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GAUGE THEORIES AND STRONG GRAVITY

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ABSTRACT

We discuss in detail the classical solutions of two field theoretical models invariant under general variable transformation. In particular we examine the case of a Yang-Mills theory and of a four-dimensional non-linear sigma model, both coupled to "strong gravitation". Instanton, meron and multimeron configurations are obtained and their properties discussed.

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1. INTRODUCTION

During the last years, quantum chromodynamics has emerged as a new promising candidate for a theory of strong interactions.

It is hoped that the attraction due to the presence of coloured vector gluons could be responsible for the fundamental phenomenon of quark confinement. Since this phenomenon is clearly of a non-perturbative nature, a great amount of interest has been devoted to the study of classical solutions of the self-coupled Yang-Mills equations, like, for instance, instantons and merons. Those solutions which share the common feature of being inversely proportional to the gauge coupling constant "e", can be able to describe the existence of large effects due to an interaction term of a modest size.

It is useful, for the sake of completeness, to summarize some of the well-known results concerning those classical solutions.

The Yang-Mills Lagrangian density is, in the case of SU(2) internal symmetry,

$$\mathcal{L} = \frac{1}{2} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} \tag{1.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e\left[A_\mu, A_\nu\right] \tag{1.2}$$

and we have defined, as usual, SU(2) field operators

$$A_\mu = \frac{1}{2i} A_\mu^\alpha \sigma_\alpha \quad . \tag{1.3}$$

The ensuing equations of motion are

$$\partial_\mu F^{\mu\nu} = e\left[F^{\mu\nu}, A_\mu\right] . \tag{1.4}$$

An elementary class of solutions is given by the general ansatz*)

$$A_\mu = \frac{i}{e} \sigma_{\mu\nu} \partial^\nu \ln h(x) \tag{1.5}$$

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*) In Eq. (1.5) the internal operator $\sigma_{\mu\nu}$ is defined as follows:
$\sigma_{\mu\nu} = \frac{1}{4i}\left(s_\mu \bar{s}_\nu - s_\nu \bar{s}_\mu\right)$; $s_\mu = (1, i\vec{\sigma})$, $\bar{s}_\mu = (1, -i\vec{\sigma})$.

provided the density function $h(x)$ obeys the constraint equation[1)2)]

$$\frac{\Box h(x)}{h^3(x)} = \text{const.} \tag{1.6}$$

The instanton solution[3)] corresponds to

$$h(x) = \frac{\text{const.}}{x^2 + a^2} \tag{1.7}$$

whereas the meron solution[4)] in its simplest form (single meron at the origin) is given by

$$h(x) = \frac{\text{Const.}}{\sqrt{x^2}} \tag{1.8}$$

The characteristic property of the meron solutions is that the axial density[*)]

$$D(x) = -\frac{e^2}{32\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma} \tag{1.9}$$

turns out to be localized at some fixed points (the positions of the merons). In particular, in the case of solution (1.8) we simply have

$$D(x) = \frac{1}{2}\,\delta^4(x). \tag{1.10}$$

Although exact multimeron solutions are known in elementary two-dimensional models[5)6)], in our case the use of conformal plus gauge transformations does not allow to go beyond exact two-meron configurations of the form[4)7)8)9)]

$$D(x) = \pm\frac{1}{2}\,\delta^4(x-a) \pm \frac{1}{2}\,\delta^4(x-b) \tag{1.11}$$

Very beautiful work beyond this stage has been done by Glimm and Jaffe[10)] who have treated the case of several merons located on the same axis and have reduced the problem to a solution of a non-linear integral equation. Progress along this line can be made by a combination of existence theorems and numerical programs[11)].

*) We adopt for $\varepsilon^{\mu\nu\rho\sigma}$ the conventions of Ref. 13).

Since it is has been suggested that the confinement problem is strongly related to the existence of a multimeron solution in a general configuration, it is important to investigate the question from a new, more general, point of view.

The idea is very simple: the two-meron solution corresponding to Eq. (1.11) has been obtained from the single-meron solution (1.8) by exploiting the invariance of the Yang-Mills equations under the well-known 15-parameter conformal group, i.e., by performing an appropriate conformal transformation leading from $h(x)$ in Eq. (1.8) to

$$h(x) = \text{const.}\ \frac{\sqrt{(a-b)^2}}{\sqrt{(x-a)^2\,(x-b)^2}}\ . \tag{1.12}$$

If we now introduce a more general Lagrangian, which both exhibits a single-meron classical solution and is invariant under the group of all reparametrizations, $x_\mu \to f_\mu(x)$, it is then conceivable that the simple solution (1.8) can be transformed in a more general multimeron solution, for which

$$D(x) = \frac{1}{2}\left\{ \sum_{i\ \text{merons}} \delta^4(x-a_i) - \sum_{j\ \text{antimerons}} \delta^4(x-b_j) \right\}. \tag{1.13}$$

This development is rather natural if we remark that the reparametrization group appearing, for example, in general relativity, has the 15 conformal generators among its integrated charges. We then recall as the study of the algebra of charges for internal symmetries leads to non-Abelian locally gauge invariant theories. In the same way, when one considers important (as we do!) the study of the algebra of the conformal generators it is hard to resist the temptation to move towards generally invariant Lagrangians.

This general programme has been sketched in a previous paper[12)] describing a generally invariant gauge theory whose equations of motion contained higher (than second) derivatives of the fundamental fields (the "vierbeine"). In that case it was indeed possible to obtain a general multimeron solution for arbitrary locations of the merons.

However, the scheme has certain shortcomings which do not make it, at this stage at least, a plausible candidate for a theory of strong interactions. First of all, the similarity with chromodynamics becomes quite remote; on the other hand, the presence of higher derivatives in the Lagrangian makes extremely likely the presence of ghosts at the quantum level.

In this paper we wish to present more conventional theoretical schemes which are invariant under general co-ordinate transformations and thus possess multi-meron solutions.

The simplest case is, of course, the equivalent of Einstein gravity, in that case, however, even in the presence of a cosmological term, only instanton solutions exist whereas merons are absent.

The most obvious generalization is the case of a Yang-Mills field coupled to strong gravity. This will be dealt with in Section 3 and we shall see that indeed a large class of solutions exists (including merons) only in the presence of a "cosmological term", provided that the cosmological constant is fixed in terms of the Yang-Mills constant.

If one wants to avoid such an unpleasant constraint, one can consider (Section 4) a non-linear sigma model in the presence of gravitation. In this case, both an instanton and a meron exist. The absence of a cosmological coupling constant allows for the free field solution $g_{\mu\nu} = \text{const}\ \delta_{\mu\nu}$.

In Section 5 we shall derive general multimeron solutions and investigate their properties. Finally, in Section 6, we shall summarize and discuss the results of this paper.

2. GENERAL PROPERTIES

In this section we wish to recall and discuss some known properties of generally invariant theories which will be relevant for our investigation.

The simplest case of a generally invariant theory (with no derivatives higher than second in the equations of motion) is the original Einstein theory, whose Lagrangian (in the presence of a cosmological term) can be written as:

$$\mathcal{L}_E = -\frac{1}{4K} \sqrt{G} \left\{ R + \frac{3}{2} \Lambda^2 \right\} \tag{2.1}$$

$G_{\mu\nu}(x)$ represents the spin-2 field and all symbols are as defined in standard textbooks[13] ($G = \det G_{\mu\nu}$ and so on. Notice that for convenience we are adopting a Euclidean metric). For reasons clearer later, a cosmological like term Λ^2 has also been introduced which represents the only difference with respect to a standard Einstein theory.

One can easily make the "strong gravitational" constant K disappear by introducing a new set of quantities:

$$g_{\mu\nu} = K^{-1} G_{\mu\nu}, \qquad g^{\mu\nu} = K\, G^{\mu\nu}$$
$$g = K^{-4} G \quad , \quad \lambda^2 = K^{-1} \Lambda^2 . \tag{2.2}$$

The new Lagrangian then reads

$$\mathcal{L} = -\frac{1}{4} \sqrt{g} \left\{ R(g) + \frac{3}{2} \lambda^2 \right\} . \tag{2.3}$$

As a consequence of the relations (2.2) we go from a dimensionless $G_{\mu\nu}(x)$ to a new field $g_{\mu\nu}(x)$ with dimensions of a $(\text{length})^{-2}$, while λ^2 is a pure number.

Equation (2.3) leads to the equation of motion*)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(R + \frac{3}{2} \lambda^2 \right) = 0 . \tag{2.4}$$

If we limit ourselves to "Weyl conformally flat" solutions,

$$g_{\mu\nu}(x) = h^2(x)\, \delta_{\mu\nu} , \tag{2.5}$$

the only such solution is the instanton one

$$h(x) = \sqrt{\frac{16 a^2}{\lambda^2}} \quad \frac{1}{x^2 + a^2} \tag{2.5'}$$

It is easy to see that no other solution of the form (2.5) exists, in particular, no meron-like configuration

$$h(x) = \frac{\text{const.}}{\sqrt{x^2}}$$

For this reason, in the next chapters we shall concentrate our attention on models containing, in addition to $g_{\mu\nu}$, also vector (Section 3) or scalar (Section 4) fields.

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*) Many authors have discussed recently solutions of this equation and we quote the main papers by Eguchi and Freund[14], Charap and Duff[15] and Drechsler and Sasaki[16] where also an exhaustive bibliography can be found.

In the previous result we see an important feature, the need for a non-vanishing cosmological constant. The presence of such a constant forbids the existence of the free solution

$$g_{\mu\nu} = C\,\delta_{\mu\nu}\,. \tag{2.6}$$

If one wishes to relate $g_{\mu\nu}$ to gravity, C should be identified with 1/K and the solution (2.6) would be the zero order approximation of the Newtonian solution of the form

$$g_{\mu\nu} = \frac{1}{K}\left(\delta_{\mu\nu} + h_{\mu\nu}\right). \tag{2.6'}$$

In this case one would have an extremely stringent (10^{-120}) bound on the size of the cosmological constant. As a consequence, the Lagrangian of Section 3 in which λ is of the order of the gauge coupling constant is quite hopeless. On the other hand, the σ model Lagrangian of Section 4, in which the cosmological term is absent might allow a slight hope. We shall come back to this point in Section 6.

Let us now discuss in some detail the special invariance properties of our Lagrangian which will allow us (Section 5) to derive multimeron solutions.

The Einstein Lagrangian is, as is well known, invariant under the Einstein group of general co-ordinate transformations, and the same will be true for all Lagrangians considered in this paper. We recall the effect of these transformations on the field $g_{\mu\nu}$, the vector field A_μ and the scalar field ϕ, and a quantity D(x) that transforms as a density (like, for example, $\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma}$):

$$\phi(x) \to \phi(f(x)) \tag{2.7}$$

$$A_\mu(x) \to A_\nu(f(x))\,\frac{\partial f^\nu}{\partial x^\mu} \tag{2.8}$$

$$g_{\mu\nu}(x) \to g_{\alpha\beta}(f(x))\,\frac{\partial f^\alpha}{\partial x^\mu}\,\frac{\partial f^\beta}{\partial x^\nu} \tag{2.9}$$

$$D(x) \to D(x)\,\det\left(\partial f^\alpha/\partial x^\beta\right) \tag{2.10}$$

where $f^\lambda(x^\mu)$ are general functions of the x^μ. For special choices of the functions $f^\lambda(x^\mu)$ the above transformations reduce to those of the 15-parameter conformal group, which has played such a distinctive role in the study of instanton and meron solutions.

The key elements of the conformal group are:

1) translations

$$f^\mu(x) = x^\mu + a^\mu \tag{2.11}$$

2) four-dimensional rotations

$$f^\mu(x) = \alpha^\mu{}_\nu \, x^\nu \tag{2.12}$$

3) dilatations

$$f^\mu(x) = c\, x^\mu \tag{2.13}$$

4) inversions

$$f^\mu(x) = x^\mu / x^2 \, . \tag{2.14}$$

We have written these formulae in such cumbersome detail in order to avoid a possible ambiguity. The conformal transformations previously defined fit with the general field theoretical definitions valid for fields of any spin. On the other hand, in the language of general relativity, the name of conformal transformations is rather used for the so-called Weyl transformations

$$\begin{aligned} g_{\mu\nu}(x) &\to \rho(x)\, g_{\mu\nu}(x) \\ A_\mu(x) &\longrightarrow A_\mu(x) \, . \end{aligned} \tag{2.15}$$

It is to be noted that whereas the action corresponding to the Lagrangian (2.1) is invariant under the general reparametrization group and thus under the conformal group, it is not Weyl invariant*). We shall actually exploit such non-Weyl invariance of our equations in order to construct the elementary solutions of the next chapter.

3. THE YANG-MILLS MODEL

In this section we shall discuss a simple general class of solutions for the dynamical system represented by a "gravitational" field $g_{\mu\nu}(x)$ with cosmological term λ^2 and by a matter gauge field $A^{\alpha}_{\mu}(x)$, we shall take to transform as an SU(2) triplet, $\alpha = 1,2,3$ **).

The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}\sqrt{g}\left(R + \frac{3}{2}\lambda^2\right) - \frac{1}{4}\sqrt{g}\,\Sigma_\alpha\, g^{\mu\nu} g^{\rho\sigma} F^{\alpha}_{\mu\rho} F^{\alpha}_{\nu\sigma} , \tag{3.1}$$

$$g = \det g_{\mu\nu} . \tag{3.2}$$

Starting from Eq. (3.1) one readily obtains the equations of motion for the vector field

$$\partial_\mu \left\{\sqrt{g}\, F^{\mu\nu}\right\} = e\sqrt{g}\left[F^{\mu\nu}, A_\mu\right] \tag{3.3}$$

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*) In the simple case of rescaling the action is invariant under

$$g_{\mu\nu}(x) \to c^2 g_{\mu\nu}(cx), \quad A_\mu(x) \to c\, A_\mu(cx)$$

whereas it is not invariant under "Weyl scaling"

$$g_{\mu\nu}(x) \to c^2 g_{\mu\nu}(x)$$

**) The extension to the case of an $O(4) \sim SU(2) \times SU(2)$ internal symmetry is well known and straightforward.

and for the tensor part

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\left(R + \frac{3}{2}\lambda^2\right) = -2\,\theta_{\mu\nu} \tag{3.4}$$

As usual, we have defined an operator

$$A_\mu(x) = \frac{1}{2i}\,\sigma^\alpha A^\alpha_\mu(x) \tag{3.5}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e\,[A_\mu, A_\nu]\,, \tag{3.6}$$

while $\theta_{\mu\nu}$ on the right-hand side of Eq. (3.4) is the energy-momentum tensor of the gauge field

$$\begin{aligned}\theta_{\mu\nu} &= \Sigma_\alpha \left\{ F^\alpha_{\mu\lambda} F^\alpha_{\nu\rho}\, g^{\lambda\rho} - \frac{1}{4} g_{\mu\nu} F^\alpha_{\rho\sigma} F^\alpha_{\tau\lambda}\, g^{\rho\tau} g^{\sigma\lambda} \right\} \\ &= -\frac{1}{2}\,\mathrm{Tr}\left\{ 4 F_{\mu\lambda} F_{\nu\rho}\, g^{\lambda\rho} - g_{\mu\nu} F_{\rho\sigma} F_{\tau\lambda}\, g^{\rho\tau} g^{\sigma\lambda} \right\}.\end{aligned} \tag{3.7}$$

From the above explicit expression we derive the trace identity

$$g^{\mu\nu}\theta_{\mu\nu} = 0 \tag{3.8}$$

which in turn, using Eq. (3.4), enables us to relate $R = g^{\mu\nu}R_{\mu\nu}$ to the "cosmological" constant λ^2, namely,

$$R = -3\lambda^2 \tag{3.9}$$

and to rewrite the Einstein equation in the traceless form

$$R_{\mu\nu} - \frac{1}{4} R\, g_{\mu\nu} = -2\,\theta_{\mu\nu} \tag{3.4'}$$

The class of solutions which will be considered are characterized by having a "conformally flat" form for $g_{\mu\nu}$:

$$g_{\mu\nu} = h^2(x)\,\delta_{\mu\nu} \tag{3.10}$$

The ansatz (3.10) has two fundamental consequences:

1) The trace equation (3.9) leads to the constraint

$$R = 6 \Box h / h^3 = - 3 \lambda^2 \tag{3.11}$$

As emphasized in Refs. 1) and 2), we recover the conformally invariant scalar equation!

2) Since

$$\sqrt{g}\, F^{\mu\nu} = \sqrt{g}\, g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} = F_{\mu\nu}$$

the equation for the Yang-Mills field reduces to the familiar one in a flat space, i.e., Eq. (4). It is thus natural to resort to the corresponding form (1.5) for A_μ so that finally our complete ansatz will be

$$g_{\mu\nu}(x) = h^2(x)\, \delta_{\mu\nu} \,, \tag{3.12a}$$

$$A_\mu(x) = \frac{i}{e} \sigma_{\mu\nu} \partial^\nu \ln h(x) \quad . \tag{3.12b}$$

h(x) is a solution of the constraint equation (3.11) which also guarantees [after (1.6)] that the Yang-Mills equation is satisfied.

Our final task is now to compute both sides of Eq. (3.4') in terms of the above ansatz and to check their consistency. A direct calculation gives the following result*):

$$R_{\mu\nu} = \delta_{\mu\nu} \left\{ 3h^2 \left(\partial_\sigma \frac{1}{h}\right)^2 - h \Box \frac{1}{h} \right\} - 2h\, \partial_\mu \partial_\nu \frac{1}{h} \tag{3.13}$$

i.e.,

$$R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = -2h \left\{ \partial_\mu \partial_\nu \frac{1}{h} - \frac{1}{4} \delta_{\mu\nu} \Box \frac{1}{h} \right\} \tag{3.14}$$

*) Computation is made easier by the use of Weyl-type transformations, see Ref. 18).

while

$$\theta_{\mu\nu} = -\frac{2}{e^2}\frac{\Box h}{h^2}\left\{\partial_\mu\partial_\nu\frac{1}{h} - \frac{1}{4}\delta_{\mu\nu}\Box\frac{1}{h}\right\}. \tag{3.15}$$

These quite simple expressions now make it easy to complete our discussion.

Let us first consider separately the exceptional case of the instanton solution:

$$h(x) = \sqrt{\frac{16a^2}{\lambda^2}}\;\frac{1}{x^2+a^2} \tag{3.16}$$

for which

$$\partial_\mu\partial_\nu\frac{1}{h} - \frac{1}{4}\delta_{\mu\nu}\Box\frac{1}{h} = 0.$$

Equation (3.4') thus reduces simply to

$$R_{\mu\nu} - \frac{1}{4}R\,g_{\mu\nu} = -2\,\theta_{\mu\nu} = 0 \tag{3.17}$$

and is verified for any value of λ^2 and e^2, even (as discussed in the previous section) when the pure gauge field is absent. This fact is a direct consequence of the self-dual nature of the instanton solution, which implies a vanishing of the energy-momentum tensor $\theta_{\mu\nu}$ and leads therefore to a decoupling of the vector system.

In the general case in which the left-hand side of (3.14) [or (3.15)] does not vanish (e.g., for the meron!) we obtain from the equation of motion (3.4') that

$$\frac{\Box h}{e^2 h^3} = 1. \tag{3.18}$$

When combined with the trace condition (3.11) this leads to a relation

$$\lambda^2 = e^2 \tag{3.19}$$

between the cosmological term and the colour charge, which has to be satisfied in order to guarantee the existence of non-instanton solutions of the form (3.12a) and (3.12b).

As already pointed out earlier, Eq. (3.19) leads to values for λ that forbid any interpretation of $g_{\mu\nu}$ as a gravitational field.

Finally, we introduce the fundamental invariant quantities of the vector theory built from the field strength $F_{\mu\nu}$ defined in Eq. (3.6):

$$S_1(x) = -e^2 \, \mathrm{Tr} \, F_{\mu\nu} F^{\mu\nu} \tag{3.20}$$

$$D_1(x) = -\frac{e^2}{32\pi^2} \, \mathrm{Tr} \, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \,. \tag{3.21}$$

$D_1(x)$ can also be written in the form of a pure divergence, namely

$$D_1(x) = \partial_\mu I^\mu(x) \tag{3.22}$$

$$I^\mu(x) = -\frac{e^2}{8\pi^2} \, \varepsilon^{\mu\nu\rho\sigma} \left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} e A_\nu A_\rho A_\sigma \right) \tag{3.23}$$

where I^μ is the familiar topological current of the Yang-Mills theory.

The space-time integrations

$$\int d^4x \sqrt{g} \; S_1(x) \tag{3.24}$$

$$\int d^4x \; D_1(x) \tag{3.25}$$

lead to invariant numbers. The over-all constants in Eqs. (3.20) and (3.21) are determined by convenience reasons.

The analogous quantities for the gravitational field are:

$$S_2(x) = \frac{1}{4} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \tag{3.26}$$

$$D_2(x) = \frac{1}{256\pi^2} \varepsilon^{\lambda\mu\lambda'\mu'} \varepsilon^{\nu\rho\nu'\rho'} R_{\lambda\mu\nu\rho} R_{\lambda'\mu'\nu'\rho'} \frac{1}{\sqrt{g}} \tag{3.27}$$

$$P(x) = \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} \tag{3.28}$$

In Appendix A we give the main details for the computation of these quantities in general.

We state the connection between the Yang-Mills and gravitational invariants:

$$S_1(x) = S_2(x) \tag{3.29}$$

$$D_1(x) = D_2(x) \ . \tag{3.30}$$

Further, for the class of solutions investigated, $P(x) = 0$. The computations and general results are given in Appendix A, while here we collect the main results for the two interesting cases of instanton and meron solutions.

A) Instanton

$$g_{\mu\nu}(x) = \frac{16a^2}{\lambda^2} \frac{\delta_{\mu\nu}}{(x^2+a^2)^2} \ ; \ A_\mu(x) = -\frac{2i}{e} \frac{\sigma_{\mu\nu} x_\nu}{x^2+a^2} \tag{3.31}$$

$$S_1(x) = \frac{3}{8} \lambda^4 \tag{3.32}$$

Integration of

$$\sqrt{g}\, S_1(x) = \frac{96\, a^4}{(x^2+a^2)^4}$$

over space-time leads to the finite Euclidean action for the Yang-Mills field:

$$\int d^4x \sqrt{g}\; S_1(x) = 16\pi^2$$

We also have

$$D_1(x) = \frac{6a^4}{\pi^2} \frac{1}{(x^2+a^2)^4} \tag{3.33}$$

which shows the customary form of the instanton topological density, diffused over the whole Euclidean domain. The unit topological number for the instanton is recovered:

$$\int d^4x\; D_1(x) = 1 . \tag{3.34}$$

Looking at the gravitational side, the so-called Pontryagin and Euler numbers are zero and one, respectively:

$$\int d^4x \sqrt{g}\; P(x) = 0 , \quad \int d^4x\; D_2(x) = 1 .$$

B) Meron*)

$$g_{\mu\nu} = \frac{2}{e^2} \frac{\delta_{\mu\nu}}{x^2} , \qquad A_\mu = -\frac{i}{e} \frac{\sigma_{\mu\nu} x_\nu}{x^2} . \tag{3.35}$$

We have

$$S_1 = \frac{3e^4}{4} \tag{3.36}$$

and the action is logarithmically divergent since

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*) In order to cope with the badly defined behaviour at $x = 0$ one has to use the device

$$h^{-1}(x) \longrightarrow \frac{2}{e^2} \sqrt{x^2+\varepsilon^2}$$

This gives for $D(x)$ the result (3.38)

$$D(x) = \frac{3}{4\pi^2} \lim_{\varepsilon \to 0} \frac{\varepsilon^2(x^2+2\varepsilon^2)}{(x^2+\varepsilon^2)^4} = \frac{1}{2} \delta^4(x) .$$

$$\sqrt{g}\, S_1 = \frac{3}{(x^2)^2} \quad . \tag{3.37}$$

In this case a finite action can be obtained in the Minkowski space after the appropriate interpretation of variables as explained in Section 2.

For the topological density we find the usual result for merons:

$$D_1(x) = \frac{1}{2}\, \delta^4(x) \tag{3.38}$$

which gives the half integer value

$$\int d^4x \; D_1(x) = \frac{1}{2} \tag{3.39}$$

Similarly the Pontryagin and Euler numbers for the gravitational partner are zero and $\frac{1}{2}$.

4. THE NON-LINEAR SIGMA MODEL

A simple model that has drawn much attention for many reasons is the non-linear sigma model in various space-time dimensions. Let us recall that the two-dimensional case is particularly interesting for us because it displays both instanton and meron solutions which can be elegantly generalized (as an elementary application of the theory of complex variables) to multi-instantons and multimerons. We shall spend a few lines in recalling the main points of the two-dimensional sigma model because the results are strikingly similar to those we will obtain for the four-dimensional generally invariant sigma model in the present section.

In the two-dimensional case the Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \sum_1^3 (\partial_\mu \phi_a)^2 - \frac{m(x)}{2} \left\{ \sum_1^3 \phi_a^2 - 1 \right\} \tag{4.1}$$

where $m(x)$ is a Lagrange multiplier so that $\Sigma_a\, \phi_a^2 = 1$. The simplest non-trivial internal symmetry group is O(3). An interesting quantity is the "topological charge" density, defined as the Jacobian of the mapping $\phi_a(x)$ of the two-dimensional space-time over the unit sphere S_2

$$D(x) = \frac{1}{8\pi}\, \varepsilon_{abc}\, \varepsilon_{\mu\nu}\, \partial_\mu \phi_a\, \partial_\nu \phi_b\, \phi_c \; . \tag{4.2}$$

The instanton solution is

$$\phi_a = \xi_a , \qquad a = 1,2,3 \tag{4.3}$$

where, as usual,

$$\xi_\mu = \frac{2 x_\mu a}{a^2 + x^2} , \qquad \mu = 1,2$$

$$\xi_3 = \frac{a^2 - x^2}{a^2 + x^2} .$$

The topological density is diffuse:

$$D(x) = \frac{1}{\pi} \frac{a^2}{(a^2 + x^2)^2} \tag{4.4}$$

$$Q = \int d^2x \, D(x) = 1 . \tag{4.5}$$

The meron solution is given by

$$\phi_\alpha = \frac{x_\alpha}{\sqrt{x^2}} , \qquad \alpha = 1,2 \tag{4.6}$$

$$\phi_3 = 0$$

so that the density D(x) vanishes wherever the solution is regular. Care must be used for $x_\mu \to 0$ in order not to draw the erroneous conclusion that D vanishes. One can regularize the meron by

$$\sqrt{x^2} \longrightarrow \sqrt{x^2 + \varepsilon^2}$$

which implies a small third component

$$\phi_3 = \frac{\varepsilon}{\sqrt{x^2 + \varepsilon^2}}$$

Then

$$D(x) = \frac{1}{4\pi} \lim_{\varepsilon \to 0} \frac{\varepsilon}{(x^2 + \varepsilon^2)^{3/2}} = \frac{1}{2} \delta^2(x) \tag{4.7}$$

and the correct $Q = \frac{1}{2}$ is recovered. Also, multimeron solutions are very easy to write. We have

$$\phi = \phi_1 + i\phi_2 = e^{i\varphi} \tag{4.8}$$

where

$$\varphi = \sum_1^N \arg(z - a_i) - \sum_1^{\bar{N}} \arg(z - b_j) ,$$

$$z = x_1 + i x_2 ;$$

the sets of points $\{a_i\}$ and $\{b_i\}$ are merons and antimerons, respectively.

Equations (4.8) can also be written in the form

$$\phi = \frac{F(z)}{|F(z)|} , \quad \text{where} \quad F(z) = \frac{(z-a_1)\cdots(z-a_m)}{(z-b_1)\cdots(z-b_m)} \tag{4.8'}$$

which will be useful for comparison with the four-dimensional case.

When trying to extend to four dimensions the pure sigma model we are faced with some difficulties[6]. In the first place, if we stick to a traditional Lagrangian, ϕ_a has non-zero dimension, so that the constraint $\Sigma\, \phi_a^2 = \lambda^2$ must introduce a dimensional constant λ that destroys even the conformal invariance. Then one is pushed to use a dimensionless ϕ_a and a higher derivative Lagrangian that makes extremely likely the presence of ghosts at the quantum level. In addition, the four-dimensional pure sigma model does not have the general elementary multimeron and multi-instanton solutions of the two-dimensional case.

The model that we shall analyze overcomes very elegantly all these drawbacks by coupling the sigma field to the field $g_{\mu\nu}$ in a generally invariant fashion so that at the same time we shall have dimensionless ϕ_a, second order equations and general invariance leading to multi-instanton and multi-meron solutions.

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}\sqrt{g}\,\left\{ R + f\, g^{\mu\nu}\, \partial_\mu \phi_a\, \partial_\nu \phi_a \right\} \tag{4.9}$$

subject to the constraint

$$\sum_1^5 \phi_a^{\,2} = 1 \tag{4.10}$$

(we take real fields and $a = 1 \ldots 5$ so that the model has an internal $O(5)$ invariance).

The fundamental invariants of the theory are

$$S_3(x) = g^{\mu\nu}\,\partial_\mu \phi_a \partial_\nu \phi_a \quad , \tag{4.11}$$

$$D_3(x) = \frac{\varepsilon^{\mu\nu\rho\sigma}}{64\,\pi^2}\,\varepsilon_{abcde}\;\phi_a \partial_\mu \phi_b \partial_\nu \phi_c \partial_\rho \phi_d \partial_\sigma \phi_e \,. \tag{4.12}$$

The only dimensionless constant appearing in (4.9) is the parameter f whose value will be fixed unambiguously if we require the presence of an instanton or a meron solution. We shall see that the values of f corresponding to the presence of an instanton or of a meron do not coincide.

We are now ready to derive the equations of motion. Using the Lagrange multiplier method we get the effective Lagrangian

$$\mathcal{L} = -\frac{1}{4}\sqrt{g}\left\{ R + f\, g^{\mu\nu}\partial_\mu \phi_a \partial_\nu \phi_a + f\,\frac{N(x)}{2}\left(\phi_a^2 - 1\right)\right\} . \tag{4.13}$$

The equations of motion which follow from the variation of the fields are

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -2\,\theta_{\mu\nu} \tag{4.14}$$

where

$$\theta_{\mu\nu} = \frac{f}{2}\left\{ \partial_\mu \phi_a \partial_\nu \phi_a - \frac{1}{2} g_{\mu\nu}\, \partial_\rho \phi_a \partial_\sigma \phi_a\, g^{\rho\sigma}\right\} . \tag{4.15}$$

Equations (4.14), (4.15) lead to the simplified equation

$$R_{\mu\nu} = -f\; \partial_\mu \phi_a \partial_\nu \phi_a \tag{4.16}$$

Further we have

$$\partial_\mu \left\{ \sqrt{g}\, g^{\mu\nu} \partial_\nu \phi_a \right\} = \sqrt{g}\, N(x)\, \phi_a . \qquad (4.17)$$

The condition $\Sigma\, \phi_a^2 = 1$ implies

$$N(x) = - g^{\mu\nu} \partial_\mu \phi_a\, \partial_\nu \phi_a . \qquad (4.18)$$

We see that the presence of the field $g_{\mu\nu}$ with dimension -2 allows the field ϕ_a to have the natural dimension zero, with no need for higher derivatives in the Lagrangian.

Another interesting property of the field equations (4.16), (4.17) is that they are invariant under the simple rescaling

$$g_{\mu\nu}(x) \longrightarrow \text{const.}\; g_{\mu\nu}(x)$$

As a consequence, in all our solutions $g_{\mu\nu}$ will contain a multiplicative constant not determined by theory.

Let us now give the explicit instanton and meron solutions:

A) Instanton

We have an instanton solution only for $f = 3$:

$$g_{\mu\nu} = \frac{c\, a^2}{(a^2+x^2)^2}\, \delta_{\mu\nu} , \qquad (4.19)$$

$$\phi_a = \xi_a \qquad (4.20)$$

with

$$\xi_\mu = \frac{2\, a\, x_\mu}{a^2+x^2} , \quad \xi_5 = \frac{a^2-x^2}{a^2+x^2} ,$$

$$N = -16/c .$$

The ξ_a are the five co-ordinates on the four-dimensional sphere S_4; the scalar field transforms as the five-dimensional representation of the internal 0(5) group.

The values of the invariants are

$$S_3 = 16/C \tag{4.21}$$

Integration of

$$\sqrt{g}\ S_3 = \frac{16\, C a^4}{(a^2+x^2)^4}$$

over the four-dimensional space-time leads to the finite Euclidean action for the instanton solution:

$$\int d^4x\ \sqrt{g}\ S_3 = \frac{2}{3}\pi^2 C$$

We have also

$$D_3(x) = \frac{6}{\pi^2}\ \frac{a^4}{(a^2+x^2)^4} \tag{4.22}$$

leading to

$$\int d^4x\ D_3(x) = 1$$

For the gravitational field we have

$$S_2 = 96/C^2 \tag{4.23}$$

$$D_2 = D_3 \tag{4.24}$$

B) <u>Meron</u>

We have a meron solution <u>only for f = 2</u>:

$$g_{\mu\nu} = \frac{C}{x^2}\ \delta_{\mu\nu} \tag{4.25}$$

$$\phi_\mu = \frac{x_\mu}{\sqrt{x^2}}\ ,\ \mu = 1,\dots,4 \tag{4.26}$$

$$\phi_5 = 0$$

and

$$\mathcal{N} = -3/C .$$

The scalar invariant S_3 has the value

$$S_3 = 12/C \tag{4.27}$$

so

$$\sqrt{g}\, S_3 = \frac{12\, C}{(x^2)^2} \tag{4.28}$$

showing the usual behaviour for the action density of the meron in the Euclidean space-time.

For the pseudoscalar invariant D_3, which vanishes wherever the solution is regular, the usual care must be used in order to reproduce the value $\frac{1}{2}$ for the topological charge, so one must replace $\sqrt{x^2} \to \sqrt{x^2 + \varepsilon^2}$ in ϕ_μ, $\mu = 1 \ldots 4$, and introduce the fifth component

$$\phi_5 = \frac{\varepsilon}{\sqrt{x^2+\varepsilon^2}} \quad ;$$

by the suitable limit process one obtains

$$D_3(x) = \frac{1}{2}\, \delta^4(x) . \tag{4.29}$$

Finally, for the gravitational field we have, as usual,

$$S_2 = \frac{3}{C^2} \quad , \quad D_2 = \frac{1}{2}\, \delta^4(x) . \tag{4.30}$$

In conclusion, instanton and meron are characterized by the quantified values of the constant f appearing in the Lagrangian, $f = 3$ and 2 respectively.

In addition to these solutions, it is obvious, but important to remark, that this model has also the "free" solution

$$\begin{aligned} g_{\mu\nu} &= C\, \delta_{\mu\nu} , \\ \phi_a &= c_a , \quad \Sigma\, c_a^2 = 1 . \end{aligned} \tag{4.31}$$

Indeed, our generalized non-linear sigma model, in which the unpleasant cosmological term is absent, exhibits the three fundamental classical solutions: instanton, meron and the free field form (4.31).

In addition, general invariance leads to a much larger class of solutions. In particular, the multimeron solution, to be discussed in the next section, is closely analogous to the two-dimensional solution (4.8), (4.8').

5. MULTIMERON SOLUTIONS

The meron solutions obtained in the last two sections have the general form

$$g_{\mu\nu} = C \frac{\delta_{\mu\nu}}{x^2} , \tag{5.1}$$

$$A_\mu = \frac{1}{2e} \frac{s \cdot x}{\sqrt{x^2}} \partial_\mu \frac{\bar{s} \cdot x}{\sqrt{x^2}} , \tag{5.2}$$

$$\phi_\alpha = \frac{x_\alpha}{\sqrt{x^2}} , \alpha = 1 \dots 4 , \quad \phi_5 = 0 . \tag{5.3}$$

The scalar invariant quantities S_i are pure numbers and the pseudoscalar densities are given by

$$D_i(x) = \frac{1}{2} \delta^4(x) . \tag{5.4}$$

Of course, Eq. (5.4) takes into account the presence of a meron at the origin but ignores the presence of an antimeron at infinity. If one wishes to account for that point too, one should write*)

$$D(x) = \frac{1}{2} \left[\delta^4(x_\mu) - \frac{1}{x^8} \delta^4(x_\mu / x^2) \right\} \tag{5.5}$$

which has the correct transformation properties under inversion.

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*) Of course, by this formula we just want to remind that the solution with a meron at $x = a$ and an antimeron at $x = b$ has

$$D(x) = \frac{1}{2} \left\{ \delta^4(x-a) - \delta^4(x-b) \right\}$$

and the formula (6.5) stands for a limit $a \to 0$, $b \to \infty$: one must not use it in formulae like $\int D \, d^3x$!

We can now use the invariance of our Lagrangian under general change of variables in order to generate from the elementary solutions (5.1), (5.2), (5.3) a more general class of solutions.

Since we shall be interested in solutions with any number of point singularities (merons and antimerons) we shall introduce the four-dimensional analogue of a fractional transformation. This is done by a straightforward use of quaternion language in which a four vector is represented by a 2 × 2 matrix for which the well-known quanternion algebraic rules are defined[19].

Our co-ordinate transformation is

$$x_\mu \longrightarrow f_\mu (x) \tag{5.6}$$

where f_μ is defined by

$$\bar{s}\cdot f = F(x) \tag{5.7}$$

and $F(x)$ is given by

$$F(x) = \mathcal{A}_1 \mathcal{B}_1 \mathcal{A}_2 \mathcal{B}_2 \ldots \mathcal{A}_n \mathcal{B}_n \tag{5.8}$$

with

$$\begin{aligned} \mathcal{A}_m &= \bar{s}\cdot(x - a_m) \\ \mathcal{B}_m &= \{\bar{s}\cdot(b_m - x)\}^{-1} \end{aligned} \tag{5.9}$$

It is useful to define

$$\bar{\mathcal{A}}_m = s\cdot(x - a_m)\ , \quad \bar{\mathcal{B}}_m = \{s\,(b_m - x)\}^{-1} \tag{5.10}$$

$$\bar{F}(x) = \bar{\mathcal{B}}_n \bar{\mathcal{A}}_n \ldots \bar{\mathcal{B}}_1 \bar{\mathcal{A}}_1 \tag{5.11}$$

$$\begin{aligned} \langle \mathcal{A}_m \rangle &= \sqrt{\mathcal{A}_m \bar{\mathcal{A}}_m} = \sqrt{(x - a_m)^2} \\ \langle \mathcal{B}_m \rangle &= \sqrt{\mathcal{B}_m \bar{\mathcal{B}}_m} = 1/\sqrt{(x - b_m)^2} \end{aligned} \tag{5.12}$$

so

$$\langle F \rangle = \sqrt{F\bar{F}} = \langle \mathcal{A}_1 \rangle \langle \mathcal{B}_1 \rangle \ldots \langle \mathcal{A}_m \rangle \langle \mathcal{B}_m \rangle . \tag{5.13}$$

Further we put

$$G(x) = \frac{F(x)}{\langle F(x) \rangle} ,$$

$$\bar{G}(x) = \frac{\bar{F}(x)}{\langle F(x) \rangle} \tag{5.14}$$

With these definitions it is easy to write down the effect of the transformation (5.6) on the elementary solutions (5.1), (5.2), (5.3). Using the transformation rule (2.7), (2.8), (2.9) given in Section 2 we get*)

$$A_\mu = \frac{1}{2} \bar{G} \partial_\mu G \tag{5.15}$$

$$\phi \cdot \bar{s} = G , \qquad \text{or} \qquad \phi_\alpha = Tr \; s_\alpha G \tag{5.16}$$

$$g_{\mu\nu} = \frac{1}{\langle F \rangle^2} Tr \frac{\partial F}{\partial x^\mu} \frac{\partial \bar{F}}{\partial x^\nu} . \tag{5.17}$$

Looking at Eqs (5.15), (5.16), (5.17) we see that, as expected, those solutions are regular everywhere except at

$x = a_i$ location of merons

$x = b_i$ location of antimerons.

It is not difficult to see that the pseudoscalar density $D_1(x)$ of (3.21) is equal to

$$D_1(x) = \frac{1}{2} \left\{ \Sigma_i \delta^4(x - a_i) - \Sigma_i \delta^4(x - b_i) \right\} . \tag{5.18}$$

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*) The definition of Tr is such that $Tr\, s_\alpha \bar{s}_\beta = \delta_{\alpha\beta}$, i.e.,
$Tr = \frac{1}{2} \sum_{diag.}$

Equation (5.18) can be understood in two ways. First, as in Ref. 12), it is easy to check from the form (5.15) that $D_1(x)$ vanishes everywhere but at the location of merons or antimerons. In the neighbourhood of the i^{th} meron or antimeron, apart from a constant phase factor, we have

$$A_\mu \sim \frac{1}{2}\,\bar{\mathcal{A}}_m\,\partial_\mu\,\mathcal{A}_m \qquad \text{meron} \tag{5.19}$$
$$A_\mu \sim \frac{1}{2}\,\bar{\mathcal{B}}_m\,\partial_\mu\,\mathcal{B}_m \qquad \text{antimeron}$$

which immediately leads to Eq. (5.18). A more direct way of getting Eq. (5.18) is to apply the transformation (2.10) to the form (5.5) of $D(x)$. One readily gets

$$D_1(x) = \frac{1}{2}\left\{\delta^4\left(f^\alpha(x)\right) - \frac{1}{(f^2)^4}\,\delta^4\left(f^\alpha(x)/f^2\right)\right\}.\det\left(\partial f^\alpha/\partial x^\beta\right) \tag{5.20}$$

The first term generates all meron contributions whereas the second term gives antimeron ones, leading again to Eq. (5.16)*).

Our present solution is the generalization of the two-dimensional meron solution discussed in Section 4. This can be seen by comparing the expressions (5.16) and (4.8') for the scalar field.

At this point we wish to discuss this class of solutions in the Minkowski framework, which is the natural one for meron solutions. It is easily seen that our solutions, translated in that language, become real when each meron and a corresponding antimeron are symmetric with respect to the x_1, x_2, x_3 "plane". We thus set

$$a_i = (\gamma_i, \vec{c}_i)$$
$$b_i = (-\gamma_i, \vec{c}_i) \tag{5.21}$$

so that the product $\mathcal{C}_i = \mathcal{A}_i\mathcal{B}_i$ is a purely unitary matrix:

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*) One can relate the presence of several merons and antimerons to the multivaluedness of the transformation $f_\mu(x) \to x_\mu$. Indeed, $x = 0$ is mapped into the set of points $\{a_i\}$ whereas $x = \infty$ is mapped into the set $\{b_i\}$.

$$C_i = \mathcal{A}_i \mathcal{B}_i = \frac{\gamma_i - i(t - \vec{\sigma}\vec{y}_i)}{\gamma_i + i(t - \vec{\sigma}\vec{y}_i)} = e^{i(\alpha_i - \beta_i \vec{\sigma}\cdot\vec{u}_i)} \tag{5.22}$$

where

$$\begin{aligned}
\vec{y}_i &= \vec{x} - \vec{c}_i \\
\psi^i_\pm &= 2 \arg\{\gamma_i + i(t \pm |y_i|)\} = -2 \,\mathrm{arctg}\, \frac{t \pm |y_i|}{\gamma_i} \\
\alpha_i &= \frac{1}{2}(\psi^i_+ + \psi^i_-) \\
\beta_i &= \frac{1}{2}(\psi^i_+ - \psi^i_-) \\
\vec{u}_i &= \vec{i}/|y_i|
\end{aligned} \tag{5.23}$$

In this case we can write

$$F = e^{i\alpha} G$$

where

$$G = e^{-i\beta_1 \vec{\sigma}\cdot\vec{u}_1} \cdots e^{-i\beta_n \vec{\sigma}\cdot\vec{u}_n} \tag{5.24}$$

with $\alpha = \Sigma\, \alpha_i$, and the expression of the field $g_{\mu\nu}$, (5.17) becomes

$$g_{\mu\nu} = \mathrm{Tr}\, \frac{\partial G}{\partial x^\mu} \frac{\partial G}{\partial x^\nu} - \frac{\partial \alpha}{\partial x^\mu} \frac{\partial \alpha}{\partial x^\nu} \tag{5.25}$$

which is indeed real.

In the particular case already considered in Ref. 12) all $c_i = 0$ and we simply get

$$G = e^{-i\beta\vec{\sigma}\cdot\vec{u}} \tag{5.26}$$

where

$$\beta = \sum \beta_i = \sum \operatorname{arctg} \frac{t-r}{\gamma_i} - \sum \operatorname{arctg} \frac{t+r}{\gamma_i} \tag{5.27}$$

and

$$\vec{u} = \vec{x}/|x|$$

We can also write for G the simple expression

$$G = w_\mu \bar{s}_\mu \quad ; \qquad w_0 = \cos\beta \,, \; w_i = \sin\beta \, u_i \tag{5.28}$$

In this case $g_{\mu\nu}$ can be factorized in the form

$$\begin{aligned} g_{\mu\nu} = \partial_\mu \beta \partial_\nu \beta - \partial_\mu \alpha \partial_\nu \alpha \; + \\ + \sin^2\beta \;\; \partial_\mu \vec{u} \cdot \partial_\nu \vec{u} \end{aligned} \tag{5.29}$$

We recall for the sake of definiteness the relevant formulae for the well-known two-meron case

$$F = s_\mu z_\mu \tag{5.30}$$

where

$$z_\mu = \sqrt{a^2} \left\{ -\frac{a_\mu}{a^2} + 2 \frac{(x+a)_\mu}{(x+a)^2} \right\} \tag{5.31}$$

and

$$\begin{aligned} & a_\mu = (\gamma, 0, 0, 0) \\ & z_\mu = e^{i\alpha} w_\mu \\ & w_\mu = (\cos\beta, \sin\beta \, \vec{u}) \\ & \alpha = \operatorname{arctg} \frac{t+r}{\gamma} + \operatorname{arctg} \frac{t-r}{\gamma} \,, \; \beta = \operatorname{arctg} \frac{t+r}{\gamma} - \operatorname{arctg} \frac{t\;r}{\gamma} \end{aligned} \tag{5.32}$$

Of course, the invariance of our Lagrangian under general transformations can be applied to use other elementary solutions like, for example, the instanton as a starting point for new sets of classical solutions.

6. CONCLUSIONS

In this paper we have discussed some properties of field theories which are invariant under the general transformations of space-time variables. This infinite parameter transformation, which is the natural generalization of the 15-parameter conformal group, might become a characteristic feature of next theories of strong interactions.

From the classical point of view one can say that, whereas the conformal group is one essential ingredient for the derivation and the understanding of elementary classical solutions like instantons and merons, the general group plays a similar role for an extended class of solutions involving an infinite number of free parameters.

In particular, the multimeron solutions discussed in Section 5 seem to play an important role in the problem of confinement[20]. Two models exhibiting different but complementary features have been considered. The first one is a Yang-Mills theory coupled with gravitation. We have shown that the elementary solutions of the usual Yang-Mills theory could be generalized to that case only in the presence of a cosmological term whose constant is fixed in terms of the gauge coupling constant. Of course, the advantage due to the appearance of gravitation is the infinite parameter invariance group which allows to go much beyond the elementary solutions.

The second model is the non-linear sigma model, coupled with gravitation. The immediate advantage is that we can now define a four-dimensional conformal invariant sigma model without the unpleasant appearance of higher-order derivatives in the equations of motion.

Again, in addition to instanton and meron, we have a more general class of solutions, in particular, the multimerons are closely analogous to the corresponding expressions already found in the case of two space-time dimensions.

This analogy might suggest that our generally invariant four dimensional sigma models might have in common with the two-dimensional one some of the very exciting properties which have recently been discovered[21].

We now come to the crucial problem of the physical interpretation of the field $g_{\mu\nu}$. The easy way, which has essentially been followed in this paper, is to deny any connection between $g_{\mu\nu}$ and gravitation, $g_{\mu\nu}$ being the field of a new spin 2 particle whose existence is important for confinement[22]. This interpretation is indeed mandatory in the case of a generalized Yang-Mills theory because of the presence of a cosmological constant which is proportional to the gauge coupling constant.

In the case of the generalized sigma model no cosmological term is present and therefore the free field solution

$$g_{\mu\nu} = \frac{1}{K} \delta_{\mu\nu} \tag{6.1}$$

is allowed.

One can thus write a Newtonian solution by expanding around the solution (6.1)

$$g_{\mu\nu} = \frac{1}{K} \left(\delta_{\mu\nu} + h_{\mu\nu} \right) \tag{6.2}$$

and identifying K with the gravitational coupling constant.

It is important to notice that, from our point of view, the gravitational equations are fully scale invariant and contain no dimensional constant. The gravitational coupling constant K (having the dimensions of the square of a length) appears only in the solution (6.2) of the gravitational equations. The presence of a fundamental scale in the gravitational phenomena is thus related to the spontaneous breaking of dilatation invariance induced by the solution (6.2).

The over-all picture is thus that our gravitational equations are fully conformally invariant and that perturbations around the instanton, meron or (6.1) solutions will break spontaneously conformal invariance in different ways. In the instanton case the $O(5)$ subgroup will be preserved, the meron will preserve $O(4) \times O(2)$ and finally the solution (6.1) will preserve Poincaré invariance.

One could thus conceive a scheme in which a single spin 2 field $g_{\mu\nu}$ exists. Because of the very structure of the gravitational Lagrangian, it is possible that different approximation methods should be applied in different regions of space-time. In particular, one could account for ordinary gravitational effects by using the expansion (6.2) at large distance. This expansion, however, would not be valid at small distances where we could rely on our instanton or meron solutions to get a reasonable zero-order approximation.

In order that this scheme could be physically relevant one would be required to exhibit a more general classical solution joining smoothly the small and large distance solutions. If this were the case, then one might be led to conclude that the usual wisdom that gravity is irrelevant in the framework of particle physics is due to an unjustified use of the perturbation method in the framework of a terribly non-linear problem.

Leaving those slightly heretical thoughts aside, we conclude that the semi-classical investigation of Einsteinian kinds of field theories affords an exciting avenue for the future.

APPENDIX A

In this Appendix we collect some general formulae and computations relevant to the class of fields dealt with in Section 3:

$$g_{\mu\nu}(x) = h^2(x)\,\delta_{\mu\nu}\,, \tag{A.1}$$

$$A_\mu(x) = \frac{i}{e}\,\sigma_{\mu\nu}\,\partial_\nu \ln h(x)\,, \tag{A.2}$$

with

$$\Box h + \frac{1}{2}\lambda^2 h^3 = 0 \tag{A.3}$$

and

$$\lambda^2 = e^2 \tag{A.4}$$

The Yang-Mills field strength $F_{\mu\nu}$ and the curvature tensor $R_{\mu\nu\lambda\rho}$ (for which we adopt the standard definition as in Weinberg's book) are given by

$$F_{\mu\nu} = \frac{ih}{e}\left\{\sigma_{\mu\lambda}\partial_\nu\partial_\lambda \frac{1}{h} - \sigma_{\nu\lambda}\partial_\mu\partial_\lambda\frac{1}{h} - \right. \\ \left. - \sigma_{\mu\nu}\, h\left(\partial_\sigma \frac{1}{h}\right)^2\right\} \tag{A.5}$$

and

$$R_{\mu\nu\lambda\rho} = h^3\left\{\delta_{\nu\lambda}\,\partial_\mu\partial_\rho\frac{1}{h} + \delta_{\mu\rho}\,\partial_\nu\partial_\lambda\frac{1}{h} - \delta_{\mu\lambda}\,\partial_\nu\partial_\rho\frac{1}{h}\right. \\ \left. - \delta_{\nu\rho}\,\partial_\mu\partial_\lambda\frac{1}{h} + h\left(\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}\right)\left(\partial_\sigma\frac{1}{h}\right)^2\right\} \tag{A.6}$$

From this result it is also immediate to obtain

$$R_{\mu\nu} = -2h\,\partial_\mu\partial_\nu\frac{1}{h} + \delta_{\mu\nu}\left\{3h^2\left(\partial_\sigma\frac{1}{h}\right)^2 - \right. \\ \left. - h\,\Box\frac{1}{h}\right)\right\} \tag{A.7}$$

as well as the expression (3.11) for R.

The similarity among the above expression of the vector and tensor quantities suggests the existence of a simple connection, which is easily found to be

$$2ie \, \mathrm{Tr} \, F_{\mu\nu} \sigma_{\lambda\rho} = \frac{1}{h^2} R_{\mu\nu\lambda\rho} + h^2 \left(\varepsilon_{\mu\sigma\lambda\rho} R^{\sigma}_{\nu} - \varepsilon_{\nu\sigma\lambda\rho} R^{\sigma}_{\mu} \right) - \frac{1}{3} h^2 \varepsilon_{\mu\nu\lambda\rho} R \quad . \tag{A.8}$$

The fundamental invariant quantities of the vector theory are

$$S_1 = - e^2 \, \mathrm{Tr} \, F_{\mu\nu} F^{\mu\nu} \tag{A.9}$$

$$D_1 = - \frac{e^2}{32 \pi^2} \, \mathrm{Tr} \, \varepsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma} \tag{A.10}$$

Explicit computation now gives the results:

$$S_1 = 6 \left(\partial_\sigma \frac{1}{h} \right)^4 - 6 \left(1/h \, \Box \, 1/h \right) \left(\partial_\sigma \frac{1}{h} \right)^2 + \frac{1}{h^2} \left(\Box \frac{1}{h} \right)^2 + 2 \frac{1}{h^2} \left(\partial_\sigma \partial_\rho \frac{1}{h} \right)^2 \tag{A.11}$$

$$D_1 = \frac{h^4}{8\pi^2} \left[\frac{1}{h^2} \left(\Box \frac{1}{h} \right)^2 + 3 \left(\partial_\sigma \frac{1}{h} \right)^4 - 3 \frac{1}{h} \left(\partial_\sigma \frac{1}{h} \right)^2 \Box \frac{1}{h} - \frac{1}{h^2} \left(\partial_\sigma \partial_\rho \frac{1}{h} \right)^2 \right. \tag{A.12}$$

$$I_\mu(x) = \frac{1}{8\pi^2} h^2 \left\{ \left(\partial_\mu \frac{1}{h} \right) \left[\Box \frac{1}{h} - h \left(\partial_\sigma \frac{1}{h} \right)^2 \right] + \partial_\mu \partial_\rho \frac{1}{h} \; \partial_\rho \frac{1}{h} \right\} \tag{A.13}$$

It is a simple matter to show now that the following invariant quantities*) of the gravitational system

$$S_2 = \frac{1}{4} R_{\mu\nu\lambda\rho} R^{\mu'\nu'\lambda'\rho'} \tag{A.14}$$

and

$$D_2(x) = \frac{1}{256\pi^2} \varepsilon^{\lambda\mu\lambda'\mu'} \varepsilon^{\nu\rho\nu'\rho'} \frac{R_{\lambda\mu\nu\rho} R_{\lambda'\mu'\nu'\rho'}}{\sqrt{g}} \tag{A.15}$$

are identical to the above vector ones, i.e.,

$$S_1 = S_2 \quad , \quad D_1 = D_2 \ . \tag{A.16}$$

[This is of course a consequence of the relation (A.8).]

The additional gravitational invariant

$$P(x) = \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu\sigma\tau} R_{\lambda\rho}{}^{\sigma\tau} \tag{A.17}$$

is vanishing for the class of solution (3.12).

The specific expressions of all these quantities in the special cases of the instanton and meron are given in the text.

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*) We remind the reader of the fact that for conformally flat metrics the vanishing of the Weyl tensor leads to the following relation between the gravitational invariants:

$$R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = 2 R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2$$

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ADDENDUM

To be inserted on page 21, after Eq. (4.30):

The difference between meron and instanton solutions emerges clearly if one adds a cosmological term in the Lagrangian (4.9) which now reads

$$\mathcal{L} = -\frac{1}{4}\sqrt{g}\left\{ R + \frac{3}{2}\lambda^2 + f\, g^{\mu\nu}\partial_\mu \phi_a \partial_\nu \phi_a \right\}. \tag{a}$$

It is easy to see that the instanton solution, Eqs. (4.19) and (4.20), leads to the following relation between the "coupling constants" λ^2 and f

$$f = 3 - \frac{3}{16} C \lambda^2. \tag{b}$$

On the other hand, in the meron case one finds the more stringent determination

$$f = 2, \quad \lambda^2 = 0. \tag{c}$$