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# Gauge Theories from Toric Geometry and Brane Tilings

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Sebastián Franco<sup>1</sup>, Amihay Hanany<sup>1</sup>, Dario Martelli<sup>2</sup>, James Sparks<sup>3</sup>, David Vegh<sup>1</sup>,  
and Brian Wecht<sup>1</sup>

1. *Center for Theoretical Physics, Massachusetts Institute of Technology,  
Cambridge, MA 02139, USA.*

2. *Department of Physics, CERN Theory Division, 1211 Geneva 23, Switzerland.*

3. *Department of Mathematics, Harvard University,  
One Oxford Street, Cambridge, MA 02318, U.S.A.*

and

*Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, U.S.A.*

sfranco@mit.edu, hanany@mit.edu, dario.martelli@cern.ch,  
sparks@math.harvard.edu, dvegh@mit.edu, bwecht@mit.edu

**ABSTRACT:** We provide a general set of rules for extracting the data defining a quiver gauge theory from a given toric Calabi–Yau singularity. Our method combines information from the geometry and topology of Sasaki–Einstein manifolds, AdS/CFT, dimers, and brane tilings. We explain how the field content, quantum numbers, and superpotential of a superconformal gauge theory on D3–branes probing a toric Calabi–Yau singularity can be deduced. The infinite family of toric singularities with known horizon Sasaki–Einstein manifolds  $L^{a,b,c}$  is used to illustrate these ideas. We construct the corresponding quiver gauge theories, which may be fully specified by giving a tiling of the plane by hexagons with certain gluing rules. As checks of this construction, we perform  $a$ -maximisation as well as  $Z$ -minimisation to compute the exact R-charges of an arbitrary such quiver. We also examine a number of examples in detail, including the infinite subfamily  $L^{a,b,a}$ , whose smallest member is the Suspended Pinch Point.

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## 1. Introduction

Gauge theories arise within string theory in a variety of different ways. One possible approach, which is particularly interesting due to its relationship with different branches of geometry, is to use D-branes to probe a singularity. The geometry of the singularity then determines the amount of supersymmetry, the gauge group structure, the matter content and the superpotential interactions on the worldvolume of the D-branes. The richest of such examples which are both tractable, using current techniques, and also non-trivial, are given by the  $d = 4$   $\mathcal{N} = 1$  gauge theories that arise on a stack of D3-branes probing a singular Calabi–Yau 3-fold. There has been quite remarkable progress over the last year in understanding these theories, especially in the case where the Calabi–Yau singularity is also toric. In this case one can use toric geometry, which by now is an extremely well-developed subject, to study the gauge theories. In fact part of this paper is devoted to pushing this further and we will show how one can use arguments in toric geometry and topology to derive much of the field content of a D3-brane probing a toric Calabi–Yau singularity in a very simple manner, using essentially only the toric diagram and associated gauged linear sigma model.

At low energies the theory on the D3-brane is expected to flow to a superconformal fixed point. The AdS/CFT correspondence [1, 2, 3] connects the strong coupling regime of such gauge theories with supergravity in a mildly curved geometry. For the case of D3-branes placed at the tips of Calabi–Yau cones over five-dimensional geometries  $Y_5$ , the gravity dual is of the form  $AdS_5 \times Y_5$ , where  $Y_5$  is a Sasaki–Einstein manifold [4, 5, 6, 7]. There has been considerable progress in this subject recently: for a long time, there was only one non-trivial Sasaki–Einstein five-manifold,  $T^{1,1}$ , where the metric was known. Thanks to recent progress, we now have an infinite family of explicit metrics which, when non-singular and simply-connected, have topology  $S^2 \times S^3$ . The most general such family is specified by 3 positive integers  $a, b, c$ , with the metrics denoted  $L^{a,b,c}$  [8, 9]<sup>1</sup>. When  $a = p - q, b =$

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<sup>1</sup>We have changed the notation to  $L^{a,b,c}$  to avoid confusion with the  $p$  and  $q$  of  $Y^{p,q}$ . In our notation,  $Y^{p,q}$  is  $L^{p+q,p-q,p}$ .

$p + q, c = p$  these reduce to the  $Y^{p,q}$  family of metrics, which have an enhanced  $SU(2)$  isometry [10, 11, 12]. Aided by the toric description in [12], the entire infinite family of gauge theories dual to these metrics was constructed in [13]. These theories have subsequently been analysed in considerable detail [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. There has also been progress on the non-conformal extensions of these theories (and others) both from the supergravity [14, 22] and gauge theory sides [20, 21, 23, 24, 26, 27]. These extensions exhibit many interesting features, such as cascades [28] and dynamical supersymmetry breaking. In addition to the  $Y^{p,q}$  spaces, there are also several other interesting infinite families of geometries which have been studied recently: the  $X^{p,q}$  spaces [29], deformations of geometries with  $U(1) \times U(1)$  isometry [18], and deformations of geometries with  $U(1)^3$  isometry [30, 31].

Another key ingredient in obtaining the gauge theories dual to singular Calabi–Yau manifolds is the principle of  $a$ -maximisation [32], which permits the determination of exact R-charges of superconformal field theories. Recall that all  $d = 4$   $\mathcal{N} = 1$  gauge theories possess a  $U(1)_R$  symmetry which is part of the superconformal group  $SU(2, 2|1)$ . If this superconformal R-symmetry is correctly identified, many properties of the gauge theory may be determined.  $a$ -maximisation [32] is a simple procedure – maximizing a cubic function – that allows one to identify the R-symmetry from among the set of global symmetries of any given gauge theory. Plugging the superconformal R-charges into this cubic function gives exactly the central charge  $a$  of the SCFT [33, 34, 35]. Although here we will focus on superconformal theories with known geometric duals,  $a$ -maximization is a general procedure which applies to any  $\mathcal{N} = 1$   $d = 4$  superconformal field theory, and has been studied in this context in a number of recent works, with much emphasis on its utility for proving the  $a$ -theorem [38, 39, 40, 41, 42, 43, 44].

In the case that the gauge theory has a geometric dual, one can use the AdS/CFT correspondence to compute the volume of the dual Sasaki–Einstein manifold, as well as the volumes of certain supersymmetric 3-dimensional submanifolds, from the R-charges. For example, remarkable agreement was found for these two computations in the case of the  $Y^{p,q}$  singularities [45, 13]. Moreover, a general geometric procedure that allows one to compute the volume of any toric Sasaki–Einstein manifold, as well as its toric supersymmetric submanifolds, was then given in [46]. In [46] it was shown that one can determine the Reeb vector field, which is dual to the R-symmetry, of any toric Sasaki–Einstein manifold by minimising a function  $Z$  that depends only on the toric data that defines the singularity. For example, the volumes of the  $Y^{p,q}$  manifolds are easily reproduced this way. Remarkably, one can also compute the volumes of manifolds for which the metric is not known explicitly. In all cases agreement has been found between the geometric and field theoretic calculations.

This was therefore interpreted as a geometric dual of  $a$ -maximisation in [46], although to date there is no general proof that the two extremal problems, within the class of superconformal gauge theories dual to toric Sasaki–Einstein manifolds, are in fact equivalent.

Another important step was achieved recently by the introduction of dimer technology as a tool for studying  $\mathcal{N} = 1$  gauge theories. Although it has been known in principle how to compute toric data dual to a given SCFT and vice versa [36, 37], these computations are often very computationally expensive, even for fairly small quivers. Dimer technology greatly simplifies this process, and turns previously intractable calculations into easily solved problems. The initial connection between toric geometry and dimers was suggested in [47]; the connection to  $\mathcal{N} = 1$  theories was proposed and explored in [48]. A crucial realization that enables one to use this tool is that all the data for an  $\mathcal{N} = 1$  theory can be simply represented as a periodic tiling (“brane tiling”) of the plane by polygons with an even number of sides: the faces represent gauge groups, the edges represent bifundamentals, and the vertices represent superpotential terms. This tiling has a physical meaning in Type IIB string theory as an NS5-brane wrapping a holomorphic curve (the edges of the tiling) with D5-branes (the faces) ending on the NS5-brane. The fact that the polygons have an even number of sides is equivalent to the requirement that the theories be anomaly free; by choosing an appropriate periodicity, one can color the vertices of such a graph with two colours (say, black and white) so that a black vertex is adjacent only to white vertices. Edges that stretch between black and white nodes are the dimers, and there is a simple prescription for computing a weighted adjacency matrix (the Kasteleyn matrix) which gives the partition function for a given graph. This partition function encodes the toric diagram for the dual geometry in a simple way, thus enabling one to have access to many properties of both the gauge theory and the geometry. The tiling construction of a gauge theory is a much more compact way of describing the theory than the process of specifying both a quiver and the superpotential, and we will use this newfound simplicity rather extensively in this paper.

In this paper, we will use this recent progress in geometry, field theory, and dimer models to obtain a lot of information about gauge theories dual to general toric Calabi–Yau cones. Our geometrical knowledge will specify many requirements of the gauge theory, and we describe how one can read off gauge theory quantities rather straightforwardly from the geometry. As a particular example of our methods, we construct the gauge theories dual to the recently discovered  $L^{a,b,c}$  geometries. We will realize the geometrically derived requirements by using the brane tiling approach. Since the  $L^{a,b,c}$  spaces are substantially more complicated than the  $Y^{p,q}$ 's, we will not give a closed form expression for the gauge theory. We will, however, specify all the necessary building blocks for the brane tiling, and

discuss how these building blocks are related to quantities derived from the geometry.

The plan of the paper is as follows: In Section 2, we discuss how to read gauge theory data from a given toric geometry. In particular, we give a detailed prescription for computing the quantum numbers (*e.g.* baryon charges, flavour charges, and R-charges) and multiplicities for the different fields in the gauge theory. Section 3 applies these results to the  $L^{a,b,c}$  spaces. We derive the toric diagram for a general  $L^{a,b,c}$  geometry, and briefly review the metrics [8, 9] for these theories. We compute the volumes of the supersymmetric 3-cycles in these Sasaki–Einstein spaces, and discuss the constraints these put on the gauge theories. In Section 4 we discuss how our geometrical computations constrain the superpotential, and describe how one may always find a phase of the gauge theory with at most only three different types of interactions. In Section 5, we prove that  $a$ -maximisation reduces to the same equations required by the geometry for computing R-charges and central charges. Thus we show that  $a$ -maximisation and the geometric computation agree. In Section 6, we construct the gauge theories dual to the  $L^{a,b,c}$  spaces by using the brane tiling perspective, and give several examples of interesting theories. In particular, we describe a particularly simple infinite subclass of theories, the  $L^{a,b,a}$  theories, for which we can simply specify the toric data and brane tiling. We check via  $Z$ -minimisation and  $a$ -maximisation that all volumes and dimensions reproduce the results expected from AdS/CFT. Finally, in the Appendix, we give some more interesting examples which use our construction.

**Note:** While this paper was being finalised, we were made aware of other work in [49], which has some overlap with our results. Similar conclusions have been reached in [50].

## 2. Quiver content from toric geometry

In this section we explain how one can extract a considerable amount of information about the gauge theories on D3-branes probing toric Calabi–Yau singularities using simple geometric methods. In particular, we show that there is always a distinguished set of fields whose multiplicities, baryon charges, and flavour charges can be computed straightforwardly using the toric data.

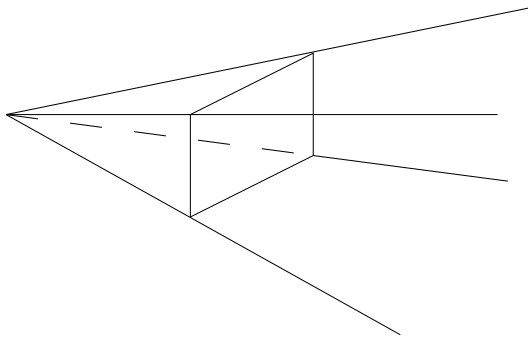
### 2.1 General geometrical set-up

Let us first review the basic geometrical set-up. For more details, the reader is referred to [46]. Let  $(X, \omega)$  be a toric Calabi–Yau cone of complex dimension  $n$ , where  $\omega$  is the Kähler form on  $X$ . In particular  $X = C(Y) \cong \mathbb{R}^+ \times Y$  has an isometry group containing an  $n$ -torus,  $T^n$ . A conical metric on  $X$  which is both Ricci-flat and Kähler then gives a Sasaki–Einstein

metric on the base of the cone,  $Y$ . The moment map for the torus action exhibits  $X$  as a Lagrangian  $T^n$  fibration over a strictly convex rational polyhedral cone  $\mathcal{C} \subset \mathbb{R}^n$ . This is a subset of  $\mathbb{R}^n$  of the form

$$\mathcal{C} = \{y \in \mathbb{R}^n \mid (y, v_A) \geq 0, A = 1, \dots, D\}. \quad (2.1)$$

Thus  $\mathcal{C}$  is made by intersecting  $D$  hyperplanes through the origin in order to make a convex polyhedral cone. Here  $y \in \mathbb{R}^n$  are coordinates on  $\mathbb{R}^n$  and  $v_A$  are the inward pointing normal vectors to the  $D$  hyperplanes, or *facets*, that define the polyhedral cone. The normals are rational and hence one can normalise them to be primitive<sup>2</sup> elements of the lattice  $\mathbb{Z}^n$ . We also assume this set of vectors is minimal in the sense that removing any vector  $v_A$  in the definition (2.1) changes  $\mathcal{C}$ . The condition that  $\mathcal{C}$  be *strictly* convex is simply the condition that it is a cone over a convex polytope.



**Figure 1:** A four-faceted polyhedral cone in  $\mathbb{R}^3$ .

The condition that  $X$  is Calabi–Yau,  $c_1(X) = 0$ , implies that the vectors  $v_A$  may, by an appropriate  $SL(n; \mathbb{Z})$  transformation of the torus, be all written as  $v_A = (1, w_A)$ . In particular, in complex dimension  $n = 3$  we may therefore represent any toric Calabi–Yau cone by a convex lattice polytope in  $\mathbb{Z}^2$ , where the vertices are simply the vectors  $w_A$ . This is usually called the toric diagram.

From the vectors  $v_A$  one can reconstruct  $X$  as a Kähler quotient or, more physically, as the classical vacuum moduli space of a gauged linear sigma model (GLSM). To explain this, denote by  $\Lambda \subset \mathbb{Z}^n$  the span of the normals  $\{v_A\}$  over  $\mathbb{Z}$ . This is a lattice of maximal rank since  $\mathcal{C}$  is strictly convex. Consider the linear map

$$\begin{aligned} A : \mathbb{R}^D &\rightarrow \mathbb{R}^n \\ e_A &\mapsto v_A \end{aligned} \quad (2.2)$$

---

<sup>2</sup>A vector  $v \in \mathbb{Z}^n$  is primitive if it cannot be written as  $mv'$  with  $v' \in \mathbb{Z}^n$  and  $\mathbb{Z} \ni m > 1$ .



which maps each standard orthonormal basis vector  $e_A$  of  $\mathbb{R}^D$  to the vector  $v_A$ . This induces a map of tori

$$T^D \cong \mathbb{R}^D / \mathbb{Z}^D \rightarrow \mathbb{R}^n / \Lambda . \quad (2.3)$$

In general the kernel of this map is  $\mathcal{A} \cong T^{D-n} \times \Gamma$  where  $\Gamma$  is a finite abelian group. Then  $X$  is given by the Kähler quotient

$$X = \mathbb{C}^D // \mathcal{A} . \quad (2.4)$$

Recall we may write this more explicitly as follows. The torus  $T^{D-n} \subset T^D$  is specified by a charge matrix  $Q_I^A$  with integer coefficients,  $I = 1, \dots, D-n$ , and we define

$$\mathcal{K} \equiv \left\{ (Z_1, \dots, Z_D) \in \mathbb{C}^D \mid \sum_A Q_I^A |Z_A|^2 = 0 \right\} \subset \mathbb{C}^D \quad (2.5)$$

where  $Z_A$  denote complex coordinates on  $\mathbb{C}^D$ . In GLSM language,  $\mathcal{K}$  is simply the space of solutions to the D-term equations. Dividing out by gauge transformations gives the quotient

$$X = \mathcal{K} / T^{D-n} \times \Gamma . \quad (2.6)$$

We also denote by  $L$  the link of  $\mathcal{K}$  with the sphere  $S^{2D-1} \subset \mathbb{C}^D$ . We then have a fibration

$$\mathcal{A} \hookrightarrow L \rightarrow Y \quad (2.7)$$

where  $Y$  is the Sasakian manifold which is the base of the cone  $X = C(Y)$ . For a general set of vectors  $v_A$ , the space  $Y$  will not be smooth. In fact typically one has orbifold singularities.  $Y$  is smooth if and only if the polyhedral cone is *good* [51], although we will not enter into the general details of this here – see, for example, [46].

Finally in this subsection we note some topological properties of  $Y$ , in the case that  $Y$  is a smooth manifold. In [52] it is shown that  $L$  has trivial homotopy groups in dimensions 0, 1 and 2. From the long exact homotopy sequence for the fibration (2.7) one concludes that [52]

$$\begin{aligned} \pi_1(Y) &\cong \pi_0(\mathcal{A}) \cong \Gamma \cong \mathbb{Z}^n / \Lambda \\ \pi_2(Y) &\cong \pi_1(\mathcal{A}) \cong \mathbb{Z}^{D-n} . \end{aligned} \quad (2.8)$$

In particular,  $Y$  is simply-connected if and only if the  $\{v_A\}$  span  $\mathbb{Z}^n$  over  $\mathbb{Z}$ . In fact we will assume this throughout this paper – any finite quotient of a toric singularity will correspond to an orbifold of the corresponding gauge theory, and this process is well-understood by now.

From now on we also restrict to the physical case of complex dimension  $n = 3$ . Moreover, throughout this section we assume that the Sasaki–Einstein manifold  $Y$  is smooth. The reason for this assumption is firstly to simplify the geometrical and topological analysis, and secondly because the physics in the case that  $Y$  is an orbifold which is *not* a global quotient of a smooth manifold is not well–understood. However, as we shall see later, one can apparently relax this assumption with the results essentially going through without modification. The various cohomology groups that we introduce would then need replacing by their appropriate orbifold versions.

## 2.2 Quantum numbers of fields

In this subsection we explain how one can deduce the quantum numbers for a certain distinguished set of fields in any toric quiver gauge theory. Recall that, quite generally,  $N$  D3–branes placed at a toric Calabi–Yau singularity have an AdS/CFT dual that may be described by a toric quiver gauge theory. In particular, the matter content is specified by giving the number of gauge groups,  $N_g$ , and number of fields  $N_f$ , together with the charge assignments of the fields. In fact these fields are always bifundamentals (or adjoints). This means that the matter content may be neatly summarised by a quiver diagram.

We may describe the toric singularity as a convex lattice polytope in  $\mathbf{Z}^2$  or by giving the GLSM charges, as described in the previous section. By setting each complex coordinate  $Z_A = 0$ ,  $A = 1, \dots, D$ , one obtains a toric divisor  $D_A$  in the Calabi–Yau cone. This is also a cone, with  $D_A = C(\Sigma_A)$  where  $\Sigma_A$  is a 3–dimensional supersymmetric submanifold of  $Y$ . Thus in particular wrapping a D3–brane over  $\Sigma_A$  gives rise to a BPS state, which via the AdS/CFT correspondence is conjectured to be dual to a dibaryonic operator in the dual gauge theory. We claim that there is always a distinguished subset of the fields, for any toric quiver gauge theory, which are associated to these dibaryonic states. To explain this, recall that given any bifundamental field  $X$ , one can construct the dibaryonic operator

$$\mathcal{B}[X] = \epsilon^{\alpha_1 \dots \alpha_N} X_{\alpha_1}^{\beta_1} \dots X_{\alpha_N}^{\beta_N} \epsilon_{\beta_1 \dots \beta_N} \quad (2.9)$$

using the epsilon tensors of the corresponding two  $SU(N)$  gauge groups. This is dual to a D3–brane wrapped on a supersymmetric submanifold, for example one of the  $\Sigma_A$ . In fact to each toric divisor  $\Sigma_A$  let us associate a bifundamental field  $X_A$  whose corresponding dibaryonic operator (2.9) is dual to a D3–brane wrapped on  $\Sigma_A$ . These fields in fact have multiplicities, as we explain momentarily. In particular each field in such a multiplet has the same baryon charge, flavour charge, and R–charge.

### 2.2.1 Multiplicities

Recall that  $D_A = \{Z_A = 0\} = C(\Sigma_A)$  where  $\Sigma_A$  is a 3-submanifold of  $Y$ . To each such submanifold we associated a bifundamental field  $X_A$ . As we now explain, these fields have multiplicities given by the simple formula

$$m_A = |(v_{A-1}, v_A, v_{A+1})| \quad (2.10)$$

where we have defined the cyclic identification  $v_{D+1} = v_1$ , and we list the normal vectors  $v_A$  in order around the polyhedral cone, or equivalently, the toric diagram. Here  $(\cdot, \cdot, \cdot)$  denotes a  $3 \times 3$  determinant, as in [46].

In fact, when  $Y$  is smooth, each  $\Sigma_A$  is a Lens space  $L(n_1, n_2)$  for appropriate  $n_1$  and  $n_2$ . To see this, note that each  $\Sigma_A$  is a principle  $T^2$  fibration over an interval, say  $[0, 1]$ . By an  $SL(2; \mathbf{Z})$  transformation one can always arrange that at 0 the  $(1, 0)$ -cycle collapses, and at 1 the  $(n_1, n_2)$ -cycle collapses. It is well-known that this can be equivalently described as the quotient of  $S^3 \subset \mathbb{C}^2$  by the  $\mathbf{Z}_{n_1}$  action

$$(z_1, z_2) \rightarrow (z_1 \omega_{n_1}, z_2 \omega_{n_1}^{n_2}) \quad (2.11)$$

where  $\text{hcf}(n_1, n_2) = 1$  and  $\omega_{n_1}$  denotes an  $n_1$ th root of unity. These spaces have a rich history, and even the classification of homeomorphism types is rather involved. We shall only need to know that  $\pi_1(L(n_1, n_2)) \cong \mathbf{Z}_{n_1}$ , which is immediate from the second definition above.

Consider now wrapping a D3-brane over some smooth  $\Sigma$ , where  $\pi_1(\Sigma) = \mathbf{Z}_m$ . As we just explained, when  $\Sigma$  is toric, it is necessarily some Lens space  $L(m, n_2)$ . In fact, when the order of the fundamental group is greater than one, there is not a single such D3-brane, but in fact  $m$  D3-branes. The reason is that, for each  $m$ , we can turn on a flat line bundle for the  $U(1)$  gauge field on the D3-brane worldvolume. Indeed, recall that line bundles on  $\Sigma$  are classified topologically by  $H^2(\Sigma; \mathbf{Z}) \cong H_1(\Sigma; \mathbf{Z}) \cong \pi_1(\Sigma)$ , where the last relation follows for abelian fundamental group. A torsion line bundle always admits a flat connection, which has zero energy. Since these D3-brane states have different charge – namely torsion D1-brane charge – they must correspond to different operators in the gauge theory. However, as will become clear, these operators all have the same baryon charge, flavour charge and R-charge. We thus learn that the multiplicity of the bifundamental field  $X_A$  associated with  $D_A$  is given by  $m$ .

It remains then to relate  $m$  to the formula (2.10). Without loss of generality, pick a facet  $A$ , and suppose that the normal vector is  $v_A = (1, 0, 0)$ . The facet is itself a polyhedral cone in the  $\mathbb{R}^2$  plane transverse to this vector. To obtain the normals that define this cone we simply project  $v_{A-1}, v_{A+1}$  onto the plane. Again, by a special linear transformation we

may take these 2–vectors to be  $(0, 1)$ ,  $(n_1, -n_2)$ , respectively, for some integers  $n_1$  and  $n_2$ . One can then verify that this toric diagram indeed corresponds to the cone over  $L(n_1, n_2)$ , as defined above. By direct calculation we now see that

$$|(v_{A-1}, v_A, v_{A+1})| = |(0, 1) \times (n_1, -n_2)| = n_1 \quad (2.12)$$

which is the order of  $\pi_1(\Sigma_A)$ . The determinant is independent of the choice of basis we have made, and thus this relation is true in general, thus proving the formula (2.10). One can verify this formula in a large number of examples where the gauge theories are already known.

### 2.2.2 Baryon charges

In this subsection we explain how one can deduce the baryonic charges of the fields  $X_A$ . Recall that, in general, the toric Sasaki–Einstein manifold  $Y$  arises from a quotient by a torus

$$T^{D-3} \hookrightarrow L \rightarrow Y . \quad (2.13)$$

This fibration can be thought of as  $D - 3$  circle fibrations over  $Y$  with total space  $L$ . Equivalently we can think of these as complex line bundles  $\mathcal{M}_I$ . Let  $C_I$ ,  $I = 1, \dots, D - 3$ , denote the Poincaré duals of the first Chern classes of these bundles. Thus they are classes in  $H_3(Y; \mathbf{Z})$ . Recall from (2.8) that  $\pi_2(Y) \cong \mathbf{Z}^{D-3}$  when  $Y$  is smooth. Provided  $Y$  is also simply–connected<sup>3</sup> one can use the Hurewicz isomorphism, Poincaré duality and the universal coefficients theorem to deduce that

$$H_3(Y; \mathbf{Z}) \cong \mathbf{Z}^{D-3} . \quad (2.14)$$

In particular note that the number of independent 3–cycles is just  $D - 3$ . A fairly straightforward calculation<sup>4</sup> in algebraic topology shows that the classes  $C_I$  above actually generate the homology group  $H_3(Y; \mathbf{Z}) \cong \mathbf{Z}^{D-3}$ . Thus  $\{C_I\}$  form a basis of 3–cycles on  $Y$ .

In Type IIB supergravity one can Kaluza–Klein reduce the Ramond–Ramond four–form potential  $C_4$  to obtain  $D - 3$  gauge fields  $A_I$  in the  $AdS_5$  space:

$$C_4 = \sum_{I=1}^{D-3} A_I \wedge \mathcal{H}_I . \quad (2.15)$$

---

<sup>3</sup>Recall this is also one of our assumptions in this section.

<sup>4</sup>For example, one can use the Gysin sequence for each circle in turn.

Here  $\mathcal{H}_I$  is a harmonic 3–form on  $Y$  that is Poincaré dual to the 3–cycle  $C_I$ . In the superconformal gauge theory, which recall may be thought of as living on the conformal boundary of  $AdS_5$ , these become  $D - 3$  global  $U(1)$  symmetries

$$U(1)_B^{D-3} . \quad (2.16)$$

These are baryonic symmetries precisely because the D3–brane is charged under  $C_4$  and a D3–brane wrapped over a supersymmetric submanifold of  $Y$  is interpreted as a dibaryonic state in the gauge theory. Indeed, the  $\Sigma_A$  are precisely such a set of submanifolds.

Again, a fairly standard calculation in toric geometry then shows that topologically

$$[\Sigma_A] = \sum_{I=1}^{D-3} Q_I^A C_I \in H_3(Y; \mathbf{Z}) . \quad (2.17)$$

This perhaps requires a little explanation. Each GLSM field  $Z_A$ ,  $A = 1, \dots, D$ , can be viewed as a section of a complex line bundle  $\mathcal{L}_A$  over  $Y$ . They are necessarily sections of line bundles, rather than functions, because the fields  $Z_A$  are charged under the torus  $T^{D-3}$ . Now  $Z_A = 0$  is the zero section of the line bundle associated to  $Z_A$ , and by definition this cuts out the submanifold  $\Sigma_A$  on  $Y$ . Moreover, the first Chern class of this line bundle is then Poincaré dual to  $[\Sigma_A]$ . Recall that the charge matrix  $Q$  specifies the embedding of the torus  $T^{D-3}$  in  $T^D$ , which then acts on the fields/coordinates  $Z_A$ ; the element  $Q_I^A$  specifies the charge of  $Z_A$ , which is a section of  $\mathcal{L}_A$ , under the circle  $\mathcal{M}_I$ . This means that the two sets of line bundles are related by

$$\mathcal{L}_A = \bigotimes_{I=1}^{D-3} \mathcal{M}_I^{Q_I^A} . \quad (2.18)$$

Taking the first Chern class of this relation and applying Poincaré duality then proves (2.17).

It follows that the baryon charges of the fields  $X_A$  are given precisely by the matrix  $Q$  that enters in defining the GLSM. Thus if  $B_I[X_A]$  denotes the baryon charge of  $X_A$  under the  $I$ th copy of  $U(1)$  in (2.16) we have

$$B_I[X_A] = Q_I^A . \quad (2.19)$$

Note that from the Calabi–Yau condition the charges of the linear sigma model sum to zero

$$\sum_A B_I[X_A] = \sum_A Q_I^A = 0 \quad I = 1, \dots, D - 3 . \quad (2.20)$$

Moreover, the statement that

$$\sum_A v_A^i [\Sigma_A] = 0 \quad (2.21)$$

may then be interpreted as saying that, for each  $i$ , one can construct a state in the gauge theory of zero baryon charge by using  $v_A^i$  copies of the field  $X_A$ , for each  $A$ .

### 2.2.3 Flavour charges

In this subsection we explain how one can compute the flavour charges of the  $X_A$ . Recall that the horizon Sasaki–Einstein manifolds have at least a  $U(1)^3$  isometry since they are toric. By definition a flavour symmetry in the gauge theory is a non–R–symmetry – that is, the supercharges are left invariant under such a symmetry. The geometric dual of this statement is that the Killing spinor  $\psi$  on the Sasaki–Einstein manifold  $Y$  is left invariant by the corresponding isometry. Thus a Killing vector field  $V_F$  is dual to a flavour symmetry in the gauge theory if and only if

$$\mathcal{L}_{V_F}\psi = 0 \tag{2.22}$$

where  $\psi$  is a Killing spinor on  $Y$ . In fact there is always precisely a  $U(1)^2$  subgroup of  $U(1)^3$  that satisfies this condition. This can be shown by considering the holomorphic  $(3, 0)$  form of the corresponding Calabi–Yau cone [46]. It is well known that this is constructed from the Killing spinors as a bilinear

$$\Omega = \psi^c \Gamma_{(3)} \psi , \tag{2.23}$$

where  $\Gamma_{(3)}$  is the totally antisymmetrised product of 3 gamma matrices in  $\text{Cliff}(6, 0)$ . In particular, in the basis in which the normal vectors of the polyhedral cone  $\mathcal{C}$  are of the form  $v_A = (1, w_A)$ , the Lie algebra elements  $(0, 1, 0), (0, 0, 1)$  generate the group  $U(1)_F^2$  of flavour isometries. Note that, for  $Y^{p,q}$ , one of these  $U(1)_F$  symmetries is enhanced to an  $SU(2)$  flavour symmetry. However,  $U(1)_F^2$  is the generic case.

We would like to determine the charges of the fields  $X_A$  under  $U(1)_F^2$ . In fact in the gauge theory this symmetry group is far from unique – one is always free to mix any flavour symmetry with part of the baryonic symmetry group  $U(1)_B^{D-3}$ . The baryonic symmetries are distinguished by the fact that mesons in the gauge theory, for example constructed from closed loops in a quiver gauge theory, should have zero baryonic charge. Thus the flavour symmetry group is unique only up to mixing with baryonic symmetries, and of course mixing with each other.

This mixing ambiguity has a beautiful geometric interpretation. Recall that the Calabi–Yau cone  $X$  is constructed as a symplectic quotient

$$X = \mathbf{C}^D // T^{D-3} \tag{2.24}$$

where the torus  $T^{D-3} \subset T^D$  is defined by the kernel of the map

$$A : \mathbb{R}^D \rightarrow \mathbb{R}^3 \tag{2.25}$$

$$e_A \mapsto v_A . \tag{2.26}$$

More precisely the kernel of  $A$  is generated by the matrix  $Q_I^A$ , which in turn defines a sublattice  $\Upsilon$  of  $\mathbb{Z}^D$  of rank  $D-3$ . The torus is then  $T^{D-3} = \mathbb{R}^{D-3}/\Upsilon$ . We may also consider the quotient  $\mathbb{Z}^D/\Upsilon$ . The map induced from  $A$  then maps this quotient space isomorphically onto  $\mathbb{Z}^3$  and the corresponding torus  $T^3 = T^D/T^{D-3}$  is then precisely the torus isometry of  $X$ .

Let us pick two elements  $\alpha_1, \alpha_2$  of  $\mathbb{Z}^D$  that map to the basis vectors  $(0, 1, 0), (0, 0, 1)$  under  $A$ . From the last paragraph these are defined only up to elements of the lattice  $\Upsilon$ , and thus may be considered as elements of the quotient  $\mathbb{Z}^D/\Upsilon$ . Geometrically,  $\alpha_1, \alpha_2$  define circle subgroups of  $T^D$  that descend to the two  $U(1)$  flavour isometries generated by  $(0, 1, 0)$  and  $(0, 0, 1)$ . The charges of the complex coordinates  $Z_A$  on  $\mathbb{C}^D$  are then simply  $\alpha_1^A, \alpha_2^A$  for each  $A = 1, \dots, D$ . However, as discussed in the last subsection, the  $Z_A$  descend to complex line bundles on  $Y$  whose Poincaré duals are precisely the submanifolds  $\Sigma_A$ . Thus the flavour charges of  $X_A$  may be identified with  $\alpha_1^A, \alpha_2^A$ . Moreover, by construction, each  $\alpha$  was unique only up to addition by some element in the lattice  $\Upsilon$  generated by  $Q_I^A$ . But as we just saw in the previous subsection, this is precisely the set of baryon charges in the gauge theory. We thus see that the ambiguity in the choice of flavour symmetries in the gauge theory is in 1–1 correspondence with the ambiguity in choosing  $\alpha_1, \alpha_2$ .

#### 2.2.4 R–charges

The R–charges were treated in reference [46], so we will be brief here. Let us begin by emphasising that all the quantities computed so far can be extracted in a simple way from the toric data, or equivalently from the charges of the gauged linear sigma model, without the need of an explicit metric. In [46], it was shown that the total volume of any toric Sasaki–Einstein manifold, as well as the volumes of its supersymmetric toric submanifolds, can be computed by solving a simple extremal problem which is defined in terms of the polyhedral cone  $\mathcal{C}$ . This toric data is encoded in a function  $Z$ , which depends on a “trial” Reeb vector living in  $\mathbb{R}^3$ . Minimising  $Z$  determines the Reeb vector for the Sasaki–Einstein metric on  $Y$  uniquely, and as a result one can compute the volumes of the  $\Sigma_A$ . This is a geometric analogue of  $a$ –maximisation [32]. Indeed, recall that the volumes are related to the R–charges of the corresponding fields  $X_A$  by the simple formula

$$R[X_A] = \frac{\pi \text{vol}(\Sigma_A)}{3 \text{vol}(Y)}. \quad (2.27)$$

This formula has been used in many AdS/CFT calculations to compare the R–charges of dibaryons with their corresponding 3–manifolds [53, 54, 55, 56].

Moreover, in [46] a general formula relating the volume of supersymmetric submanifolds to the total volume of the toric Sasaki–Einstein manifold was given. This reads

$$\pi \sum_{A=1}^D \text{vol}(\Sigma_A) = 6 \text{vol}(Y) . \quad (2.28)$$

Then the physical interpretation of (2.28) is that the R–charges of the bifundamental fields  $X_A$  sum to 2:

$$\sum_A R[X_A] = 2 . \quad (2.29)$$

This is related to the fact that each term in the superpotential is necessarily the sum

$$\sum_{A=1}^D \Sigma_A \quad (2.30)$$

and the superpotential has R–charge 2 by definition. We shall discuss this further in Section 4.

### 3. The $L^{a,b,c}$ toric singularities

In the remainder of this paper we will be interested in the specific GLSM with charges

$$Q = (a, -c, b, -d) \quad (3.1)$$

where of course  $d = a + b - c$  in order to satisfy the Calabi–Yau condition. We will define this singularity to be  $L^{a,b,c}$ . The reason we choose this family is two–fold: firstly, the Sasaki–Einstein metrics are known explicitly in this case [8, 9] and, secondly, this family is sufficiently simple that we will be able to give a general prescription for constructing the gauge theories.

Let us begin by noting that this is essentially the most general GLSM with four charges, and hence the most general toric quiver gauge theories with a single  $U(1)_B$  symmetry, up to orbifolding. Indeed, provided all the charges are non–zero, either two have the same sign or else three have the same sign. The latter are in fact just orbifolds of  $S^5$ , and this case where all but one of the charges have the same sign is slightly degenerate. Specifically, the charges  $(e, f, g, -e - f - g)$  describe the orbifold of  $S^5 \subset \mathbb{C}^3$  by  $\mathbb{Z}_{e+f+g}$  with weights  $(e, f, g)$ . The polyhedral cones therefore have *three* facets, and not four, or equivalently the  $(p, q)$  web has 3 external legs. By our general analysis there is therefore no  $U(1)$  baryonic symmetry, as expected. Indeed, note that setting  $Z_4 = 0$  does *not* give a divisor in this case, since the remaining charges are all positive and there is no solution to the remaining D–terms. The



Sasaki–Einstein metrics are just the quotients of the round metric on  $S^5$  and these theories are therefore not particularly interesting. In the case that one of the charges is zero, we instead obtain  $\mathcal{N} = 2$  orbifolds of  $S^5$ , which are also well–studied.

We are therefore left with the case that two charges have the same sign. In (3.1) we therefore take all integers to be positive. Without loss of generality we may of course take  $0 < a \leq b$ . Also, by swapping  $c$  and  $d$  if necessary, we can always arrange that  $c \leq b$ . By definition  $\text{hcf}(a, b, c, d) = 1$  in order that the  $U(1)$  action specified by (3.1) is effective, and it then follows that any three integers are coprime. The explicit Sasaki–Einstein metrics on the horizons of these singularities were constructed in [8]. The toric description above was then given in [9]. The manifolds were named  $L^{p,q,r}$  in reference [8] but, following [9], we have renamed these  $L^{a,b,c}$  in order to avoid confusion with  $Y^{p,q}$ . Indeed, notice that these spaces reduce to  $Y^{p,q}$  when  $c = d = p$ , and then  $a = p - q$ ,  $b = p + q$ . In particular there is an enhanced  $SU(2)$  symmetry in the metric in this limit. It is straightforward to determine when the space  $Y = L^{a,b,c}$  is non–singular: each of the pair  $a, b$  must be coprime to each of  $c, d$ . This condition is necessary to avoid codimension *four* orbifold singularities on  $Y$ . To see this, consider setting  $Z_1 = Z_4 = 0$ . If  $b$  and  $c$  had a common factor  $h$ , then the circle action specified by (3.1) would factor through a cyclic group  $\mathbf{Z}_h$  of order  $h$ , and this would descend to a local orbifold group on the quotient space. In fact it is simple to see that this subspace is just an  $S^1$  family of  $\mathbf{Z}_h$  orbifold singularities. All such singularities arise in this way. When  $Y = L^{a,b,c}$  is non–singular it follows from the last section that  $\pi_2(Y) \cong \mathbf{Z}$  and hence  $H_2(Y; \mathbf{Z}) \cong \mathbf{Z}$ . By Smale’s theorem  $Y$  is therefore diffeomorphic to  $S^2 \times S^3$ . In particular there is one 3–cycle and hence one  $U(1)_B$  for these theories.

The toric diagram can be described by an appropriate set of four vectors  $v_A = (1, w_A)$ . We take the following set

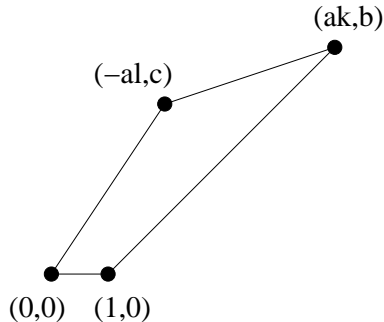
$$w_1 = [1, 0] \quad w_2 = [ak, b] \quad w_3 = [-al, c] \quad w_4 = [0, 0] \quad (3.2)$$

where  $k$  and  $l$  are two integers satisfying

$$ck + bl = 1 \quad (3.3)$$

and we have assumed for simplicity of exposition that  $\text{hcf}(b, c) = 1$ . This toric diagram is depicted in Figure 2.

The solution to the above equation always exists by Euclid’s algorithm. Moreover, there is a countable infinity of solutions to this equation, where one shifts  $k$  and  $l$  by  $-tb$  and  $tc$ , respectively, for any integer  $t$ . However, it is simple to check that different solutions are



**Figure 2:** Toric diagram for the  $L^{a,b,c}$  geometries.

related by the  $SL(2; \mathbf{Z})$  transformation

$$\begin{pmatrix} 1 & -ta \\ 0 & 1 \end{pmatrix} \tag{3.4}$$

acting on the  $w_A$ , as must be the case of course. The kernel of the linear map (2.2) is then generated by the charge vector  $Q$  in (3.1).

It is now simple to see that the toric diagram for  $L^{a,b,c}$  always admits a triangulation with  $a + b$  triangles. It is well known that this gives the number of gauge groups  $N_g$  in the gauge theory. To see this one uses the fact that the area of the toric diagram is the Euler number of the (any) completely resolved Calabi–Yau  $\tilde{X}$  obtained by toric crepant resolution, and then for toric manifolds this is the dimension of the even cohomology of  $\tilde{X}$ . Now on 0, 2 and 4–cycles in  $\tilde{X}$  one can wrap space–filling D3, D5 and D7–branes, respectively, and these then form a basis of fractional branes. The gauge groups may then be viewed as the gauge groups on these fractional branes. By varying the Kähler moduli of  $\tilde{X}$  one can blow down to the conical singularity  $X$ . The holomorphic part of the gauge theory is independent of the Kähler moduli, which is why the matter content of the superconformal gauge theory can be computed at large volume in this way. To summarise, we have

$$N_g = a + b . \tag{3.5}$$

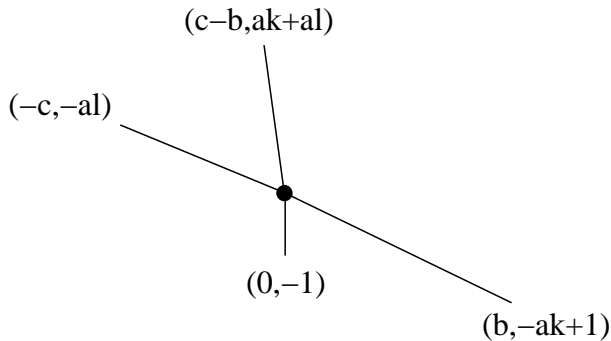
Note that, different from  $Y^{p,q}$ , the number of gauge groups for  $L^{a,b,c}$  can be *odd*.

We may now draw the  $(p, q)$  web [57, 58, 59]. Recall that this is simply the graph–theoretic dual to the toric diagram, or, completely equivalently, is the projection of the polyhedral cone  $\mathcal{C}$  onto the plane with normal vector  $(1, 0, 0)$ . The external legs of the  $(p, q)$

web are easily computed to be

$$\begin{aligned}
(p_1, q_1) &= (-c, -al) \\
(p_2, q_2) &= (c - b, a(k + l)) \\
(p_3, q_3) &= (b, -ak + 1) \\
(p_4, q_4) &= (0, -1)
\end{aligned}
\tag{3.6}$$

This  $(p, q)$ -web is pictured in Figure 3. Using this information we can compute the total



**Figure 3:**  $(p, q)$ -web for the  $L^{a,b,c}$  theories.

number of fields in the gauge theory. Specifically we have

$$N_f = \frac{1}{2} \sum_{i,j \in \text{legs}}^4 \left| \det \begin{pmatrix} p_i & q_i \\ p_j & q_j \end{pmatrix} \right|.
\tag{3.7}$$

This formula comes from computing intersection numbers of 3-cycles in the mirror geometry [60]. In fact the four adjacent legs each contribute  $a, b, c, d$  fields, which are simply the GLSM charges, up to sign. The two cross terms then contribute  $c - a$  and  $b - c$  fields, giving

$$N_f = a + 3b.
\tag{3.8}$$

To summarise this section so far, the gauge theory for  $L^{a,b,c}$  has  $N_g = a + b$  gauge groups, and  $N_f = a + 3b$  fields in total. In Section 3.2 we will determine the multiplicities of the fields, as well as their baryon and flavour charges, using the results of the previous section.

### 3.1 The sub-family $L^{a,b,a}$

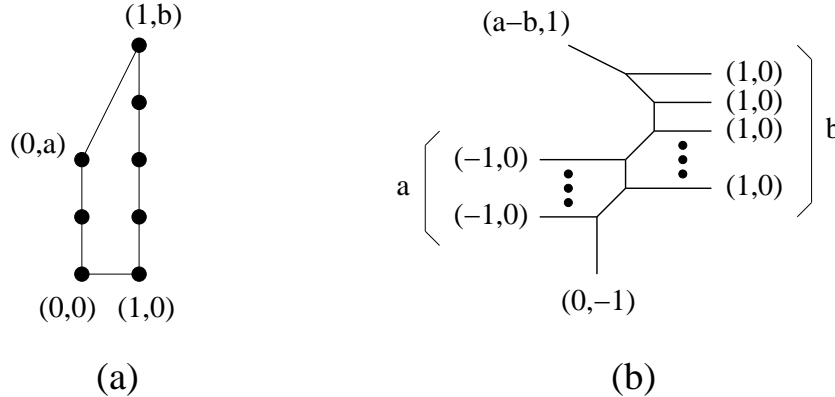
The observant reader will have noticed that the charges in (3.6) are not always primitive. In fact this is a consequence of orbifold singularities in the Sasaki–Einstein space. In such singular cases one can have some number of lattice points, say  $m - 1$ , on the edges of the

toric diagram, and then the corresponding leg of the  $(p, q)$  web in (3.6) is not a primitive vector. One should then really write the primitive vector, and associate to that leg the label, or multiplicity,  $m$ . Each leg of the  $(p, q)$  web corresponds to a circle on  $Y$  which is a locus of singular points if  $m > 1$ , where  $m$  gives the order of the orbifold group. Nevertheless, the charges (3.6) as written above give the correct numbers of fields. In fact this discussion is rather similar to the classification of *compact* toric orbifolds in [61], where each facet is assigned a positive integer label that describes the order of an orbifold group. Moreover, the non-primitive vectors are then used in the symplectic quotient.

Rather than explain this point in generality, it is easier to give an example. Here we consider the family  $L^{a,b,a}$ , which are always singular if one of  $a$  or  $b$  is greater than 1. In fact by the  $SL(2; \mathbb{Z})$  transformation

$$\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \tag{3.9}$$

one maps the toric diagram to an isosceles trapezoid as shown in Figure 4.a.



**Figure 4:** a) Toric diagram and b)  $(p, q)$  web for the  $L^{a,b,a}$  sub-family.

Notice that there are  $a - 1$  lattice points on one external edge, and  $b - 1$  lattice points on the opposite edge. This is indicative of the singular nature of these spaces. Correspondingly, the  $(p, q)$  web has non-primitive charges (or else one can assign positive labels  $b$  and  $a$  to primitive charges). Indeed, the leg with label  $a$  is just the submanifold obtained by setting  $Z_3 = Z_4 = 0$ . On  $Y$  the D-terms, modulo the  $U(1)$  gauge transformation, just give a circle  $S^1$ . However, the  $U(1)$  group factors through  $a$  times, due to the charges of  $Z_1$  and  $Z_2$  being both equal to  $a$ . This means that the  $S^1$  is a locus of  $\mathbb{Z}_a$  orbifold singularities. Obviously, similar remarks apply to  $Z_1 = Z_2 = 0$ . The singular nature of these spaces will also show up in the gauge theory: certain types of fields will be absent, and there will be adjoints,

as well as bifundamentals. The  $L^{a,b,a}$  family will be revisited in Section 6.2.3, where we will construct their associated brane tilings, gauge theories and compare the computations performed in the field theories with those in the dual supergravity backgrounds.

### 3.2 Quantum numbers of fields

Let us denote the distinguished fields as

$$X_1 = Y \quad X_2 = U_1 \quad X_3 = Z \quad X_4 = U_2 . \quad (3.10)$$

In the limit  $c = d = p$  we have that  $L^{a,b,c}$  reduces to a  $Y^{p,q}$ . Specifically,  $b = p + q$  and  $a = p - q$ . Then this notation for the fields coincides with that of reference [13]. In particular, the  $U_i$  become a doublet under the  $SU(2)$  isometry/flavour symmetry in this limit.

The multiplicities of the fields can be read off from the results of the last section:

$$\text{mult}[Y] = b \quad \text{mult}[U_1] = d \quad \text{mult}[Z] = a \quad \text{mult}[U_2] = c . \quad (3.11)$$

This accounts for  $2(a + b)$  fields, which means that there are  $b - a$  fields missing. The  $(p, q)$  web suggests that there are two more fields  $V_1$  and  $V_2$  with multiplicities

$$\text{mult}[V_1] = c - a \quad \text{mult}[V_2] = b - c . \quad (3.12)$$

Indeed, this also reproduces a  $Y^{p,q}$  theory in the limit  $c = d$ , where the fields  $V_i$  again become an  $SU(2)$  doublet.

It is now simple to work out which toric divisors these additional fields are associated to. As will be explained later, each divisor must appear precisely  $b$  times in the list of fields. Roughly, this is because there are necessarily  $a + 3b - (a + b) = 2b$  terms in the superpotential, and every field must appear precisely twice by the quiver toric condition [62]. From this we deduce that we may view the remaining fields  $V_1, V_2$  as “composites” — more precisely, we identify them with unions of adjacent toric divisors  $D_i \cup D_j$  in the Calabi–Yau, or equivalently in terms of supersymmetric 3–submanifolds in the Sasaki–Einstein space:

$$\begin{aligned} V_1 : & \quad \Sigma_3 \cup \Sigma_4 \\ V_2 : & \quad \Sigma_2 \cup \Sigma_3 . \end{aligned} \quad (3.13)$$

We may now compute the baryon and flavour charges of all the fields. The charges for the fields  $V_1, V_2$  can be read off from their relation to the divisors  $\Sigma_A$  above. We summarise the various quantum numbers in Table 1.

Field	SUSY submanifold	number	$U(1)_B$	$U(1)_{F_1}$	$U(1)_{F_2}$
$Y$	$\Sigma_1$	$b$	$a$	1	0
$U_1$	$\Sigma_2$	$d$	$-c$	0	$l$
$Z$	$\Sigma_3$	$a$	$b$	0	$k$
$U_2$	$\Sigma_4$	$c$	$-d$	-1	$-k-l$
$V_1$	$\Sigma_3 \cup \Sigma_4$	$b-c$	$c-a$	-1	$-l$
$V_2$	$\Sigma_2 \cup \Sigma_3$	$c-a$	$b-c$	0	$k+l$

**Table 1:** Charge assignments for the six different types of fields present in the general quiver diagram for  $L^{a,b,c}$ .

Notice that the  $SL(2; \mathbb{Z})$  transformation (3.4) that shifts  $k$  and  $l$  is equivalent to redefining the flavour symmetry

$$U(1)_{F_2} \rightarrow U(1)_{F_2} - tU(1)_B + taU(1)_{F_1} . \quad (3.14)$$

Note also that each toric divisor appears precisely  $b$  times in the table. This fact automatically ensures that the linear traces vanish

$$\text{Tr}U(1)_B = 0 \quad \text{and} \quad \text{Tr}U(1)_{F_1} = \text{Tr}U(1)_{F_2} = 0 , \quad (3.15)$$

as must be the case. As a non-trivial check of these assignments, one can compute that the cubic baryonic trace vanishes as well

$$\text{Tr}U(1)_B^3 = ba^3 - dc^3 + ab^3 - cd^3 + (b-c)(c-a)^3 + (c-a)(b-c)^3 = 0 . \quad (3.16)$$

### 3.3 The geometry

In this subsection we summarise some aspects of the geometry of the toric Sasaki–Einstein manifolds  $L^{a,b,c}$ . First, we recall the metrics [8], and how these are associated to the toric singularities discussed earlier [9]. We also discuss supersymmetric submanifolds, compute their volumes, and use these results to extract the R-charges of the dual field theory.

The local metrics were given in [8] in the form

$$\begin{aligned} ds^2 = & \frac{\rho^2 dx^2}{4\Delta_x} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\Delta_x}{\rho^2} \left( \frac{\sin^2 \theta}{\alpha} d\phi + \frac{\cos^2 \theta}{\beta} d\psi \right)^2 \\ & + \frac{\Delta_\theta \sin^2 \theta \cos^2 \theta}{\rho^2} \left( \frac{\alpha - x}{\alpha} d\phi - \frac{\beta - x}{\beta} d\psi \right)^2 + (d\tau + \sigma)^2 \end{aligned} \quad (3.17)$$

where

$$\begin{aligned}
\sigma &= \frac{\alpha - x}{\alpha} \sin^2 \theta d\phi + \frac{\beta - x}{\beta} \cos^2 \theta d\psi \\
\Delta_x &= x(\alpha - x)(\beta - x) - \mu, \quad \rho^2 = \Delta_\theta - x \\
\Delta_\theta &= \alpha \cos^2 \theta + \beta \sin^2 \theta .
\end{aligned} \tag{3.18}$$

Here  $\alpha, \beta, \mu$  are *a priori* arbitrary constants. These local metrics are Sasaki–Einstein which can be equivalently stated by saying that the metric cone  $dr^2 + r^2 ds^2$  is Ricci–flat and Kähler, or that the four–dimensional part of the metric (suppressing the  $\tau$  direction) is a local Kähler–Einstein metric of positive curvature. These local metrics were also found in [9]. The coordinates in (3.17) have the following ranges:  $0 \leq \theta \leq \pi/2$ ,  $0 \leq \phi \leq 2\pi$ ,  $0 \leq \psi \leq 2\pi$ , and  $x_1 \leq x \leq x_2$ , where  $x_1, x_2$  are the smallest two roots of the cubic polynomial  $\Delta_x$ . The coordinate  $\tau$ , which parameterises the orbits of the Reeb Killing vector  $\partial/\partial\tau$  is generically non–periodic. In particular, generically the orbits of the Reeb vector field do not close, implying that the Sasaki–Einstein manifolds are in general *irregular*.

The metrics are clearly toric, meaning that there is a  $U(1)^3$  contained in the isometry group. Three commuting Killing vectors are simply given by  $\partial/\partial\psi, \partial/\partial\phi, \partial/\partial\tau$ . The global properties of the spaces are then conveniently described in terms of those linear combinations of the vector fields that vanish over real codimension two fixed point sets. This will correspond to toric divisors in the Calabi–Yau cone — see *e.g.* [12]. It is shown in [8] that there are precisely *four* such vector fields, and in particular these are  $\partial/\partial\phi$  and  $\partial/\partial\psi$ , vanishing on  $\theta = 0$  and  $\theta = \pi/2$  respectively, and two additional vectors

$$\ell_i = a_i \frac{\partial}{\partial\phi} + b_i \frac{\partial}{\partial\psi} + c_i \frac{\partial}{\partial\tau} \quad i = 1, 2 \tag{3.19}$$

which vanish over  $x = x_1$  and  $x = x_2$ , respectively. The constants are given by [8]

$$\begin{aligned}
a_i &= \frac{\alpha c_i}{x_i - \alpha}, \quad b_i = \frac{\beta c_i}{x_i - \beta}, \\
c_i &= \frac{(\alpha - x_i)(\beta - x_i)}{2(\alpha + \beta)x_i - \alpha\beta - 3x_i^2}.
\end{aligned} \tag{3.20}$$

In order that the corresponding space is globally well–defined, there must be a linear relation between the four Killing vector fields

$$a \ell_1 + b \ell_2 + c \frac{\partial}{\partial\phi} + d \frac{\partial}{\partial\psi} = 0 \tag{3.21}$$

where  $(a, b, c, d)$  are relatively prime integers. It is shown in [8] that for appropriately chosen coefficients  $a_i, b_i, c_i$  there are then countably infinite families of complete Sasaki–Einstein manifolds.

The fact that there are four Killing vector fields that vanish on codimension 2 submanifolds implies that the image of the Calabi–Yau cone under the moment map for the  $T^3$  action is a four faceted polyhedral cone in  $\mathbb{R}^3$  [12]. Using the linear relation among the vectors (3.21) one can show that the normal vectors to this polyhedral cone satisfy the relation

$$a v_1 - c v_2 + b v_3 - (a + b - c) v_4 = 0 \quad (3.22)$$

where  $v_A$ ,  $A = 1, 2, 3, 4$  are the primitive vectors in  $\mathbb{R}^3$  that define the cone. Note that we have listed the vectors according to the order of the facets of the polyhedral cone. As explained in [9], it follows that, for  $a, b, c$  relatively prime, the Sasaki–Einstein manifolds arise from the symplectic quotient

$$\mathbb{C}^4 // (a, -c, b, -a - b + c) \quad (3.23)$$

which is precisely the gauged linear sigma model considered in the previous subsection.

The volume of the Sasaki–Einstein manifolds/orbifolds is given by [8]

$$\text{vol}(Y) = \frac{\pi^2}{2k\alpha\beta} (x_2 - x_1)(\alpha + \beta - x_1 - x_2)\Delta\tau \quad (3.24)$$

where here  $k = \text{gcd}(a, b)$  and

$$\Delta\tau = \frac{2\pi k |c_1|}{b} . \quad (3.25)$$

This can also be written as

$$\text{vol}(Y) = \frac{\pi^3 (a + b)^3}{8abcd} W \quad (3.26)$$

where  $W$  is a root of certain quartic polynomial given in [8]. This shows that the central charges of the dual conformal field theory will be generically *quartic irrational*.

In order to compute the R–charges from the metric, we need to know the volumes of the four supersymmetric 3–submanifolds  $\Sigma_A$ . These volumes were not given in [8] but it is straightforward to compute them. We obtain

$$\begin{aligned} \text{vol}(\Sigma_1) &= \frac{\pi}{k} \left| \frac{c_1}{a_1 b_1} \right| \Delta\tau & \text{vol}(\Sigma_2) &= \frac{\pi}{k\beta} (x_2 - x_1) \Delta\tau \\ \text{vol}(\Sigma_3) &= \frac{\pi}{k} \left| \frac{c_2}{a_2 b_2} \right| \Delta\tau & \text{vol}(\Sigma_4) &= \frac{\pi}{k\alpha} (x_2 - x_1) \Delta\tau . \end{aligned} \quad (3.27)$$



We can now complete the charge assignments of all the fields in the quiver by giving their R-charges purely from the geometry. The charges of the distinguished fields  $Y, U_1, Z, U_2$  are obtained from the geometry using the formula

$$R[X_A] = \frac{\pi \text{vol}(\Sigma_A)}{3 \text{vol}(Y)} , \quad (3.28)$$

while those of the  $V_1, V_2$  fields are simply deduced from (3.13). In particular

$$R[V_1] = R[Z] + R[U_2] \quad R[V_2] = R[Z] + R[U_1] . \quad (3.29)$$

It will be convenient to note that the constants  $\alpha, \beta, \mu$  appearing in  $\Delta_x$  are related to its roots as follows

$$\begin{aligned} \mu &= x_1 x_2 x_3 \\ \alpha + \beta &= x_1 + x_2 + x_3 \\ \alpha\beta &= x_1 x_2 + x_1 x_3 + x_2 x_3 , \end{aligned} \quad (3.30)$$

where  $x_3$  is the third root of the cubic, and  $x_3 \geq x_2 \geq x_1 \geq 0$ . Using the volumes in (3.27), we then obtain the following set of R-charges

$$\begin{aligned} R[Y] &= \frac{2}{3x_3}(x_3 - x_1) & R[U_1] &= \frac{2\alpha}{3x_3} \\ R[Z] &= \frac{2}{3x_3}(x_3 - x_2) & R[U_2] &= \frac{2\beta}{3x_3} . \end{aligned} \quad (3.31)$$

To obtain explicit expressions, one should now write the constants  $x_i, \alpha, \beta$  in terms of the integers  $a, b, c$ . This can be done, using the equations (9) in [8]. We have

$$\frac{x_1(x_3 - x_1)}{x_2(x_3 - x_2)} = \frac{a}{b} \quad (3.32)$$

$$\frac{\alpha(x_3 - \alpha)}{\beta(x_3 - \beta)} = \frac{c}{d} . \quad (3.33)$$

Notice that  $\alpha = \beta$  implies  $c = d$ , as claimed in [8]. Combining these two equations with (3.30), one obtains a complicated system of quartic polynomials, which in principle can be solved. However, we will proceed differently. Our aim is simply to show that the resulting R-charges will match with the  $a$ -maximisation computation in the field theory. Therefore, using the relations above, we can write down a system of equations involving the R-charges

and the integers  $a, b, c, d$ . We obtain the following:

$$\begin{aligned}\frac{R[Y](2 - 3R[Y])}{R[Z](2 - 3R[Z])} &= \frac{a}{b} \\ \frac{R[U_1](2 - 3R[U_1])}{R[U_2](2 - 3R[U_2])} &= \frac{c}{d} \\ \frac{3}{4}(R[U_1]R[U_2] - R[Y]R[Z]) + R[Z] + R[Y] &= 1 \\ R[Y] + R[U_1] + R[Z] + R[U_2] &= 2.\end{aligned}\tag{3.34}$$

With the aid of a computer program, one can check that the solutions to this system are given in terms of roots of various quartic polynomials involving  $a, b, c$ . For the case of  $L^{a,b,a}$  the polynomials reduce to quadratics and the R-charges can be given in closed form. These in fact match precisely with the values that we will compute later using  $a$ -maximisation, as well as  $Z$ -minimisation. Therefore we won't record them here.

In the general case, instead of giving the charges in terms of unwieldy quartic roots, we can more elegantly show that the system (3.34) can be recast into an equivalent form which is obtained from  $a$ -maximisation. In order to do so, we can use the last equation to solve for  $R[U_1]$ . Expressing the first three equations in terms of  $R[U_2] = x$ ,  $R[Y] = y$  and  $R[z] = z$ , we have

$$\begin{aligned}b(2 - 3y) + az(3z - 2) &= 0 \\ c(x + y - 2)(3x + 3y - 4) - (a + b - c)(x - z)(3x - 3z - 2) &= 0 \\ 3x^2 - 4y + 2(z + 2) + x(3y - 3z - 6) &= 0.\end{aligned}\tag{3.35}$$

Interestingly, the third equation does not involve any of the parameters. For later comparison with the results coming from  $a$ -maximisation, it is important to find a way to reduce this system of three coupled quadratic equations in three variables to a standard form. The simplest way of doing so is to 'solve' for one of the variables  $x$ ,  $y$  or  $z$  and two of the parameters. A particularly simple choice is to solve for  $y$ ,  $a$  and  $b$ . The simplicity follows from the fact that it is possible to use the third equation to solve for  $y$  and the parameters then appear linearly in all the equations. Doing this, we obtain

$$\begin{aligned}a &= \frac{c(3x - 2)(3x(x - z - 2) + 2(2 + z))}{(3x - 4)^2(x - z)} \\ b &= \frac{cz(3z - 2)}{(3x - 3z - 2)(x - z)} \\ y &= \frac{-2(2 + z) - 3x(x - z - 2)}{(3x - 4)}.\end{aligned}\tag{3.36}$$

This system of equations is equivalent to the original one, and is the one we will compare with the results of  $a$ -maximisation.

Of course, one could also compute these R-charges using  $Z$ -minimisation [46]. The algebra encountered in tackling the minimisation problem is rather involved, but it is straightforward to check agreement of explicit results on a case by case basis.

## 4. Superpotential and gauge groups

In the previous sections we have already described how rather generally one can obtain the number of gauge groups, and the field content of a quiver whose vacuum moduli space should reproduce the given toric variety. In particular, we have listed the multiplicities of every field and their complete charge assignments, namely their baryonic, flavour, and R-charges. In the following we go further and predict the form of the superpotential as well as the nature of the gauge groups, that is, the types of nodes appearing in the quivers.

### 4.1 The superpotential

First, we recall that in [48] a general formula was derived relating the number of gauge groups  $N_g$ , the number of fields  $N_f$ , and the number of terms in the superpotential  $N_W$ . This follows from applying Euler's formula to a brane tiling that lives on the surface of a 2-torus, and reads

$$N_W = N_f - N_g . \tag{4.1}$$

Using this we find that the number of superpotential terms for  $L^{a,b,c}$  is  $N_W = 2b$ . Now we use the fact each term in the superpotential  $W$  must be

$$\cup_{A=1}^4 \Sigma_A . \tag{4.2}$$

In fact this is just the canonical class of  $X$  – a standard result in toric geometry. One can justify the above form as follows. Each term in  $W$  is a product of fields, and each field is associated to a union of toric divisors. The superpotential has R-charge 2, and is uncharged under the baryonic and flavour symmetries. This is true, using the results of Section 2 and (4.2).

A quick inspection of Table 1 then allows us to identify three types of monomials that may appear in the superpotential

$$W_q = \text{Tr } YU_1ZU_2 \quad W_{c_1} = \text{Tr } YU_1V_1 \quad W_{c_2} = \text{Tr } YU_2V_2 . \tag{4.3}$$

Furthermore, their number is uniquely fixed by the multiplicities of the fields, and the fact that  $N_W = 2b$ . The schematic form of the superpotential for a general  $L^{a,b,c}$  quiver theory is then

$$W = 2[a W_q + (b - c) W_{c_1} + (c - a) W_{c_2}] . \quad (4.4)$$

In the language of dimer models, this is telling us the types of vertices in the brane tilings [48]. In particular, in each fundamental domain of the tiling we must have  $2a$  four-valent vertices,  $2(b - c)$  three-valent vertices of type 1, and  $2(c - a)$  three-valent vertices of type 2.

## 4.2 The gauge groups

Finally, we discuss the nature of the  $N_g = a + b$  gauge groups of the gauge theory, i.e. we determine the types of nodes in the quiver. This information, together with the above, will be used to construct the brane tilings. First, we will identify the allowed types of nodes, and then we will determine the number of times each node appears in the quiver.

The allowed types of nodes can be deduced by requiring that at any given node

1. the total baryonic and flavour charge is zero:  $\sum_{i \in \text{node}} U(1)_i = 0$
2. the beta function vanishes:  $\sum_{i \in \text{node}} (R_i - 1) + 2 = 0$
3. there are an even number of legs.

These requirements are physically rather obvious. The first property is satisfied if we construct a node out of products (and powers) of the building blocks of the superpotential (4.3). Moreover, using (2.29), this also guarantees that the total R-charge at the node is even.

Imposing these three requirements turns out to be rather restrictive, and we obtain *four* different types of nodes that we list below:

$$A : U_1 Y V_1 \cdot U_1 Y V_1 \quad B : V_2 Y V_1 \cdot U_1 Y U_2 \quad D : U_2 Y V_2 \cdot U_2 Y V_2 \quad C : U_1 Y U_2 Z \quad (4.5)$$

Next, we determine the number of times each node appears in the quiver. Denote these numbers  $n_A, n_B, n_D, 2n_C$  respectively. Taking into account the multiplicities of all the fields imposes six linear relations. However, it turns out that these *do not* uniquely fix the number of different nodes. We have

$$\begin{aligned} n_C &= a \\ n_B + 2n_A &= 2(b - c) \\ n_B + 2n_D &= 2(c - a) . \end{aligned} \quad (4.6)$$

Although the number of fields and schematic form of superpotential terms are fixed by the geometry, the number of  $A$ ,  $B$  and  $D$  nodes are not. We can then have different types of quivers that are nevertheless described by the same toric singularity. This suggests that the theories with different types of nodes are related by Seiberg dualities. We will show that this is the case in Section 6.

It is interesting to see what happens for the  $L^{a,b,a}$  geometries discussed in Section 3.1. In this case  $c - a = 0$  (the case that  $b - c = 0$  is symmetric with this) and the theory has some peculiar properties. Recall that this corresponds to a linear sigma model with charges  $(a, -a, b, -b)$ . These theories have no  $V_2$  fields, while the  $b - a$   $V_1$  fields have zero baryonic charge, and must therefore be *adjoints*. Moreover, from (4.6), we see that  $n_B = n_D = 0$ , so that there aren't any  $B$  and  $D$  type nodes, while there are  $b - a$   $A$ -type nodes. In terms of tilings, these theories are then just constructed out of  $C$ -type quadrilaterals and  $A$ -type hexagons. We will consider in detail these models in Section 6.2.3.

Finally, we note that the general conclusions derived for  $L^{a,b,c}$  quivers in Sections 4.1 and 4.2 are based on the underlying assumption that we are dealing with a generic theory (*i.e.* one in which the R-charges of different types of fields are not degenerate). It is always possible to find at least one generic phase for a given  $L^{a,b,c}$ , and thus the results discussed so far apply. Non-generic phases can be generated by Seiberg duality transformations. In these cases, new types of superpotential interactions and quiver nodes may emerge, as well as new types of fields. This was for instance the case for the toric phases of the  $Y^{p,q}$  theories [15].

## 5. R-charges from $a$ -maximisation

A remarkable check of the AdS/CFT correspondence consists of matching the gauge theory computation of R-charges and central charge with the corresponding calculations of volumes of the dual Sasaki-Einstein manifold and supersymmetric submanifolds on the gravity side. This is perhaps the most convincing evidence that the dual field theory is the correct one. Since explicit expressions for the Sasaki-Einstein metrics are available, it is natural to attempt such a check. Actually, the volumes of toric manifolds and supersymmetric submanifolds thereof can also be computed from the toric data [46], without using a metric. This gives a third independent check that everything is indeed consistent.

Here we will calculate the R-charges and central charge  $a$  for an arbitrary  $L^{a,b,c}$  quiver gauge theory using  $a$ -maximisation. From the field theory point of view, initially, there are six different R-charges, corresponding to the six types of bifundamental fields  $U_1, U_2, V_1, V_2,$

$Y$  and  $Z$ . Since the field theories are superconformal, these R-charges are such that the beta functions for the gauge and superpotential couplings vanish. Using the constraints (4.3) and (4.5) it is possible to see that these conditions always leave us with a three-dimensional space of possible R-charges. This is in precise agreement with the fact that the non-R abelian global symmetry is  $U(1)^3 \simeq U(1)_F^2 \times U(1)_B$ . It is convenient to adopt the parametrization of R-charges of Section 3.3:

$$\begin{aligned} R[U_1] &= x - z & R[U_2] &= 2 - x - y \\ R[V_1] &= 2 - x - y + z & R[V_2] &= x \\ R[Y] &= y & R[Z] &= z. \end{aligned} \tag{5.1}$$

This guarantees that all beta functions vanish. Using the multiplicities in Table 1, we can check that  $\text{tr}R(x, y, z) = 0$ . This is expected, since this trace is proportional to the sum of all the beta functions. In addition, the trial  $a$  central charge can be written as

$$\begin{aligned} \text{tr}R^3(x, y, z) &= \frac{1}{3} [a(9(2-x)(x-z)z - 2) + b(9xy(2-x-y) - 2) \\ &\quad + 9(b-c)yz(2x+y-z-2)]. \end{aligned} \tag{5.2}$$

The R-charges are determined by maximising (5.2) with respect to  $x$ ,  $y$  and  $z$ . This corresponds to the following equations

$$\begin{aligned} \partial_x \text{tr}R^3(x, y, z) &= 0 = -3by(2x+y-2z-2) + 3z(a(2-2x+z-2cy)) \\ \partial_y \text{tr}R^3(x, y, z) &= 0 = -3b(x-z)(x+2y-z-2) - 3cz(2x+2y-z-2) . \\ \partial_z \text{tr}R^3(x, y, z) &= 0 = -3a(x-2)(x-2z) + 3(b-c)yz(2x+y-2z-2) \end{aligned} \tag{5.3}$$

It is straightforward to show that this system of equations is equivalent to (3.36). In fact, proceeding as in Section 3.3, we reduce (5.3) to an equivalent system by ‘solving’ for  $y$ ,  $a$  and  $b$ . In this case, there are three solutions, although only one of them does not produce zero R-charges for some of the fields, and indeed corresponds to the local maximum of (5.2). This solution corresponds to the following system of equations

$$\begin{aligned} a &= \frac{c(3x-2)(3x(x-z-2) + 2(2+z))}{(3x-4)^2(x-z)} \\ b &= \frac{cz(3z-2)}{(3x-3z-2)(x-z)} \\ y &= \frac{-2(2+z) - 3x(x-z-2)}{(3x-4)} \end{aligned} \tag{5.4}$$

which is identical to (3.36). We conclude that, for the entire  $L^{a,b,c}$  family, the gauge theory computation of R-charges and central charge using a-maximisation agrees precisely with the

values determined using geometric methods on the gravity side of the AdS/CFT correspondence.

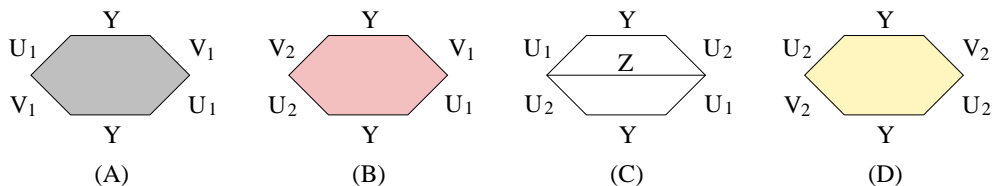
## 6. Constructing the gauge theories using brane tilings

In Sections 3 and 4 we derived detailed information regarding the gauge theory on D3-branes transverse to the cone over an arbitrary  $L^{a,b,c}$  space. Table 1 gives the types of fields along with their multiplicities and global  $U(1)$  charges, (4.3) presents the possible superpotential interactions and (4.5) and (4.6) give the types of nodes in the quiver along with some constraints on their multiplicities.

This information is sufficient for constructing the corresponding gauge theories. We have used it in Section 5 to prove perfect agreement between the geometric and gauge theory computations of R-charges and central charges. Nevertheless, it is usually a formidable task to combine all these pieces of information to generate the gauge theory. In this section we introduce a simple set of rules for the construction of the gauge theories for the  $L^{a,b,c}$  geometries. In particular, our goal is to find a simple procedure in the spirit of the ‘impurity idea’ of [13, 15].

Our approach uses the concept of a **brane tiling**, which was introduced in [48], following the discovery of the connection between toric geometry and dimer models of [47]. Brane tilings encode both the quiver diagram and the superpotential of gauge theories on D-branes probing toric singularities. Because of this simplicity, they provide the most suitable language for describing complicated gauge theories associated with toric geometries. We refer the reader to [48] for a detailed explanation of brane tilings and their relation to dimer models.

All the conditions of Sections 3 and 4 can be encoded in the properties of four elementary building blocks. These blocks are shown in Figure 5. We denote them A, B, C and D, following the corresponding labeling of gauge groups in (4.5). It is important to note that a C hexagon contains two nodes of type C.



**Figure 5:** The four building blocks for the construction of brane tilings for  $L^{a,b,c}$ .

Every edge in the elementary hexagons is associated with a particular type of field. These edge labels fully determine the way in which hexagons can be glued together along their edges to form a periodic tiling. The quiver diagram and superpotential can then be read off from the resulting tiling using the results of [48]. The elementary hexagons automatically incorporate the three superpotential interactions of (4.3). The number of A, B, C and D hexagons is  $n_A$ ,  $n_B$ ,  $n_C$  and  $n_D$ , respectively. Taking their values as given by (4.6), the multiplicities in Table 1 are reproduced.

Following the discussion in Section 4, the number of A, B, C and D hexagons is not fixed for a given  $L^{a,b,c}$  geometry. There is a one parameter space of solutions to (4.6), which we can take to be indexed by  $n_D$ . It is possible to go from one solution of (4.6) to another one by decreasing the number of B hexagons by two and introducing one A and one D, *i.e.*  $(n_A, n_B, n_C, n_D) \rightarrow (n_A + 1, n_B - 2, n_C, n_D + 1)$ . We show in the next section that this freedom in the number of each type of hexagon is associated with Seiberg duality.

## 6.1 Seiberg duality and transformations of the tiling

We now study Seiberg duality [63] transformations that produce ‘toric quivers’<sup>5</sup>. We can go from one toric quiver to another one by applying Seiberg duality to the so-called **self-dual** nodes. These are nodes for which the number of flavours is twice the number of colours, thus ensuring that the rank of the dual gauge group does not change after Seiberg duality. Such nodes are represented by squares in the brane tiling [48]. Hence, for  $L^{a,b,c}$  theories, we only have to consider dualizing C nodes. Seiberg duality on a self-dual node corresponds to a **local** transformation of the brane tiling [48]. This is important, since it means that we can focus on the sub-tilings surrounding the nodes of interest in order to analyse the possible behavior of the tiling.

We will focus on cases in which the tiling that results from dualizing a self-dual node can also be described in terms of A, B, C and D hexagons. There are some cases in which Seiberg duality generates tilings that are not constructed using the elementary building blocks of Figure 5. In these cases, the assumption of the six types of fields being non-degenerate does not hold. We present an example of this non-generic case below, corresponding to  $L^{2,6,3}$ .

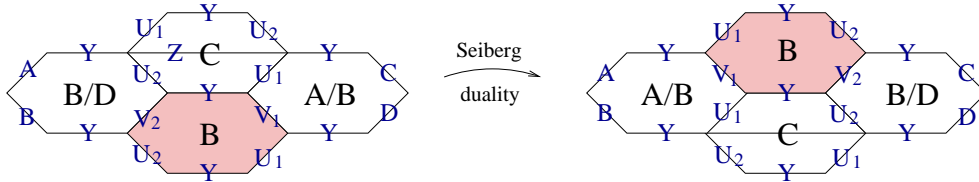
Leaving aside non-generic cases, we see that we only need to consider two possibilities. They are presented in Figures 6 and 7. Figure 6 shows the case in which dualization of the central square does not change the number of hexagons of each type in the tiling. The

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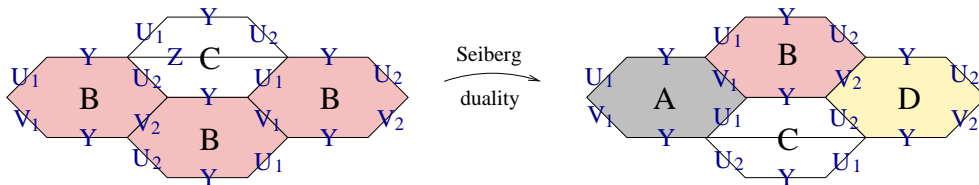
<sup>5</sup>This term was introduced in [36] and refers to quivers in which the ranks of all the gauge groups are equal. Toric quivers are a subset of the infinite set of Seiberg dual theories associated to a given toric singularity, *i.e.* it is possible to obtain quivers that are not toric on D-branes probing toric singularities.



new tiling is identical to the original one up to a shift. Figure 7 shows a situation in which dualization of the central square removes two type B hexagons and adds an A and a D.



**Figure 6:** Seiberg duality on a self-dual node that does not change the hexagon content.



**Figure 7:** Seiberg duality on a self-dual node under which  $(n_A, n_B, n_C, n_D) \rightarrow (n_A + 1, n_B - 2, n_C, n_D + 1)$ .

The operations discussed above leave the labels of the edges on the boundary of the sub-tiling invariant<sup>6</sup>. Hence, the types of hexagons outside the sub-tilings are not modified. The above discussion answers the question of how to interpret the different solutions of (4.6): they just describe Seiberg dual theories.

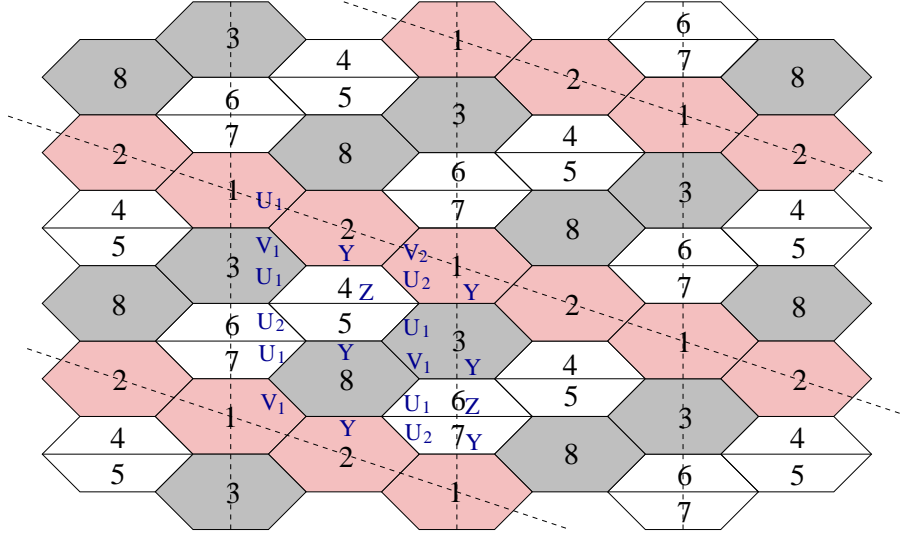
## 6.2 Explicit examples

Having presented the rules for constructing tilings for a given  $L^{a,b,c}$ , we now illustrate their application with several examples. We first consider  $L^{2,6,3}$ , which is interesting since it has eight gauge groups and involves A, B and C elementary hexagons. We also discuss how its tiling is transformed under the action of Seiberg duality. We then present tilings for  $L^{2,6,4}$ , showing how D hexagons are generated by Seiberg duality. Finally, we classify all sub-families whose brane tilings can be constructed using only two types of elementary hexagons. These theories are particularly simple and it is straightforward to match the geometric and gauge theory computations of R-charges and central charges explicitly. We analyse the  $L^{a,b,a}$  sub-family in detail, and present other interesting examples in the appendix.

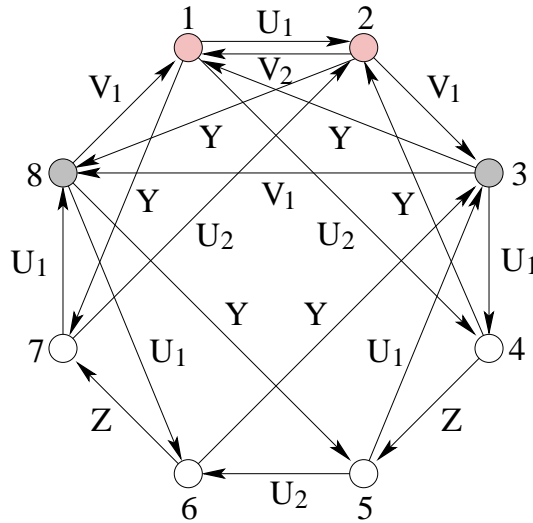
<sup>6</sup>There is a small subtlety in this argument: in some cases, identifications of faces due to the periodicity of the tiling can be such that the boundary of the sub-tiling is actually modified when performing Seiberg duality.

### 6.2.1 Gauge theory for $L^{2,6,3}$

Let us construct the brane tiling for  $L^{2,6,3}$ . We consider the  $(n_A, n_B, n_C, n_D) = (2, 2, 2, 0)$  solution to (4.6). Hence, we have two A, two B and two C hexagons. Using the gluing rules given by the edge labeling in Figure 5, it is straightforward to construct the brane tiling shown in Figure 8.



**Figure 8:** Brane tiling for  $L^{2,6,3}$ .



**Figure 9:** Quiver diagram for  $L^{2,6,3}$ .

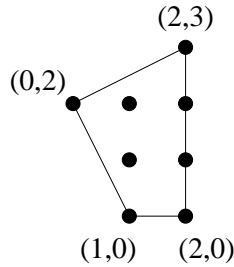
From the tiling we determine the quiver diagram shown in Figure 9. The multiplicities

of each type of field are in agreement with the values in Table 2. In addition, we can also read off the corresponding superpotential

$$\begin{aligned}
W = & Y_{31}U_{12}^{(1)}V_{23}^{(1)} - Y_{42}V_{23}^{(1)}U_{34}^{(1)} + Y_{42}V_{21}^{(2)}U_{12}^{(2)} + Y_{85}U_{53}^{(1)}V_{38}^{(1)} \\
& + Y_{17}U_{78}^{(1)}V_{81}^{(1)} - Y_{63}V_{38}^{(1)}U_{86}^{(1)} - Y_{28}V_{81}^{(1)}U_{12}^{(1)} - Y_{17}U_{72}^{(2)}V_{21}^{(2)} \\
& + Z_{45}U_{56}^{(2)}Y_{63}U_{34}^{(1)} - Z_{45}U_{53}^{(1)}Y_{31}U_{14}^{(2)} - Z_{67}U_{78}^{(1)}Y_{85}U_{56}^{(2)} + Z_{67}U_{72}^{(2)}Y_{28}U_{86}^{(1)}
\end{aligned} \tag{6.1}$$

where for simplicity we have indicated the type of  $U$  and  $V$  fields with a superscript and have used subscripts for the gauge groups under which the bifundamental fields are charged.

Having the brane tiling for a gauge theory at hand makes the derivation of its moduli space straightforward. The corresponding toric diagram is determined from the characteristic polynomial of the Kasteleyn matrix of the tiling [47, 48]. In this case, we obtain the toric diagram shown in Figure 10. This is an additional check of our construction.

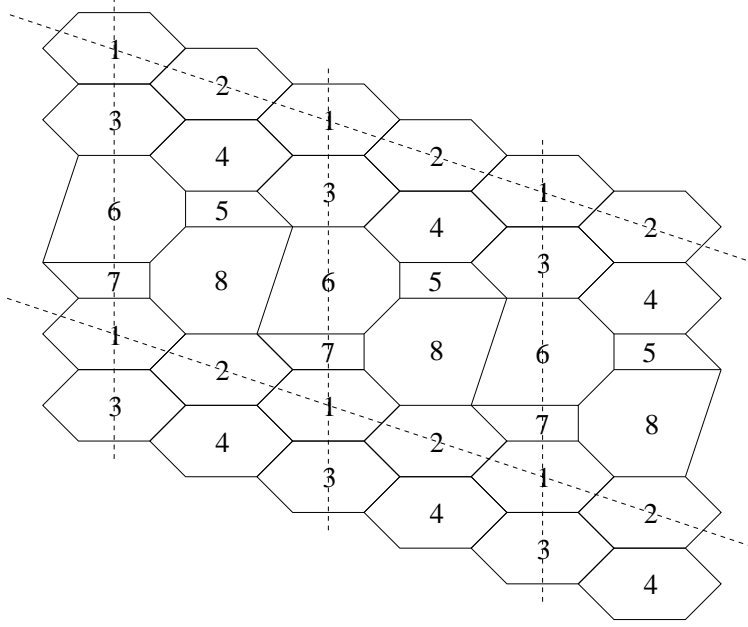


**Figure 10:** Toric diagram for  $L^{2,6,3}$  determined using the characteristic polynomial of the Kasteleyn matrix for the tiling in Figure 8.

Let us now consider how Seiberg duality on self-dual nodes acts on this tiling. Dualization of nodes 4 or 7 corresponds to the situation in Figure 6. The resulting tiling is identical to the original one up to an upward or downward shift, respectively. The situation is different when we dualize node 5 or 6. In these cases, Seiberg duality ‘splits apart’ the two squares corresponding to the C nodes forming C hexagons. Figure 11 shows the tiling after dualizing node 5. This tiling seems to violate the classification of possible gauge groups given in Section 4.3. In particular, some of the hexagons would have at least one edge corresponding to a  $Z$  field. As we discussed in Section 4.2, this is not a contradiction, but just indicates that we are in a non-generic situation in which some of the six types of fields are degenerate.

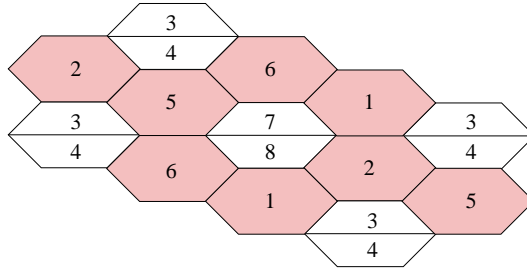
### 6.2.2 Generating D hexagons by Seiberg duality: $L^{2,6,4}$

We now construct brane tilings for  $L^{2,6,4}$ . This geometry is actually a  $\mathbf{Z}_2$  orbifold of  $L^{1,3,2}$ . This example illustrates how D hexagons are generated by Seiberg duality. We start with



**Figure 11:** Brane tiling for a Seiberg dual phase of  $L^{2,6,3}$ .

the  $(n_A, n_B, n_C, n_D) = (0, 4, 2, 0)$  solution to (4.6), whose corresponding tiling is shown in Figure 12.

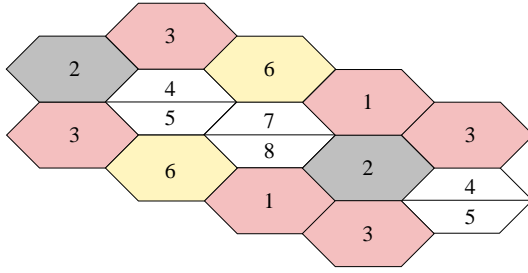


**Figure 12:** Brane tiling for  $L^{2,6,4}$ .

We see that all self-dual nodes are of the form presented in Figure 7. Seiberg duality on node 4 leads to a tiling with  $(n_A, n_B, n_C, n_D) = (1, 2, 2, 1)$ , which we show in Figure 13.

### 6.2.3 The $L^{a,b,a}$ sub-family

It is possible to use brane tilings to identify infinite sub-families of the  $L^{a,b,c}$  theories whose study is considerably simpler than the generic case. In particular, classifying the geometries whose corresponding tilings can be constructed using only two different types of hexagons is straightforward. We now proceed with such a classification.



**Figure 13:** Brane tiling for  $L^{2,6,4}$ .

Let us first consider those models that do not involve C type hexagons. These tilings consist entirely of ‘pure’ hexagons and thus correspond to orbifolds [47, 48]. We have already discussed them in Section 3, where we mentioned the case in which  $a$ , and thus  $n_C$ , is equal to zero. The orbifold action is determined by the choice of a fundamental cell (equivalently, by the choice of labeling of faces in the tiling).

We now focus on theories for which one of the two types of hexagons is of type C. There are only three possibilities of this form:

	<b>Hexagon types</b>	<b>Sub-family</b>	
$n_B = n_D = 0$	A and C	$L^{a,b,a}$	(6.2)
$n_A = n_D = 0$	B and C	$L^{a,b,\frac{a+b}{2}} = Y^{\frac{a+b}{2},\frac{a-b}{2}}$	
$n_A = n_B = 0$	C and D	$L^{a,b,b}$	

It is interesting to see that the  $Y^{p,q}$  theories emerge naturally from this classification of simple models. In addition to orbifolds and  $Y^{p,q}$ ’s, the only new family is that of  $L^{a,b,a}$ . The  $L^{a,b,b}$  family is equivalent to the latter by a trivial reordering of the GLSM charges, which in the gauge theory exchanges  $U_1 \leftrightarrow U_2$  and  $V_1 \leftrightarrow V_2$ .

Let us study the gauge theories for the  $L^{a,b,a}$  manifolds. These theories were first studied in [64] using Type IIA configurations of relatively rotated NS5–branes and D4–branes (see also [65, 66, 67] for early work on these models). The simplest example of this family is the SPP theory [7], which in our notation is  $L^{1,2,1}$ , and has GLSM charges  $(1, -1, 2, -2)$ . The brane tiling for this theory was constructed in [48] and indeed uses one  $A$  and one  $C$  building block. We will see that it is possible to construct the entire family of gauge theories. We have already shown in Section 5 that the computation of R–charges and central charge using a–maximisation agrees with the geometric calculation for an arbitrary  $L^{a,b,c}$ . We now compute these values explicitly for this sub–family and show agreement with the results derived using the metric [8] and the toric diagram [46]. These types of checks have already been performed for another infinite sub–family of the  $L^{a,b,c}$  geometries, namely the  $Y^{p,q}$  manifolds, in [13]

and [46].

Let us first compute the volume of  $L^{a,b,a}$  from the metric. The quartic equation in [8] from which the value of  $W$  entering (3.26) is determined becomes

$$W^2 \left( \frac{1024 a^2 (a-b)^2 b^2}{(a+b)^6} + \frac{64 (2a-b)(2b-a)W}{(a+b)^2} - 27 W^2 \right) = 0 . \quad (6.3)$$

Taking the positive solution to this equation, we obtain

$$\text{vol}(L^{a,b,a}) = \frac{4\pi^3}{27 a^2 b^2} \left[ (2b-a)(2a-b)(a+b) + 2 (a^2 - ab + b^2)^{3/2} \right] . \quad (6.4)$$

There is an alternative geometric approach to computing the volume of  $L^{a,b,a}$  which uses the toric diagram instead of the metric:  $Z$ -minimisation. This method for calculating the volume of the base of a toric cone from its toric diagram was introduced in [46]. For 3-complex dimensional cones, it corresponds to the minimisation of a two variable function  $Z[y, t]$ . The toric diagram for  $L^{a,b,a}$ , as we presented in Section 3.1, has vertices

$$[0, 0] \quad [1, 0] \quad [1, b] \quad [0, a] . \quad (6.5)$$

We then have

$$Z[y, t] = 3 \frac{y(b-a) + 3a}{t(y-3)y(t-y(b-a)-3a)} . \quad (6.6)$$

The values of  $t$  and  $y$  that minimize  $Z[y, t]$  are

$$t_{min} = \frac{1}{2}(a+b+w) \quad y_{min} = \frac{2a-b-w}{a-b} \quad (6.7)$$

where

$$w = \sqrt{a^2 - ab + b^2} . \quad (6.8)$$

Computing  $\text{vol}(Y) = \pi^3 Z_{min}/3$ , we recover (6.4), which was determined using the metric. We now show how this result is reproduced by a gauge theory computation. The unique solution to (4.6) for the case of  $L^{a,b,a}$  is  $(n_A, n_B, n_C, n_D) = (b-a, 0, a, 0)$ , so the brane tiling consists of  $(b-a)$  A and  $a$  C hexagons. This tiling is shown in Figure 14. First, we note that these theories are non-chiral. Figure 15 shows their quiver diagram.

Their superpotential can be easily read from the tiling in Figure 14. These models do not have  $V_2$  fields. Nevertheless, the parametrization of R-charges given in (5.1) is applicable to this case. Using it, we have

$$\text{tr} R^3(x, y, z) = \frac{1}{3} [b(9y(z-x)(x+y-z-2) - 2) + a(9(2-x-y)(x+y-z)z - 2)] . \quad (6.9)$$

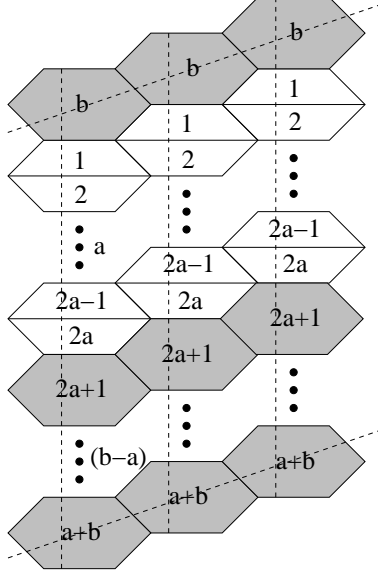


Figure 14: Brane tiling for  $L^{a,b,a}$ .

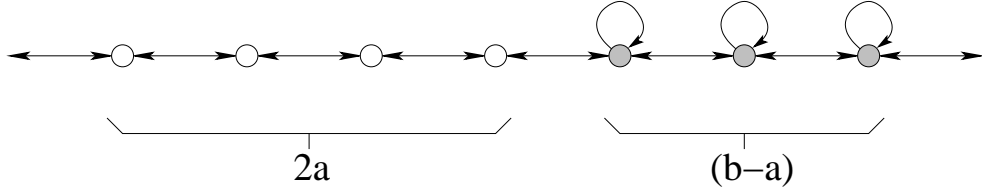


Figure 15: Quiver diagram for  $L^{a,b,a}$ . It consists of  $2a$  C nodes and  $(b-a)$  A nodes. The last node is connected to the first one by a bidirectional arrow.

Maximising (6.9), we obtain the R-charges

$$\begin{aligned}
 R[U_1] &= \frac{1}{3} \frac{b-2a+w}{b-a} & R[U_2] &= \frac{1}{3} \frac{2b-a-w}{b-a} \\
 R[V_1] &= \frac{2}{3} \frac{2b-a-w}{b-a} & R[Y] &= \frac{1}{3} \frac{b-2a+w}{b-a} . \\
 R[Z] &= \frac{1}{3} \frac{2b-a-w}{b-a}
 \end{aligned} \tag{6.10}$$

The central charge  $a$  is then

$$a(L^{aba}) = \frac{27}{16} \frac{a^2 b^2}{\left[ (2b-a)(2a-b)(a+b) + 2(a^2 - ab + b^2)^{3/2} \right]^{-1}} \tag{6.11}$$

which reproduces (6.4) on using  $a = \pi^3/4 \text{ vol}(Y)$ .

## 7. Conclusions

The main result of this paper is the development of a combination of techniques which allow one to extract the data defining a (superconformal) quiver gauge theory purely from toric and Sasaki–Einstein geometry. We have shown that the brane tiling method provides a rather powerful organizing principle for these theories, which generically have very intricate quivers. We emphasise that, in the spirit of [46], our results *do not* rely on knowledge of explicit metrics, and are therefore applicable in principle to an arbitrary toric singularity. It is nevertheless interesting, for a variety of reasons, to know the corresponding Sasaki–Einstein metrics in explicit form.

For illustrating these general principles, we have discussed an infinite family of toric singularities denoted  $L^{a,b,c}$ . These generalise the  $Y^{p,q}$  family, which have been the subject of much attention; the corresponding  $L^{a,b,c}$  Sasaki–Einstein metrics have been recently constructed in [8] (see also [9]). The main input into constructing these theories came from the geometrical data, which strongly restricts the allowed gauge theories. Subsequently, the brane tiling technique provides a very elegant way of organizing the data of the gauge theory. In contrast to the quivers and superpotentials, which are very complicated to write down in general, it is comparatively easy to describe the building blocks of the brane tiling associated to any given  $L^{a,b,c}$ . We have computed the exact R–charges of the entire family using three different methods and found perfect agreement of the results, thus confirming the validity of our construction.

There are several possible directions for future research suggested by this work. One obvious question is how to extend these results to more general toric geometries. The smooth  $L^{a,b,c}$  spaces are always described by toric diagrams with four external legs, implying that the Sasaki–Einstein spaces (when smooth) have topology  $S^2 \times S^3$ . In fact, they are the most general such metrics. This is reflected in the field theory by the existence of only one global baryonic  $U(1)$  symmetry, giving a total of  $U(1)^3$  non–R global symmetries for these theories. But of course this is not the most general toric geometry one might consider. One class of examples which do not fit into the  $L^{a,b,c}$  family are the  $X^{p,q}$  [29] spaces, whose toric diagrams have five external legs. These include for instance the complex cone over  $dP_2$ , and the Sasaki–Einstein metrics are not known explicitly. It would be very interesting (and probably a very difficult feat) to write down the gauge theories dual to any toric diagram. However, this may be possible: Thanks to such advances as  $Z$ –minimisation [46], and from the results of this paper, we now know that it is possible to read off a large amount of information from the toric data alone. In principle one could use this, along with the brane tilings, to obtain non-trivial information about the gauge theory dual to any toric geometry.



One could also proceed by analogy to the progress made on the  $Y^{p,q}$  spaces. Since we know the  $L^{a,b,c}$  metrics, it is possible to seek deformations of these theories on both the supergravity and gauge theory sides. Indeed there have been many exciting new developments thanks to the study of the  $Y^{p,q}$  theories and their cascading solutions. For the  $L^{a,b,c}$  family, the study of the supergravity side of the cascades has been initiated in [9]. Here we have not attempted to elaborate on these deformations, but doing so would be very interesting. Additionally, the recent progress in Sasaki-Einstein spaces opens up the question of what other metrics we can derive for other infinite families of spaces.

Although the toric data specifies much information, it is still important in many cases to know the metric. This is true for example if one wishes to compute the Kaluza-Klein spectrum of a given background. Another example is given by the problem of constructing non-conformal deformations, or cascading solutions. It will be interesting to see whether the program initiated in [46] can be developed further, so that for instance one can extract information on the spectrum of the Laplacian operator purely from the toric data.

Another interesting question that arises in the context of our work is how to prove the equivalence of  $a$ -maximisation and  $Z$ -minimisation. Although both procedures yield the same numbers in all known examples (of which there are infinitely many), it is nevertheless not *a priori* obvious that this should be the case. In particular, the function one extremizes during  $a$ -maximisation is simply a cubic, whereas the function used in  $Z$ -minimisation is a rational function. Furthermore,  $Z$ -minimisation is a statement about Sasaki-Einstein spaces in any dimension, while  $a$ -maximisation appears to be unique to four dimensions. Intuitively, we suspect (thanks to AdS/CFT) that these two procedures are equivalent, but this statement is far from obvious. It would be wonderful to have a general proof that  $Z$ -minimisation is the same as  $a$ -maximisation.

## 8. Acknowledgements

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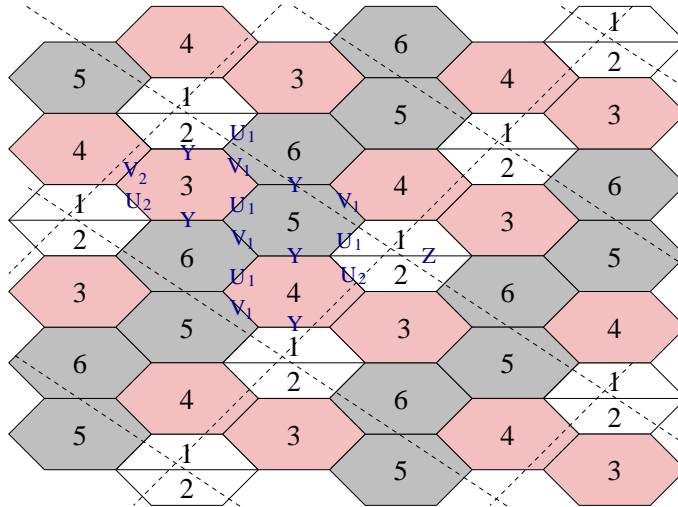
00-96515. A.H. is additionally supported by the BSF American-Israeli Bi-National Science Foundation and a DOE OJI Award. D.V. is also supported by the MIT Praecis Presidential Fellowship.

## 9. Appendix: More examples

In this appendix, we include additional examples that illustrate the simplicity of our approach to the construction of brane tilings and gauge theories.

### 9.1 Brane tiling and quiver for $L^{1,5,2}$

Figure 16 shows a brane tiling for  $L^{1,5,2}$  with  $(n_A, n_B, n_C, n_D) = (2, 2, 1, 0)$ .



**Figure 16:** Brane tiling for  $L^{1,5,2}$ .

The corresponding quiver diagram is shown Figure 17.

The toric diagram computed from the tiling according to the prescription in [47, 48] is presented in Figure 18.

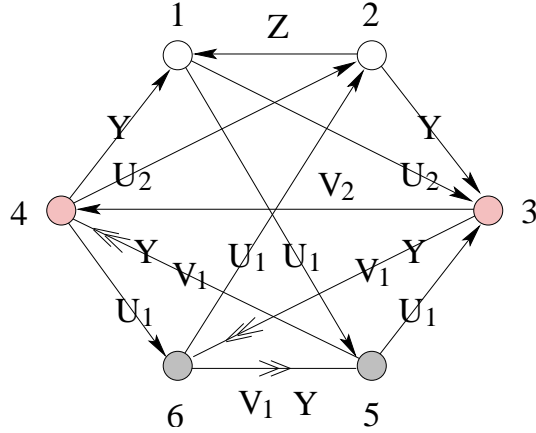


Figure 17: Quiver diagram for  $L^{1,5,2}$ .

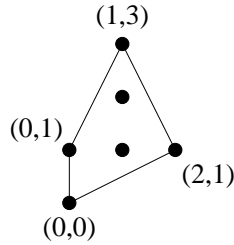


Figure 18: Toric diagram for  $L^{1,5,2}$ .

## 9.2 Brane tiling and quiver for $L^{1,7,3}$

The brane tiling for  $L^{1,7,3}$  corresponding to  $(n_A, n_B, n_C, n_D) = (2, 4, 1, 0)$  is presented in Figure 19.

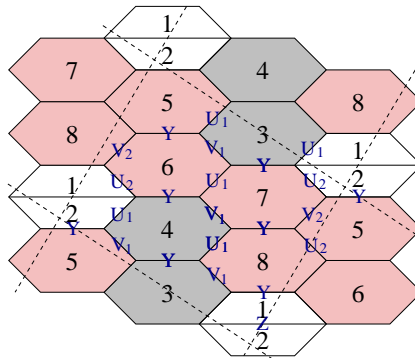
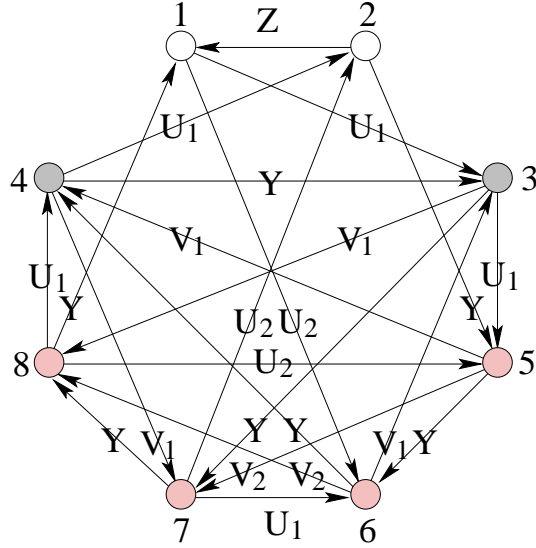


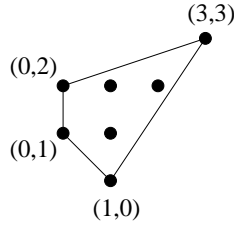
Figure 19: Brane tiling for  $L^{1,7,3}$ .

Figure 20 shows the quiver diagram for this phase.

The toric diagram is given in Figure 21.



**Figure 20:** Quiver diagram for  $L^{1,7,3}$ .



**Figure 21:** Toric diagram for  $L^{1,7,3}$ .

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