambda}\dot{r}^2 + r^2\dot{\varphi}^2. \tag{94}$$

Using the static Killing field $\xi^a = (\partial/\partial t)^a$ and the rotational Killing field $\psi^a = (\partial/\partial\varphi)^a$, two conserved quantities $\mathcal{E}$ and L can be defined as

$$\mathcal{E} = -g_{ab}\xi^a u^b = e^{2\Phi}\dot{t}, \quad L = g_{ab}\psi^a u^b = r^2\dot{\varphi}. \tag{95}$$

Then the geodesic equation reads

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}e^{-2\Lambda}\left(\frac{L^2}{r^2} + 1\right) = \frac{1}{2}e^{-2\Lambda-2\Phi}\mathcal{E}^2. \tag{96}$$

Considering that the particle moves along a circular orbit with radio R, which is determined by $\partial V/\partial r = 0$, the four velocity of the particle can be written as

$$u_a = (-\mathcal{E}, 0, 0, L), \tag{97}$$

and now $\mathcal{E}$ and L represent the energy and angular momentum of the particle, given respectively by,

$$\mathcal{E} = \frac{e^{\Phi|_R}}{\sqrt{R\Lambda'|_R + 1}}, \quad L = R\sqrt{\frac{-R\Lambda'|_R}{R\Lambda'|_R + 1}}. \tag{98}$$

where $|_R$ denotes the corresponding function taking its value along the orbit. The orbital frequency can also be defined as

$$\Omega^2 = \left(\frac{u^\varphi}{u^t}\right)^2 = -\frac{e^{2\Phi}\Lambda'|_R}{R}, \tag{99}$$

which gives two useful relations

$$\mathcal{E} = \frac{e^{2\Phi|_R}}{\Omega R^2}L, \quad \Omega L = -R\Lambda'|_R \cdot \mathcal{E}. \tag{100}$$

Projecting the stress-energy tensor T_{ab} to the harmonic basis, one can obtain E_A-E_K. The results reveal that E_D, E_H, E_J and E_K vanish automatically, and the non-vanishing projections of the stress-energy tensor are given as follows. For $l \geq 0$, we have

$$E_A^{lm} = -16\pi e^{-\Lambda+\Phi}\frac{\mu\mathcal{E}}{r^2}\delta(r-R)Y_{lm}^*\left(\frac{\pi}{2}, \Omega t\right), \tag{101}$$

$$E_E^{lm} = -8\pi e^{-\Lambda-\Phi}\frac{\mu L\Omega R^2}{r^4}\delta(r-R)Y_{lm}^*\left(\frac{\pi}{2}, \Omega t\right), \tag{102}$$

where E_A and E_E satisfy the relation $E_A = -\frac{2e^{2\Phi}r^2}{R^3\Lambda'}E_E$. For $l \geq 1$, we find

$$E_B^{lm} = -\frac{16\pi}{l(l+1)}e^{-\Lambda-\Phi}\frac{\mu\mathcal{E}\Omega R^2}{r^3}\delta(r-R)\left(\frac{\partial}{\partial\varphi}Y_{lm}^*(\theta, \varphi)\right)\Big|_{\theta=\frac{\pi}{2},\varphi=\Omega t}, \tag{103}$$

$$E_C^{lm} = -\frac{16\pi}{l(l+1)}\frac{\mu\mathcal{E}\Omega R^2}{r^3}\delta(r-R)\left(\frac{\partial}{\partial\theta}Y_{lm}^*(\theta, \varphi)\right)\Big|_{\theta=\frac{\pi}{2},\varphi=\Omega t}, \tag{104}$$

and for $l \geq 2$,

$$E_F^{lm} = \frac{16\pi(l-2)!}{(l+2)!}e^{-\Lambda-\Phi}\frac{\mu L\Omega R^2}{r^4}\delta(r-R)\left[2\frac{\partial^2}{\partial\theta^2}Y_{lm}^*(\theta, \varphi) + l(l+1)Y_{lm}^*(\theta, \varphi)\right]\Big|_{\theta=\frac{\pi}{2},\varphi=\Omega t}, \tag{105}$$

$$E_G^{lm} = -\frac{32\pi(l-2)!}{(l+2)!}\frac{\mu L\Omega R^2}{r^4}\delta(r-R)\left(\frac{\partial^2}{\partial\theta\partial\varphi}Y_{l,m}^*(\theta, \varphi)\right)\Big|_{\theta=\frac{\pi}{2},\varphi=\Omega t}. \tag{106}$$

A. Perturbations for $l = 0$

For $l = 0$, substituting Eqs.(102) into Eq.(78), we have

$$\left(\iota_0\frac{\partial}{\partial r} - \sigma_0\right)\psi_0 = -\frac{r}{2}e^{4\Lambda}E_E^{00}. \tag{107}$$

Using the standard method to solve the above equation, one can obtain

$$\psi_0 = -\exp\left(\int^r \frac{\sigma_0(r)}{\iota_0(r)}dr\right)\int^r \left[\frac{r'e^{4\Lambda(r')}}{2\iota_0(r')}\exp\left(-\int^{r'}\frac{\sigma_0(r'')}{\iota_0(r'')}dr''\right)E_E^{00}(r')\right]dr'$$

$$= -\exp\left(\int^r \frac{\sigma_0}{\iota_0}dr - \int^R \frac{\sigma_0}{\iota_0}dr\right)\frac{Re^{4\Lambda|_R}}{2\iota_0|_R}\bar{E}_E^{00}\Theta(r-R)$$

$$= -\exp\left(\int_R^r \frac{\sigma_0}{\iota_0}dr\right)\frac{Re^{4\Lambda|_R}}{2\iota_0|_R}\bar{E}_E^{00}\Theta(r-R), \tag{108}$$

where $\Theta(r-R)$ is the unit step function, and $\bar{E}_E^{00}$ denote the evaluated coefficient given by

$$\bar{E}_E^{00} = -4\sqrt{\pi}e^{-\Lambda|_R-\Phi|_R}\frac{\mu L\Omega}{R^2} = 4\sqrt{\pi}e^{-\Lambda|_R-\Phi|_R}\frac{\mu\Lambda'|_R \cdot \mathcal{E}}{R}. \tag{109}$$

From Eq.(73), we have

$$K_0 = 2\psi_0 = -4\sqrt{\pi}\frac{\Lambda'|_R \cdot \mu\mathcal{E}}{\iota_0|_R}\exp\left(\int_R^r \frac{\sigma_0}{\iota_0}dr + 3\Lambda|_R - \Phi|_R\right)\Theta(r-R). \tag{110}$$

Using Eq.(77), we can get o_0, and then solve Eq.(73) to obtain A_0

$$A_0 = 4\sqrt{\pi}e^{2\Phi}\frac{\Lambda'|_R \cdot \mu\mathcal{E}}{\iota_0|_R}\cdot\mathcal{A}\cdot\exp\left[-\int^R \frac{\sigma_0}{\iota_0}dr + 3\Lambda|_R - \Phi|_R\right]\Theta(r-R), \tag{111}$$

where

$$\mathcal{A} = \int r^{-1}\exp\left[\int^r \frac{\sigma_0(r')}{\iota_0(r')}dr'\right]dr. \tag{112}$$

With A_0 and K_0 in hand, the metric perturbation can be directly written out. Since both A_0 and K_0 contain step function, it is obvious that inside the orbit, i.e. $r < R$, the perturbation would vanishes. While outside the orbit, i.e. $r > R$, the perturbed metric can be given by

$$h_{ab}^{00} = \frac{1}{2\sqrt{\pi}}(A_0 v_a v_b + K_0 n_a n_b)$$

$$= 2e^{2\Phi}\frac{\Lambda'|_R \cdot \mu\mathcal{E}}{\iota_0|_R}\cdot\mathcal{A}\cdot\exp\left[-\int^R \frac{\sigma_0}{\iota_0}dr + 3\Lambda|_R - \Phi|_R\right]v_a v_b$$

$$- 2\frac{\Lambda'|_R \cdot \mu\mathcal{E}}{\iota_0|_R}\exp\left[\int_R^r \frac{\sigma_0}{\iota_0}dr + 3\Lambda|_R - \Phi|_R\right]n_a n_b. \tag{113}$$

Unlike the Schwarzschild spacetime, there is no clear definition of the mass M for general spherically symmetric spacetimes. It is obvious that the perturbation for $l=0$ only affects g_{tt} and g_{rr}. In the Schwarzschild spacetime, this perturbation is equivalent to adding an extra mass δm to the system [4].

B. Perturbations for $l=1$

1. Even-parity Perturbations

For the even-parity perturbations with $l=1$, we find that we can set directly $A_1 = E_1 = H_1 = 0$. From Eq.(81), we find that K_1 can be determined by E_A. Note that $Y_{1m}^*(\frac{\pi}{2},\varphi)$ would vanish for $m=0$, hence

$$K_1 = -8\pi R\frac{\mu\mathcal{E}}{r^2}Y_{1,\pm1}^*\left(\frac{\pi}{2},\Omega t\right)\exp\left(-\int_R^r \frac{1+e^{2\Lambda}-4r\Lambda'}{r}dr + 3\Lambda|_R - \Phi|_R\right)\Theta(r-R). \tag{114}$$

Substituting K_1 into Eq.(83), together with $E_H = E_K = 0$, we have

$$\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t}B_1\right) - r^{-1}(1-e^{2\Lambda})\frac{\partial}{\partial t}B_1 = \mathcal{K}K_1, \tag{115}$$

where

$$\mathcal{K} = \frac{1}{2}r^{-2}e^{2\Phi-2\Lambda}(-2 + e^{2\Lambda} - 2r\Phi').$$ (116)

Hence, B_1 can be solved by Eq.(115). Then, considering Eq.(82) with $E_D = 0$, in principle, the metric perturbed component D_1 can also be obtained.

2. Odd-parity Perturbations

To study the metric perturbation components C_1 and J_1, we should first solve the master variable α_1 from Eq.(89). Then using Eqs.(91) and (92), C_1 and J_1 can be obtained. However, G automatically vanish for the $l = 1$ case, there exists one freedom for the odd-parity. If we impose the gauge $C_1 = 0$, the master variable becomes

$$\alpha_1 = r^2 \frac{\partial}{\partial t}J_1.$$ (117)

Then, Eq.(91) becomes

$$\frac{\partial}{\partial r}\alpha_1 + \left(\frac{1}{r} - 2\Lambda' - 2\Phi'\right)\alpha_1 = e^{2\Lambda}r^2 E_{\mathrm{C}}.$$ (118)

The solution of this equation is

$$\begin{aligned}
\alpha_1 &= -\frac{16\pi}{l(l+1)} \cdot \frac{\mu\mathcal{E}\Omega R^2}{r} \exp(2\Lambda + 2\Phi - 2\Phi|_R) \cdot \left(\frac{\partial}{\partial\theta}Y_{lm}^*(\theta,\varphi)\right)|_{\theta=\frac{\pi}{2},\varphi=\Omega t} \\
&= -8\pi\frac{\mu L}{r}e^{2\Lambda+2\Phi}\left(\frac{\partial}{\partial\theta}Y_{lm}^*(\theta,\varphi)\right)|_{\theta=\frac{\pi}{2},\varphi=\Omega t},
\end{aligned}$$ (119)

where we have usedamour the relation Eq.(100) and the fact $l = 1$. From α_1, one can solve J_1 with a undetermined function that only depend on r. However, the metric perturbation component J_1 yields a contribution to $h_{r\theta}^{1m}$ and $h_{r\varphi}^{1m}$.

Next, we impose the gauge $J_1 = 0$, and now the gauge invariant α_1 becomes

$$\alpha_1 = r^2\frac{\partial}{\partial r}C_1 - r(1 + r\Lambda' + r\Phi')C_1.$$ (120)

Substituting this relation into Eqs.(91) and (92), we know that α_1 should only be a function of r, and C_1 satisfies the second-order differential equation

$$r^2\frac{\partial^2}{\partial r^2}C_1 + r(2 - 3r\Lambda' - 3r\Phi')\frac{\partial}{\partial r}C_1 - \left(2 - 2r^2\Lambda'^2 + 3r\Phi' - r\Lambda'(1 + 6r\Phi') + r^2\Lambda'' + 3r^2\Phi''\right)C_1 = e^{2\Lambda}r^2 E_{\mathrm{C}},$$ (121)

which is Eq.(87) with $J_1 = 0$. Considering that E_{C} contains the delta function, we predict that the solution of C_1 contain the step function $\Theta(r-R)$, with two arbitrary constants of integration. One can be determined by considering that the background spacetime without intrinsic angular momentum, and the other can be determined by the fact that perturbation C_1 would be convergent at infinity [28]. If the background metric degenerates to the Schwarzschild spacetime, our equation would become Eq. (10.21) of [28], which gives a non-vanishing $h_{t\varphi}^{01}$ outside the circular orbit to describe the adding angular momentum of the system.

C. Perturbations For $l \geq 2$

Generally speaking, for $l \geq 2$ the problem is much more mathematically involved, and normally we have to seek help from numerical computations. From the source term, Eqs.(45), (56) or (65), we know that they depend only on Eqs.(101)-(106). All these terms are proportional to $e^{-im\Omega t}$, which implies that our master equations, Eqs.(44), (55) and (64), can be written as the second-order ordinary differential equations in r for each l and m. And the distributional sources of these differential equations would be vanish everywhere except at the circular orbit radius $r = R$.

An basic outline is as follows. To solve the second-order equations, one can study two regions separately. One region is from the circular orbit radius to infinity, i.e. $r \in (R, \infty)$. In this region, a naturally boundary condition is considering an appropriate radiation at spatial infinity. In the other region there are several different situations. For example, if the background spacetime has an event horizon, then one should determine this region from the event horizon to the circular orbit radius, i.e., $r \in (r_+, R)$. Then solve the differential equation with the boundary condition at the event horizon r_+. Another example is that the background spacetime is a perfect fluid star without event horizons, then one should determine the other region from the center of the star to the circular orbit radius, i.e., $r \in (0, R)$. Then solve the differential equation with some boundary conditions at the center of the star. Finally, matching the solutions obtained in the two separate regions properly across the boundary $r = R$, we obtain the perturbations valid over the whole spacetime.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we systematically study the gauge invariant perturbations of a general spherically symmetric background. First, we find that, in general spherically symmetric spacetimes, for even-parity, there are several gauge choices. One is the well-known RW gauge, and the other is the EZ gauge. For odd-parity perturbation, only one gauge choice exists, that is, by setting G = 0. Then, we mainly focus on the construction of master equations for $l \geq 2$. For even-parity perturbation, under the EZ gauge or the RW gauge, the master equation (44) or (55) are the third-order equations. However, after the separation of variables, the equations all reduce to a second-order equation. For odd-parity perturbation, the master equation is also constructed as Eq.(64). Next, the cases for $l = 0$ and $l = 1$ are discussed. For the $l = 0$ case, we present the equations that the gauge invariants satisfy. For even-parity perturbations with $l = 1$, we find that the metric perturbation components are still determined when the source is specified. And for odd-parity perturbations with $l = 1$, the master equation (89) is still a wave-like equation. Finally, using our general results, we investigate a point particle moving along a circular orbit in general spherically symmetric spacetimes. In particular, the form of the solutions for $l = 0$ and $l = 1$ are carefully discussed.

Our results can be applied to various modified theories of gravity, in which the background is described by the general static metric (2), instead of the particular one (1). In particular, it can be applied to the EOB system. Jing et. al. pointed out that one should consider the Hamilton equations for an EOB system self-consistently [42]. Specifically, considering that the EOB system takes the spinless effective metric $g_{\mu\nu}^{eff}$ [40, 48, 49], the Hamiltonian $H[g_{\mu\nu}^{eff}]$ and the radiation-reaction force $\mathcal{F}_{\varphi}^{circ}[g_{\mu\nu}^{eff}]$ should be both based on the same effective metric. Under the quasi-circular approximation, the radiation-reaction force can be obtained by the energy-loss rate,

$$\mathcal{F}_{\varphi}^{circ}[g_{\mu\nu}^{eff}] \simeq \frac{1}{\dot{\varphi}} \frac{dE[g_{\mu\nu}^{eff}]}{dt}. \tag{122}$$

Considering that the perturbed Weyl tensor ψ_4^B can be divided into the even-parity ψ_4^{BE} and the odd-parity ψ_4^{BO} parts [42], then the energy-loss rate can be calculated from ψ_4^{BE} and ψ_4^{BO} via the relation,

$$\frac{dE[g_{\mu\nu}^{eff}]}{dt} = \frac{c^3}{16\pi G \omega^2} \int \left\{ \left[Re(\psi_4^{BE} + \psi_4^{BO}) \right]^2 + \left[Im(\psi_4^{BE} + \psi_4^{BO}) \right]^2 \right\} r^2 d\Omega^2. \tag{123}$$

So, the key step to obtain a self-consistent radiation-reaction force $\mathcal{F}_{\varphi}^{circ}[g_{\mu\nu}^{eff}]$ is to solve the solution of ψ_4^{BE} and ψ_4^{BO}. In [42], the authors constructed the decoupled equations for both even-parity ψ_4^{BE} and odd-parity ψ_4^{BO} in the effective metric spacetime rather than in the Schwarzschild spacetime. Generally speaking, ψ_4^{BO} is only related to the odd-parity perturbation, i.e. the C and J terms defined in Eq.(17). When Eq.(64) is solved, one can determine C and J by Eqs.(66) and (67), and then ψ_4^{BO} can be calculated. Similar processing can be done for the even-parity perturbation. However, Ref. [42] only considered the background effective metric that takes the form as Eq.(1), which can be applied to the Post-Minkowskian (PM) approximation [48–50], if the undeterminated parameters d_i are well constrained. While in this paper we consider the most general spherical symmetric metric, which can be applied to the EOB theory with either the PN approximation or the PM approximation. We wish to come back to this important issue soon.

Acknowledgments

Appendix A: Normalization Factors And A-K Decomposition

The normalization factors $N^{(vec)}(A,r,l)$ and $N^{(ten)}(A,r,l)$ in Sec.II can be calculated as

$$\oint Y_a^{E,lm}(Y_{E,l'm'}^a)d\Omega = N^{(vec)}(E,r,l)\delta_{ll'}\delta_{mm'} = l(l+1)\delta_{ll'}\delta_{mm'},$$

$$\oint Y_a^{B,lm}(Y_{B,l'm'}^a)d\Omega = N^{(vec)}(B,r,l)\delta_{ll'}\delta_{mm'} = e^{-2\Lambda-2\Phi}l(l+1)\delta_{ll'}\delta_{mm'},$$

$$\oint Y_a^{R,lm}(Y_{R,l'm'}^a)d\Omega = N^{(vec)}(R,r,l)\delta_{ll'}\delta_{mm'} = e^{-2\Lambda}\delta_{ll'}\delta_{mm'},$$

and

$$\oint T_{ab}^{T0,lm}(T_{T0,l'm'}^{ab})^*d\Omega = N^{(ten)}(T0,r,l)\delta_{ll'}\delta_{mm'} = 2\delta_{ll'}\delta_{mm'},$$

$$\oint T_{ab}^{L0,lm}(T_{L0,l'm'}^{ab})^*d\Omega = N^{(ten)}(L0,r,l)\delta_{ll'}\delta_{mm'} = e^{-4\Lambda}\delta_{ll'}\delta_{mm'},$$

$$\oint T_{ab}^{E1,lm}(T_{E1,l'm'}^{ab})^*d\Omega = N^{(ten)}(E1,r,l)\delta_{ll'}\delta_{mm'} = \frac{1}{2}e^{-2\Lambda}l(l+1)\delta_{ll'}\delta_{mm'},$$

$$\oint T_{ab}^{B1,lm}(T_{B1,l'm'}^{ab})^*d\Omega = N^{(ten)}(B1,r,l)\delta_{ll'}\delta_{mm'} = \frac{1}{2}e^{-4\Lambda-2\Phi}l(l+1)\delta_{ll'}\delta_{mm'},$$

$$\oint T_{ab}^{E2,lm}(T_{E2,l'm'}^{ab})^*d\Omega = N^{(ten)}(E2,r,l)\delta_{ll'}\delta_{mm'} = \frac{(l+2)!}{2(l-2)!}\delta_{ll'}\delta_{mm'},$$

$$\oint T_{ab}^{B2,lm}(T_{B2,l'm'}^{ab})^*d\Omega = N^{(ten)}(B2,r,l)\delta_{ll'}\delta_{mm'} = e^{-2\Lambda-2\Phi}\frac{(l+2)!}{2(l-2)!}\delta_{ll'}\delta_{mm'}.$$

Projecting the perturbed metric h_{ab} onto these basis, one can obtain the expressions of coefficients A-K

$$\text{A} = e^{4\Phi}\oint h_{ab}^{lm}(v^a v^b Y_{lm}^*)d\Omega, \qquad \text{B} = -\frac{e^{2\Phi}}{l(l+1)}\oint h_{ab}^{lm}v^a Y_{E,lm}^{b*}d\Omega,$$

$$\text{C} = -\frac{e^{4\Phi+2\Lambda}}{l(l+1)}\oint h_{ab}^{lm}v^a Y_{B,lm}^{b*}d\Omega, \qquad \text{D} = -e^{2\Phi+2\Lambda}\oint h_{ab}^{lm}v^a Y_{R,lm}^{b*}d\Omega,$$

$$\text{E} = \frac{1}{2}\oint h_{ab}^{lm}T_{T0,lm}^{ab*}d\Omega, \qquad \text{F} = \frac{2(l-2)!}{(l+2)!}\oint h_{ab}^{lm}T_{E2,lm}^{ab*}d\Omega,$$

$$\text{G} = \frac{2(l-2)!}{(l+2)!}e^{2\Phi+2\Lambda}\oint h_{ab}^{lm}T_{B2,lm}^{ab*}d\Omega, \qquad \text{H} = \frac{e^{2\Lambda}}{l(l+1)}\oint h_{ab}^{lm}T_{E1,lm}^{ab*}d\Omega,$$

$$\text{J} = \frac{e^{4\Lambda+2\Phi}}{l(l+1)}\oint h_{ab}^{lm}T_{B1,lm}^{ab*}d\Omega, \qquad \text{K} = e^{4\Lambda}\oint h_{ab}^{lm}T_{L0,lm}^{ab*}d\Omega.$$

The above results are only valid for $l \geq 2$.

Appendix B: Decomposition of Linearized Einstein Equations

In this appendix, we show the decomposition of the linearized Einstein tensor into the A-K components. Note that these expressions are only valid for $l \geq 2$.

1. Even Parity

$$
\begin{aligned}
E_{\text{A}} =& 2r^{-2}e^{-2\Lambda}(-1+e^{2\Lambda}+2r\Lambda')\text{A} - r^{-2}e^{2\Phi}(l^2+l-2)\text{E} \\
& - 2r^{-1}e^{-2\Lambda+2\Phi}(r\Lambda'-3)\frac{\partial}{\partial r}\text{E} + 2e^{-2\Lambda+2\Phi}\frac{\partial^2}{\partial r^2}\text{E} - \frac{1}{2}r^{-2}e^{2\Phi}l(l+2)(l+1)(l-1)\text{F} \\
& - 2r^{-2}e^{-2\Lambda+2\Phi}l(l+1)(r\Lambda'-2)\text{H} + 2r^{-1}e^{-2\Lambda+2\Phi}l(l+1)\frac{\partial}{\partial r}\text{H} \\
& - r^{-2}e^{-4\Lambda+2\Phi}\left[2+e^{2\Lambda}l(l+1)-8r\Lambda'\right]\text{K} - 2r^{-1}e^{-4\Lambda+2\Phi}\frac{\partial}{\partial r}\text{K},
\end{aligned}
\tag{B1}
$$

$$
\begin{aligned}
E_{\text{B}} =& r^{-2}e^{-2\Lambda}(-2+2e^{2\Lambda}+3r\Lambda'-r\Phi'+2r^2\Lambda'\Phi'-2r^2\Phi'^2-2r^2\Phi'')\text{B} \\
& - r^{-1}e^{-2\Lambda}(-2+r\Lambda'+r\Phi')\frac{\partial}{\partial r}\text{B} + e^{-2\Lambda}\frac{\partial^2}{\partial r^2}\text{B} + r^{-1}e^{-2\Lambda}(\Lambda'-\Phi')\text{D} - r^{-1}e^{-2\Lambda}\frac{\partial}{\partial r}\text{D} \\
& - r^{-1}\frac{\partial}{\partial t}\text{E} - \frac{1}{2}r^{-1}(l^2+l-2)\frac{\partial}{\partial t}\text{F} - r^{-1}e^{-2\Lambda}(-3+r\Lambda'+r\Phi')\frac{\partial}{\partial t}\text{H} + e^{-2\Lambda}\frac{\partial^2}{\partial t\partial r}\text{H} \\
& - r^{-1}e^{-2\Lambda}\frac{\partial}{\partial t}\text{K},
\end{aligned}
\tag{B2}
$$

$$
\begin{aligned}
E_{\text{D}} =& r^{-2}l(l+1)(1-2r\Phi')\text{B} + r^{-1}l(l+1)\frac{\partial}{\partial r}\text{B} + 2r^{-1}(-1+r\Phi')\frac{\partial}{\partial t}\text{E} - 2\frac{\partial}{\partial t\partial r}\text{E} \\
& - r^{-2}e^{-2\Lambda}(2-2e^{2\Lambda}+e^{2\Lambda}l+e^{2\Lambda}l^2-4r\Lambda')\text{D} - r^{-1}l(l+1)\frac{\partial}{\partial t}\text{H} + 2r^{-1}e^{-2\Lambda}\frac{\partial}{\partial t}\text{K},
\end{aligned}
\tag{B3}
$$

$$
\begin{aligned}
E_{\text{E}} =& -\frac{1}{2}r^{-2}e^{-2\Lambda-2\Phi}(e^{2\Lambda}l+e^{2\Lambda}l^2+4r\Phi'-4r^2\Lambda'\Phi'+4r^2\Phi'')\text{A} - r^{-1}e^{-2\Lambda-2\Phi}(-1+r\Lambda'+2r\Phi')\frac{\partial}{\partial r}\text{A} \\
& + e^{-2\Lambda-2\Phi}\frac{\partial^2}{\partial r^2}\text{A} - r^{-1}e^{-2\Phi}l(l+1)\frac{\partial}{\partial t}\text{B} - 2r^{-1}e^{-2\Lambda-2\Phi}(-1+r\Lambda')\frac{\partial}{\partial t}\text{D} + 2e^{-2\Lambda-2\Phi}\frac{\partial^2}{\partial t\partial r}\text{D} \\
& - 2r^{-1}e^{-2\Lambda}(\Phi'-\Lambda'-r\Lambda'\Phi'+r\Phi'^2+r^2\Phi'')\text{E} + r^{-1}e^{-2\Lambda}(-2+r\Lambda'-r\Phi')\frac{\partial}{\partial r}\text{E} \\
& - e^{-2\Lambda}\frac{\partial^2}{\partial r^2}\text{E} + e^{-2\Phi}\frac{\partial^2}{\partial t^2}\text{E} + r^{-2}e^{-2\Lambda}l(l+1)(-1+r\Lambda'-r\Phi')\text{H} - r^{-1}e^{-2\Lambda}l(l+1)\frac{\partial}{\partial r}\text{H} \\
& + \frac{1}{2}r^{-2}e^{-4\Lambda}(e^{2\Lambda}l+e^{2\Lambda}l^2+4r\Phi'-8r\Lambda'+4r^2\Phi'^2-8r^2\Lambda'\Phi'+4r^2\Phi'')\text{K} + r^{-1}e^{-4\Lambda}(1+r\Phi')\frac{\partial}{\partial r}\text{K} \\
& + e^{-2\Lambda-2\Phi}\frac{\partial^2}{\partial t^2}\text{K},
\end{aligned}
\tag{B4}
$$

$$
\begin{aligned}
E_{\text{F}} =& -r^{-2}e^{-2\Phi}\text{A} - 2r^{-1}e^{-2\Phi}\frac{\partial}{\partial t}\text{B} - 2r^{-1}e^{-2\Lambda}(\Phi'+r\Phi'^2-\Lambda'(1+r\Phi')+r\Phi'')\text{F} \\
& - r^{-1}e^{-2\Lambda}(-2+r\Lambda'-r\Phi')\frac{\partial}{\partial r}\text{F} + e^{-2\Lambda}\frac{\partial^2}{\partial r^2}\text{F} - e^{-2\Phi}\frac{\partial^2}{\partial t^2}\text{F} + 2r^{-2}e^{-2\Lambda}(-1+r\Lambda'-r\Phi')\text{H} \\
& - 2r^{-1}e^{-2\Lambda}\frac{\partial}{\partial r}\text{H} + r^{-2}e^{-2\Lambda}\text{K},
\end{aligned}
\tag{B5}
$$

$$
\begin{aligned}
E_{\text{H}} =& r^{-2}e^{-2\Phi}(1+r\Phi')\text{A} - r^{-1}e^{-2\Phi}\frac{\partial}{\partial r}\text{A} + r^{-1}e^{-2\Phi}\frac{\partial}{\partial t}\text{B} - e^{-2\Phi}\frac{\partial^2}{\partial t\partial r}\text{B} - r^{-1}e^{-2\Phi}\frac{\partial}{\partial t}\text{D} + r^{-1}\frac{\partial}{\partial r}\text{E} \\
& + \frac{1}{2}r^{-1}(l+2)(l-1)\frac{\partial}{\partial r}\text{F} + 2r^{-2}e^{-2\Lambda}(e^{2\Lambda}-r\Phi'-r^2\Phi'^2+r\Lambda'(1+r\Phi')-r^2\Phi'')\text{H} \\
& - e^{-2\Phi}\frac{\partial^2}{\partial t^2}\text{H} - r^{-2}e^{-2\Lambda}(1+r\Phi')\text{K},
\end{aligned}
\tag{B6}
$$

$$E_{\mathrm{K}} = - r^{-2}e^{-2\Phi}(e^{2\Lambda}l + e^{2\Lambda}l^2 + 4r\Phi')\mathrm{A} + 2r^{-1}e^{-2\Phi}\frac{\partial}{\partial r}\mathrm{A} - 2r^{-1}e^{2\Lambda-2\Phi}l(l+1)\frac{\partial}{\partial t}\mathrm{B} + 4r^{-1}e^{-2\Phi}\frac{\partial}{\partial t}\mathrm{D}$$
$$+ r^{-2}e^{2\Lambda}(l^2 + l - 2)\mathrm{E} - 2r^{-1}(1 + r\Phi')\frac{\partial}{\partial r}\mathrm{E} + 2e^{2\Lambda-2\Phi}\frac{\partial^2}{\partial t^2}\mathrm{E} + \frac{1}{2}r^{-2}e^{2\Lambda}l(l+2)(l+1)(l-1)\mathrm{F}$$
$$- 2r^{-2}l(l+1)(1 + r\Phi')\mathrm{H} + 2r^{-2}\mathrm{K}. \tag{B7}$$

2. Odd Parity

$$E_{\mathrm{C}} = - r^{-2}e^{-2\Lambda}\left(2 - 2e^{2\Lambda} + e^{2\Lambda}l + e^{2\Lambda}l^2 - 2r^2\Lambda'^2 + 3r\Phi' - r\Lambda'(1 + 6r\Phi') + r^2\Lambda'' + 3r^2\Phi''\right)\mathrm{C}$$
$$- r^{-1}e^{-2\Lambda}(-2 + 3r\Lambda' + 3r\Phi')\frac{\partial}{\partial r}\mathrm{C} + e^{-2\Lambda}\frac{\partial^2}{\partial r^2}\mathrm{C} - \frac{1}{2}r^{-1}(l+2)(l-1)\frac{\partial}{\partial t}\mathrm{G}$$
$$- r^{-1}e^{-2\Lambda}(-3 + 2r\Lambda' + 2r\Phi')\frac{\partial}{\partial t}\mathrm{J} + e^{-2\Lambda}\frac{\partial^2}{\partial t\partial r}\mathrm{J}, \tag{B8}$$

$$E_{\mathrm{G}} = -2r^{-1}e^{-2\Phi}\frac{\partial}{\partial t}\mathrm{C} + r^{-1}e^{-2\Lambda}(2r\Lambda'^2 - 4\Phi' + 4r\Lambda'\Phi' - 2r\Phi'^2 - r\Lambda'' - 3r\Phi'')\mathrm{G}$$
$$- r^{-1}e^{-2\Lambda}(-2 + 3r\Lambda' + r\Phi')\frac{\partial}{\partial r}\mathrm{G} + e^{-2\Lambda}\frac{\partial^2}{\partial r^2}\mathrm{G} - e^{-2\Phi}\frac{\partial^2}{\partial t^2}\mathrm{G} + 2r^{-2}e^{-2\Lambda}(-1 + 2r\Lambda')\mathrm{J} - 2r^{-1}e^{-2\Lambda}\frac{\partial}{\partial r}\mathrm{J}, \tag{B9}$$

$$E_{\mathrm{J}} = r^{-1}e^{-2\Phi}(1 + r\Lambda' + r\Phi')\frac{\partial}{\partial t}\mathrm{C} - e^{-2\Phi}\frac{\partial^2}{\partial t\partial r}\mathrm{C} - \frac{1}{2}r^{-1}(l+2)(l-1)(\Lambda' + \Phi')\mathrm{G} + \frac{1}{2}r^{-1}(l+2)(l-1)\frac{\partial}{\partial r}\mathrm{G}$$
$$- r^{-2}e^{-2\Lambda}(-2e^{2\Lambda} + e^{2\Lambda}l + e^{2\Lambda}l^2 - 2r\Lambda' + 2r\Phi' + 2r^2\Phi'^2 + 2r^2\Phi'' - 2r^2\Lambda'\Phi')\mathrm{J} - e^{-2\Phi}\frac{\partial^2}{\partial t^2}\mathrm{J}. \tag{B10}$$

Appendix C: Linearized Einstein Tensor In Terms of Gauge Invariants

In this appendix, we rewrite the linearized Einstein tensor in terms of the gauge invariants under the EZ gauge or the RW gauge. Note that for even-parity perturbations, we use the superscripts to distinguish expressions under the EZ and RW gauges, respectively. These relations are only valid for $l \geq 2$ case. The relation $\lambda = l(l+1)$ also has been used in this appendix.

1. Even Parity and the EZ Gauge

$$E_{\mathrm{A}}^{\mathrm{I}} = - 4r^{-2}e^{-2\Lambda}(-1 + e^{2\Lambda} + 2r\Lambda')\epsilon^{\mathrm{I}} - 2r^{-2}e^{-2\Lambda+2\Phi}\lambda(r\Lambda' - 2)\chi^{\mathrm{I}} + 2r^{-1}e^{-2\Lambda+2\Phi}\lambda\frac{\partial}{\partial r}\chi^{\mathrm{I}}$$
$$- 2r^{-2}e^{-4\Lambda+2\Phi}\left(2 + e^{2\Lambda}\lambda - 8r\Lambda'\right)\psi^{\mathrm{I}} - 4r^{-1}e^{-4\Lambda+2\Phi}\frac{\partial}{\partial r}\psi^{\mathrm{I}}, \tag{C1}$$

$$E_{\mathrm{B}}^{\mathrm{I}} = r^{-1}e^{-2\Lambda}(\Lambda' - \Phi')\delta^{\mathrm{I}} - r^{-1}e^{-2\Lambda}\frac{\partial}{\partial r}\delta^{\mathrm{I}} - r^{-1}e^{-2\Lambda}(-3 + r\Lambda' + r\Phi')\frac{\partial}{\partial t}\chi^{\mathrm{I}} + e^{-2\Lambda}\frac{\partial^2}{\partial t\partial r}\chi^{\mathrm{I}} - 2r^{-1}e^{-2\Lambda}\frac{\partial}{\partial t}\psi^{\mathrm{I}}, \tag{C2}$$

$$E_{\mathrm{D}}^{\mathrm{I}} = - r^{-2}e^{-2\Lambda}\left[2 + e^{2\Lambda}(\lambda - 2) - 4r\Lambda'\right]\delta^{\mathrm{I}} - r^{-1}\lambda\frac{\partial}{\partial t}\chi^{\mathrm{I}} + 4r^{-1}e^{-2\Lambda}\frac{\partial}{\partial t}\psi^{\mathrm{I}}, \tag{C3}$$

$$
\begin{aligned}
E_{\mathrm{E}}^{\mathrm{I}} =& r^{-2}e^{-2\Lambda-2\Phi}\left(e^{2\Lambda}\lambda + 4r\Phi' - 4r^2\Lambda'\Phi' + 4r^2\Phi''\right)\epsilon^{\mathrm{I}} + 2r^{-1}e^{-2\Lambda-2\Phi}(-1 + r\Lambda' + 2r\Phi')\frac{\partial}{\partial r}\epsilon^{\mathrm{I}} \\
& - 2e^{-2\Lambda-2\Phi}\frac{\partial^2}{\partial r^2}\epsilon^{\mathrm{I}} - 2r^{-1}e^{-2\Lambda-2\Phi}(-1 + r\Lambda')\frac{\partial}{\partial t}\delta^{\mathrm{I}} + 2e^{-2\Lambda-2\Phi}\frac{\partial^2}{\partial t\partial r}\delta^{\mathrm{I}} + r^{-2}e^{-2\Lambda}\lambda(-1 + r\Lambda' - r\Phi')\chi^{\mathrm{I}} \\
& - r^{-1}e^{-2\Lambda}\lambda\frac{\partial}{\partial r}\chi^{\mathrm{I}} + r^{-2}e^{-4\Lambda}\left(e^{2\Lambda}\lambda + 4r\Phi' - 8r\Lambda' + 4r^2\Phi'^2 - 8r^2\Lambda'\Phi' + 4r^2\Phi''\right)\psi^{\mathrm{I}} \\
& + 2r^{-1}e^{-4\Lambda}(1 + r\Phi')\frac{\partial}{\partial r}\psi^{\mathrm{I}} + 2e^{-2\Lambda-2\Phi}\frac{\partial^2}{\partial t^2}\psi^{\mathrm{I}},
\end{aligned}
\tag{C4}
$$

$$
E_{\mathrm{F}}^{\mathrm{I}} = 2r^{-2}e^{-2\Phi}\epsilon^{\mathrm{I}} + 2r^{-2}e^{-2\Lambda}(-1 + r\Lambda' - r\Phi')\chi^{\mathrm{I}} - 2r^{-1}e^{-2\Lambda}\frac{\partial}{\partial r}\chi^{\mathrm{I}} + 2r^{-2}e^{-2\Lambda}\psi^{\mathrm{I}},
\tag{C5}
$$

$$
\begin{aligned}
E_{\mathrm{H}}^{\mathrm{I}} =& -2r^{-2}e^{-2\Phi}(1 + r\Phi')\epsilon^{\mathrm{I}} + 2r^{-1}e^{-2\Phi}\frac{\partial}{\partial r}\epsilon^{\mathrm{I}} - r^{-1}e^{-2\Phi}\frac{\partial}{\partial t}\delta^{\mathrm{I}} - 2r^{-2}e^{-2\Lambda}(1 + r\Phi')\psi^{\mathrm{I}} \\
& + 2r^{-2}e^{-2\Lambda}(e^{2\Lambda} - r\Phi' - r^2\Phi'^2 + r\Lambda'(1 + r\Phi') - r^2\Phi'')\chi^{\mathrm{I}} - e^{-2\Phi}\frac{\partial^2}{\partial t^2}\chi^{\mathrm{I}},
\end{aligned}
\tag{C6}
$$

$$
E_{\mathrm{K}}^{\mathrm{I}} = 2r^{-2}e^{-2\Phi}\left(e^{2\Lambda}\lambda + 4r\Phi'\right)\epsilon^{\mathrm{I}} - 4r^{-1}e^{-2\Phi}\frac{\partial}{\partial r}\epsilon^{\mathrm{I}} + 4r^{-1}e^{-2\Phi}\frac{\partial}{\partial t}\delta^{\mathrm{I}} - 2r^{-2}\lambda(1 + r\Phi')\chi^{\mathrm{I}} + 4r^{-2}\psi^{\mathrm{I}}.
\tag{C7}
$$

2. Even Parity and the RW Gauge

$$
\begin{aligned}
E_{\mathrm{A}}^{\mathrm{II}} =& -4r^{-2}e^{-2\Lambda}\left(-1 + e^{2\Lambda} + 2r\Lambda'\right)\epsilon^{\mathrm{II}} \\
& + r^{-2}e^{-4\Lambda+2\Phi}\left[4 - 4e^{2\Lambda} + 4e^{2\Lambda}\lambda - 8r^2\Lambda'^2 - 2r\Lambda'\left(e^{2\Lambda}\lambda - 4r\Phi'\right) + 4r^2\Lambda'' - 4r\Phi'(1 - e^{2\Lambda})\right]\chi^{\mathrm{II}} \\
& + 2r^{-1}e^{-2\Lambda+2\Phi}\lambda\frac{\partial}{\partial r}\chi - 2r^{-2}e^{-4\Lambda+2\Phi}(2 + e^{2\Lambda}\lambda - 8r\Lambda')\psi^{\mathrm{II}} - 4r^{-1}e^{-4\Lambda+2\Phi}\frac{\partial}{\partial r}\psi^{\mathrm{II}},
\end{aligned}
\tag{C8}
$$

$$
E_{\mathrm{B}}^{\mathrm{II}} = r^{-1}e^{-2\Lambda}(\Lambda' - \Phi')\delta^{\mathrm{II}} - r^{-1}e^{-2\Lambda}\frac{\partial}{\partial r}\delta^{\mathrm{II}} - r^{-1}e^{-2\Lambda}(-3 + r\Lambda' + r\Phi')\frac{\partial}{\partial t}\chi^{\mathrm{II}} + e^{-2\Lambda}\frac{\partial^2}{\partial t\partial r}\chi^{\mathrm{II}} - 2r^{-1}e^{-2\Lambda}\frac{\partial}{\partial t}\psi^{\mathrm{II}},
\tag{C9}
$$

$$
E_{\mathrm{D}}^{\mathrm{II}} = -r^{-2}e^{-2\Lambda}\left(2 - 2e^{2\Lambda} + e^{2\Lambda}\lambda - 4r\Lambda'\right)\delta^{\mathrm{II}} - r^{-1}e^{-2\Lambda}\left(2 - 2e^{2\Lambda} + e^{2\Lambda}\lambda + 4r\Phi'\right)\frac{\partial}{\partial t}\chi^{\mathrm{II}} + 4r^{-1}e^{-2\Lambda}\frac{\partial}{\partial t}\psi^{\mathrm{II}},
\tag{C10}
$$

$$
\begin{aligned}
E_{\mathrm{E}}^{\mathrm{II}} =& r^{-2}e^{-2\Lambda-2\Phi}\left(e^{2\Lambda}\lambda + 4r\Phi' - 4r^2\Lambda'\Phi' + 4r^2\Phi''\right)\epsilon^{\mathrm{II}} + 2r^{-1}e^{-2\Lambda-2\Phi}(-1 + r\Lambda' + 2r\Phi')\frac{\partial}{\partial r}\epsilon^{\mathrm{II}} \\
& - 2e^{-2\Lambda-2\Phi}\frac{\partial^2}{\partial r^2}\epsilon^{\mathrm{II}} - 2r^{-1}e^{-2\Lambda-2\Phi}(-1 + r\Lambda')\frac{\partial}{\partial t}\delta^{\mathrm{II}} + 2e^{-2\Lambda-2\Phi}\frac{\partial^2}{\partial t\partial r}\delta^{\mathrm{II}} \\
& + r^{-2}e^{-4\Lambda}\left[-e^{2\Lambda}\lambda + 4r^2\Phi'^2 + 4r^2\Lambda'^2(1 + r\Phi') - 2r^2\Lambda'' + 6r^2\Phi''\right. \\
& \left. -r\Phi'(-2 + e^{2\Lambda}\lambda + 2r^2\Lambda'' - 4r^2\Phi'') - r\Lambda'(2 - e^{2\Lambda}\lambda + 8r\Phi' + 4r^2\Phi'^2 + 6r^2\Phi'') + 2r^3\Phi'''\right]\chi^{\mathrm{II}} \\
& - r^{-1}e^{-2\Lambda}\lambda\frac{\partial}{\partial r}\chi^{\mathrm{II}} + r^{-2}e^{-4\Lambda}\left(e^{2\Lambda}\lambda + 4r\Phi' - 8r\Lambda' + 4r^2\Phi'^2 - 8r^2\Lambda'\Phi' + 4r^2\Phi''\right)\psi^{\mathrm{II}} \\
& + 2r^{-1}e^{-4\Lambda}(1 + r\Phi')\frac{\partial}{\partial r}\psi^{\mathrm{II}} + 2e^{-2\Lambda-2\Phi}\frac{\partial^2}{\partial t^2}\psi^{\mathrm{II}},
\end{aligned}
\tag{C11}
$$

$$E_{\mathrm{F}}^{\mathrm{II}} =2r^{-2}e^{-2\Phi}\epsilon^{\mathrm{II}} + 2r^{-2}e^{-2\Lambda}(-1 + r\Lambda' - r\Phi')\chi^{\mathrm{II}} - 2r^{-1}e^{-2\Lambda}\frac{\partial}{\partial r}\chi^{\mathrm{II}} + 2r^{-2}e^{-2\Lambda}\psi^{\mathrm{II}}, \tag{C12}$$

$$\begin{aligned}
E_{\mathrm{H}}^{\mathrm{II}} = &-2r^{-2}e^{-2\Phi}(1 + r\Phi')\epsilon^{\mathrm{II}} + 2r^{-1}e^{-2\Phi}\frac{\partial}{\partial r}\epsilon^{\mathrm{II}} - r^{-1}e^{-2\Phi}\frac{\partial}{\partial t}\delta^{\mathrm{II}} - 2r^{-2}e^{-2\Lambda}(1 + r\Phi')\psi^{\mathrm{II}} \\
&+ 2r^{-2}e^{-2\Lambda}\left(1 + r\Phi' + r\Lambda' - r^2\Phi'^2 + r^2\Lambda'\Phi' - r^2\Phi''\right)\chi^{\mathrm{II}} - e^{-2\Phi}\frac{\partial^2}{\partial t^2}\chi^{\mathrm{II}},
\end{aligned} \tag{C13}$$

$$\begin{aligned}
E_{\mathrm{K}}^{\mathrm{II}} =&2r^{-2}e^{-2\Phi}(e^{2\Lambda}\lambda + 4r\Phi')\epsilon^{\mathrm{II}} - 4r^{-1}e^{-2\Phi}\frac{\partial}{\partial r}\epsilon^{\mathrm{II}} + 4r^{-1}e^{-2\Phi}\frac{\partial}{\partial t}\delta^{\mathrm{II}} \\
&- 2r^{-2}e^{-2\Lambda}\left[e^{2\Lambda}\lambda + r\Phi'(e^{2\Lambda}\lambda - 2) + 2r\Lambda'(2 - e^{2\Lambda} + 4r\Phi') - 2r^2\Phi''\right]\chi^{\mathrm{II}} \\
&+ 4r^{-1}e^{-2\Lambda}\left(1 - e^{2\Lambda} + 2r\Phi'\right)\frac{\partial}{\partial r}\chi^{\mathrm{II}} + 4r^{-2}e^{-2\Lambda}\left(e^{2\Lambda} + 2r\Lambda' + r^2\Lambda'\Phi' - r^2\Phi'^2 - r^2\Phi''\right)\psi^{\mathrm{II}}.
\end{aligned} \tag{C14}$$

3. Odd Parity

$$\begin{aligned}
E_{\mathrm{C}} =&r^{-2}e^{-2\Lambda}\left[2 - 2e^{2\Lambda} + e^{2\Lambda}\lambda - 2r^2\Lambda'^2 + 3r\Phi' - r\Lambda'(1 + 6r\Phi') + r^2\Lambda'' + 3r^2\Phi''\right]\beta \\
&+ r^{-1}e^{-2\Lambda}(-2 + 3r\Lambda' + 3r\Phi')\frac{\partial}{\partial r}\beta - e^{-2\Lambda}\frac{\partial^2}{\partial r^2}\beta - r^{-1}e^{-2\Lambda}(-3 + 2r\Lambda' + 2r\Phi')\frac{\partial}{\partial t}\alpha + e^{-2\Lambda}\frac{\partial^2}{\partial t\partial r}\alpha,
\end{aligned} \tag{C15}$$

$$E_{\mathrm{G}} =2r^{-1}e^{-2\Phi}\frac{\partial}{\partial t}\beta + 2r^{-2}e^{-2\Lambda}(-1 + 2r\Lambda')\alpha - 2r^{-1}e^{-2\Lambda}\frac{\partial}{\partial r}\alpha, \tag{C16}$$

$$\begin{aligned}
E_{\mathrm{J}} =&-r^{-2}e^{-2\Lambda}\left[e^{2\Lambda}\lambda - 2e^{2\Lambda} + 2r\Phi' + 2r^2\Phi'^2 + 2r^2\Phi'' - 2r\Lambda'(1 + r\Phi')\right]\alpha \\
&- e^{-2\Phi}\frac{\partial^2}{\partial t^2}\alpha - r^{-1}e^{-2\Phi}(1 + r\Lambda' + r\Phi')\frac{\partial}{\partial t}\beta + e^{-2\Phi}\frac{\partial^2}{\partial t\partial r}\beta.
\end{aligned} \tag{C17}$$

Appendix D: The specific structure of the parameters

1. Even-parity and the EZ gauge for $l \geq 2$

The functions σ^{I}, τ^{I}, η^{I}, γ^{I}, ρ^{I}, μ^{I}, ν^{I}, κ^{I} in Eqs.(41) and (42) have the following form

$$\sigma^{\mathrm{I}} =1 + e^{2\Lambda}(\lambda - 1) - 2r\Lambda', \tag{D1}$$
$$\tau^{\mathrm{I}} =-2 + e^{2\Lambda}\lambda + 2r\Phi', \tag{D2}$$
$$\eta^{\mathrm{I}} =2 + e^{2\Lambda}(\lambda - 2) - 4r\Lambda', \tag{D3}$$
$$\gamma^{\mathrm{I}} =2 - 2e^{2\Lambda} + r\left(e^{2\Lambda}\lambda - 4\right)\Lambda' + 2r\Phi' + 2r^2\Phi'', \tag{D4}$$
$$\rho^{\mathrm{I}} =e^{2\Lambda}(\lambda - 2) + 4r\Phi', \tag{D5}$$
$$\begin{aligned}
\mu^{\mathrm{I}} =&-2e^{2\Lambda} + 2e^{4\Lambda} - 2e^{4\Lambda}\lambda + e^{4\Lambda}\lambda^2 - r\left(-2 + e^{2\Lambda}(2 - 4\lambda) + e^{4\Lambda}\lambda\right)\Phi' - 2\left(-1 + e^{2\Lambda}\right)r^2\Phi'^2 + 4r^2\Lambda'^2\left(1 + r\Phi'\right) \\
&+ 2r^2\Phi'' - 2e^{2\Lambda}r^2\Phi'' + e^{2\Lambda}r^2\lambda\Phi'' - r\Lambda'\left(2 - 6e^{2\Lambda} + 3e^{2\Lambda}\lambda + r\left(6 + e^{2\Lambda}(-2 + 3\lambda)\right)\Phi' + 4r^2\Phi'^2 + 4r^2\Phi''\right),
\end{aligned} \tag{D6}$$
$$\nu^{\mathrm{I}} =4 - 4e^{2\Lambda} - e^{4\Lambda}(\lambda - 2) + e^{2\Lambda}r(\lambda - 2)\Phi' + 4r\Lambda'\left(-3 + e^{2\Lambda}(1 + \lambda) + r\Phi'\right), \tag{D7}$$
$$\kappa^{\mathrm{I}} =-e^{2\Lambda}\lambda\left(e^{2\Lambda}\lambda - 2 + 2r\Phi'\right). \tag{D8}$$

To simplify the expression of N_R and N_Z, we first introduce two auxiliary functions M^I and M^{II}

$$M_1^I = -2\nu^I + \rho^I\sigma^I, \tag{D9}$$

$$M_2^I = \frac{2}{\sigma^I}\left(-2\mu^I + \gamma^I\sigma^I + r\tau^I\left(\sigma^I\right)'\right). \tag{D10}$$

Then the functions in Eq.(44) are given by

$$N^I = -4\mu^I + \sigma^I(2\gamma^I - 2\nu^I + \rho^I\sigma^I) + 2r\tau^I\left(\sigma^I\right)', \tag{D11}$$

$$N_R^I = -\left(N^I\right)^2 - 2r\sigma^I\tau^I\left(N^I\right)' + N^I\left(-4\mu^I + M_1^I\sigma^I - 2\nu^I\sigma^I + 2\sigma^I\tau^I + 2r\tau^I\left(\sigma^I\right)' + 2r\sigma^I\left(\tau^I\right)'\right), \tag{D12}$$

$$N_T^I = -\left(N^I\right)^2\left(\eta^I\right)^2 + 2N^I\left(r\left(\sigma^I\right)^2\tau^I\left(\eta^I\right)' + \eta^I\sigma^I\left(2\mu^I + \nu^I\sigma^I - 2\tau^I\left(r\left(\sigma^I\right)' + \sigma^I\left(1 + r\Lambda' - r\Phi'\right)\right)\right)\right)$$
$$+ 2r\eta^I\sigma^I(-\eta^I + \sigma^I)\tau^I\left(N^I\right)' + \left(\eta^I\right)^2\left(-2\mu^I - \nu^I\sigma^I + \tau^I\left(r\left(\sigma^I\right)' + 2\sigma^I\left(1 + r\Lambda' - r\Phi'\right)\right)\right)\right), \tag{D13}$$

$$N_Z^I = N^I\sigma^I\left(M_2^I\nu^I + r\tau^I\left(M_1^I\right)'\right) + M_1^I\left(-r\sigma^I\tau^I\left(N^I\right)' + N^I\left(-2\mu^I + r\tau^I\left(\sigma^I\right)'\right)\right), \tag{D14}$$

and the functions in Eq.(45) are given by

$$N_A^I = -\left(N^I\right)^2 - 2r\sigma^I\tau^I\left(N^I\right)' + 2N^I\left(-2\mu^I - \nu^I\sigma^I + 2\sigma^I\tau^I + 4r\sigma^I\tau^I\Lambda' + r\tau^I\left(\sigma^I\right)' + r\sigma^I\left(\tau^I\right)' - 2r\sigma^I\tau^I\Phi'\right), \tag{D15}$$

$$N_D^I = \left(N^I\right)^2\left(\eta^I\right)^2 + 2N^I\left(-r\left(\sigma^I\right)^2\tau^I\left(\eta^I\right)' + \left(\eta^I\right)^2\left(2\mu^I + \nu^I\sigma^I - \tau^I\left(r\left(\sigma^I\right)' + \sigma^I\left(3 + 4r\Lambda' - 2r\Phi'\right)\right)\right)\right)$$
$$+ 2r\eta^I(\eta^I - \sigma^I)\sigma^I\tau^I\left(N^I\right)' + \eta^I\sigma^I\left(-2\mu^I - \nu^I\sigma^I + \tau^I\left(2r\left(\sigma^I\right)' + \sigma^I\left(3 + 4r\Lambda' - 2r\Phi'\right)\right)\right)\right), \tag{D16}$$

$$M_F^I = \frac{1}{2}\left(\kappa^I + \sigma^I\left(-2 + e^{2\Lambda}\lambda + 2r\Phi'\right)\right), \tag{D17}$$

$$N_F^I = \left(N^I\right)^2\kappa^I + 4rM_F^I\sigma^I\tau^I\left(N^I\right)' + 4N^I\left(-r\sigma^I\tau^I\left(M_F^I\right)' + M_F^I\left(2\mu^I + \nu^I\sigma^I - \tau^I\left(2\sigma^I\left(1 + r\Lambda'\right) + r\left(\sigma^I\right)'\right)\right)\right), \tag{D18}$$

$$N_{HK}^I(r) = \left(N^I\right)^2\eta^I + 2r(\eta^I - \sigma^I)\sigma^I\tau^I\left(N^I\right)' + 2N^I\left(-\sigma^I\left(2\mu^I + \nu^I\sigma^I + \tau^I\left(-2\sigma^I\left(1 + r\Lambda'\right) + r\left(\left(\eta^I\right)' - 2\left(\sigma^I\right)'\right)\right)\right)\right)$$
$$+ \eta^I\left(2\mu^I + \nu^I\sigma^I - \tau^I\left(2\sigma^I\left(1 + r\Lambda'\right) + r\left(\sigma^I\right)'\right)\right)\right). \tag{D19}$$

2. Even-parity and the RW gauge for $l \geq 2$

The functions σ^{II}, τ^{II}, η^{II}, γ^{II}, ρ^{II}, μ^{II}, ν^{II} and κ^{II} in Eqs.(52) and (53) have the following form

$$\sigma^{II} = 2 - 2e^{2\Lambda} + e^{2\Lambda}\lambda - 2r\Lambda' + 2r\Phi', \tag{D20}$$

$$\tau^{II} = e^{2\Lambda}(-2 + \lambda) + 6r\Phi', \tag{D21}$$

$$\eta^{II} = 2 + e^{2\Lambda}(-2 + \lambda) - 4r\Lambda', \tag{D22}$$

$$\gamma^{II} = r\left(-4\Phi' + \Lambda'\left(e^{2\Lambda}(-2 + \lambda) + 8r\Phi'\right)\right), \tag{D23}$$

$$\rho^{II} = e^{2\Lambda}(-2 + \lambda) + 4r\Phi', \tag{D24}$$

$$\mu^{II} = -2r^2\Lambda'^2\left(e^{2\Lambda}(-2 + \lambda) + 8r\Phi'\right) + r\Lambda'\left(-2e^{2\Lambda}(-2 + \lambda) + r\left(-2 + e^{2\Lambda}(-2 + \lambda)\right)\Phi'\right)$$
$$+ e^{2\Lambda}(-2 + \lambda)\left(2 + e^{2\Lambda}(-2 + \lambda) + r^2\Lambda''\right) + 2r\Phi'\left(4 + 2e^{2\Lambda}(-2 + \lambda) + 3r^2\Lambda''\right), \tag{D25}$$

$$\nu^{II} = -e^{2\Lambda}(-2 + \lambda) - r\left(8 + e^{2\Lambda}(-2 + \lambda)\right)\Phi' + 4r\Lambda'\left(e^{2\Lambda}(-2 + \lambda) + 7r\Phi'\right), \tag{D26}$$

$$\kappa^{II} = 4 - 8e^{2\Lambda} + 4e^{4\Lambda} + 2e^{2\Lambda}\lambda - e^{4\Lambda}\lambda^2 - 2r\left(-4 + e^{2\Lambda}(4 + \lambda)\right)\Phi' - 8r\Lambda'\left(1 - e^{2\Lambda} + 2r\Phi'\right). \tag{D27}$$

To simplify the expressions of N_T^{II}, N_R^{II} and N_Z^{II}, two auxiliary functions M_1^{II} and M_2^{II} are introduced

$$M_1^{II} = 2\nu^{II} - \rho^{II}\sigma^{II}, \tag{D28}$$

$$M_2^{II} = 2\left(-2\mu^{II} + \gamma^{II}\sigma^{II} + r\tau^{II}\left(\sigma^{II}\right)'\right). \tag{D29}$$

Then the functions in Eq.(55) are given by

$$N^{\rm II} = 4\mu^{\rm II} - \sigma^{\rm II}(2\gamma^{\rm II} - 2\nu^{\rm II} + \rho^{\rm II}\sigma^{\rm II}) - 2r\tau^{\rm II}\left(\sigma^{\rm II}\right)', \tag{D30}$$

$$N_{\rm R}^{\rm II} = -\left(N^{\rm II}\right)^2 + 2r\sigma^{\rm II}\tau^{\rm II}\left(N^{\rm II}\right)' + N^{\rm II}\left(4\mu^{\rm II} + M_1^{\rm II}\sigma^{\rm II} + 2\nu^{\rm II}\sigma^{\rm II} - 2r\tau^{\rm II}\left(\sigma^{\rm II}\right)' - 2\sigma^{\rm II}\left(\tau^{\rm II} + r\left(\tau^{\rm II}\right)'\right)\right), \tag{D31}$$

$$N_{\rm T}^{\rm II} = -\left(N^{\rm II}\right)^2\left(\eta^{\rm II}\right)^2 - 4r^2\eta^{\rm II}\sigma^{\rm II}\tau^{\rm II}\left(N^{\rm II}\right)'(\Lambda' + \Phi') + 4rN^{\rm II}\left(-r\sigma^{\rm II}\tau^{\rm II}\left(\eta^{\rm II}\right)'(\Lambda' + \Phi')\right.$$

$$+\eta^{\rm II}\left(-2\mu^{\rm II}(\Lambda' + \Phi') - \nu^{\rm II}\sigma^{\rm II}(\Lambda' + \Phi') + \tau^{\rm II}\left(r\left(\sigma^{\rm II}\right)'(\Lambda' + \Phi')\right.\right.$$

$$\left.\left.\left.+\sigma^{\rm II}\left(3\Lambda' + 2r\Lambda'^2 + 3\Phi' - 2r\Phi'^2 + r\left(\Lambda'' + \Phi''\right)\right)\right)\right)\right), \tag{D32}$$

$$N_{\rm Z}^{\rm II} = N^{\rm II}\left(-M_2^{\rm II}\nu^{\rm II} + r\sigma^{\rm II}\tau^{\rm II}\left(M_1^{\rm II}\right)'\right) + M_1^{\rm II}\left(-r\sigma^{\rm II}\tau^{\rm II}\left(N^{\rm II}\right)' + N^{\rm II}\left(-2\mu^{\rm II} + r\tau^{\rm II}\left(\sigma^{\rm II}\right)'\right)\right). \tag{D33}$$

and the functions in Eq.(56) are given by

$$N_{\rm A}^{\rm II} = 2N^{\rm II}\left(2\mu^{\rm II} + \nu^{\rm II}\sigma^{\rm II} - 2\sigma^{\rm II}\tau^{\rm II} - 4r\sigma^{\rm II}\tau^{\rm II}\Lambda' - r\tau^{\rm II}\left(\sigma^{\rm II}\right)' - r\sigma^{\rm II}\left(\tau^{\rm II}\right)' + 2r\sigma^{\rm II}\tau^{\rm II}\Phi'\right)$$

$$- \left(N^{\rm II}\right)^2 + 2r\sigma^{\rm II}\tau^{\rm II}\left(N^{\rm II}\right)', \tag{D34}$$

$$N_{\rm D}^{\rm II} = -4rN^{\rm II}\left(-r\sigma^{\rm II}\tau^{\rm II}\left(\eta^{\rm II}\right)'(\Lambda' + \Phi') + \eta^{\rm II}\left(-2\mu^{\rm II}(\Lambda' + \Phi') - \nu^{\rm II}\sigma^{\rm II}(\Lambda' + \Phi')\right.\right.$$

$$\left.\left.+\tau^{\rm II}\left(r\left(\sigma^{\rm II}\right)'(\Lambda' + \Phi') + \sigma^{\rm II}\left(4r\Lambda'^2 + 4\Phi' - 2r\Phi'^2 + 2\Lambda'\left(2 + r\Phi'\right) + r\left(\Lambda'' + \Phi''\right)\right)\right)\right)\right)$$

$$+ \left(N^{\rm II}\right)^2\left(\eta^{\rm II}\right)^2 + 4r^2\eta^{\rm II}\sigma^{\rm II}\tau^{\rm II}\left(N^{\rm II}\right)'(\Lambda' + \Phi'), \tag{D35}$$

$$M_{\rm F}^{\rm II} = -\frac{1}{2}\kappa^{\rm II} - \frac{1}{2}\sigma^{\rm II}\left(-2 + e^{2\Lambda}\lambda + 2r\Phi'\right), \tag{D36}$$

$$N_{\rm F}^{\rm II} = 4N^{\rm II}\left(-r\sigma^{\rm II}\tau^{\rm II}\left(M_{\rm F}^{\rm II}\right)' + M_{\rm F}^{\rm II}\left(2\mu^{\rm II} + \nu^{\rm II}\sigma^{\rm II} - \tau^{\rm II}\left(2\sigma^{\rm II}\left(1 + r\Lambda'\right) + r\left(\sigma^{\rm II}\right)'\right)\right)\right)$$

$$+ \left(N^{\rm II}\right)^2\kappa^{\rm II} + 4rM_{\rm F}^{\rm II}\sigma^{\rm II}\tau^{\rm II}\left(N^{\rm II}\right)', \tag{D37}$$

$$N_{\rm HK}^{\rm II} = \left(N^{\rm II}\right)^2\eta^{\rm II} + 4r^2\sigma^{\rm II}\tau^{\rm II}\left(N^{\rm II}\right)'(\Lambda' + \Phi') - 4rN^{\rm II}\left(-2\mu^{\rm II}(\Lambda' + \Phi') - \nu^{\rm II}\sigma^{\rm II}(\Lambda' + \Phi')\right.$$

$$\left.+\tau^{\rm II}\left(r\left(\sigma^{\rm II}\right)'(\Lambda' + \Phi') + \sigma^{\rm II}\left(2r\Lambda'^2 + 3\Phi' + \Lambda'\left(3 + 2r\Phi'\right) + r\left(\Lambda'' + \Phi''\right)\right)\right)\right). \tag{D38}$$

3. Odd-parity for $l \geq 2$

The function X in the Eq.(64) takes the form

$$X = e^{2\Lambda}\lambda - 2r\Lambda' + 2r\Phi' - 2r^2\Lambda'\Phi' + 2r^2\Phi'^2 + 2r^2\Phi'', \tag{D39}$$

And the expression of $N_{\rm odd}$ is given by

$$N_{\rm odd} = -X + r\frac{X'}{X}\left(-1 + 2r\Lambda' + 2r\Phi'\right) + \left(2r^2\Lambda'^2 + r\Phi' + 2r^2\Phi'^2 + r\Lambda'\left(1 + 4r\Phi'\right) - 2\left(1 + r^2\Lambda'' + r^2\Phi''\right)\right). \tag{D40}$$

4. The case for $l = 0$

In this case, the functions μ_0, ν_0 and τ_0 are given by

$$\iota_0(r) = -1 + e^{2\Lambda} - r\Phi',$$

$$\sigma_0(r) = 2\left(1 - e^{2\Lambda}\right)\Phi' + 2r\Phi'^2 + \Lambda'\left(-4 + e^{2\Lambda} - 4r\Phi'\right) + 2r\Phi''. \tag{D41}$$

5. Odd parity for $l = 1$

The function X_1 in the Eq.(89) takes the form

$$X_1 = -2r\Lambda' + 2r\Phi' - 2r^2\Lambda'\Phi' + 2r^2\Phi'^2 + 2r^2\Phi'', \tag{D42}$$

And the expression of $N_{\text{odd}}^{(l=1)}$ is given by

$$N_{\text{odd}}^{(l=1)} = -X_1 + r\frac{X_1'}{X_1}\left(-1 + 2r\Lambda' + 2r\Phi'\right) + \left(2r^2\Lambda'^2 + r\Phi' + 2r^2\Phi'^2 + r\Lambda'\left(1 + 4r\Phi'\right) - 2\left(1 + r^2\Lambda'' + r^2\Phi''\right)\right). \quad (D43)$$

which is similar to the expression Eq.(D40).

Appendix E: The Degeneration Of Eqs.(44) and (55)

Considering that the metric of the background spacetime degenerates to the Schwarzschild spacetime, i.e.

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (E1)$$

where the Riemann curvature R_{ab} and the scalar curvature R all vanish. If we set

$$\tilde{Z}^{(+)} = \frac{(r - 2M)^2}{6M + r\tilde{\lambda}}Z^{(+)}, \quad (E2)$$

The equations Eq.(44) and (55) would become

$$\left[\frac{(r - 2M)^2}{r^2}\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2} + \left(\frac{r - 2M}{r}\right)\frac{2M}{r}\frac{\partial}{\partial r} - V_Z\right]\tilde{Z}^{(+)} = S_Z^{\text{w}}, \quad (E3)$$

where V_Z is the Zerilli effective potential

$$V_Z = \left(\frac{r - 2M}{r}\right)\left[\frac{72M^3 + 36M^2r\tilde{\lambda} + 6Mr^2\tilde{\lambda}^2 + r^3\tilde{\lambda}^2(\tilde{\lambda} + 2)}{r^3(6M + r\tilde{\lambda})^2}\right], \quad (E4)$$

with the source term

$$S_Z^{\text{w}} = \frac{r - 2M}{2(6M + r\tilde{\lambda})}\left[\frac{96M^2r + r^3\tilde{\lambda}(\tilde{\lambda} - 2) + 2Mr^2(7\tilde{\lambda} - 18)}{2(2M - r)(6M + r\tilde{\lambda})}E_{\text{A}} + r^2\left(\frac{\partial}{\partial r}E_{\text{A}} + \frac{\partial}{\partial t}E_{\text{D}}\right)\right.$$
$$\left. - \left(2 + \tilde{\lambda}\right)\left(\frac{(6M + r\tilde{\lambda})}{2}E_{\text{D}} - (r - 2M)E_{\text{H}} - \frac{(r - 2M)}{2}E_{\text{K}}\right)\right], \quad (E5)$$

where $\tilde{\lambda} = \lambda - 2 = (l + 2)(l - 1)$. It is obvious that our result reduces to that of the Schwarzschild spacetime [28].

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