Gauge String Field Theory for Torus Compactified Closed String[*]

Hiroyuki HATA, Katsumi ITOH, Taichiro KUGO,
Hiroshi KUNITOMO and Kaku OGAWA

Department of Physics, Kyoto University, Kyoto 606

Gauge-invariant string field theory is constructed for the closed string compactified on a torus in a generalized manner of Narain. On the basis of the observation that the coordinates of the compactified dimensions cannot be connected smoothly on the 3-string vertex, we show that it is necessary to multiply the vertex functional by the two-cocycle phase factor to achieve the gauge-invariance of the action. With this modified 3-string vertex used, all the identities concerning the ∗-product of string fields (e.g., Jacobi identity) are shown to hold in the same forms as in the ordinary non-compactified case.

§ 1. Introduction

We construct a gauge string field theory in this paper for the closed string compactified on a torus. This is one of the necessary steps toward the construction of heterotic string[1] field theory.

The gauge-invariant action, or its gauge-fixed BRS-invariant version, for such a torus compactified closed string takes essentially the same form as that for the ordinary non-compactified closed string which we have presented in our previous papers.[2]~[4] Nevertheless there appears one very novel feature in this compactified case. We will find it *impossible* to construct a 3-string vertex realizing the smooth connection conditions for the coordinates of the compactified dimensions. This problem of disconnectedness occurs only for the zero-mode parts of those coordinates, and in fact causes a trouble invalidiating the gauge-invariance proof at the order g^2. That is, the important Jacobi identity and the commutativity for the ∗-product of strings are violated. This enforces us to introduce a phase factor $\varepsilon(\boldsymbol{p}_1, \boldsymbol{p}_2)$ by which to multiply the 3-string vertex. It turns out to be exactly the two-cocycle factor well-known in the literature,[1],[5] and we can actually prove the full gauge-invariance of the action based on this modified 3-string vertex.

In § 2 we explain what type of torus compactifications we consider in this paper and fix the notations and conventions concerning the string coordinates and the string field. In § 3 we present the gauge-invariant action and the various identities for the ∗-product of the same forms as in the ordinary closed string case. We explain there how the right- and left-moving coordinates in the compactified dimensions are disconnected on the ordinary 3-string vertex, and how it causes the violation of the Jacobi identity. In § 4 it is proved that the modification of the 3-string vertex by multiplying the two-cocycle phase factor actually cures all the defects and hence, in particular, the full gauge invariance is established. The explicit form of the phase factor is also

given there.　In §5 we calculate explicitly the gauge transformation of string field in this theory for the functional transformation parameter corresponding to the global Yang-Mills gauge transformation and confirm that their generators indeed reproduce the correct commutation relations of the Lie algebra.　The final section is devoted to the discussion.

§ 2.　Closed string field on a torus

We consider the closed string in $d+D$-dimensional space-time, whose first d dimensions are "external" flat Minkowski space and the rest D dimensions are "internal" torus space.　We denote the "external" coordinates by $X^\mu(\sigma)$ $(\mu=0, 1, 2, \cdots, d-1)$ and the "internal" ones by $X^I(\sigma)(I=1, 2, \cdots, D)$, both of which are periodic functions of σ with period 2π.　The former d-dimensional coordinates $X^\mu(\sigma)$ are expanded into the oscillator modes as usual:[2),3),*)]

$$X^\mu(\sigma)=\frac{1}{\sqrt{\pi}}\left\{x^\mu+\frac{i}{2}\sum_{n\neq0}\frac{1}{n}(a_n{}^{\mu(+)}e^{in\sigma}+a_n{}^{\mu(-)}e^{-in\sigma})\right\},$$

$$A_\pm{}^\mu(\sigma)\equiv\eta^{\mu\nu}P_\nu(\sigma)\mp X^{\mu\prime}(\sigma)=\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^\infty a_n{}^{\mu(\pm)}e^{\pm in\sigma},$$

$$a_0{}^{\mu(+)}=a_0{}^{\mu(-)}=\frac{1}{2}p^\mu,\quad [a_n{}^{\mu(\varepsilon)}, a_m{}^{\nu(\varepsilon')}]=n\delta_{n+m,0}\eta^{\mu\nu}\delta^{\varepsilon\varepsilon'},\quad (\varepsilon, \varepsilon'=\pm) \tag{1}$$

where $P_\mu(\sigma)$ stand for the momenta $-i\delta/\delta X^\mu(\sigma)$ and prime denotes $\partial/\partial\sigma$.　Note that only the non-zero mode parts have split into the right-movers $a_n{}^{\mu(+)}(n\neq0)$ and the left-movers $a_n{}^{\mu(-)}(n\neq0)$ while the zero-modes x^μ or their conjugates p_μ are common. On the contrary, the "internal" coordinates $X^I(\sigma)$ should completely split into the left- and right-moving sectors including the zero-modes, as was first recongnized by Gross et al.:[1)]

$$X^I(\sigma)=X_+{}^I(\sigma)+X_-{}^I(\sigma),$$

$$X_\pm{}^I(\sigma)=\frac{1}{\sqrt{\pi}}\left\{x_\pm{}^I\mp\frac{1}{2}p_\pm{}^I\sigma+\frac{i}{2}\sum_{n\neq0}\frac{1}{n}a_n{}^{I(\pm)}e^{\pm in\sigma}\right\},$$

$$P_\pm{}^I(\sigma)=\frac{1}{2\sqrt{\pi}}\sum_{n=-\infty}^\infty a_n{}^{I(\pm)}e^{\pm in\sigma}=\mp X_\pm{}^{I\prime}(\sigma)\equiv\frac{1}{2}A_\pm{}^I(\sigma),$$

$$a_0{}^{I(\pm)}=p_\pm{}^I,\quad [x_\varepsilon{}^I, p_{\varepsilon'}{}^J]=\frac{1}{2}i\delta^{IJ}\delta_{\varepsilon\varepsilon'},\quad [a_n{}^{I(\varepsilon)}, a_m{}^{J(\varepsilon')}]=n\delta_{n+m,0}\delta^{IJ}\delta^{\varepsilon\varepsilon'}.\quad (\varepsilon, \varepsilon'=\pm) \tag{2}$$

Note that we are taking the convention[1)] that $2p_\pm{}^I$ are the translation operators of $x_\pm{}^I$.

We discuss the compactifications of closed string in a general scheme considered by Narain.[6)]　Combining the two D-dimensional coordinates $X_\pm{}^I(\sigma)$ of left- and right-movers, we regard the interal space $(X_+{}^I, X_-{}^I)$ as a $2D$-dimensional one and consider its compactification on a $2D$-dimensional torus R^{2D}/Γ, where Γ is a lattice generated by $2D$ independent vectors $e_i=(e_{i+}^I, e_{i-}^I)(i=1, 2, \cdots, 2D)$.　[Henceforce we

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*) $\eta^{\mu\nu}=\mathrm{diag}(-1, +1, +1, \cdots, +1)$.

often denote the $2D$-dimensional vectors in the internal space by bold face letters, e.g., $\boldsymbol{x}=(x_+{}^I, x_-{}^I)$.] So we identify the center-of-mass coordinate $\boldsymbol{x}=(x_+{}^I, x_-{}^I)$ with its translation by π in the direction $\boldsymbol{e}_i$:

$$\boldsymbol{x}\equiv\boldsymbol{x}+\pi\sum_{i=1}^{2D}n_i\boldsymbol{e}_i. \qquad (n_i\,;\text{integer}) \qquad (3)$$

For the center-of-mass momenta $p_+{}^I$ and $p_-{}^I$, we denote by $\boldsymbol{p}$ the $2D$-dimensional vector $(p_+{}^I, -p_-{}^I)$ instead of $(p_+{}^I, p_-{}^I)$ for convenience. Then $\boldsymbol{p}$ takes the form

$$\boldsymbol{p}=\sum_{i=1}^{2D}m_i\tilde{\boldsymbol{e}}_i, \qquad (m_i\,;\text{integer}) \qquad (4)$$

since the translation operator by an amount $\pi\sum_i n_i\boldsymbol{e}_i$ should be equal to 1, where $\tilde{\boldsymbol{e}}_i =(\tilde{e}_{i+}^I, \tilde{e}_{i-}^I)(i=1,\cdots,2D)$ are the basis vectors of the dual lattice $\tilde{\Gamma}$ in the sense of Lorentzian metric:

$$\boldsymbol{e}_i\cdot\tilde{\boldsymbol{e}}_j=\sum_I(e_{i+}^I\tilde{e}_{j+}^I-e_{i-}^I\tilde{e}_{j-}^I)=\delta_{ij}. \qquad (5)$$

The periodicity of closed string (with period $\sigma=2\pi$), on the other hand, demands a constraint on the allowed momenta $\boldsymbol{p}$:

$$2\pi\cdot\frac{\boldsymbol{p}}{2}=\pi\sum_i n_i\boldsymbol{e}_i \quad\text{for}\quad {}^\exists n_i, \qquad (6)$$

since $X_\pm^I(\sigma)$ in (2) contains the zero-mode in the form $\boldsymbol{x}-(1/2)\boldsymbol{p}\sigma=(x_+{}^I-(1/2)p_+{}^I\sigma,$ $x_-{}^I+(1/2)p_-{}^I\sigma)$. Note that, if the description of string field in the $\boldsymbol{p}$-representation can be equivalent to that in the $\boldsymbol{x}$-representation, *all* the points on the dual lattice $\tilde{\Gamma}$ (4) have to be the allowed momenta. Therefore the constraint (6) with (4) says

$$\sum_j m_j\tilde{\boldsymbol{e}}_j=\sum_i n_i\boldsymbol{e}_i \quad {}^\exists n_i \text{ for } {}^\forall m_j, \qquad (7)$$

implying $\tilde{\Gamma}\subseteqq\Gamma$; that is, the lattice Γ must be self-dual, $\Gamma=\tilde{\Gamma}$, or contain $\tilde{\Gamma}$ as its sublattice.

This condition (7) is, however, satisfied automatically if we demand a constraint

$$\frac{1}{2}\sum_I(p_+{}^I)^2+N_+=\frac{1}{2}\sum_I(p_-{}^I)^2+N_-, \qquad (8)$$

where $N_\pm$ are the right- and left-moving oscillator mode numbers (defined below in (14)). [This is a constraint $L_+-L_-=0$ implying the invariance of closed string under the shift of the origin of σ-coordinate, which we impose on our string field shortly in (13).] Since the mode number operators $N_\pm$ take integer numbers, the constraint (8) demands

$$\boldsymbol{p}^2\equiv\sum_I[(p_+{}^I)^2-(p_-{}^I)^2]=\text{even integers} \qquad (9)$$

for any momenta $\boldsymbol{p}$. Therefore $\tilde{\Gamma}$ must be a (Lorentzian) even lattice with a metric of signature (D, D). The even lattice $\tilde{\Gamma}$ is of course an integral lattice and hence automatically satisfies the above condition (7): $\tilde{\Gamma}\subseteqq(\tilde{\tilde{\Gamma}})=\Gamma$. In the present context of constructing gauge-invariant closed-string field theory, however, we shall find no more

constraint, in particular the self-duality of the lattice as found by Gross et al.[1] and Narain.[6] This is not unnatural because we are working in the *classical* string field theory framework and will probably find the self-duality requirement at the one-loop level as an anomaly free[7] condition.

In addition to the above bosonic coordinates $X^\mu(\sigma)$ and $X^I(\sigma)$, we need further Grassmann Faddeev-Popov (FP) ghost and anti-ghost coordinates, $c(\sigma)$ and $\bar{c}(\sigma)$, in a covariant formulation.[8],[9] They have the same mode-expansions as in the ordinary closed string:[2),4)]

$$C_\pm(\sigma)=i\pi_{\bar{c}}(\sigma)\mp c(\sigma)=\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}c_n^{(\pm)}e^{\pm in\sigma},$$

$$\bar{C}_\pm(\sigma)=\bar{c}(\sigma)\mp i\pi_c(\sigma)=\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\bar{c}_n^{(\pm)}e^{\pm in\sigma},$$

$$\bar{c}_0^{(\pm)}=\frac{1}{2}\bar{c}_0\mp i\pi_c^0,\quad c_0^{(\pm)}=\frac{\partial}{\partial\bar{c}_0}\pm\frac{i}{2}\frac{\partial}{\partial\pi_c^0},\quad \{c_n^{(\varepsilon)},\bar{c}_m^{(\varepsilon')}\}=\delta_{n+m,0}\delta^{\varepsilon\varepsilon'}(\varepsilon,\varepsilon'=\pm).$$

$$(10)$$

The BRS charge Q_B in this compactified case also remains the same as in the ordinary closed string and is given by the sum of two open-string's BRS charges $Q_{B\pm}$ of the right- and left-moving modes:

$$Q_B=Q_{B+}+Q_{B-},$$

$$Q_{B\pm}=\frac{\sqrt{\pi}}{2}\int_{-\pi}^{\pi}d\sigma C_\pm\left(-A_\pm\cdot A_\pm\mp 2i\frac{dC_\pm}{d\sigma}\bar{C}_\pm\right),\qquad(11)$$

where the scalar product of $(d+D)$-dimensional vectors means $x\cdot y=\sum_\mu x_\mu y^\mu+\sum_I x^I y^I$. From this form (11) it is clear that $Q_{B\pm}$ and hence Q_B satisfy the nilpotency separately,

$$Q_B{}^2=Q_{B+}^2=Q_{B-}^2=0,\qquad(12)$$

provided that the total dimension $d+D$ is 26.[8]

For simplicity, hereafter, we adopt the "π_c^0-omitted formulation"[3),4)] in this paper, in which one of the ghost zero-modes, π_c^0, is discarded by imposing a constraint

$$\mathscr{P}\Phi=\Phi\quad\text{or}\quad(L_+-L_-)\Phi=0\qquad(13)$$

on the string field Φ. Here L_+-L_- is the σ-coordinate translation operator and $\mathscr{P}$ is the projection operator into the $L_+-L_-=0$ sector:

$$\mathscr{P}=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{i\theta(L_+-L_-)},\qquad L_+-L_-=\int_{-\pi}^{\pi}d\theta(X'\cdot P+c'\pi_c+\bar{c}'\pi_{\bar{c}}),$$

$$L_\pm=-\frac{1}{8}p_\mu^2-\frac{1}{2}(p_\pm^I)^2+1-N_\pm,$$

$$N_\pm=\sum_{n\geq 1}\{a_{-n}^{(\pm)}\cdot a_n^{(\pm)}+n(c_{-n}^{(\pm)}\bar{c}_n^{(\pm)}+\bar{c}_{-n}^{(\pm)}c_n^{(\pm)})\}.\qquad(14)$$

This constraint (13) is identical with (8) as announced already. The closed-string

field Φ is a functional $\Phi[Z]$ of coordinates

$$Z = (X^\mu(\sigma), X^I(\sigma), c(\sigma), \bar{c}(\sigma); \alpha) \qquad \text{(zero-mode is omitted in } c(\sigma))$$

with α being the "string-length" parameter, and is subject to a reality condition:

$$\Phi^\dagger[Z] = \Phi[\tilde{Z}], \qquad \tilde{Z} \equiv (X^\mu(-\sigma), X^I(-\sigma), -c(-\sigma), \bar{c}(-\sigma); -\alpha). \tag{15}$$

The field Φ carries the (net) ghost number $N_{\text{FP}} = -1$ and hence is Grassmann-odd. Separating out the ghost zero-mode (which is only $\bar{c}_0$ now) explicitly and using Dirac's ket representation for the non-zero modes of Z, we can write it as[4]

$$|\Phi(x, \bar{c}_0, \alpha)\rangle = -\bar{c}_0 |\phi(x, \alpha)\rangle + |\psi(x, \alpha)\rangle, \qquad x \equiv (x^\mu, x_+^I, x_-^I). \tag{16}$$

The physical modes of the string are contained in the bosonic ϕ-component carrying $N_{\text{FP}} = 0$. In the case of the gauge-invariant action, Φ is further restricted to the internal ghost number $n_{\text{FP}} = -1$ sector[*] so as for all its component (local) fields to have vanishing ghost number.

§ 3. Gauge-invariant action and disconnectedness of $X_\pm^I(\sigma)$

Hereafter we concentrate on constructing gauge-invariant action and shall discuss the gauge-fixed action only at the final section briefly. We expect that no changes occur in the forms of the gauge-invariant action and the gauge transformation even in the present closed string compactified on a torus, compared with the ordinary closed string on a flat space. Hence, as in the latter, they should take the forms[3],[4]

$$S = \Phi \cdot Q_B \Phi + \frac{2}{3} g \Phi^3, \qquad \Phi^3 \equiv \Phi \cdot (\Phi * \Phi), \tag{17}$$

$$\delta \Phi = Q_B \Lambda + g(\Phi * \Lambda - \Lambda * \Phi), \tag{18}$$

respectively, where Λ is a functional gauge-transformation parameter carrying internal and net ghost numbers $n_{\text{FP}} = N_{\text{FP}} = -2$ and subject to the constraints $\mathcal{P}\Lambda = \Lambda$ and $\Lambda^\dagger[Z] = -\Lambda[\tilde{Z}]$. Here the dot implies an inner product

$$\Phi \cdot \Psi = \int [dZ] \Phi[\tilde{Z}] \Psi[Z] \tag{19}$$

and Q_B represents the BRS operator given by (11). The $*$-product, yielding a string field $\Phi * \Psi$ from two arbitrary fields Φ and Ψ, is defined by referring to a 3-string vertex functional V as

$$(\Phi * \Psi)[Z_3] = \int [dZ_1 dZ_2] \Phi[Z_1] \Psi[Z_2] V[Z_1, Z_2, \tilde{Z}_3], \tag{20·a}$$

or equivalently, in terms of the bra-ket notation,[4]

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[*] The internal ghost number n_{FP} is the number of excited ghost modes $c_{\pm n}^{(\pm)}(n \geqq 1)$ minus that of excited anti-ghost modes $\bar{c}_{\pm n}^{(\pm)}(n \geqq 1)$ [see Ref. 4)].

$$|(\varPhi * \varPsi)(3)\rangle = \varepsilon_\phi\varepsilon_\psi \int d1d2 \langle \varPhi(1)|\langle \varPsi(2)\| V(1,2,3)\rangle , \tag{20·b}$$

where r and dr denote a set of zero-mode variables $(x^\mu, x_+{}^I, x_-{}^I, \bar{c}_0, \alpha)$ of the r-th string and its integration measure $d^d x^\mu d^D x_+{}^I d^D x_-{}^I d\bar{c}_0 (d\alpha/2\pi)$, respectively, and ε_ϕ specifies whether $\varPhi$ is hermitian ($\varepsilon_\phi = +1$) or anti-hermitian ($\varepsilon_\phi = -1$), i.e., $\varPhi^\dagger[Z] = \varepsilon_\phi \varPhi[\tilde{Z}]$.

The gauge-invariance of the action (17) under (18), as well as the closure property of the gauge-transformation algebra[3]

$$[\delta(\Lambda_1), \delta(\Lambda_2)] = \delta(2g\Lambda_1 * \Lambda_2) , \tag{21}$$

was assured in the ordinary closed-string case by the following identities for Q_B and the $*$-product:

$$O(g^0): \quad Q_B{}^2 = 0 , \qquad\qquad \text{(nilpotency)} \tag{22}$$

$$O(g^1): \quad Q_B(\varPhi * \varPsi) = Q_B\varPhi * \varPsi + (-)^{|\varPhi|}\varPhi * Q_B\varPsi , \quad \text{(distributive law)} \tag{23}$$

$$O(g^2): \quad (\varPhi * \varPsi) * \Lambda + (-)^{|\varPhi|(|\varPsi|+|\Lambda|)}(\varPsi * \Lambda) * \varPhi + (-)^{|\Lambda|(|\varPhi|+|\varPsi|)}(\Lambda * \varPhi) * \varPsi = 0 ,$$

$$\text{(Jacobi identity)} \tag{24}$$

$$\varPhi * \varPsi = (-)^{|\varPhi||\varPsi|+1}\varPsi * \varPhi , \qquad \text{(commutativity)} \tag{25}$$

at each order level of the coupling constant g. The nilpotency (22) holds also in this case as noted before in (12) and hence there is no problem at $O(g^0)$.

The $O(g^1)$ identity (23) (distributive law) is equivalent to the BRS invariance of the 3-string vertex functional V:

$$\left(\sum_{r=1}^{3} Q_B{}^{(r)}\right)|V(1,2,3)\rangle = 0 . \tag{26}$$

This also can be satisfied if we simply adopt the same form of the vertex as in the ordinary closed string:[2],[4],[*]

$$|V^{\text{ord}}(1,2,3)\rangle = \mathcal{P}^{(1)}\mathcal{P}^{(2)}\mathcal{P}^{(3)}\mu^2(\alpha_1, \alpha_2, \alpha_3) G(\sigma_I)|V_0(1,2,3)\rangle ,$$

$$|V_0(1,2,3)\rangle = \exp\{E(1,2,3)\}|0\rangle \bar{\delta}(1,2,3) ,$$

$$E(1,2,3) = \sum_{\pm}\sum_{r,s=1}^{3}\sum_{n,m=0}^{\infty} \bar{N}_{nm}^{rs}\left(\frac{1}{2}a_{-n}^{(\pm)(r)}\cdot a_{-m}^{(\pm)(s)} + i\gamma_{-n}^{(\pm)(r)}\bar{\gamma}_{-m}^{(\pm)(s)}\right) ,$$

$$\bar{\delta}(1,2,3) = (2\pi)^{d+1}\delta^d(\sum_r p_r^\mu)\delta^D(\sum_r p_{r+}^I)\delta^D(\sum_r p_{r-}^I)\delta(\sum_r \alpha_r)\delta(\sum_r \alpha_r^{-1}\bar{c}_0^{(r)}) ,$$

$$\gamma_n^{(r)} = in\alpha_r c_n^{(r)} , \qquad \bar{\gamma}_n^{(r)} = \bar{c}_n^{(r)}/\alpha_r ,$$

$$\mu(\alpha_1, \alpha_2, \alpha_3) = \exp(-\tau_0\sum_{r=1}^{3}1/\alpha_r) , \qquad \tau_0 = \sum_{r=1}^{3}\alpha_r\ln|\alpha_r| , \tag{27}$$

where $|V_0(1,2,3)\rangle$ is essentially a product of two open-string's δ-functionals of the right- and left-moving modes and $\bar{N}_{nm}^{rs}$'s are the same as defined in I and are the Fourier components of the Neumann function for the open-string diagrams in Fig. 1 which in

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[*] The δ-functions for the discrete quantities $p_{r\pm}^I$ should of course be understood to be Kronecker's deltas.
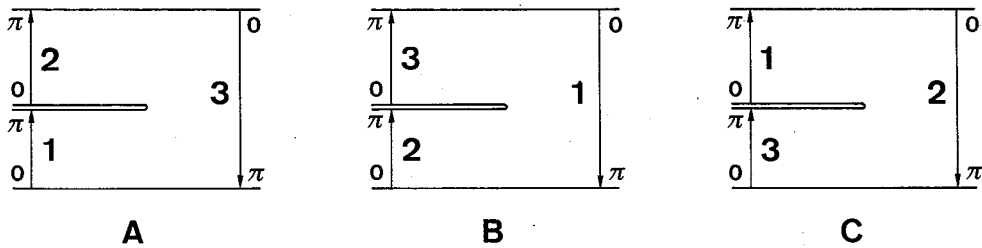
Fig. 1. The open-string diagrams (or the half-portions of closed-string diagrams with $0 \leq \sigma_r \leq \pi$) to which the Neumann function in the 3-string vertex corresponds for the cases A) $|\alpha_1|+|\alpha_2|=|\alpha_3|$, B) $|\alpha_2|+|\alpha_3|=|\alpha_1|$ and C) $|\alpha_3|+|\alpha_1|=|\alpha_2|$.

fact stand for the half-portions of the closed-string diagram with $0 \leq \sigma_r \leq \pi$. $G(\sigma_I)$ is the ghost factor at the interaction point σ_I:

$$G(\sigma_I) = i\sqrt{\pi}\, \alpha_r \pi_{\bar{c}}^{(r)}(\sigma_I^{(r)}). \qquad (r=1, 2 \text{ or } 3) \tag{28}$$

The overlapping δ-functional $|V_0\rangle$ in (27) satisfies the connection conditions (for instance for the case $\alpha_{1,2}>0$, $\alpha_3<0$)

$$(\Theta_1 \mathcal{A}_\pm^{M(1)}(\sigma_1) + \Theta_2 \mathcal{A}_\pm^{M(2)}(\sigma_2) - \mathcal{A}_\pm^{M(3)}(\sigma_3))| V_0(1, 2, 3)\rangle = 0, \tag{29}$$

$$\Theta_1(\sigma) \equiv \theta(\pi\alpha_1 - |\sigma|), \qquad \Theta_2(\sigma) \equiv \theta(|\sigma| - \pi\alpha_1),$$

$$\sigma_1(\sigma) \equiv \frac{\sigma}{\alpha_1}, \qquad \sigma_2(\sigma) \equiv \frac{\sigma - \pi\alpha_1\,\text{sgn}(\sigma)}{\alpha_2}, \qquad \sigma_3(\sigma) \equiv \frac{\pi|\alpha_3|\text{sgn}(\sigma) - \sigma}{|\alpha_3|}$$

by construction, for the coordinates

$$\mathcal{A}_\pm^M(\sigma) = (\alpha^{-1}A_\pm^\mu(\sigma), \ \alpha^{-1}A_\pm^I(\sigma), \ \alpha C_\pm(\sigma), \ \alpha^{-2}\bar{C}_\pm(\sigma)). \tag{30}$$

Since all the coordinates appearing in the expression (11) of Q_B are those of (30), it is clear that the previous proof[4],[10] of $\sum_r Q_B^{(r)}| V\rangle = 0$ for the ordinary case applies also here without any changes and Eq. (26) is proved to hold for the vertex (27).

Therefore, the problem is now only the $O(g^2)$ gauge-invariance, i.e., the Jacobi identity (24) and the commutativity (25). First of all we should note that the connection conditions (29) imply only the connection of the σ-derivatives $X'(\sigma)$ of the string coordinates $X^\mu(\sigma)$ and $X^I(\sigma)$. Indeed the "internal" coordinates $X_\pm^I(\sigma)$ are not connected smoothly contrary to the "external" coordinates $X^\mu(\sigma)$. As is easily seen by letting $X_\pm^I(\sigma)$ operate on the vertex (27), they are actually disconnected by the following amounts, for instance, for the case $\alpha_1, \alpha_2 > 0$, $\alpha_3 < 0$ with the configuration shown in Fig. 2:

$$X_\pm^{I(1)}(\sigma_1) - X_\pm^{I(3)}(\sigma_3) = \pm \frac{\sqrt{\pi}}{2} p_{3\pm}^I,$$

$$X_\pm^{I(2)}(\sigma_2) - X_\pm^{I(3)}(\sigma_3) = \mp \frac{\sqrt{\pi}}{2} p_{2\pm}^I = \pm \frac{\sqrt{\pi}}{2}(p_{3\pm}^I + p_{1\pm}^I) \qquad \text{on} \quad | V^{\text{ord}}(1, 2, 3)\rangle. \tag{31}$$

[Since X contains the center-of-mass coordinate x in the form $(1/\sqrt{\pi})x$ as seen in (2) in our convention, these equations are understood to be the equalities modulo the

rescaled Γ-lattice vectors $(1/\sqrt{\pi})\cdot(\pi\sum n_i e_i)$ (see (3)).] It is important to note that this disconnectedness occurs even for the genuine internal coordinates $X^I = X_+^I + X_-^I$ since $p_+ \neq p_-$ generally. These jumps of $X_\pm^I$-coordinates should be constants (independent of σ), since $X_\pm^{\prime\prime}(\sigma)$ are connected smoothly, and come from the presence of the term $\mp(1/2\sqrt{\pi})p_\pm^I\sigma$ in $X_\pm^I(\sigma)$ of (2) which is characteristic to the compactified case.

This disconnectedness causes a violation of the $O(g^2)$ gauge-invariance, in particular, of the Jacobi identity (24). In order to see this, we should recall how the Jacobi identity (24) held in the case of ordinary closed string. For the three terms in (24) the relevant configurations are the diagrams P, Q and R in Fig. 3 with the time interval T set equal to zero: (For definiteness we are considering the case $\alpha_1, \alpha_2, \alpha_3 > 0$.) For instance, the P term can be rewritten into the form[4]

$$((\Phi^{(1)} * \Phi^{(2)}) * \Phi^{(3)})[Z_4] = \alpha_4^{-1} \int [dZ_1 dZ_2 dZ_3] \Phi^{(1)}[\tilde{Z}_1] \Phi^{(2)}[\tilde{Z}_2] \Phi^{(3)}[\tilde{Z}_3]$$

$$\times \left\{ \int_{-\pi}^{\pi} \frac{d\theta_P}{2\pi} D(\alpha_{1-4}, \theta_P) \mathscr{P}^{(1)} \mathscr{P}^{(2)} \mathscr{P}^{(3)} \mathscr{P}^{(4)} G(\sigma_I^{126}) G(\sigma_I^{534}) V_{0,\theta_P}^{(4)}[Z_1, Z_2, Z_3, Z_4] \right\}, \tag{32}$$
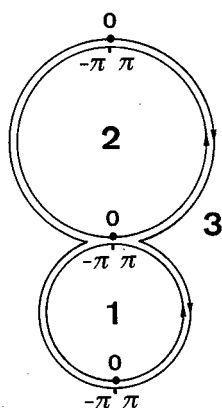

Fig. 2. The string-configuration of the overlapping δ-functional in the 3-string vertex for the case $\alpha_1, \alpha_2 > 0$, $\alpha_3 < 0$.

where $V_{0,\theta_P}^{(4)}$ is the 4-string overlapping δ-functional corresponding to the $T=0$ diagram P of Fig. 3. The diagram P itself represents the configuration at $\theta_P = 0$ and the configuration with non-zero θ_P is given by rotating the intermediate string 6 combined with the strings 1 and 2 by an angle θ_P as indicated in Fig. 3. $G(\sigma_I^{126}) G(\sigma_I^{534})$ are the ghost factors at the two 3-string interaction points, and $D(\alpha_{1-4}, \theta_P)$ is a certain determinant factor of a matrix consisting of the Neumann functions $\bar{N}_{nm}^{rs}$ ($n, m \geq 1$) of the two 3-string vertices. Similar formulas to (32) are obtained also for the Q and R terms by a suitable cyclic permuta-
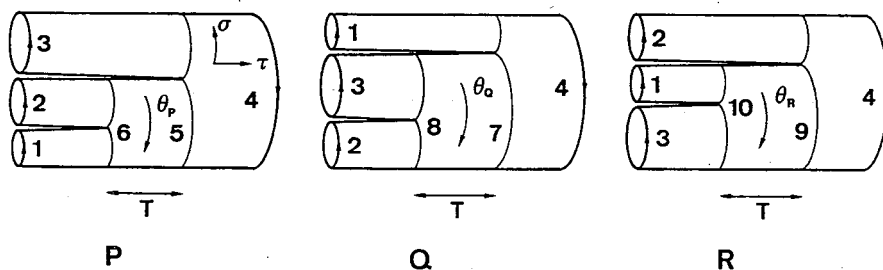

Fig. 3. The closed-string diagrams which represent the three configurations corresponding to the first (P), second (Q) and third (R) terms in the Jacobi identity (24), respectively, for the case $\alpha_1, \alpha_2, \alpha_3 > 0$. The numbers 1, 2 and 3 stand for the string fields Φ, Ψ and Λ.

tions of the indices 1, 2 and 3. The cancellations occur between the P and Q, Q and R and R and P terms according to the regions of the twisting angles θ_P, θ_Q and θ_R, and the sum of the three terms vanish as a whole so that the Jacobi identity (24) holds. Let us exemplify this cancellation by considering the P term with $\theta_P = 0$ and the Q term with $\theta_Q = \pi$. For these two terms, we immediately understand the following equality of the 4-string overlapping δ-functionals by comparing the diagrams P with $\theta_P = 0$ and Q with $\theta_Q = \pi$ in Fig. 3:

i) $\quad V_{0,\theta_P=0}^{(4)}|_P = (\prod_{r=1,4} e^{-i\pi(L_+^{(r)}-L_-^{(r)})}) V_{0,\theta_Q=\pi}^{(4)}|_Q$. $\hfill (33)$

The σ-translation operators $e^{-i\pi(L_+^{(r)}-L_-^{(r)})}$ can be dropped out owing to the presence of $\mathscr{P}^{(1)}\mathscr{P}^{(2)}\mathscr{P}^{(3)}\mathscr{P}^{(4)}$ in (32). In addition, the two ghost factors attached to the two 3-string interaction points are of course common between these two coincidental 4-string configurations of P and Q terms, i.e., from Fig. 3 we see $G(\sigma_I^{126})|_P = G(\sigma_I^{714})|_Q$ and $G(\sigma_I^{534})|_P = G(\sigma_I^{238})|_Q$. However, since the ghost factors in the Q term appear in the order $G(\sigma_I^{238})G(\sigma_I^{714})$ (as is understandable from the cyclic permutation $(1,2,3)$ $\to (2,3,1)$ in Eq. (32)), the ghost factors coincide with *negative* sign between the P and Q terms:

ii) $\quad G(\sigma_I^{126})G(\sigma_I^{534})|_P = -G(\sigma_I^{238})G(\sigma_I^{714})|_Q$. $\hfill (34)$

Furthermore, also for the determinant factor D, we have an equality

iii) $\quad |d\theta_P|D(\alpha_1, \alpha_2, \alpha_3, \alpha_4; \theta_P=0) = |d\theta_Q|D(\alpha_2, \alpha_3, \alpha_1, \alpha_4; \theta_Q=\pi)$, $\hfill (35)$

if the total dimension $d+D$ is 26, as a result of the generalized Cremmer-Gervais identity.[11] These three equalities i)~iii) guarantee the cancellation of the P term with an interval $d\theta_P$ around $\theta_P=0$ and the Q term with an interval $d\theta_Q$ around $\theta_Q=\pi$.

In the present case also, the equalities ii) and iii) hold with no problem. (Recall here that the determinant factor contains the Neumann functions $\bar{N}_{nm}^{rs}$ with $n, m \geq 1$ which are relevant only to the non-zero modes parts.) However a trouble occurs in the equality (33). In the preceding case, (33) resulted from the fact that the connection condition was complete, i.e. they hold including the zero-modes of the $X(\sigma)$ coordinates. But in the present case they are violated in the zero modes parts of the "internal" coordinates $X_{\pm}^I(\sigma)$. Indeed, taking account of the "disconnectedness" relations (31) at the 3-string vertices, we easily find from the diagram P in Fig. 3 the following connection conditions of $X_{\pm}^I$ on the 4-string δ-functional $V_{0,\theta_P=0}^{(4)}$:

$$X_{\pm}^{I(1)}(\sigma_1) - X_{\pm}^{I(4)}(\sigma_4) = \pm\frac{\sqrt{\pi}}{2}p_{4\pm}^I ,$$

$$X_{\pm}^{I(2)}(\sigma_2) - X_{\pm}^{I(4)}(\sigma_4) = \pm\frac{\sqrt{\pi}}{2}(p_{4\pm}^I + p_{1\pm}^I) , \qquad \text{on } V_{0,\theta_P=0}^{(4)} ,$$

$$X_{\pm}^{I(3)}(\sigma_3) - X_{\pm}^{I(4)}(\sigma_4) = \pm\frac{\sqrt{\pi}}{2}(p_{4\pm}^I + p_{1\pm}^I + p_{2\pm}^I) = \mp\frac{\sqrt{\pi}}{2}p_{3\pm}^{\prime I} . \qquad (36)$$

Similarly, by considering the 4-string overlapping δ-functional on the RHS of (33) and the diagram Q in Fig. 3, we find

$$X_{\pm}^{I(1)}(\sigma_1) - X_{\pm}^{I(4)}(\sigma_4) = \mp \frac{\sqrt{\pi}}{2} p_{1\pm}^I ,$$

$$X_{\pm}^{I(2)}(\sigma_2) - X_{\pm}^{I(4)}(\sigma_4) = \pm \frac{\sqrt{\pi}}{2} p_{4\pm}^I , \qquad \text{on } \left(\prod_{r=1,4} e^{-i\pi(L_+^{(r)} - L_-^{(r)})} \right) V_{0,\theta_Q=\pi}^{(4)} ,$$

$$X_{\pm}^{I(3)}(\sigma_3) - X_{\pm}^{I(4)}(\sigma_4) = \pm \frac{\sqrt{\pi}}{2} (p_{4\pm}^I + p_{2\pm}^I) , \tag{37}$$

which coincides with the result obtained simply by a cyclic permutation $(1, 2, 3)$ $\rightarrow (2, 3, 1)$ from (36). This is understandable if we notice that since $e^{i\theta(L_+^{(r)} - L_-^{(r)})}$ only changes the argument σ_r of $X_{\pm}^{I(r)}$ and commutes with $p_{r\pm}^I$, these values of the jumps of $X_{\pm}^I$ coordinates, themselves, are in fact independent of the presence of the σ-translation operators $e^{i\theta(L_+^{(r)} - L_-^{(r)})}$ and hence they are also independent of the twisting angle θ_Q. Now Eqs. (36) and (37) show that the values of the jumps of $X_{\pm}^I$ coordinates are different between on the vertices $V_{0,\theta_P=0}^{(4)}$ and $\left(\prod_{r=1,4} e^{-i\pi(L_+^{(r)} - L_-^{(r)})} \right)$ $\times V_{0,\theta_Q=\pi}^{(4)}$; that is, the connection conditions satisfied by those vertices are different (although only in the zero-mode parts), and hence the desired equality (33) is violated in the present case.

One might suspect that these violations of connection conditions for the zero-mode parts of $X_{\pm}^I(\sigma)$ could be remedied by multiplying the vertex $|V^{\mathrm{ord}}\rangle$ of (27) further by the factor of the form

$$\exp\{i \sum_{\pm IJrs} p_{r\pm}^I C_{IJ\pm}^{rs} p_{s\pm}^J\} . \qquad (C_{IJ\pm}^{rs}: \text{real constants}) \tag{38}$$

Actually this type of factor is the unique candidate which can modify the connection conditions only for the zero-modes of $X_{\pm}^I$ and with the amounts proportional to $p_{\pm}^I$. Nevertheless it is impossible to realize the complete connectedness for all the $X_{\pm}^{I(r)}$'s by multiplying such a factor, as is easily seen by actually trying it.

We should note, however, that all we need now is not to realize such complete connectedness of $X_{\pm}^{I(r)}$'s on the 3-string vertex but simply to realize the equality (33) between the 4-string δ-functionals $V_{0,\theta}^{(4)}$ resulting from joining two 3-string vertices. So it is enough if we can make the two sets of connection conditions (36) and (37) have a common form by modifying the 3-string vertex (27) by multiplying a suitable factor (38). This can be achieved relatively easily as follows.

§ 4.   Modified 3-string vertex and gauge-invariance proof

Modifying the ordinary 3-string vertex $|V^{\mathrm{ord}}(1, 2, 3)\rangle$ in (27), we now take the following 3-string vertex functional:

$$|V(1, 2, 3)\rangle = \varepsilon(\boldsymbol{p}_1, \boldsymbol{p}_2)|V^{\mathrm{ord}}(1, 2, 3)\rangle , \tag{39}$$

where $\varepsilon(\boldsymbol{p}_1, \boldsymbol{p}_2)$ denote the phase factor of the form (38) and its dependence on $\boldsymbol{p}_3$ is suppressed by taking account of the conservation $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 = 0$. The factor $\varepsilon(\boldsymbol{p}, \boldsymbol{q})$ needs in the following to satisfy the properties:

a) $\varepsilon(\boldsymbol{p}, \boldsymbol{q}) \varepsilon(\boldsymbol{p} + \boldsymbol{q}, \boldsymbol{r}) = \varepsilon(\boldsymbol{p}, \boldsymbol{q} + \boldsymbol{r}) \varepsilon(\boldsymbol{q}, \boldsymbol{r}) , \tag{40·a}$

b) $\varepsilon(\boldsymbol{p}, \boldsymbol{q}) = (-)^{\boldsymbol{p} \cdot \boldsymbol{q}} \varepsilon(\boldsymbol{q}, \boldsymbol{p})$, $\qquad\qquad$ (40·b)

where $\boldsymbol{p} \cdot \boldsymbol{q}$ denote the inner product with Lorentzian metric: $\boldsymbol{p} \cdot \boldsymbol{q} = \sum_I (p_+{}^I q_+{}^I - p^I_- q^I_-)$. The first property a) says that ε is a two-cocycle. The phase factor $\varepsilon(\boldsymbol{p}, \boldsymbol{q})$ satisfying (40) is indeed well known in the literature,[5] and we shall give its explicit form later.

Before that, let us first show that the modified vertex (39) actually realizes the desired equality (33). Since we now have the additional multiplicative factors $\varepsilon(\boldsymbol{p}_1, \boldsymbol{p}_2)$ in the 3-string vertex (39), the resultant 4-string δ-functional $V_{0,\theta_P}^{(4)}$ for the P-configuration (Fig. 3, P) becomes

$$V_{0,\theta_P}^{(4)} = \varepsilon(\boldsymbol{p}_1, \boldsymbol{p}_2)\varepsilon(\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}_3) V_{0,\theta_P}^{(4)\mathrm{ord}}, \qquad\qquad (41)$$

and similarly, $V_{0,\theta_Q}^{(4)}$ for the Q-configuration (Fig. 3, Q) becomes

$$V_{0,\theta_Q}^{(4)} = \varepsilon(\boldsymbol{p}_2, \boldsymbol{p}_3)\varepsilon(\boldsymbol{p}_2 + \boldsymbol{p}_3, \boldsymbol{p}_1) V_{0,\theta_Q}^{(4)\mathrm{ord}}. \qquad\qquad (42)$$

If we use the properties (40) of the phase factors ε, Eq. (41) can be rewritten as

$$V_{0,\theta_P}^{(4)} = \varepsilon(\boldsymbol{p}_2, \boldsymbol{p}_3)\varepsilon(\boldsymbol{p}_2 + \boldsymbol{p}_3, \boldsymbol{p}_1)(-)^{\boldsymbol{p}_1 \cdot (\boldsymbol{p}_2 + \boldsymbol{p}_3)} V_{0,\theta_P}^{(4)\mathrm{ord}}. \qquad\qquad (43)$$

So $V_{0,\theta_P}^{(4)}$ has an additional factor $(-)^{\boldsymbol{p}_1 \cdot (\boldsymbol{p}_2 + \boldsymbol{p}_3)}$ besides the common phase factors with $V_{0,\theta_Q}^{(4)}$ in (42). The previous connection conditions (36) on the ordinary 4-string δ-functional $V_{0,\theta_P=0}^{(4)\mathrm{ord}}$ are now changed into the following ones on the new δ-functional $(-)^{\boldsymbol{p}_1 \cdot (\boldsymbol{p}_2 + \boldsymbol{p}_3)} V_{0,\theta_P}^{(4)\mathrm{ord}}$ since the additional factor displaces the "center-of-mass coordinates" $x_\pm^{I(r)}$ in $X_\pm^{I(r)}(\sigma)$ (2):

$$X_\pm^{I(1)}(\sigma_1) - X_\pm^{I(4)}(\sigma_4) = \pm \frac{\sqrt{\pi}}{2}[p^I_{4\pm} - (p^I_{2\pm} + p^I_{3\pm})] = \mp \frac{\sqrt{\pi}}{2} p^I_{1\pm},$$

$$X_\pm^{I(2)}(\sigma_2) - X_\pm^{I(4)}(\sigma_4) = \pm \frac{\sqrt{\pi}}{2}[p^I_{4\pm} + p^I_{1\pm} - p^I_{1\pm}] = \pm \frac{\sqrt{\pi}}{2} p^I_{4\pm}, \;\; \text{on} \;\; (-)^{\boldsymbol{p}_1 \cdot (\boldsymbol{p}_2 + \boldsymbol{p}_3)} V_{0,\theta_P=0}^{\mathrm{ord}},$$

$$X_\pm^{I(3)}(\sigma_3) - X_\pm^{I(4)}(\sigma_4) = \mp \frac{\sqrt{\pi}}{2}(p^I_{3\pm} + p^I_{1\pm}) = \pm \frac{\sqrt{\pi}}{2}(p^I_{4\pm} + p^I_{2\pm}), \qquad\qquad (44)$$

where we have used in the first equation the fact that $(1/\sqrt{\pi}) \cdot (\boldsymbol{p}/2) \cdot 2\pi = \sqrt{\pi}\boldsymbol{p} = \sqrt{\pi}(p_+{}^I, -p_-{}^I) = 0$ modulo the rescaled Γ-lattice vector $(1/\sqrt{\pi})\sum_i n_i e_i$ because of the periodicity condition (6). But these connection conditions are exactly the same as those of (37) for $(\prod_{r=1,4} e^{-i\pi(L_+^{(r)} - L_-^{(r)})}) V_{0,\theta_Q=\pi}^{(4)\mathrm{ord}}$, and therefore, from (42) and (43), we can conclude the desired equality

$$V_{0,\theta_P=0}^{(4)} = (\prod_{r=1,4} e^{-i\pi(L_+^{(r)} - L_-^{(r)})}) V_{0,\theta_Q=\pi}^{(4)}. \qquad\qquad (45)$$

Since the center-of-mass momentum $\boldsymbol{p}$ commutes with the σ-translation operators $e^{i\theta(L_+ - L_-)}$, it is clear that the necessary equalities similar to (45) hold between the δ-functionals of P- and Q-configurations with general corresponding angles ($\theta_P = \varphi/(\alpha_1 + \alpha_2)$ and $\theta_Q = \pi - \varphi/(\alpha_2 + \alpha_3)$ in the region $|\varphi| < \alpha_3\pi$ [see Sec. IV in Ref. 4)]). Other necessary equalities between the Q- and R-configurations (with $|{}^{\vee}\varphi = (\alpha_2 + \alpha_3)\theta_Q = (\alpha_3 + \alpha_1)(\pi - \theta_R)| \leq \alpha_3\pi$) and between the R- and P-configurations (with $|{}^{\vee}\varphi = (\alpha_3 + \alpha_1)\theta_R = (\alpha_1 + \alpha_2)(\pi - \theta_P)| \leq \alpha_1\pi$) follow similarly, and thus the Jacobi identity (24) is proved to

hold for the modified 3-string vertex (39). [It is easy to see that this is true also for the other cases than $\alpha_1, \alpha_2, \alpha_3 > 0$.]

We need next consider whether the commutativity (25), $\Phi * \Psi = (-)^{|\Phi||\Psi|+1} \Psi * \Phi$, holds with our modified 3-string vertex (39). In the previous case of the ordinary closed string, it resulted from the following property of the δ-functional part V_0 of the 3-string vertex V (see Eq. (3·31) in Ref. 4)):

$$|V_0(1,2,3)\rangle = (\prod_{r=1}^{3} e^{i\pi(L_+^{(r)} - L_-^{(r)})})|V_0(2,1,3)\rangle . \qquad (46)$$

However, this property is also violated for the δ-functional part $|V_0\rangle$ of the ordinary 3-string vertex $|V^{ord}\rangle$ in (27) again owing to the disconnectedness of the zero-mode parts in the internal space. Indeed the connection conditions realized by $|V_0(1,2,3)\rangle$ is given by (31) (for the case $\alpha_1, \alpha_2 > 0, \alpha_3 < 0$), but they do not coincide with those realized by $(\prod_{r=1}^{3} e^{i\pi(L_+^{(r)} - L_-^{(r)})})|V_0(2,1,3)\rangle$, which are given by

$$X_\pm^{I(1)}(\sigma_1) - X_\pm^{I(3)}(\sigma_3) = \mp \frac{\sqrt{\pi}}{2} p_{1\pm}^I ,$$

$$X_\pm^{I(2)}(\sigma_2) - X_\pm^{I(3)}(\sigma_3) = \pm \frac{\sqrt{\pi}}{2} p_{3\pm}^I \qquad \text{on } (\prod_{r=1}^{3} e^{i\pi(L_+^{(r)} - L_-^{(r)})})|V_0(2,1,3)\rangle . \qquad (47)$$

[These values of disconnectedness are obtained from (31) simply by exchanging the string names 1 and 2, being independent of the presence of the operators $\prod_{r=1}^{3} e^{i\pi(L_+^{(r)} - L_-^{(r)})}$ which just adjust the σ-coordinates of the corresponding points on the strings.] But in this case of the modified 3-string vertex (39), the δ-functional $|V_0(1,2,3)\rangle$ is multiplied by $\varepsilon(\boldsymbol{p}_1, \boldsymbol{p}_2)$, while $|V_0(2,1,3)\rangle$ by $\varepsilon(\boldsymbol{p}_2, \boldsymbol{p}_1) = (-)^{p_1 \cdot p_2} \varepsilon(\boldsymbol{p}_1, \boldsymbol{p}_2)$ by (40·b). Therefore the connection conditions to be compared with (31) are those on the vertex in (47) multiplied by an additional factor $(-)^{p_1 \cdot p_2}$. This factor actually convert the connection conditions (47) into the same ones as in (31). Thus we can conclude the equality

$$\varepsilon(\boldsymbol{p}_1 \cdot \boldsymbol{p}_2)|V_0(1,2,3)\rangle = \varepsilon(\boldsymbol{p}_2, \boldsymbol{p}_1) \prod_{r=1}^{3} e^{i\pi(L_+^{(r)} - L_-^{(r)})}|V_0(2,1,3)\rangle , \qquad (48)$$

from which the commutativity (25), $\Phi * \Psi = (-)^{|\Phi||\Psi|+1} \Psi * \Phi$ follows for the *-product with our modified 3-string vertex.

Although we have finished the proofs of the Jacobi identity (24) and the commutativity (25), we still need take care of the following two important properties which the ordinary 3-string vertex satisfied;[4] one is the hermiticity

$$V^{ord}[\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3] = V^{ord\dagger}[Z_2, Z_1, Z_3] \qquad (49)$$

and the other is the cyclic symmetry

$$V^{ord}[Z_1, Z_2, Z_3] = V^{ord}[Z_2, Z_3, Z_1] . \qquad (50)$$

The former property guaranteed the consistency that the gauge transformed field $\Phi + Q_B \Lambda + g(\Phi * \Lambda - \Lambda * \Phi)$ has the same hermiticity as the original field Φ, and the latter led to the important cyclic property of tri-linear form $\Phi \cdot (\Psi * \Lambda)$:

$$\Phi\cdot(\Psi*\Lambda)=(-)^{|\Phi|(|\Psi|+|\Lambda|)}\Psi\cdot(\Lambda*\Phi)\,. \tag{51}$$

Therefore both of these properties (49) and (50) are also indispensable for the gauge-invariance of the theory and should not be violated by the present modification of the vertex. Since we are multiplying the phase factor $\varepsilon(\boldsymbol{p}_1,\boldsymbol{p}_2)$, those properties (49) and (50) remain to hold, if the function $\varepsilon(\boldsymbol{p},\boldsymbol{q})$ satisfies

$$\varepsilon(-\boldsymbol{p}_1,-\boldsymbol{p}_2)=\varepsilon^*(\boldsymbol{p}_2,\boldsymbol{p}_1)\,, \tag{52}$$

$$\varepsilon(\boldsymbol{p}_1,\boldsymbol{p}_2)=\varepsilon(\boldsymbol{p}_2,\boldsymbol{p}_3=-\boldsymbol{p}_1-\boldsymbol{p}_2)\,, \tag{53}$$

respectively. It is easy to see by using the two-cocycle condition (40·a) that these are satisfied if

$$\varepsilon(\boldsymbol{p},-\boldsymbol{p})=\frac{1}{\varepsilon(0,\boldsymbol{p})}=\boldsymbol{p}\text{-independent}\,,$$

or equivalently, if

$$\varepsilon(\boldsymbol{p},-\boldsymbol{p})\varepsilon(0,0)=1 \qquad \text{for} \quad {}^\forall\boldsymbol{p}\,, \tag{54}$$

since also the $\boldsymbol{p}$-independence of $\varepsilon(0,\boldsymbol{p})=\varepsilon(\boldsymbol{p},0)$ follows from (40).

However the condition (54) can always be realized by redefining the phase of the string field Φ: Indeed suppose that a certain phase function $\bar{\varepsilon}(\boldsymbol{p},\boldsymbol{q})$ satisfies the properties (40) but not the condition (54), and then consider the string field redefinition from the field $\bar{\Phi}$ to

$$\Phi[\boldsymbol{p},\cdots]=U(\boldsymbol{p})\bar{\Phi}[\boldsymbol{p}\cdots] \qquad (|U(\boldsymbol{p})|=1) \tag{55}$$

in the $\boldsymbol{p}$-representation. On this new field basis Φ, the phase function $\bar{\varepsilon}(\boldsymbol{p},\boldsymbol{q})$ is changed into

$$\varepsilon(\boldsymbol{p},\boldsymbol{q})=\bar{\varepsilon}(\boldsymbol{p},\boldsymbol{q})U^{-1}(\boldsymbol{p})U^{-1}(\boldsymbol{q})U(\boldsymbol{p}+\boldsymbol{q}) \tag{56}$$

as is seen from (39) and (20·a), and still satisfies (40). If we take

$$U(\boldsymbol{p})=[\,\bar{\varepsilon}(0,0)\,\bar{\varepsilon}(\boldsymbol{p},-\boldsymbol{p})]^{1/2}\,, \tag{57}$$

the new phase factor $\varepsilon(\boldsymbol{p},\boldsymbol{q})$ defined by (56) clearly satisfies the condition (54) also (since $\bar{\varepsilon}(\boldsymbol{p},-\boldsymbol{p})=\bar{\varepsilon}(-\boldsymbol{p},\boldsymbol{p})$ follows from (40·a)). This, in turn, implies that we need not necessarily do such a field redefinition (55) actually and may keep to use the phase function $\bar{\varepsilon}(\boldsymbol{p},\boldsymbol{q})$ and the corresponding field $\bar{\Phi}$. Then we have only to adopt the follwing definitions of hermiticity and inner-product:

$$\bar{\Phi}^{\dagger}[\boldsymbol{p},Z']=(U(\boldsymbol{p})U(-\boldsymbol{p}))\bar{\Phi}[-\boldsymbol{p},\tilde{Z}']$$

$$=[\,\bar{\varepsilon}(0,0)\,\bar{\varepsilon}(\boldsymbol{p},-\boldsymbol{p})]\bar{\Phi}[-\boldsymbol{p},\tilde{Z}']\,,$$

$$\bar{\Phi}\cdot\bar{\Psi}=\int[dZ]\bar{\Phi}[-\boldsymbol{p},\tilde{Z}'][\,\bar{\varepsilon}(0,0)\,\bar{\varepsilon}(\boldsymbol{p},-\boldsymbol{p})]\bar{\Psi}[\boldsymbol{p},Z'] \tag{58}$$

with Z' denoting Z other than $\boldsymbol{p}$. These are of course equivalent to the usual definitions (15) and (19) for Φ by the identification (55).

Let us now give the explicit expression of the phase function $\varepsilon(\boldsymbol{p},\boldsymbol{q})$ satisfying the

properties (40) and the condition (54).   First of all we should note that the property
(40·b) for $p=q$ demands

$$(-)^{p^2}=1 \rightarrow p^2=\text{even integer for } {}^\forall p, \tag{59}$$

the same condition as was required already in (9) from the constraint $(L_+ - L_-)\Phi=0$.
That is, the dual lattice must be a (Lorentzian) *even lattice* and hence the basis vectors
$\tilde{e}_i$ of $\tilde{\Gamma}$ satisfy

$$\tilde{e}_i \cdot \tilde{e}_j = \begin{cases} \text{even integer}, & (i=j) \\ \text{integer}. & (i \neq j) \end{cases} \tag{60}$$

We can now find a phase function $\varepsilon(p, q)$ satisfying the two properties (40) by
demanding a stronger condition, bilinearity:

$$\text{a')} \quad \varepsilon(p+q, r)=\varepsilon(p, r)\varepsilon(q, r),$$

$$\varepsilon(p, q+r)=\varepsilon(p, q)\varepsilon(p, r) \tag{61}$$

from which the two-cocycle condition (40·a) follows trivially.   With this bilinearity
condition (61) imposed, the second property (40·b) is satisfied if it holds only for the
basis vectors $\tilde{e}_i$ of the dual lattice $\tilde{\Gamma}$:

$$\varepsilon(\tilde{e}_i, \tilde{e}_j)=(-)^{\tilde{e}_i \cdot \tilde{e}_j}\varepsilon(\tilde{e}_j, \tilde{e}_i). \tag{62}$$

A solution to this equation is clearly given by

$$\bar{\varepsilon}(\tilde{e}_i, \tilde{e}_j)=\begin{cases} (-)^{\tilde{e}_i \cdot \tilde{e}_j}, & (i>j) \\ 1, & (i \leq j) \end{cases} \tag{63}$$

with an arbitrary choice of the ordering of the basis vectors $\tilde{e}_i$, which thus gives
generally

$$\bar{\varepsilon}(p=\sum_i n_i \tilde{e}_i, q=\sum_j m_j \tilde{e}_j)=\exp i\pi \sum_{i>j} n_i m_j \tilde{e}_i \cdot \tilde{e}_j \equiv \exp i\pi \sum_> p \cdot q. \tag{64}$$

This solution has an advantage that $\bar{\varepsilon}(p, q)$ takes only the real values ± 1 and may
be convenient.   With this $\bar{\varepsilon}(p, q)$, however, since it does not satisfy the condition (54),
we have to use the above-mentioned unconventional form of definitions (58) of the
hermiticity and inner-product.
   Another solution to (62) which satisfies also the condition (54) is given by

$$\varepsilon(\tilde{e}_i, \tilde{e}_j)=\begin{cases} (-)^{1/2 \tilde{e}_i \cdot \tilde{e}_j}, & (i>j) \\ 1, & (i=j) \\ (-)^{-1/2 \tilde{e}_i \cdot \tilde{e}_j}, & (i<j) \end{cases} \tag{65}$$

from which we have generally

$$\varepsilon(p, q)=\exp i\pi \frac{1}{2}(\sum_{i>j} - \sum_{i<j})n_i m_j \tilde{e}_i \cdot \tilde{e}_j=\exp i\pi \frac{1}{2}(\sum_> - \sum_<)p \cdot q. \tag{66}$$

This indeed satisfies (54) since $\varepsilon(\boldsymbol{p}, -\boldsymbol{p}) = \varepsilon(\boldsymbol{p}, \boldsymbol{p}) = 1$. It is easy to see that this solution $\varepsilon(\boldsymbol{p}, \boldsymbol{q})$ (66) can be obtained also from the above solution $\bar{\varepsilon}(\boldsymbol{p}, \boldsymbol{q})$ (64) by the procedure (56) if we use the even-lattice condition (60).

§ 5. Global Yang-Mills gauge transformation

It would be instructive to study the gauge transformation

$$\delta\Phi = Q_B \Lambda + 2g\Phi * \Lambda = Q_B \Lambda - 2g\Lambda * \Phi \tag{67}$$

more explicitly for the case of *global* Yang-Mills gauge transformation (rigid 'color' rotation), for which Λ is given by

$$|\Lambda(p^\mu, \boldsymbol{p}_\pm, \bar{c}_0, a)\rangle = -\bar{c}_0 \sum_\pm \Big\{ \sum_I \theta_\pm{}^I \delta(\boldsymbol{p}_+)\delta(\boldsymbol{p}_-)\bar{c}_{-1}^{(\mp)}a_{-1}^{(\pm)I}|0\rangle$$

$$+ \sum_{\boldsymbol{k}_\pm} \theta_\pm(\boldsymbol{k}_\pm)\delta(\boldsymbol{p}_\pm - \boldsymbol{k}_\pm)\delta(\boldsymbol{p}_\mp)\bar{c}_{-1}^{(\mp)}|0\rangle\Big\}(2\pi)^{d+1}\delta(p^\mu)\delta(a) ,\tag{68}$$

where $\theta_\pm{}^I$ and $\theta_\pm(\boldsymbol{k}_\pm)$ are constant parameters which satisfy the reality conditions $\theta_\pm{}^{I*} = \theta_\pm{}^I$ and $\theta_\pm{}^*(\boldsymbol{k}_\pm) = -\theta_\pm(-\boldsymbol{k}_\pm)$ implied by $\Lambda^\dagger[Z] = -\Lambda[\tilde{Z}]$. Here and henceforth $\boldsymbol{p}_\pm$ denotes a D-dimensional vector with Euclidian metric and the summation $\sum_{\boldsymbol{k}_\pm}$ in (68) is taken over the internal momenta $\boldsymbol{k}_\pm$ satisfying

$$\boldsymbol{k}_\pm{}^2 = 2 \tag{69}$$

so that the constraint $\mathcal{P}\Lambda = \Lambda$ or Eq. (8) on $|\Lambda\rangle$ is met. The factor $(2\pi)^{d+1}\delta(p^\mu)\delta(a)$ in (68) represents that the transformation is a global one in the x^μ and $\tilde{a}$ (Fourier-conjugate of a) spaces. Note also that all the states in (68) are the massless states possessing the internal ghost number $n_{\mathrm{FP}} = -2$ (which is required for Λ), and that

$$Q_B|\Lambda\rangle = 0 . \tag{70}$$

So we have to evaluate only

$$\delta|\Phi(3)\rangle = -2g|(\Lambda * \Phi)(3)\rangle = 2g\int d1 d2 \langle\Lambda(1)|\langle\Phi(2)\|V(1, 2, 3)\rangle . \tag{71}$$

For the vertex given by (39) with $|V^{\mathrm{ord}}\rangle$ of (27), we use the expression (see Eqs. (3·23) and (3·24) in Ref. 4))

$$|V(1, 2, 3)\rangle = \mathcal{P}^{(1)}\mathcal{P}^{(2)}\mathcal{P}^{(3)}\mu^2(\alpha_1, \alpha_2, \alpha_3)\varepsilon(\boldsymbol{p}_3, \boldsymbol{p}_1)$$

$$\times\Big[\prod_{r=1}^3\Big(1 - \bar{c}_0^{(r)}\frac{1}{\sqrt{2}}w_1^{(r)}\Big)\Big]\exp\{E(1, 2, 3)|_{\bar{r}_0=0}\}|0\rangle\delta(1, 2, 3) ,$$

$$\delta(1, 2, 3) = (2\pi)^{d+1}\delta(\textstyle\sum_r p_r^\mu)\delta(\textstyle\sum_r \alpha_r)\delta(\textstyle\sum_r \boldsymbol{p}_{r+})\delta(\textstyle\sum_r \boldsymbol{p}_{r-}) \tag{72}$$

with $w_1^{(r)}/\sqrt{2} = (w_1^{(+)(r)} + w_1^{(-)(r)})/2$ defined in (3·14b) in Ref. 4. We are now working in the momentum representation for the zero-modes and so the $\int dr$ in (71) is understood to be $\int d^d p_r^\mu da_r/(2\pi)^{d+1}\sum_{p_{r+}}\sum_{p_{r-}}$. As a matter of fact, in order to evaluate (71) properly, we need to replace the δ-function $\delta(\alpha_1)$ in $\langle\Lambda(1)|$ by a limit $\lim_{\varepsilon\to0}\delta(\alpha_1 - \varepsilon)$. Various formulas for the Neumann functions $\bar{N}_{nm}^{rs}$ and other formulas useful in this

limit $\alpha_1 = \varepsilon \to 0$ can be found in Appendix E of Ref. 12). By using them and the hermiticity condition of Φ written as (cf. Eq. (2·8) in Ref. 4))

$$\mathcal{P}^{(3)}|\Phi(3)\rangle = \int d2(\bar{c}_0^{(2)} - \bar{c}_0^{(3)})\langle\Phi(2)|\mathcal{P}^{(2)}|r(2,3)\rangle\,,$$

$$|r(2,3)\rangle = \exp\left\{\sum_{n\geq 1}(-)^{n-1}\sum_{\pm}\left[\frac{1}{n}\alpha_{-n}^{(\pm)(2)}\cdot\alpha_{-n}^{(\pm)(3)} - \bar{c}_{-n}^{(\pm)(2)}c_{-n}^{(\pm)(3)} + c_{-n}^{(\pm)(2)}\bar{c}_{-n}^{(\pm)(3)}\right]\right\}|0\rangle$$

$$\times(2\pi)^{d+1}\delta(p_2^\mu + p_3^\mu)\delta(\alpha_2 + \alpha_3)\delta(\boldsymbol{p}_{2+} + \boldsymbol{p}_{3+})\delta(\boldsymbol{p}_{2-} + \boldsymbol{p}_{3-})\,, \tag{73}$$

and also noting that only the part

$$\prod_{r=1}^{3}\left(1 - \bar{c}_0^{(r)}\frac{1}{\sqrt{2}}w^{(r)}\right) \to (\bar{c}_0^{(2)} - \bar{c}_0^{(3)})\frac{1}{2}(c_{-1}^{(+)(1)} + c_{-1}^{(-)(1)}) \tag{74}$$

contributes to (71), we easily reach the expression

$$\delta|\Phi\rangle = -2g|\Lambda * \Phi\rangle$$

$$= g\sum_{\pm}\left\{\sum_I \theta_\pm^{\,I}H_\pm^{\,I} + \sum_{\boldsymbol{k}_\pm}\theta_\pm(\boldsymbol{k}_\pm)E_\pm(\boldsymbol{k}_\pm)\right\}|\Phi\rangle\,, \tag{75}$$

$$H_\pm^{\,I} = \int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}2\sqrt{\pi}P_\pm^{\,I}(\sigma) = \mp\int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}2\sqrt{\pi}X_\pm^{\,I\prime}(\sigma) = p_\pm^{\,I}\,,$$

$$E_\pm(\boldsymbol{k}_\pm) = \varepsilon(\boldsymbol{p}_\pm, -\boldsymbol{k}_\pm)\int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}:\exp(2\sqrt{\pi}\,i\boldsymbol{k}_\pm\cdot\boldsymbol{X}_\pm(\sigma)):$$

$$:\exp(2\sqrt{\pi}\,i\boldsymbol{k}\cdot\boldsymbol{X}_\pm(\sigma)): = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\boldsymbol{k}\cdot\boldsymbol{\alpha}_{-n}^{(\pm)}e^{\mp in\sigma}\right)\exp\left(2i\boldsymbol{k}\cdot\left(\boldsymbol{x}_\pm \mp \frac{1}{2}\boldsymbol{p}_\pm\sigma\right)\right)$$

$$\times\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\boldsymbol{k}\cdot\boldsymbol{\alpha}_n^{(\pm)}e^{\pm in\sigma}\right)\,, \tag{76}$$

where $X_\pm^{\,I}(\sigma)$ and $P_\pm^{\,I}(\sigma)$ are the "internal" coordinate and momentum of $|\Phi\rangle$ defined in (2) and $\boldsymbol{x}_\pm$ and $\boldsymbol{p}_\pm$ are their zero-modes. The phase factor $\varepsilon(\boldsymbol{p}_\pm, -\boldsymbol{k}_\pm)$ is understood to be $\varepsilon(\boldsymbol{p}, -\boldsymbol{k})$ with $\boldsymbol{p} = (p_+, 0)$, $\boldsymbol{k} = (k_+, 0)$ or $\boldsymbol{p} = (0, -p_-)$, $\boldsymbol{k} = (0, -k_-)$ substituted. From (75) we see that $\boldsymbol{H} = (H_\pm^{\,I})$ and $E_\pm(\boldsymbol{k}_\pm)$ stand for the generators of the global YM gauge transformation represented on the string field $|\Phi\rangle$, which correspond to a Lie group $G_+ \otimes G_-$. Actually the commutation relations

$$[X_\pm^{\,I}(\sigma), P_\pm^{\,J}(\sigma')] = \frac{i}{2}\delta^{IJ}\delta(\sigma - \sigma')\,,$$

$$[P_\pm^{\,I}(\sigma), P_\pm^{\,J}(\sigma')] = [\mp X_\pm^{\,I\prime}(\sigma), P_\pm^{\,J}(\sigma')] = \mp\frac{i}{2}\delta^{IJ}\delta'(\sigma - \sigma')\,, \tag{77}$$

immediately lead to

$$[H_\pm^{\,I}, H_\pm^{\,J}] = 0\,, \tag{78}$$

$$[\boldsymbol{H}, E_\pm(\boldsymbol{k}_\pm)] = \boldsymbol{k}_\pm E_\pm(\boldsymbol{k}_\pm)\,. \tag{79}$$

Further, for the commutator

$$[E_\pm(\boldsymbol{k}_\pm), E_\pm(\boldsymbol{l}_\pm)] = \varepsilon(\boldsymbol{p}_\pm, -\boldsymbol{k}_\pm)A_\pm(\boldsymbol{k}_\pm)\varepsilon(\boldsymbol{p}_\pm, -\boldsymbol{l}_\pm)A_\pm(\boldsymbol{l}_\pm) - (\boldsymbol{k}\leftrightarrow\boldsymbol{l}) \tag{80}$$

with $A_\pm(\boldsymbol{k})\equiv\int(d\sigma/2\pi){:}\exp(2\pi i\boldsymbol{k}\cdot\boldsymbol{X}_\pm(\sigma)){:}$, we use an equation

$$A_\pm(\boldsymbol{k}_\pm)\varepsilon(\boldsymbol{p}_\pm, -\boldsymbol{l}_\pm) = \varepsilon(\boldsymbol{p}_\pm - \boldsymbol{k}_\pm, -\boldsymbol{l}_\pm)A_\pm(\boldsymbol{k}_\pm), \tag{81}$$

since $A_\pm(\boldsymbol{k})$ contains the part $\exp2i\boldsymbol{k}\cdot\boldsymbol{x}_\pm=\exp(-\boldsymbol{k}\cdot\partial/\partial\boldsymbol{p}_\pm)$ non-commutative with $\boldsymbol{p}_\pm$. With the help of the properties (40) of $\varepsilon(\boldsymbol{p}, \boldsymbol{k})$ and the known formula[3]

$$A_\pm(\boldsymbol{k})A_\pm(\boldsymbol{l}) - (-)^{\boldsymbol{k}\cdot\boldsymbol{l}}A_\pm(\boldsymbol{l})A_\pm(\boldsymbol{k}) = \begin{cases} A_\pm(\boldsymbol{k}+\boldsymbol{l}) & (\boldsymbol{k}+\boldsymbol{l})^2=2, \\ \boldsymbol{k}\cdot\boldsymbol{p}_\pm & \boldsymbol{k}+\boldsymbol{l}=0, \\ 0 & \text{otherwise}, \end{cases} \tag{82}$$

we find

$$[E_\pm(\boldsymbol{k}_\pm), E_\pm(\boldsymbol{l}_\pm)] = \varepsilon(\boldsymbol{p}_\pm, -\boldsymbol{k}_\pm-\boldsymbol{l}_\pm)\varepsilon(-\boldsymbol{k}_\pm, -\boldsymbol{l}_\pm)A_\pm(\boldsymbol{k}_\pm)A_\pm(\boldsymbol{l}_\pm) - (\boldsymbol{k}\leftrightarrow\boldsymbol{l})$$

$$= \varepsilon(\boldsymbol{p}_\pm, -\boldsymbol{k}_\pm-\boldsymbol{l}_\pm)\varepsilon(-\boldsymbol{k}_\pm, -\boldsymbol{l}_\pm)[A_\pm(\boldsymbol{k}_\pm)A_\pm(\boldsymbol{l}_\pm) - (-)^{\boldsymbol{k}\cdot\boldsymbol{l}}A_\pm(\boldsymbol{l}_\pm)A_\pm(\boldsymbol{k}_\pm)]$$

$$= \begin{cases} \varepsilon(\boldsymbol{k}_\pm, \boldsymbol{l}_\pm)E_\pm(\boldsymbol{k}_\pm+\boldsymbol{l}_\pm) & (\boldsymbol{k}_\pm+\boldsymbol{l}_\pm)^2=2, \\ \boldsymbol{k}_\pm\cdot\boldsymbol{H}_\pm & \boldsymbol{k}_\pm+\boldsymbol{l}_\pm=0, \\ 0 & \text{otherwise}. \end{cases} \tag{83}$$

[We have used $\varepsilon(\boldsymbol{p}, 0)=\varepsilon(\boldsymbol{k}, -\boldsymbol{k})=1$ and $\varepsilon(-\boldsymbol{k}, -\boldsymbol{l})=\varepsilon(\boldsymbol{k}, \boldsymbol{l})$.] Thus, from (78), (79) and (83) we see that our generators of global YM gauge transformation indeed reproduce the correct commutation relations and that $H_\pm{}'$ are the generators of the Cartan subalgebra and $E_\pm(\boldsymbol{k}_\pm)$ are those of non-zero roots $\boldsymbol{k}_\pm$.

§ 6. Discussion

We thus have shown that the gauge-invariant action for the closed string field compactified on a torus is also given by the same form as (17) for the ordinary non-compactified closed string, if the 3-string vertex is modified by the two-cocycle phase factor $\varepsilon(\boldsymbol{p}_1, \boldsymbol{p}_2)$ as given in (39).

Since all the formal identities concerning the BRS operator Q_B and the $*$ and $\cdot$ products remain the same as in the ordinary case, it is clear that the gauge-fixed action $\hat{S}$ of the torus compactified closed string field also takes the same form and is given simply by putting the ψ-component of $\varPhi=-\bar{c}_0\phi+\psi$ equal to zero in the gauge-invariant action:

$$\hat{S} = \left[\varPhi\cdot Q_B\varPhi + \frac{2}{3}g\varPhi^3\right]_{\psi=0} = \phi\cdot L\phi + \frac{2}{3}g\phi^3 \tag{84}$$

with $L=2(L_++L_-)$. The BRS transformation $\hat{\delta}_B\phi$ in this gauge-fixed system is given by setting $\psi=0$ in the "original" BRS transformation $\delta_B\varPhi=Q_B\varPhi+g\varPhi*\varPhi$ of the ϕ-component:

$$\hat{\delta}_B\phi = \delta_B\phi|_{\psi=0} = \int d\bar{c}_0(Q_B\varPhi+g\varPhi*\varPhi)|_{\psi=0}, \tag{85}$$

which is on-shell nilpotent and leaves the gauge-fixed action (84) invariant.

We have restricted our discussion in this paper to the compactifications of D right moving and D left moving coordinates since we have started from the ordinary 26-dimensional closed string. However, as is clear from the present discussion, it is not necessary that the compactified dimension of the right- and left-moving coordinates are equal. Also for the general case when D_+ right-moving and D_- left-moving coordinates are compactified, our arguments in this paper apply as they stand if the inner product $\boldsymbol{p} \cdot \boldsymbol{q}$ in the phase factor formula (64),

$$\varepsilon(\boldsymbol{p}, \boldsymbol{q}) = \exp i\pi \frac{1}{2} (\sum_{>} - \sum_{<}) \boldsymbol{p} \cdot \boldsymbol{q} , \tag{86}$$

is understood to be the one with a Lorentzian metric of signature (D_+, D_-):[6]

$$\boldsymbol{p} \cdot \boldsymbol{q} = \sum_{I=1}^{D_+} p_+{}^I q_+{}^I - \sum_{I=1}^{D_-} p_-{}^I q_-{}^I . \tag{87}$$

Therfore, for instance, our 3-string vertex already gives a correct answer to the bosonic coordinate part of the heterotic string field theory by taking $(D_+, D_-) = (0, 16)$ and $d = 10$. If a further compactification would occur in the heterotic string, then we should take $(D_+, D_-) = (D, D+16)$.

In this connection with superstrings, it would be appropriate here to mention the odd lattice (=non-even integral lattice) which generally appears when the fermionic coordinates are bosonized. The extension of the present formulation to the odd lattice case is straightforward. In this case, (40·b) is replaced by

$$\text{b'}) \quad \varepsilon(\boldsymbol{p}, \boldsymbol{q}) = (-)^{\boldsymbol{p} \cdot \boldsymbol{q} + \boldsymbol{p}^2 \boldsymbol{q}^2} \varepsilon(\boldsymbol{q}, \boldsymbol{p}) , \tag{40·b'}$$

which then becomes consistent also with non-even $\boldsymbol{p}^2$. It is well known that the phase function $\varepsilon(\boldsymbol{p}, \boldsymbol{q})$ satisfying the properties (40·a), (40·b') and (54) exists.[13] The added factor $(-)^{\boldsymbol{p}^2 \boldsymbol{q}^2}$, which equals $+1$ in the even lattice case, does not affect the proof of the Jacobi identity (24) since the additional deviations of the connection condition for X^μ's induced by this factor can always be obliterated by the periodicity of X^μ.

It is interesting to note that the phase factor $\varepsilon(\boldsymbol{p}, \boldsymbol{q})$ (86) in the case $D_+ = D_-$ becomes 1 in the zero-winding number sector in which $\boldsymbol{p}_+ = \boldsymbol{p}_-$ always holds as is seen from (87). Therefore, if the radii R_i of the torus tend to infinity, the present theory reduces smoothly to the previous one for the ordinary closed string since the non-zero winding number sectors with $\varepsilon(\boldsymbol{p}, \boldsymbol{q}) \neq 1$ decouple in this limit. In this sense it is more general to think that the phase factor $\varepsilon(\boldsymbol{p}, \boldsymbol{q})$ *always* exists in the 3-string vertex. Thinking this way would make it more natural to regard that the compactification occurs *spontaneously* in the closed string field theory.

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Note added: The authors are aware of two recent articles by I. Senda[14] and by S. Thomas[15] which deal with a similar subject to ours, though the relations to ours are unclear.