Gauge Theories and Macdonald Polynomials

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Exact Methods in Gauge/String Theories
PCTS, 11/11/11

4d $\mathcal{N} = 2$ susy theories of class $\mathcal{S}$(ix)

(Gaiotto, Gaiotto Moore Neitzke, . . .)

"Partially twisted" compactification of the (2, 0) 6d theory on a 2d surface $\mathcal{C}$ with punctures $\Longrightarrow \mathcal{N} = 2$ superconformal theories in four dimensions.

- Space of complex structures $\mathcal{C} \cong$ marginal gauge couplings of the 4d theory.
- Conformal factor of the metric on $\mathcal{C}$ believed to be RG-irrelevant.

  Recent check at large N.
  Holographic RG equation (Φ is related to conformal factor)

$$\partial_\rho^2 e^\Phi + (\partial_x^2 + \partial_y^2)\Phi = e^\Phi \, .$$

  Global existence proof of regular flows interpolating between arbitrary UV metric and canonical IR metric of constant negative curvature (for $g > 1$).
  (Anderson Beem Bobev LR)
- Moore-Seiberg groupoid of $\mathcal{C}$ = (generalized) 4d S-duality
  Vast generalization of "$\mathcal{N} = 4$ S-duality as modular group of T^2".

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6=4+2: beautiful and unexpected 4d/2d connections. Most famous example,

- Correlators of Liouville/Toda on $\mathcal{C}$ compute the 4d partition functions (on S^4) (Alday Gaiotto Tachikawa)

In this talk we focus on another surprising connection:

- The superconformal index $\mathcal{I}(q, p, t; , \mathbf{x}_i)$ is computed by topological QFT[1] on $\mathcal{C}$.

Index = twisted partition function on $S^3 \times S^1$.
Encodes the protected spectrum of the 4d theory: independent of the gauge theory moduli.

Simpler $4d/2d$ relation (topological): may hope to derive it from $6d$.

Still very non-trivial. Index of generic theory unknown.

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The superconformal index

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- It is evaluated by a trace formula of the schematic form

$$\mathcal{I}(\mu_i) = \mathrm{Tr}(-1)^F \, e^{-\sum_i \mu_i T_i} \, e^{-\beta\,\delta} \,, \qquad \delta = 2\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\} \; (\geq 0) \,,$$

  where $\mathcal{Q}$ is the supercharge "with respect to which" the index is calculated and $\{T_i\}$ a complete set of generators that commute with $\mathcal{Q}$ and with each other.
- The trace is over the states of the theory on S^3 (in radial quantization). States with $\delta \neq 0$ cancel pairwise, so the index counts states with $\delta = 0$ and it is independent of β.
- For a theory with a Lagrangian description one can compute the index in the free limit by counting the gauge-invariant operators in terms of a matrix integral. (Unlike the S^4 partition function, which is sensitive to non-perturbative physics.)

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$\mathcal{N} = 2$ index

- $\mathcal{N} = 2$ SCFTs have 8 supercharges (+8 superconformal charges): $\mathcal{Q}_{I\alpha}, \quad \tilde{\mathcal{Q}}_{I\dot\alpha}$.
  Here $I = 1, 2$ are $SU(2)_R$ indices and $\alpha = \pm$, $\dot\alpha = \pm$ Lorentz indices.
  We choose to compute the index "with respect to" $\tilde{Q}_{1\dot-}$ (all other choices are equivalent).
  The commutant subalgebra to $\tilde{Q}_{1\dot-}$ and $(\tilde{Q}_{1\dot-})^\dagger$ has δ_{1-}, δ_{1+} $\tilde\delta_{2\dot-}$ as Cartan generators,

$$\delta_{1-} \equiv 2\left\{Q_{1-}, (Q_{1-})^\dagger\right\} = E - 2j_1 - 2R - r,$$

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  E is the conformal dimension, (j_1, j_2) the Cartan generators of the $SU(2)_1 \otimes SU(2)_2$ isometry group, and $(R\,, r)$, the Cartan generators of the $SU(2)_R \otimes U(1)_r$ R-symmetry.
- The index is defined as

$$\mathcal{I}(\sigma, \rho, \tau, \dots) = \mathrm{Tr}(-1)^F\, \sigma^{\frac{1}{2}\delta_{1+}}\, \rho^{\frac{1}{2}\delta_{1-}}\, \tau^{\frac{1}{2}\tilde\delta_{2\dot-}}\, e^{-\beta\,\tilde\delta_{1\dot-}}\ \dots\,.$$

  or equivalently

$$\mathcal{I}(p, q, t, \dots) = \mathrm{Tr}(-1)^F\, p^{\frac{1}{2}\delta_{1+}}\, q^{\frac{1}{2}\delta_{1-}}\, t^{R+r}\, e^{-\beta'\,\tilde\delta_{1\dot-}}\ \dots\,.$$

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$\mathcal{N} = 2$ SCFTs of class $\mathcal{S}$ (of type A)

- Defined as the IR limit of the A_{k-1} (2,0) theory on $\mathbb{R}^4 \times \mathcal{C}$, where $\mathcal{C}$ a Riemann surface with appropriate punctures (defects).
- Complex moduli of $\mathcal{C} \cong 4d$ gauge couplings
- Punctures are associated with flavor symmetries. Flavor symmetries are classified by "auxiliary Young diagrams" with k boxes (embeddings of $SU(2)$ into $SU(k)$).
- Basic building blocks: theories corresponding to spheres with three punctures

  (no moduli=no tunable couplings)
  - Free hypermultiplets of $SU(k)$ theories correspond to spheres with two "maximal" punctures ($SU(k)$ flavor symmetry) and one "minimal" ($U(1)$ flavor symmetry).
  - All other three-punctured spheres do not have Lagrangian description. Simplest example: $SU(3)$ theory with three maximal punctures $\cong E_6$ SCFT
- "Gluing" three-punctured spheres at two maximal punctures corresponds to gauging the diagonal $SU(k)$
- Different "pair-of-pants" decompositions correspond to different S-duality frames.

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TQFT structure

- The superconformal index is independent of the marginal couplings. For theories of class $\mathcal{S}$ this means that the index does not depend on the complex moduli of $\mathcal{C}$: it must be computed by a $2d$ TQFT correlator on $\mathcal{C}$.
- The index of a generic theory of class $\mathcal{S}$ can be written in terms of the index of the basic constituents: three-puncture spheres and propagators. We parametrize the indices of the three-punctured spheres as

$$\mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$$

where $\mathbf{a}_i$ are fugacities dual to the Cartan subgroup of the flavor symmetry. In general these are a priori unknown functions. The propagators are known explicitly,

$$\eta(\mathbf{a}, \mathbf{b}) = \Delta(\mathbf{a})\mathcal{I}^V(\mathbf{a})\,\delta(\mathbf{a}, \mathbf{b}^{-1})$$

where $\Delta(\mathbf{a})$ is the Haar measure and $\mathcal{I}^V(\mathbf{a})$ the index of the vector multiplet.
- As the simplest example of gluing,

$$\begin{aligned}\mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) &= \oint [d\mathbf{a}] \oint [d\mathbf{b}]\, \mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a})\, \eta(\mathbf{a}, \mathbf{b})\, \mathcal{I}(\mathbf{b}, \mathbf{a}_3, \mathbf{a}_4) \\ &= \oint [d\mathbf{a}]\, \Delta(\mathbf{a})\, \mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a})\, \mathcal{I}^V(\mathbf{a})\, \mathcal{I}(\mathbf{a}^{-1}, \mathbf{a}_3, \mathbf{a}_4)\,,\end{aligned}$$

S-duality implies that this index is invariant under permutations of $\mathbf{x}_i$, which translates into associativity of the TQFT structure constants.

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Expanding in a convenient basis of functions $\{f^\alpha(\mathbf{a})\}$, labeled by $SU(k)$ representations $\{\alpha\}$,

$$\begin{aligned}\mathcal{I}(\mathbf{a},\mathbf{b},\mathbf{c}) &= \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\, f^\alpha(\mathbf{a})\, f^\beta(\mathbf{b})\, f^\gamma(\mathbf{c}) \\ \eta^{\alpha\beta} &= \oint [d\mathbf{a}] \oint [d\mathbf{b}]\, \eta(\mathbf{a},\mathbf{b})\, f^\alpha(\mathbf{a})\, f^\beta(\mathbf{b})\,.\end{aligned}$$

Invariance of the index under the different pairs-of-pants decomposition of $\mathcal{C}$ is equivalent to the associativity of the structure constants,

$$C_{\alpha\beta\gamma}\, C^\gamma{}_{\delta\epsilon} = C_{\alpha\delta\gamma}\, C^\gamma{}_{\beta\epsilon}\,,$$

where indices are raised with the metric $\eta^{\alpha\beta}$ and lowered with the inverse metric $\eta_{\alpha\beta}$.

The full ("elliptic") index for A_1 and A_2 theories

- The "full index" is elegantly expressed in terms of elliptic Gamma functions. For example, the index of a free hypermultiplet is given by a product of eight elliptic Gamma functions (Dolan-Osborn)

$$\mathcal{I}(a,b,c) = \Gamma(t^{\frac{1}{2}}\, a^{\pm 1} b^{\pm 1} c^{\pm 1}; p, q), \qquad \Gamma(z;p,q) = \prod_{i,j=0}^{\infty} \frac{1 - p^{i+1} q^{j+1}/z}{1 - p^i q^j z} \,.$$

- For A_1 quivers everything is explicit and associativity (S-duality) can be checked (Gadde-Pomoni-LR-Razamat) by recently-found non-trivial integral identities (van de Bult).
- One can in principle use dualities to obtain the indices of the strongly-coupled building blocks for higher rank theories.
- Indeed Argyres-Seiberg duality was used to compute the index of the E_6 SCFT (the structure constants of A_2 quivers with three maximal punctures) (Gadde-Razamat-LR-Yan),

$$\int \frac{da}{a} \widetilde{\Delta}(a,c)\, \mathcal{I}_{E_6}((a,b),\mathbf{x},\mathbf{y}) \sim \mathcal{I}_{N_f=6,SU(3)}(\mathbf{x},\mathbf{y},b,c) \,.$$

- This strategy is hard to generalize to A_n, $n > 2$.

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- One can in principle use dualities to obtain the indices of the strongly-coupled building blocks for higher rank theories.
- Indeed Argyres-Seiberg duality was used to compute the index of the E_6 SCFT (the structure constants of A_2 quivers with three maximal punctures) (Gadde-Razamat-LR-Yan),

$$\int \frac{da}{a} \widetilde{\Delta}(a,c)\, \mathcal{I}_{E_6}((a,b),\mathbf{x},\mathbf{y}) \sim \mathcal{I}_{N_f=6,SU(3)}(\mathbf{x},\mathbf{y},b,c)\,.$$

- This strategy is hard to generalize to A_n, $n > 2$.

The full ("elliptic") index for A_1 and A_2 theories

- The "full index" is elegantly expressed in terms of elliptic Gamma functions. For example, the index of a free hypermultiplet is given by a product of eight elliptic Gamma functions (Dolan-Osborn)

$$\mathcal{I}(a,b,c) = \Gamma(t^{\frac{1}{2}}\, a^{\pm 1} b^{\pm 1} c^{\pm 1}; p, q), \qquad \Gamma(z;p,q) = \prod_{i,j=0}^{\infty} \frac{1 - p^{i+1} q^{j+1}/z}{1 - p^i q^j z}\,.$$

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- Find an algorithm to calculate the index for *all* class $\mathcal{S}$ theories
- Identify explicitly the $2d$ TQFT

Strategy:

- "Bottom-up", experimental approach.
- Extrapolate the results for Lagrangian theories (A_1 quivers) to higher rank.

Results:

We succeeded for a slice $(q, 0, t)$ of the (q, p, t) fugacity space.

- TQFT $\sim$ (q, t)-deformation of $2d$ Yang-Mills in the zero-area limit.
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Strategy

- Choose $\{f^\alpha(\mathbf{a})\}$ to be orthonormal under the propagator measure, that is,

$$\eta^{\alpha\beta} = \delta^{\alpha\beta}\,. \tag{1}$$

- Perform further orthogonal transformation to basis where structure constants are diagonal,

$$C_{\alpha\beta\gamma} \neq 0 \quad \rightarrow \quad \alpha = \beta = \gamma\,.$$

  Associativity is then automatic.
  Finding the diagonal basis in principle always possible: challenge is describe it concretely.
- Useful to consider the ansatz

$$f^\alpha(\mathbf{a}) = \mathcal{K}(\mathbf{a}) P^\alpha(\mathbf{a})\,, \tag{2}$$

  for some cleverly chosen $\mathcal{K}(\mathbf{a})$.
- Focus on A_1 quivers, which are Lagrangian.
  In special limits, able to diagonalize the structure constants: $\{P^\alpha(a)\}$ turn out to be well-known orthogonal polynomials (Macdonald, Schur, Hall-Littlewood)
- These polynomials are defined for any root system.
  Immediate to formulate compelling conjectures for A_n quiver. Test against expected dualities.
- Then immediate to evaluate index for genus $\mathfrak{g}$ surface with s (maximal) punctures,

$$\mathcal{I}_{\mathfrak{g},s}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s) = \sum_\alpha (C_{\alpha\alpha\alpha})^{2\mathfrak{g}-2+s} \prod_{I=1}^{s} f^\alpha(\mathbf{a}_I)\,.$$

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Interesting limits

Use susy enhancement to select special limits

$$\mathcal{I}(p, q, t, \dots) = \mathrm{Tr}(-1)^F \, p^{\frac{1}{2}\delta_{1+}} \, q^{\frac{1}{2}\delta_{1-}} \, t^{R+r} \, e^{-\beta' \tilde{\delta}_{1\dot{-}}} \dots$$

- $p \to 0$: Macdonald $\mathcal{I}_M(q, t)$ $\quad \tilde{\mathcal{Q}}_{1\dot{-}}, \mathcal{Q}_{1+}$
- $p \to 0$, $q \to 0$: Hall-Littlewood $\mathcal{I}_{HL}(t)$ $\quad \tilde{\mathcal{Q}}_{1\dot{-}}, \mathcal{Q}_{1+}, \mathcal{Q}_{1-}$
- $q = t$ (independent of p it turns out): Schur $\mathcal{I}_S(q)$ $\quad \tilde{\mathcal{Q}}_{1\dot{-}}, \mathcal{Q}_{1+}$

Macdonald index

- $p \longrightarrow 0$ limit of the full index,

$$\mathcal{I}_M = \mathrm{Tr}_M (-1)^F \, q^{-2j_1} \, t^{R+r} \,,$$

where the trace in over states with $E + 2j_1 - 2R - r = 0$.
$\frac{1}{4}$BPS: one chiral and one antichiral supercharge.

- Basic ansatz for complete set of functions that diagonalize the structure constants:

$$f^{\lambda}_{q,t}(\mathbf{a}) = \mathcal{K}_{q,t}(\mathbf{a}) \, P^{\lambda}(\mathbf{a}|q,t) \,.$$

Macdonald polynomials $P_\lambda(\mathbf{a}; q, t)$ associated to the root system A_{k-1} are labeled by representations λ of $SU(k)$. They are orthogonal with respect to the measure

$$\Delta_{q,t}(\mathbf{a}) = \frac{1}{k!} \prod_{n=0}^{\infty} \prod_{i \neq j} \frac{1 - q^n a_i/a_j}{1 - t \, q^n a_i/a_j} \,,$$

- $q = t$ gives Schur (still $\frac{1}{4}$ BPS) while $q = 0$ gives Hall-Littlewood ($\frac{3}{8}$ BPS)

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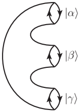
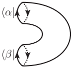
Symbol	Surface	Value
$C_{\alpha\beta\gamma}$		$\frac{\mathcal{A}(q,t)}{dim_{q,t}(\alpha)}\ \delta_{\alpha\beta}\ \delta_{\alpha\gamma}$
V^{α}		$\frac{dim_{q,t}(\alpha)}{\mathcal{A}(q,t)}$
$\eta^{\alpha\beta}$		$\delta^{\alpha\beta}$

Table: The structure constants, the cap, and the metric for the TQFT of the Macdonald index for A_{k-1} quivers.

$$
\begin{aligned}
dim_{q,t}(\lambda) &= P^{\lambda}(t^{\frac{k-1}{2}}, .., t^{\frac{1-k}{2}} \,|\, q,\ t) \\
\mathcal{A}(q,t) &= PE\left[\frac{1}{2}(k-1)\frac{t-q}{1-q}\right]\prod_{j=2}^{k}(t^j;q)\ .
\end{aligned}
$$

The Macdonald index of the theory corresponding to a sphere with generic punctures is

$$\boxed{\mathcal{I}_{\Lambda_1,\Lambda_2,\Lambda_3} = (t;q)^{k+2} \prod_{j=2}^{k} \frac{(t^j;q)}{(q;q)} \prod_{i=1}^{3} \hat{\mathcal{K}}_{\Lambda_i}(\mathbf{a}_i) \sum_{\lambda} \frac{\prod_{i=1}^{3} P_\lambda(\mathbf{a_i}(\Lambda_i)|q,t)}{P_\lambda(t^{\frac{k-1}{2}}, t^{\frac{k-3}{2}}, \ldots, t^{\frac{1-k}{2}}|q,t)}}\,.$$

There is well-defined rule $\mathbf{a_i}(\Lambda_i)$ to "partially-close" punctures by specializing the flavor fugacities.

The index and 2d Yang-Mills

- Consider the genus $\mathfrak{g}$ partition function in the Schur limit, $q = t$,

$$\mathcal{I}_{\mathfrak{g}}(q) = \left[(q;q)^{2\mathfrak{g}-2}\right]^{k-1} \, S_{00}(q)^{2-2\mathfrak{g}} \sum_{\lambda} \frac{1}{\left[dim_q(\lambda)\right]^{2\mathfrak{g}-2}} \, .$$

where S_{00} is the partition function of $SU(k)$ level ℓ Chern-Simons theory on S^3 if one formally identifies $q = e^{\frac{2\pi i}{\ell+k}}$,

$$S_{00}(q) = \prod_{j=2}^{k} \frac{(q;q)}{(q^j;q)} \, . \tag{3}$$

Up to a simple prefactor, this is the genus $\mathfrak{g}$ partition function of a q-deformed $2d$ Yang-Mills in the zero area limit (equivalently, the analytic continuation of CS partition function on $\mathcal{C}_g \times S^1$)

- In the more general case of $q \neq t$

$$\mathcal{I}_{\mathfrak{g}}(q,t) = \left[(t;q)^{\mathfrak{g}-1}\,(q;q)^{\mathfrak{g}-1}\right]^{k-1} \, \hat{S}_{00}(q,t)^{2-2\mathfrak{g}} \sum_{\lambda} \frac{1}{\left[dim_{q,t}(\lambda)\right]^{2\mathfrak{g}-2}} \, , \tag{4}$$

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Closely related to the "refinement" of Chern-Simons theory recently discussed by Aganagic and Shakirov
Possible $2d$ interpretation: $2d$ YM with modified (Macdonald) measure. Such modification might arise by integrating out degrees of freedom.

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Hall-Littlewood index

- We take the limit $\sigma, \rho \to 0$ of the full index

$$\mathcal{I} = \mathrm{Tr}(-1)^F \sigma^{\frac{1}{2}\delta_{1+}} \rho^{\frac{1}{2}\delta_{1-}} \tau^{\frac{1}{2}\bar{\delta}_{2\dot{+}}} e^{-\frac{1}{2}\beta\bar{\delta}_{1\dot{-}}} \, .$$

- Alternatively can state that it is given by

$$\mathcal{I} = \mathrm{Tr}_{HL}(-1)^F \tau^{2E-2R} \, ,$$

where the trace is over states satisfying $j_1 = 0$ and $E - 2R - r = 0$.

- The states contributing to this index are annihilated by three supercharges, two chiral and one anti-chiral.
- For Lagrangian theories the only "letters" contributing to this index are a scalar q (τ) from the hypermultiplet and a gaugino $\bar{\lambda}_{1\dot{+}}$ $(-\tau^2)$ from the vector multiplet.
- For genus-zero quivers, HL index $\cong$ Hanany's "Hilbert series of the Higgs branch".

HL index - $SU(2)$ quivers

- All these theories have a Lagrangian description.
- The basic building block (sphere with three punctures) is a trifundamental free hypermultiplet.
- The HL index of the free hypermultiplet is given by

$$\mathcal{I}(a_1, a_2, a_3) = \frac{1}{\prod_{\pm 1}(1 - \tau\, a_1^{\pm 1}\, a_2^{\pm 1}\, a_3^{\pm 1})}\,.$$

- By explicit diagonalization, it can be rewritten

$$\begin{aligned}\mathcal{I}(a_1, a_2, a_3) &= \frac{1+\tau^2}{1-\tau^2} \prod_{i=1}^{3} \frac{1}{\left(1-\tau^2 a_i^2\right)\left(1-\tau^2/a_i^2\right)} \sum_{\lambda=0}^{\infty} \frac{1}{P_\lambda^{HL}(\tau, \tau^{-1}|\,\tau)} \prod_{i=1}^{3} P_\lambda^{HL}(a_i, a_i^{-1}|\,\tau) \\ &= \mathcal{N}(\tau) \prod_{i=1}^{3} \mathcal{K}(a_i) \sum_{\lambda=0}^{\infty} C_{\lambda\lambda\lambda} \prod_{i=1}^{3} f^{\lambda}(a_i)\,.\end{aligned}$$

where

$$P_\lambda^{HL}(a, a^{-1}|\tau) = \mathcal{N}_\lambda(\tau)\left(\chi_\lambda(a) - \tau^2 \chi_{\lambda-2}(a)\right)$$

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HL index - higher rank generalization

- The HL polynomials can be defined for $U(k)$ groups

$$P_\lambda^{HL}(x_1,\ldots,x_k|\,\tau)=\mathcal{N}_\lambda(\tau)\sum_{\sigma\in S_k}\sigma\left(x_1^{\lambda_1}\ldots x_k^{\lambda_k}\prod_{i<j}\frac{x_i-\tau^2x_j}{x_i-x_j}\right).$$

and thus for higher rank building blocks, the T_k theories, we conjecture

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- For arbitrary punctures

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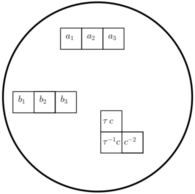
HL index - $SU(3)$ quivers - Closing punctures

The index of the $SU(3) \times SU(3) \times U(1)$ free hypermultiplet is given by

$$\mathcal{I}(\mathbf{a_1}, \mathbf{a_2}, c) = \prod_{i,j=1}^{3} \frac{1}{1 - \tau a_i b_j c} \, \frac{1}{1 - \tau \frac{1}{a_i b_j c}} \, .$$



- It can be rewritten as

$$\mathcal{I}(\mathbf{a_1}, \mathbf{a_2}, c) = \frac{1-\tau^6}{(1-\tau^2)^3} \frac{\mathcal{K}(\mathbf{a}_1)\mathcal{K}(\mathbf{a}_2)}{(1-\tau^3 c^3)(1-\tau^3 c^{-3})} \sum_{\lambda_1,\lambda_2} \frac{P^{HL}_{\lambda_1,\lambda_2}(\tau c, \tau^{-1} c, c^{-2} |\, \tau)}{P^{HL}_{\lambda_1,\lambda_2}(\tau^2, \tau^{-2}, 1 |\, \tau)} \prod_{i=1}^{2} P^{HL}_{\lambda_1,\lambda_2}(\mathbf{a_i} |\, \tau) \, .$$

- Further setting $c = \tau$ we completely close one puncture to obtain a cylinder (propagator)

$$\delta^{HL}(\mathbf{a}, \mathbf{b}) \sim \sum_{\lambda_1,\lambda_2} P^{HL}_{\lambda_1,\lambda_2}(\mathbf{a} |\, \tau) \, P^{HL}_{\lambda_1,\lambda_2}(\mathbf{b} |\, \tau) \, .$$

HL index - $SU(3)$ quivers - Closing punctures

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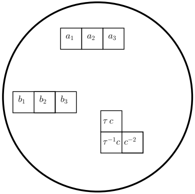

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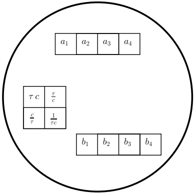
HL index - E_7 and E_8 SCFTs

The index of the E_7 SCFT is given by

$$\mathcal{I}_{E_7}(\mathbf{a_1}, \mathbf{a_2}, c) = \frac{(1+\tau^2+\tau^4)(1+\tau^4)}{(1-\tau^2)^3} \frac{\mathcal{K}(\mathbf{a}_1)\mathcal{K}(\mathbf{a}_2)}{(1-\tau^2 c^{\pm 2})(1-\tau^4 c^{\pm 2})} \times$$
$$\sum_{\lambda_1,\lambda_2,\lambda_3} \frac{P^{HL}_{\lambda_1,\lambda_2,\lambda_3}(\tau c, \frac{c}{\tau}, \frac{\tau}{c}, \frac{1}{\tau c} |\, \tau)}{P^{HL}_{\lambda_1,\lambda_2,\lambda_3}(\tau^3, \tau, \tau^{-1}, \tau^{-3} |\, \tau)} \prod_{i=1}^{2} P^{HL}_{\lambda_1,\lambda_2,\lambda_3}(\mathbf{a_i} |\, \tau)\,.$$
$$= \sum_{k=0}^{\infty} [k,0,0,0,0,0,0]_{\mathbf{z}}\, \tau^{2k}\,.$$



The index of the E_8 SCFT is given by

$$\mathcal{I}_{E_8}(\mathbf{a}, (b_1, b_2), c) = \frac{(1-\tau^8)(1-\tau^{10})(1-\tau^{12})}{(1-\tau^2)^3(1-\tau^4)^4(1-\tau^6)} \times$$
$$\frac{\mathcal{K}(\mathbf{a})}{(1-\tau^2 c^{\pm 2})(1-\tau^4 c^{\pm 2})(1-\tau^6 c^{\pm 2}) \prod_{i\neq j}(1-\tau^2 b_i/b_j)(1-\tau^4 b_i/b_j)} \times$$
$$\sum_{\lambda_1,\ldots,\lambda_5 \equiv \lambda} \frac{P^{HL}_{\lambda}(\tau b_1, \tau b_2, \tau b_3, \frac{b_1}{t}, \frac{b_2}{\tau}, \frac{b_3}{\tau} |\, \tau) P^{HL}_{\lambda}(\tau^2 c, c, \frac{c}{\tau^2}, \frac{\tau^2}{c}, \frac{1}{c}, \frac{1}{\tau^2 c}, |\, \tau) P^{HL}_{\lambda}(\mathbf{a} |\, \tau)}{P^{HL}_{\lambda}(\tau^5, \tau^3, \tau, \tau^{-1}, \tau^{-3}, \tau^{-5} |\, \tau)}$$
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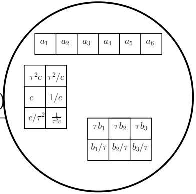

Outlook

- Possible to include line and surface operators (in progress)
- Macdonald index closely related to refined CS: hint of a relation to topological strings? (2,0) theory on $S^3 \times S^1 \times \mathcal{C}$ versus $(\mathbb{C} \times S^1 \times M_3)_{q,t}$ with $M_3 = S^1 \times \mathcal{C}$...
- Full three-parameter index?
  Natural to try elliptic generalizations of Macdonald polynomials.
  This idea can be sharpened by making recalling the relations between 2d YM, Macdonald polynomials an a family of integrable models:
  - The reduction of 2d YM on a cylinder gives the Calogero-Moser model (Gorsky-Nekrasov).
  - Macdonald polynomials are eigenfunctions of a relativistic version of the trigonometric Calogero-Moser model.

  The relevant elliptic generalizations of Macdonald polynomials should be eigenfunctions of the relativistic elliptic Calogero-Moser model. Not much is explicitly known about them ...
- Microscopic derivation of the 2d TQFT from the (2,0) theory?
  Perhaps easiest to derive quantum-mechanical model obtained by reduction of 2d TQFT to a graph $\mathcal{G}$. (From 5d SYM on $S^3 \times S^1 \times \mathcal{G}$?)

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