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Gauge theory and calibrated geometry, I

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0.1. Introduction

The geometry of submanifolds is intimately related to the theory of functions and vector bundles. It has been of fundamental importance to find out how those two objects interact in many geometric and physical problems. A typical example of this relation is that the Picard group of line bundles on an algebraic manifold is isomorphic to the group of divisors, which is generated by holomorphic hypersurfaces modulo linear equivalence. A similar correspondence can be made between the K-group of sheaves and the Chow ring of holomorphic cycles. There are two more very recent examples of such a relation. The mirror symmetry in string theory has revealed a deeper phenomenon involving special Lagrangian cycles (cf. [SYZ]). On the other hand, C. Taubes has shown that the Seiberg-Witten invariant coincides with the Gromov-Witten invariant on any symplectic 4-manifolds.

In this paper, we will show another natural interaction between Yang-Mills connections, which are critical points of a Yang-Mills action associated to a vector bundle, and minimal submanifolds, which have been studied extensively for years in classical differential geometry and the calculus of variations.

Let M be a manifold with a Riemannian metric g. Let E be a vector bundle over M with a compact Lie group as its structure group. For instance, E may be a complex bundle and G is then a unitary group. A connection A of E can be given by specifying a covariant derivative

$$D_A : C^\infty(E) \mapsto C^\infty(E \otimes \Omega^1 M).$$

In local trivializations of E, D_A is of the form $d + a$ for some Lie(G)-valued 1-form a. The curvature of A is a Lie(G)-valued 2-form F_A, which is equal to D_A^2. As usual, it measures deviation from the symmetry of second derivatives. Such a connection A is Yang-Mills if $D_A^* F_A = 0$, where D_A^* is the adjoint of D_A with respect to the metric g. By the second Bianchi identity, we also have $D_A F_A = 0$. The system $D_A^* F_A = 0, D_A F_A = 0$ is called the Yang-Mills equation and is invariant under so-called gauge transformations, which are locally made of G-valued functions.

The moduli space of Yang-Mills connections is the quotient of the set of solutions of the Yang-Mills equation by the gauge group, which consists of all gauge transformations. It is well-known that this moduli space may not be compact. Given any sequence of Yang-Mills connections $\{A_i\}$ with a uniformly bounded L^2-norm of curvature, Uhlenbeck (also see [Na]) proved that by taking a subsequence if necessary, A_i converges to, modulo gauge transformations, a Yang-Mills connection A in smooth topology outside a closed subset $S_b(\{A_i\})$ of Hausdorff codimension at least 4. In fact, for any compact $K \subset M$, $S_b(\{A_i\}) \cap K$ has finite $(n-4)$-dimensional Hausdorff measure. Furthermore, by taking subsequences if necessary, we may assume that as measures, $|F_{A_i}|^2 dV_g$ converges weakly to $|F_A|^2 dV_g + \Theta H^{n-4} \lfloor S_b(\{A_i\})$, where $\Theta \geq 0$ is a function and is called the multiplicity of $S_b(\{A_i\})$, and $H^{n-4} \lfloor S_b(\{A_i\})$ is the $(n-4)$-dimensional Hausdorff measure restricted to $S_b(\{A_i\})$. The set $S_b(\{A_i\})$ is the union of two closed subsets S_b and $S([A])$, where $S([A])$ consists of all points in M where the $(n-4)$-dimensional density of $|F_A|^2 dV_g$ is positive, and S_b is the closure of $S_b(\{A_i\}) \backslash S([A])$. One can show that $\Theta H^{n-4} \lfloor S_b(\{A_i\})$ coincides with $\Theta H^{n-4} \lfloor S_b$ and $S([A])$ has vanishing $(n-4)$-dimensional Hausdorff measure. Presumably, $S([A])$ is the singular set of A modulo gauge transformations. We will call S_b with multiplicity Θ the blow-up locus of $\{A_i\}$ converging to A. If M is a 4-dimensional compact manifold, the blow-up locus S_b consists of finitely many points, $S([A]) = \emptyset$ and the limiting connection A can be extended to be a Yang-Mills connection on the whole manifold with smaller L^2-norm of curvature [Uh1]. In particular, it follows that the moduli space of anti-self-dual instantons on a 4-manifold (see the following for the definition) can be compactified by adding all smaller anti-self-dual instantons together with finitely many points on M. This compactified moduli space plays a fundamental role in the theory of Donaldson invariants.

With M of higher dimension, little has been known about the blow-up locus S_b itself. Without further knowledge on the structure of S_b, one can not achieve a reasonable compactification of the moduli space of Yang-Mills connections as we had in the case of 4-manifolds. The main theme of this paper is to show that blow-up loci of Yang-Mills connections have natural geometric structures and introduce a natural compactification for moduli space of anti-self-dual instantons on higher dimensional manifolds by adding cycles with appropriate geometric structure. We believe that such a compactification will play an important role in our searching for new invariants of Donaldson type for higher dimensional manifolds.

In this paper, we will first show that *any blow-up locus S_b is rectifiable*; *i.e., except for a subset of $(n-4)$-dimensional Hausdorff measure zero, it is contained in a countable union of C^1-smooth submanifolds of dimension $n-4$* (cf. Proposition 3.3.3). It is equivalent to saying that S_b *has a unique tangent space $T_x S_b$ for H^{n-4}-a.e. x in S_b*. It can be thought of as a rough regularity for S_b. We will show that S_b inherits a nice geometric structure (Chapter 4). We will also prove a removable singularity theorem for the limiting Yang-Mills connection A (Chapter 5). It follows that A can be extended smoothly to the complement of $S([A])$ modulo gauge transformations.

Let Ω be a closed differential form of degree $n-4$ on M. Then one can define a linear operator $T = - * \Omega \wedge$ acting on 2-forms, where $*$ denotes the Hodge operator of the metric g. A connection A is Ω-anti-self-dual if its curvature form F_A is annihilated by $T - \mathrm{Id}$. One can also define the Ω-anti-self-duality for more general connections (cf. Section 1.2). We observe that the

Ω-anti-self-duality implies the Yang-Mills equation and is invariant under gauge transformations. Furthermore, if M is a compact manifold without boundary and A is Ω-anti-self-dual, there is an *a priori* L^2 bound on F_A, which depends only on E, M and Ω.

We will prove that *if* $\{A_i\}$ *is a sequence of* Ω*-anti-self-dual instantons converging to* A *with blow-up locus* S_b *with multiplicity* Θ*, then* (S_b, Θ) *defines a closed integral current calibrated by* Ω (Theorem 4.2.3). In particular, Θ is integer-valued and Ω restricts to the induced volume form on each tangent space $T_x S_b$. If Ω has co-mass one, then this implies that the blow-up locus (S_b, Θ) is area-minimizing (cf. [HL]). Known regularity theorems in geometry measure theory further imply that S is the closure of a smooth submanifold calibrated by Ω. We will also prove a removable singularity theorem for any stationary Yang-Mills connections (Theorem 5.2.1). Particularly, this implies that *the limiting connection* A *extends to become a smooth connection on* $M \backslash S$ *for a closed subset* S *with vanishing* $(n-4)$*-dimensional Hausdorff measure* $H^{n-4}(S) = 0$ (Theorem 5.2.2).

Now we can introduce a natural compactification of the moduli space $\mathcal{M}_{\Omega,E}$ of Ω-anti-self-dual instantons of E on M.

A generalized Ω-anti-self-dual instanton is a pair (A, C) satisfying: (1) A is Ω-anti-self-dual on $M \backslash S(A)$ with $(n-4)$-dimensional Hausdorff measure $H^{n-4}(S(A)) = 0$; (2) $C = (S, \Theta)$ is a closed, integral current calibrated by Ω; (3) The second Chern class $C_2(E)$ of E is the same as $[C_2(A)] + [C_2(S, \Theta)]$, where $C_2(A)$ denotes the second Chern form of A and $[C_2(S, \Theta)]$ denotes the Poincaré dual of the homology class represented by the current (S, Θ). If the co-norm $|\Omega| \leq 1$, it follows from a result of F. Almgren that C is of the form $\sum_{a=1}^{l(C)} m_a C_a$ ($l(C)$ may be zero), such that each m_a is a positive integer and C_a is the closure of a submanifold calibrated by Ω.

Two generalized Ω-anti-self-dual instantons (A, C), (A', C') are equivalent if and only if $C = C'$ and there is a gauge transformation σ such that $\sigma(A) = A'$ on $M \backslash S(A) \cup S(A')$. We denote by $[A, C]$ the equivalence class represented by (A, C). Clearly, $[A, C] \in \mathcal{M}_{\Omega,E}$ if and only if $C = 0$ and A extends smoothly to M modulo a gauge transformation.

We define $\overline{\mathcal{M}}_{\Omega,E}$ to be the set of all equivalence classes of generalized Ω-anti-self-dual instantons of E.

The topology of $\overline{\mathcal{M}}_{\Omega,E}$ can be defined as follows: a sequence $[A_i, C_i]$ converges to $[A, C]$ in $\overline{\mathcal{M}}_{\Omega,E}$ if and only if (1) C_i converges to a closed integral current $C_\infty \subset C$ with respect to the weak topology for currents; (2) There are gauge transformations σ_i such that $\sigma_i(A_i)$ converges to A outside $S(A) \cup (C \backslash C_\infty)$. One can show that this topology makes $\overline{\mathcal{M}}_{\Omega,E}$ a Hausdorff space.

It follows from results in Chapters 4 and 5 that $\overline{\mathcal{M}}_{\Omega,E}$ *is compact with respect to this topology on any compact manifold* M (Theorem 6.1.1).

Clearly, $\overline{\mathcal{M}}_{\Omega,E}$ coincides with Uhlenbeck's compactification of the moduli space of anti-self-dual instantons on a 4-manifold M.

There are two important cases of such Ω-anti-self-dual instantons, which are worth being mentioned. In the first case, let (M,ω) be a complex m-dimensional Kähler manifold with the Kähler form ω. For any connection A, its curvature F_A decomposes into (2,0), (1,1) and (0,2)-parts $F_A^{2,0}$, $F_A^{1,1}$ and $F_A^{0,2}$. Put $\Omega = \frac{\omega^{m-2}}{(m-2)!}$. Then A is Ω-anti-self-dual if and only if $F_A^{0,2} = 0$ and $F_A^{1,1} \cdot \omega = 0$; i.e., A is a Hermitian-Yang-Mills connection. Combining Theorem 4.2.3 with a result of King or Harvey and Shiffman, we obtain that *blow-up loci of Hermitian-Yang-Mills connections are effective holomorphic integral cycles consisting of complex subvarieties of codimension two* (Theorem 4.3.3). Consequently, the compactification $\overline{\mathcal{M}}_{\frac{\omega^{m-2}}{(m-2)!},E}$ is the collection of equivalence classes $[A,C]$, where A is a Hermitian-Yang-Mills connection and C is a holomorphic integral cycle of complex dimension $m-2$. A holomorphic integral cycle is a formal sum of irreducible subvarities with positive coefficients. In view of the Donaldson-Uhlenbeck-Yau theorem that each (irreducible) Hermitian-Yang-Mills connection corresponds to a stable bundle, our generalized Hermitian-Yang-Mills connection $[A,C]$ should correspond to a stable sheaf. We would like to point out that our method can be applied to more general situations where the connections are not necessarily Hermitian-Yang-Mills. In order to conclude the holomorphic property of the blow-up locus, we only need that the $(0,2)$-part of curvature is much smaller compared to the full curvature tensor during the limiting process.

One of our motivations in this work is to carry out part of the program proposed in [DT] in a rigorous way. The program is to build up a gauge theory in higher dimensions. If one is less ambitious, one may just want to construct new holomorphic invariants for Calabi-Yau 4-folds in terms of complex anti-self-dual instantons. Complex anti-self-dual instantons are anti-self-dual with respect to appropriate 4-form Ω on M. Since they have been discussed before by Donaldson and Thomas, we refer the readers to [DT] and its references. In contrast to the previous case, we can prove that *blow-up loci of complex anti-self-dual instantons are Cayley cycles* (cf. Theorem 4.4.3). A Cayley cycle is a rectifiable set such that its tangent spaces are Cayley with respect to the given Kähler form and the holomorphic (4,0)-form on the underlying Calabi-Yau 4-fold (cf. [HL]). Notice that special Lagrangian submanifolds used in [SYZ] are special cases of Cayley cycles. This allows us to compactify the moduli space of complex anti-self-dual instantons in terms of Cayley cycles as we did in the above. Our methods may also be used to produce Cayley cycles, which seem to be elusive with our existing knowledge.

One implication of our results here is that minimal submanifolds can be considered as limiting solutions of the Yang-Mills equation. Bearing this in mind, we may expect to construct Yang-Mills connections from minimal submanifolds in general position. Indeed, near a minimal submanifold, one can construct approximated solutions of the Yang-Mills equation, whose curvature concentrates near the submanifold, in a suitable sense.

An outline of this paper is as follows: In Chapter 1, we give general discussions on Yang-Mills connections, particularly, Ω-anti-self-dual instantons. We analyze the Ω-anti-self-duality in a few important cases. In Chapter 2, we will derive a slight generalization of the mononicity formula of P. Price, a basic curvature estimate of K. Uhlenbeck. Then we apply Uhlenbeck's estimate to defining Chern-Weil forms for admissible Yang-Mills connections, which are kinds of singular connections. In Chapter 3, we prove rectifiability of blow-up loci. In Chapter 4, we prove that blow-up loci of anti-self-dual instantons are calibrated, closed integral currents. We will also analyze a few important special cases. Chapter 5 contains a new removable singularity theorem. In the last chapter, we discuss compactification of moduli space of anti-self-dual instantons and some related problems.

All the results of this paper can be generalized to the case of the Yang-Mills-Higgs equation. The details will appear elsewhere.

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1. Preliminaries

1.1. *The Yang-Mills functional.* Let $\pi : E \to M$ be a vector bundle of rank r over a differentiable manifold M with a Lie group G as its structure group. Then there is an open covering U_α of M, such that for each α, there is a local trivialization

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times \mathbb{R}^r \\ \pi \downarrow & & \downarrow p_1 \\ U_\alpha & \xrightarrow{=} & U_\alpha \end{array} \tag{1.1.1}$$

where p_1 is the projection onto the first factor. Note that each φ_α is a diffeomorphism. Furthermore, if $U_\alpha \cap U_\beta \neq \emptyset$, then one can write

$$\begin{aligned} \varphi_\alpha \cdot \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r &\longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^r, \\ (x, v) &\longrightarrow (x, g_{\alpha\beta}(x) v) \end{aligned} \tag{1.1.2}$$

for some function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G \subset \mathrm{GL}(r, \mathbb{R})$. Such a function $g_{\alpha\beta}$ is called a transition function of $\pi : E \to M$.

Examples we often use in this paper include complex vector bundles with a hermitian structure. For those bundles, the structure group G is $U(r/2)$.

A connection A on E is defined by specifying a covariant derivative

$$D = D_A : C^\infty(E) \to C^\infty(E \otimes \Omega^1 M).$$

Here $C^\infty(E)$ denotes the space of C^∞ sections of the bundle E. In a local trivialization $(U_\alpha, \varphi_\alpha)$ of E, the covariant derivative takes the form

$$D = d + A_\alpha, \quad A_\alpha : U_\alpha \to T^*U_\alpha \otimes \mathrm{Lie}(G), \tag{1.1.3}$$

where $\mathrm{Lie}(G)$ denotes the Lie algebra of the structure group G. If G is a unitary group, Condition 1.1.3 is equivalent to saying that D preserves the corresponding hermitian structure of E.

Note that A_α usually has no global description on M. If (U_β, φ_β) is another local trivialization and $g_{\alpha\beta}$ is the corresponding transition function, then

$$A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}. \tag{1.1.4}$$

The curvature of the connection A is determined by $D^2 : \Omega^0(E) \to \Omega^2(E)$. It is a tensor, usually denoted by F_A or simply F if no confusion occurs. Formally, the curvature tensor F_A can be written as

$$F_A = dA + A \wedge A,$$

which actually means that in each local trivialization $(U_\alpha, \varphi_\alpha)$,

$$F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha. \tag{1.1.5}$$

If $\{x_1, \cdots, x_n\}$ is a local coordinate system for U_α, then we have

$$A_\alpha = A_{\alpha,i} dx_i, \quad A_{\alpha,i} \in \mathrm{Lie}(G), \tag{1.1.6}$$

and

$$\begin{aligned} F_\alpha &= \frac{1}{2} \sum_{i,j} F_{\alpha,ij} dx_i \wedge dx_j, \\ F_{\alpha,ij} &= \frac{\partial A_{\alpha,j}}{\partial x_i} - \frac{\partial A_{\alpha,i}}{\partial x_j} + [A_{\alpha i}, A_{\alpha j}]. \end{aligned} \tag{1.1.7}$$

It follows that

$$F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}. \tag{1.1.8}$$

Hence, $F_A \in \Omega^2(\mathrm{End}(E))$.

From now on, we assume that G is a compact Lie group. We denote by $\langle\cdot,\cdot\rangle$ the Killing form of its Lie algebra Lie(G). If $G = U(r/2)$, we have

$$\langle a, b\rangle = -\operatorname{tr}(ab)\,, \qquad a, b \in u\,(r/2) = \operatorname{Lie}\,(U(r/2))\,. \tag{1.1.9}$$

We can easily extend $\langle\cdot,\cdot\rangle$ to a product on differential forms with values in Lie(G) as follows: if ϕ and ψ are differential forms of degree p and q, respectively, we define

$$\langle\phi,\psi\rangle = \sum_{i_1,\cdots,i_p,j_1,\cdots,j_q} \langle\phi_{i_1\cdots i_p}, \psi_{j_1\cdots j_q}\rangle dx_{i_1}\wedge\cdots\wedge dx_{i_p}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_q},$$

where

$$\begin{aligned}\phi &= \sum_{i_1,\cdots,i_p} \phi_{i_1\cdots i_p} dx_{i_1}\wedge\cdots\wedge dx_{i_p},\ \phi_{i_1\cdots i_p}\in \operatorname{Lie}(G),\\ \psi &= \sum_{j_1,\cdots,j_q} \psi_{j_1\cdots j_q} dx_{j_1}\wedge\cdots\wedge dx_{j_q},\ \psi_{j_1\cdots j_q}\in \operatorname{Lie}(G).\end{aligned}$$

Let us also fix a Riemannian metric g on M and denote by dV its volume form. Then we can define

$$|F_A|^2 = \sum_{i,j,k,l} \langle F_{\alpha ij}, F_{\alpha kl}\rangle g^{ik} g^{jl}$$

in terms of local trivializations, where (g_{ij}) is the metric tensor of g in $x_1,\ldots,x_n$ and (g^{ij}) is its inverse matrix.

The Yang-Mills functional of E is defined by

$$YM(A) = \frac{1}{4\pi^2}\int_M |F_A|^2 dV_g. \tag{1.1.10}$$

Let $\mathcal{G}$ be the gauge group of E, which consists of all smooth sections of the bundle $P(E)\times_{\mathrm{Ad}} G$ associated to the adjoint representation Ad of G, where $P(E)$ denotes the principal bundle of E. In terms of those trivializations $\{U_\alpha,\varphi_\alpha\}$, any σ in $\mathcal{G}$ is given by a family of G-valued functions σ_α satisfying:

$$\sigma_\alpha = g_{\alpha\beta}\cdot\sigma_\beta\cdot g_{\alpha\beta}^{-1} \ \text{ on } \ U_\alpha\cap U_\beta.$$

Let $\sigma(A)$ be the connection with $D_{\sigma(A)} = \sigma\cdot D_A\cdot\sigma^{-1}$; i.e., in each U_α,

$$D_{\sigma(A)} = d - d\sigma_\alpha\cdot\sigma_\alpha^{-1} + \sigma_\alpha\cdot A_\alpha\cdot\sigma_\alpha^{-1}.$$

Two smooth connections A_1 and A_2 of E are equivalent if there is a gauge transformation σ such that $A_2 = \sigma(A_1)$. A simple observation is: if there is a gauge transformation τ of E over an open-dense subset U such that $A_2 = \tau(A_1)$ in U, then τ extends to M and A_1, A_2 are equivalent.

One can easily show

$$YM(\sigma(A)) = YM(A), \tag{1.1.11}$$

where $\sigma(A)$ is the connection with $\sigma(D_A) = \sigma \cdot D_A \cdot \sigma^{-1}$.

The Euler-Lagrange equation of YM is

$$D_A^* F_A = 0, \tag{1.1.12}$$

where D_A^* denotes the adjoint operator of D_A with respect to the Killing form of G and the Riemannian metric g on M. On the other hand, by the second Bianchi identity, we have

$$D_A F_A = 0. \tag{1.1.13}$$

This, together with (1.1.12), implies that if A is a critical point of YM, then F_A is harmonic. In this case, we say the A is a Yang-Mills connection. It follows from (1.11) that if A is a Yang-Mills connection, so is $\sigma(A)$ for any gauge transformation σ. In other words, both equations (1.1.12) and (1.1.13) are invariant under the action of the gauge group.

1.2. *Anti-self-dual instantons.* In this section, we discuss a special class of solutions to the Yang-Mills equation (1.1.12). This class includes Hermitian-Yang-Mills connections on a Kähler manifold.

Let $\pi : E \mapsto M$ be a unitary bundle of complex rank r, and Ω be a closed form of degree $n-4$, where $n = \dim M$. As before, we fix a Riemannian metric g on M. We denote by $*$ the Hodge operator acting on forms with values in $\mathrm{Lie}(G)$; i.e., for any ϕ, ψ in $\Omega^p(\mathrm{Lie}(G))$, $*\psi \in \Omega^{n-p}(\mathrm{Lie}(G))$ and

$$\langle \phi \wedge *\psi \rangle = (\phi, \psi) dV_g, \tag{1.2.1}$$

where $(\cdot, \cdot)$ denotes the inner product on $\Omega^p(\mathrm{Lie}(G))$ induced by g and the Killing form $\langle \cdot, \cdot \rangle$.

Let tr be the standard trace on unitary matrices. For any unitary connection A of the bundle E over M, we have a well-defined $\mathrm{tr}(F_A)$ in $\Omega^2(M)$. It follows from the second Bianchi identity that $\mathrm{tr}(F_A)$ is in fact a closed 2-form. In fact, $\frac{\sqrt{-1}}{2\pi} \mathrm{tr}(F_A)$ represents the first Chern class $C_1(E)$ in $H^2(M, \mathbb{R})$.

Lemma 1.2.1. *Let A be a unitary connection of E over M such that $\mathrm{tr}(F_A)$ is a harmonic 2-form and*

$$\Omega \wedge (F_A - \frac{1}{r} \mathrm{tr}(F_A)\mathrm{Id}) = - * (F_A - \frac{1}{r} \mathrm{tr}(F_A)\mathrm{Id}), \tag{1.2.2}$$

then A is a Yang-Mills connection. Moreover, if M is a compact manifold without boundary,

$$\begin{aligned}(1.2.3)\qquad & \frac{1}{4\pi^2}\int_M |F_A|^2 dV_g - \frac{1}{4r\pi^2}\int_M |\operatorname{tr}(F_A)|^2 dV_g \\ = \; & \left(2C_2(E) - \frac{r-1}{r}C_1(E)^2\right)\cdot[\Omega],\end{aligned}$$

where $[\Omega]$ *denotes the cohomology class of* Ω.

Proof. Recall that $D_A^* = - * D_A *$, so that

$$\begin{aligned} D_A^* F_A &= \frac{1}{r}D_A^*(\operatorname{tr}(F_A)\mathrm{Id}) + *D_A(\Omega\wedge(F_A - \frac{1}{r}\operatorname{tr}(F_A)\mathrm{Id})) \\ &= \frac{1}{r}d^*(\operatorname{tr}(F_A))Id + *(\Omega\wedge(D_AF_A - \frac{1}{r}d(\operatorname{tr}(F_A))\mathrm{Id}) \\ &= 0.\end{aligned}$$

Hence, A is a Yang-Mills connection.

Next, multiplying (1.2.2) by F_A and integrating the resulting identity over M, we get

$$\begin{aligned} & \left(2C_2(E) - \frac{r-1}{r}C_1(E)^2\right)\cdot[\Omega] \\ = \; & \left(-Ch_2(E) + \frac{1}{r}C_1(E)^2\right)\cdot[\Omega] \\ = \; & \frac{1}{4\pi^2}\int_M \operatorname{tr}\left((F_A - \frac{1}{r}\operatorname{tr}(F_A)\mathrm{Id})\wedge(F_A - \frac{1}{r}\operatorname{tr}(F_A)\mathrm{Id})\right)\wedge\Omega \\ = \; & -\frac{1}{4\pi^2}\int_M \operatorname{tr}\left((F_A - \frac{1}{r}\operatorname{tr}(F_A)\mathrm{Id})\wedge *(F_A - \frac{1}{r}\operatorname{tr}(F_A)\mathrm{Id})\right) \\ = \; & \frac{1}{4\pi^2}\int_M \left(|F_A|^2 - \frac{1}{r}|\operatorname{tr}(F_A)|^2\right)dV_g,\end{aligned}$$

where $C_i(E)$ denotes the i^{th} Chern class of E and $Ch_i(E)$ denotes the i^{th} Chern character of E. Then (1.2.3) follows. □

In general, (1.2.2) is an over-determined system and has no solutions. However, if A is a solution of (1.2.2) and the co-norm of Ω is less than one, then A is an absolute minimizer of YM (cf. [HL]).

We will call any solution A of (1.2.2) an Ω-anti-self-dual instanton. If there is no possible confusion, we will simply say that A is an anti-self-dual instanton.

Remark 1. For a general compact Lie group, we can also define the Ω-anti-self-duality instantons simply as the solutions of $- * (F_A\wedge\Omega) = F_A$.

In the following and next two sections, we will give some solutions of (1.2.2).

Now we let M be a complex m-dimensional Kähler manifold with a Kähler metric g. As usual, we denote by $\omega = \omega_g$ the associated Kähler form. Then

$$dV_g = \frac{\omega^m}{m!}\,. \tag{1.2.4}$$

For any $U(r)$-connection A of a complex bundle E over M, we can decompose

$$F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2} \tag{1.2.5}$$

where $F_A^{0,2}$ denotes the $(0,2)$-part of F_A, $F_A^{2,0} = -(F_A^{0,2})^*$ and $F_A^{1,1}$ denotes the $(1,1)$-part of F_A.

By the Newlander-Nirenberg theorem, the vanishing of $F_A^{0,2}$ is equivalent to the integrability of $\bar\partial_A = D_A^{0,1}$, which is the $(0,1)$-part of D_A; that is, $\pi : E \to M$ has a holomorphic structure induced by $D_A^{0,1}$.

Since A is unitary,

$$F_A^{1,1} = -(F_A^{1,1})^* \quad \text{and} \quad |F_A|^2 = |F_A^{1,1}|^2 + 2|F_A^{0,2}|^2. \tag{1.2.6}$$

We introduce notation:

$$H_A = (F_A^{1,1} \cdot \omega), \qquad \overset{\circ}{F}{}_A^{1,1} = F_A^{1,1} - \frac{1}{m} H_A \omega \tag{1.2.7}$$

where $F_A^{1,1} \cdot \omega$ denotes the orthogonal projection of $F_A^{1,1}$ in the ω-direction.

Now we set

$$\Omega \;=\; \frac{\omega^{m-2}}{(m-2)!}$$

and we have:

Proposition 1.2.2. *The unitary connection A satisfies* (1.2.2) *if and only if* $\mathrm{tr}(F_A)$ *is harmonic and*

$$F_A^{0,2} = \frac{1}{r}\,\mathrm{tr}(F^{0,2})\mathrm{Id}, \quad H_A - \frac{1}{r}\,\mathrm{tr}(F_A^{1,1} \cdot \omega)\mathrm{Id} = 0.$$

If $C_1(E)$ is of the type $(1,1)$, *then A satisfies* (1.2.2) *if and only if*

$$F_A^{0,2} = 0, \quad H_A = \lambda \mathrm{Id},$$

where $\lambda = \frac{m(C_1(E)\cdot[\omega]^{m-1})}{r[\omega]^m}$.

Furthermore, A is the absolute minimum of the Yang-Mills functional if

$$F_A^{0,2} = 0, \quad H_A = \lambda \mathrm{Id}.$$

In this case,

$$YM(A) = (2C_2(E) - C_1(E)^2) \cdot \frac{[\omega]^{m-2}}{(m-2)!} + \frac{m(C_1(E)\cdot[\omega]^{m-1})^2}{r(m-1)![\omega]^m}, \tag{1.2.8}$$

where $[\omega]$ denotes the cohomology class represented by ω.

The proof follows from (1.2.3) and direct computations,

$$4\pi^2(2C_2(E) - C_1(E)^2)\cdot[\Omega] \tag{1.2.9}$$

$$= \int_M \left(|\overset{\circ}{F}{}_A^{1,1}|^2 - 2|F_A^{0,2}|^2 - \frac{m-1}{m}|H_A|^2 \right) \frac{\omega^m}{m!}$$

$$= \int_M \left(|F_A|^2 - 4|F_A^{0,2}|^2 - |H_A|^2 \right) \frac{\omega^m}{m!}.$$

Definition 1.2.3. We call A a *Hermitian-Yang-Mills connection* of E if A is unitary and

$$F_A^{1,1}\cdot\omega = \lambda \mathrm{Id}, \quad F_A^{0,2} = 0,$$

where $\lambda = \frac{m(C_1(E)\cdot[\omega]^{m-1})}{r[\omega]^m}$.

It follows from Proposition 1.2.2 that the action $YM(A)$ of any Hermitian-Yang-Mills connection A is uniquely determined by E and the Kähler class $[\omega]$.

As we said, each Hermitian-Yang-Mills connection gives rise to a natural holomorphic structure on E. In fact, by the Donaldson-Uhlenbeck-Yau theorem, irreducible Hermitian-Yang-Mills connections are in one-to-one correspondence with stable holomorphic bundles over M.

1.3. *Complex anti-self-dual instantons.* In this section, we will discuss complex anti-self-dual instantons on 4-dimensional Calabi-Yau manifolds, as well as instantons on manifolds with special holonomy. Complex anti-self-dual instantons were previously studied by both mathematicians and physicists, notably Donaldson and Thomas. We recommend the readers to the excellent reference [DT] for a more complete history.

First we assume that M is a Calabi-Yau 4-fold with a Kähler metric ω and a holomorphic $(4,0)$-form θ. Furthermore, we normalize

$$\theta\wedge\bar\theta = \frac{\omega^4}{4!}. \tag{1.3.1}$$

Note that such a θ is only unique modulo multiplication by units in $\mathbb{C}$.

We now choose Ω to be the parallel form

$$4\mathrm{Re}(\theta) + \frac{1}{2}\omega^2.$$

Then solutions of (1.2.2) can be described as follows.

Let h be a fixed hermitian metric of $\pi: E\to M$. Then one can define a complex Hodge operator

$$*_\theta : \Omega^{0,2}(\mathrm{End}(E)) \to \Omega^{0,2}(\mathrm{End}(E)) \tag{1.3.2}$$

by the equation

$$-\operatorname{tr}(\varphi\wedge *_\theta\psi) = (\varphi,\psi)\,\bar\theta, \quad \forall\varphi,\psi\in\Omega^{0,2}(\mathrm{End}(E)), \tag{1.3.3}$$

where $(\cdot,\cdot)$ denotes the inner product on $\Omega^{0,2}(\mathrm{End}(E))$ induced by the Kähler metric ω and the hermitian metric h on E. More explicitly, let $\{\varphi_1,\varphi_2,\varphi_3,\varphi_4\}$ be any unitary coframe of ω such that

$$\begin{aligned}\omega &= \frac{\sqrt{-1}}{2}\sum_i \varphi_i\wedge\bar\varphi_i,\\ \theta &= -\frac{1}{4}\varphi_1\wedge\varphi_2\wedge\varphi_3\wedge\varphi_4.\end{aligned}$$

Then

$$\begin{aligned}*_\theta(\sigma\bar\varphi_1\wedge\bar\varphi_2) &= \sigma^*\bar\varphi_3\wedge\bar\varphi_4,\\ *_\theta(\sigma\bar\varphi_1\wedge\bar\varphi_3) &= \sigma^*\bar\varphi_4\wedge\bar\varphi_2,\\ *_\theta(\sigma\bar\varphi_1\wedge\bar\varphi_4) &= \sigma^*\bar\varphi_2\wedge\bar\varphi_3,\end{aligned}$$

where $\sigma\in\mathrm{End}(E)$ and σ^* denotes its adjoint with respect to the Hermitian metric h on E.

Let A be an Ω-anti-self-dual connection, i.e., $\mathrm{tr}(F_A)$ is harmonic and

$$\Omega\wedge(F_A-\frac{1}{r}\,\mathrm{tr}(F_A)\mathrm{Id}) = -*(F_A-\frac{1}{r}\,\mathrm{tr}(F_A)\mathrm{Id}).$$

As in last section, we decompose

$$F_A = F_A^{2,0}+F_A^{1,1}+F_A^{0,2}.$$

Then by direct computations, one can show that the above is equivalent to the system

$$\begin{aligned}F_A^{1,1}\cdot\omega &= \lambda\mathrm{Id},\\ (d+d^*)\,\mathrm{tr}(F_A^{0,2}) &= 0,\\ (1+*_\theta)(F_A^{0,2}-\frac{1}{r}\,\mathrm{tr}(F_A^{0,2})\mathrm{Id}) &= 0,\end{aligned}\tag{1.3.4}$$

where

$$\lambda = \frac{4C_1(E)\cdot[\omega]^3}{r[\omega]^4}\tag{1.3.5}$$

Note that $*_\theta$ induces a decomposition of $H^{0,2}(M,\mathbb{C})$ into the self-dual part and anti-self-dual part. For any solution A of (1.3.4), $(1+*_\theta)\,\mathrm{tr}(F_A^{0,2})$ is harmonic and represents the self-dual part of $C_1(E)^{0,2}$. In particular, if $C_1(E)^{0,2}$ is anti-self-dual, then (1.3.4) reduces to

$$(1+*_\theta)F_A^{0,2}=0,\quad F_A^{1,1}\cdot\omega=\lambda\mathrm{Id}.\tag{1.3.6}$$

Following [DT], we say that A is a complex anti-self-dual instanton associated to (E,h), if $D_A h = 0$ and F_A satisfies (1.3.4).

For such a connection A, we observe

$$(1.3.7)\qquad [\theta]\wedge(2C_2(E)-\frac{r-1}{r}C_1(E)^2)=\frac{1}{4\pi^2}\int_M\left|F_A^{0,2}-\frac{1}{r}\operatorname{tr}(F_A^{0,2})\mathrm{Id}\right|^2\frac{\omega^4}{4!},$$

where $[\theta]$ denotes the cohomology class of θ in $H^4(M,\mathbb{C})$. Hence, (1.3.4) has no solutions if $[\theta]\wedge(2C_2(E)-\frac{r-1}{r}C_1(E)^2)$ is not a nonnegative real number. Since θ is only unique modulo multiplication by units in $\mathbb{C}$, for any given complex bundle $\pi: E\to M$, we should normalize θ such that

$$(1.3.8)\qquad [\theta]\wedge(2C_2(E)-\frac{r-1}{r}C_1(E)^2)\geq 0.$$

Clearly, if this is not zero, then such a θ is unique once ω is fixed. Moreover, if $C_1(E)^2\cdot[\theta]=0$ and $C_2(E)\cdot[\theta]=0$, then any complex anti-self-dual instanton of E is automatically a Hermitian-Yang-Mills connection, which can be thought of as holomorphically flat. The readers may compare it to the Chern number conditions on the flatness of Hermitian-Yangs-Mills connections.

The following proposition can be proved by straightforward computations.

Proposition 1.3.1. *Assume that θ is chosen so that* (1.3.8) *holds. Let A be any complex anti-self-dual instanton, then*

(1.3.9)

$$\begin{aligned} YM(A) &= (2C_2(E)-C_1(E)^2)\cdot\frac{[\omega]^2}{2}+\frac{4(C_1(E)\cdot[\omega]^3)^2}{6r[\omega]^4}\\ &\quad + 4(2C_2(E)-\frac{r-1}{r}C_1(E)^2)\cdot[\theta]+\frac{1}{r\pi^2}\int_M|\operatorname{tr}(F_A^{0,2})|^2dV_g.\end{aligned}$$

It follows that each complex anti-self-dual instanton attains the absolute minimum of the Yang-Mills functional. Moreover, its action depends only on E, $[\omega]$ and $[\theta]$.

Calabi-Yau 4-folds have holonomy group SU(4), which is contained in Spin(7). It turns out that complex anti-self-dual instantons can also be defined on Spin(7)-manifolds, which have Spin(7) as their holonomy group (cf. [DT]).

Now let (M,g) be a Spin(7)-manifold. Then Spin(7), acting on $\wedge^4(M)$, the space of 4-forms, leaves invariant a parallel 4-form $\Omega\neq 0$. More explicitly, in terms of an orthonormal basis $\{e_i\}$, the form

$$\begin{aligned}\Omega &= e_1\wedge e_2\wedge e_5\wedge e_6+e_1\wedge e_2\wedge e_7\wedge e_8+e_3\wedge e_4\wedge e_5\wedge e_6\\ &\quad + e_3\wedge e_4\wedge e_7\wedge e_8+e_1\wedge e_3\wedge e_5\wedge e_7-e_1\wedge e_3\wedge e_6\wedge e_8\\ &\quad - e_1\wedge e_4\wedge e_5\wedge e_7+e_2\wedge e_4\wedge e_6\wedge e_8-e_1\wedge e_4\wedge e_5\wedge e_8\end{aligned}$$

$$- e_1 \wedge e_4 \wedge e_6 \wedge e_7 - e_2 \wedge e_3 \wedge e_5 \wedge e_8 - e_2 \wedge e_3 \wedge e_6 \wedge e_7$$

$$+ e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_5 \wedge e_6 \wedge e_7 \wedge e_8.$$

If M happens to be a Calabi-Yau 4-fold, then it is the same as the one given above.

One observes that the operator $\phi \mapsto -*(\Omega \wedge \phi)$ is self-adjoint on 2-forms and has eigenvalues 1 and -3. Following [BKS], we let $\Omega^2_{21}(M, \mathrm{End}(E))$ and $\Omega^2_+(M, \mathrm{End}(E))$ be its eigenspaces corresponding to eigenvalues 1 and -3. Given any connection A, we write $F_A = F_{A,-} + F_{A,+}$ according to this decomposition. Then A solves (1.2.2) if and only if $F_{A,+} = \frac{1}{r}\operatorname{tr}(F_{A,+})Id$ and $\operatorname{tr}(F_A)$ is harmonic. Moreover, we have the identity

$$\begin{aligned}&(2C_2(E) - \frac{r-1}{r}C_1(E)^2) \cdot [\Omega] \\ = \quad & \frac{1}{4\pi^2}\int_M \left(|F_{A,-} - \frac{1}{r}\operatorname{tr}(F_{A,-})Id|^2 - 3|F_{A,+} - \frac{1}{r}\operatorname{tr}(F_{A,+})Id|^2\right) dV_g.\end{aligned}$$

Therefore:

PROPOSITION 1.3.2. *Let (M,g) be a* Spin(7)*-manifold, and A be an Ω-anti-self-dual instanton. Then $F_{A,+} = \frac{1}{r}\operatorname{tr}(F_{A,+})\mathrm{Id}$ and $YM(A)$ depends only on M and E. In fact,*

$$YM(A) = (2C_2(E) - \frac{r-1}{r}C_1(E)^2)\cdot[\Omega] + \frac{1}{4r\pi^2}\int_M |\operatorname{tr}(F_A)|^2 dV_g. \tag{1.3.10}$$

1.4. *Instantons on G_2-manifolds.* Let (M,g) be a Riemannian manifold with holonomy group being the exceptional group G_2. Then there is a parallel, hence closed, 3-form Ω which is invariant under the action of G_2. In terms of an orthonormal basis $\{e_i\}$, this form

$$\begin{aligned}\Omega \quad = \quad & e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5 - e_1 \wedge e_6 \wedge e_7 \\ & + e_2 \wedge e_4 \wedge e_6 + e_2 \wedge e_5 \wedge e_7 + e_3 \wedge e_4 \wedge e_7 - e_3 \wedge e_5 \wedge e_6.\end{aligned}$$

The operator $\phi \mapsto -*(\Omega \wedge \phi)$ is self-adjoint on 2-forms and has eigenvalues 1 and -2. We denote by $\Omega^2_{12}(M, \mathrm{End}(E))$ and $\Omega^2_+(M, \mathrm{End}(E))$ its eigenspaces corresponding to eigenvalues 1 and -2. Given any connection A, we write $F_A = F_{A,-} + F_{A,+}$ according to this eigenspace decomposition. Then A is an Ω-anti-self-dual instanton if and only if $F_{A,+} = \frac{1}{r}\operatorname{tr}(F_{A,+})Id$ and $\operatorname{tr}(F_A)$ is harmonic. Moreover, we have the identity

$$\begin{aligned}&(2C_2(E) - \frac{r-1}{r}C_1(E)^2) \cdot [\Omega] \\ = \quad & \frac{1}{4\pi^2}\int_M \left(|F_{A,-} - \frac{1}{r}\operatorname{tr}(F_{A,-})Id|^2 - 2|F_{A,+} - \frac{1}{r}\operatorname{tr}(F_{A,+})\mathrm{Id}|^2\right) dV_g.\end{aligned}$$

Therefore:

PROPOSITION 1.4.1. *Let (M, g) be a G_2-manifold, and A be an Ω-anti-self-dual instanton, where Ω is the above 3-form defining the G_2-structure. Then $F_{A,+} = \frac{1}{r}\operatorname{tr}(F_{A,+})\mathrm{Id}$ and $YM(A)$ depends only on M and E. In fact,*

$$YM(A) = (2C_2(E) - \frac{r-1}{r}C_1(E)^2)\cdot[\Omega] + \frac{1}{4r\pi^2}\int_M |\operatorname{tr}(F_A)|^2 dV_g. \tag{1.4.1}$$

2. Consequences of a monotonicity formula

In this chapter, we discuss Price's monotonicity formula, Uhlenbeck's curvature estimate and singular Yang-Mills connections of a certain type.

2.1. *A monotonicity formula.* In this section, we will derive a monotonicity formula for Yang-Mills connections, which is essentially due to Price [Pr]. This formula will be used in establishing cone properties of blow-up loci. Its proof follows Price's arguments with some modifications.

As before, M denotes a Riemannian manifold with a metric g and E is a vector bundle over M with compact structure group G.

For any connection A of E, its curvature form F_A takes values in $\mathrm{Lie}(G)$. The norm of F_A at any $p \in M$ is given by

$$|F_A|^2 = \sum_{i,j=1}^n \langle F_A(e_i, e_j), F_A(e_i, e_j)\rangle\,, \tag{2.1.1}$$

where $\{e_i\}$ is any orthonormal basis of T_pM, and $\langle\cdot,\cdot\rangle$ is the Killing form of $\mathrm{Lie}(G)$.

Let $\{\phi_t\}_{|t|<\infty}$ be a one-parameter family of diffeomorphisms of M, and A_0 be a fixed smooth connection of E and D be its associated covariant derivative. Then for any connection A, we can define a family of connections $\phi_t^*(A)$ as follows: Denote by τ_t^0 the parallel transport of E associated to A_0 along the path $\phi_s(x)_{0\le s\le t}$, where $x \in M$. More precisely, for any $u \in E_x$ over $x \in M$, let $\tau_s^0(u)$ be the section of E over the path $\phi_s(x)_{0\le s\le t}$ such that

$$D_{\frac{\partial}{\partial s}}\tau_s^0(u) = 0, \quad \tau_0^0(u) = u. \tag{2.1.2}$$

We define $A^t = \phi_t^*(A)$ by defining its associated covariant derivative

$$D_X^t v = (\tau_t^0)^{-1}\left(D_{d\phi_t(X)}(\tau_t^0(v))\right) \tag{2.1.3}$$

for any $X \in TM, v \in \Gamma(M, E)$, where $\Gamma(M, E)$ is the space of sections of E over M.

To see that A^t is indeed a connection, it is sufficient to check

$$\begin{aligned} D^t_X(fv)(x) &\\ = \quad & (\tau^0_t)^{-1}\left(D_{d\phi_t(X)}((\phi_t^{-1})^*f\cdot\phi_t)\tau^0_t(v))\right)(x)\\ = \quad & (\tau^0_t)^{-1}\left(f(x)D_{d\phi_t(X)}\tau^0_t(v)(\phi_t(x)) + d\phi_t(x)\left((\phi_t^{-1})^*f\right)\tau^0_t(v)(\phi_t(x))\right)\\ = \quad & f(x)D^t_Xv(x) + X(f)(x)v(x). \end{aligned}$$

The curvature form of A^t is then given by

$$F_{A^t}(X,Y) = (\tau^0_t)^{-1}\cdot F_A(d\phi_t(X), d\phi_t(Y))\cdot\tau^0_t. \tag{2.1.4}$$

It follows that

$$\begin{aligned} YM(A^t) &= \frac{1}{4\pi^2}\int_M |F_{A^t}|^2 dV_g \\ &= \frac{1}{4\pi^2}\int_M \sum_{i,j=1}^n |F_A(d\phi_t(e_i), d\phi_t(e_j))|^2(\phi_t(x))dV_g(x)\,, \end{aligned} \tag{2.1.5}$$

where dV_g denotes the volume form of g, and $\{e_i\}$ is any local orthonormal basis of TM.

By changing variables, we obtain

$$YM(A^t) = \frac{1}{4\pi^2}\int_M \sum_{i,j=1}^n |F_A(d\phi_t(e_i(\phi_t^{-1}(x))), d\phi_t(e_j(\phi_t^{-1}(x))))|^2\mathrm{Jac}(\phi_t^{-1})dV_g.$$

Let X be the vector field $\frac{\partial\phi_t}{\partial t}|_{t=0}$ on M. Then we deduce from the above that

$$\begin{aligned} &\frac{d}{dt}YM(A^t)|_{t=0}\\ &\qquad = -\frac{1}{4\pi^2}\int_M\left(|F_A|^2\mathrm{div}X + 4\sum_{i,j=1}^n\langle F_A([X,e_i],e_j), F_A(e_i,e_j)\rangle\right)dV_g. \end{aligned} \tag{2.1.6}$$

Here we have used the formula

$$\frac{d}{dt}\left(d\phi_t(e_i(\phi_t^{-1}(x)))\right)|_{x=0} = -[X,e_i].$$

Since $[X,e_i] = \nabla_X e_i - \nabla_{e_i}X$, where ∇ denotes the Levi-Civita connection of g, we obtain

$$\begin{aligned} &\sum_{i,j=1}^n\langle F_A([X,e_i],e_j), F_A(e_i,e_j)\rangle\\ &\qquad = -\sum_{i,j=1}^n\left(\langle F_A(\nabla_{e_i}X,e_j), F_A(e_i,e_j)\rangle - \langle F_A(\nabla_X e_i,e_j),F_A(e_i,e_j)\rangle\right) \end{aligned} \tag{2.1.7}$$

$$= -\sum_{i,j=1}^{n}\Big(\langle F_A(\nabla_{e_i}X, e_j), F_A(e_i,e_j)\rangle$$
$$-g(\nabla_X e_i, e_k)\langle F_A(e_k,e_j), F_A(e_i,e_j)\rangle\Big).$$

Since
$$g(\nabla_X e_i, e_k) = -g(e_i, \nabla_X e_k) = -g(\nabla_X e_k, e_i),$$
the second term in (2.1.7) vanishes.

Now suppose that A is a Yang-Mills connection; then

$$0 = \int_M \left(|F_A|^2 \operatorname{div} X - 4\sum_{i,j=1}^{n}\langle F_A(\nabla_{e_i}X, e_j), F_A(e_i,e_j)\rangle\right) dV_g. \tag{2.1.8}$$

The required monotonicity will follow from this variational formula.

Fix any $p \in M$, let r_p be a positive number with properties: there are normal coordinates $x_1, \cdots, x_n$ in the geodesic ball $B_{r_p}(p)$ of (M, g), such that $p = (0, \cdots, 0)$ and for some constant $c(p)$,

$$|g_{ij} - \delta_{ij}| \leq c(p)(|x_1|^2 + \cdots + |x_n|^2), \tag{2.1.9}$$

$$|dg_{ij}| \leq c(p)\sqrt{|x_1|^2 + \cdots + |x_n|^2}, \tag{2.1.10}$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right). \tag{2.1.11}$$

Remark 2. The constants r_p and $c(p)$ can be chosen depending only on the injective radius at p and the curvature of g. If $M = \mathbb{R}^n$ and g is flat, we can take $r_p = \infty$ and $c(p) = 0$.

Let $r(x)$ be the distance function from p; i.e.,

$$r(x) = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Let ϕ be a positive function on the unit sphere S^{n-1}. Define

$$X(x) = \xi(r)\phi(\frac{x}{r})r\frac{\partial}{\partial r} = \xi(r)\phi(\frac{x}{r})\left(\sum_i x_i \frac{\partial}{\partial x_i}\right), \tag{2.1.12}$$

where ξ is some smooth function with compact support in $B_{r_p}(p)$.

Let $\{e_1, \cdots, e_n\}$ be any orthonormal basis near p such that $e_1 = \frac{\partial}{\partial r}$. Since $x_1, \cdots, x_n$ are normal coordinates, we have

$$\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0.$$

It follows that

$$\nabla_{\frac{\partial}{\partial r}} X = (\xi r)'\phi(\theta)\frac{\partial}{\partial r} = (\xi' r + \xi)\phi(\theta)\frac{\partial}{\partial r}, \tag{2.1.13}$$

where $\theta = \frac{x}{r}$. Moreover, for $i \geq 2$,

$$\nabla_{e_i} X = \xi r \nabla_{e_i}(\phi\frac{\partial}{\partial r}) = \xi r e_i(\phi)\frac{\partial}{\partial r} + \ \xi\phi\sum_{j=1}^{n} b_{ij}e_j, \tag{2.1.14}$$

where $|b_{ij} - \delta_{ij}| = O(1)c(p)r^2$. We will always denote by $O(1)$ a quantity bounded by a constant depending only on n.

Applying (2.1.13) and (2.1.14) to the first variational formula (2.1.8), we obtain

$$\begin{aligned} &\int_M |F_A|^2(\xi' r + (n-4)\xi + O(1)c(p)r^2\xi)\phi dV_g \\ &\quad = \ 4\int_M \left(\xi' r\phi|\frac{\partial}{\partial r}\rfloor F_A|^2 + \xi r\langle\frac{\partial}{\partial r}\rfloor F_A, \nabla\phi\rfloor F_A\rangle\right) dV_g, \end{aligned} \tag{2.1.15}$$

where $\frac{\partial}{\partial r}\rfloor F_A = F_A(\frac{\partial}{\partial r}, \cdot)$.

We choose, for any τ small enough, $\xi(r) = \xi_\tau(r) = \eta(\frac{r}{\tau})$, where η is smooth and satisfies: $\eta(r) = 1$ for $r \in [0,1], \eta(r) = 0$ for $r \in [1+\varepsilon, \infty), \varepsilon > 0$ and $\eta'(r) \leq 0$. Then

$$\tau\frac{\partial}{\partial \tau}(\xi_\tau(r)) = -r\xi_\tau'(r). \tag{2.1.16}$$

Plugging this into (2.1.15), we obtain

$$\begin{aligned} &\tau\frac{\partial}{\partial\tau}\left(\int_M \xi_\tau\phi|F_A|^2 dV_g\right) + \left((4-n) + O(1)c(p)\tau^2\right)\int_M \xi_\tau\phi|F_A|^2 dV_g \\ &\quad = 4\tau\frac{\partial}{\partial\tau}\left(\int_M \xi_\tau\phi|\frac{\partial}{\partial r}\rfloor F_A|^2 dV_g\right) - 4\int_M \xi_\tau r\langle\frac{\partial}{\partial r}\rfloor F_A, \nabla\phi\rfloor F_A\rangle dV_g. \end{aligned} \tag{2.1.17}$$

Choose a nonnegative number $a \geq O(1)c(p)$. Then we deduce from the above

$$\begin{aligned} &\frac{\partial}{\partial\tau}\left(\tau^{4-n}e^{\pm a\tau^2}\int_M \xi_\tau\phi|F_A|^2 dV_g\right) \\ = \ \ & 4\tau^{4-n}e^{\pm a\tau^2}\left(\frac{\partial}{\partial\tau}\left(\int_M \xi_\tau\phi|\frac{\partial}{\partial r}\rfloor F_A|^2 dV_g\right)\right. \\ &\left. + \ (-O(1)c(p) \pm 2a)\tau\int_M \xi_\tau\phi|F_A|^2 dV_g - \ \tau^{-1}\int_M \xi_\tau\langle\frac{\partial}{\partial r}\rfloor F_A, \nabla\phi\rfloor F_A\rangle dV_g\right). \end{aligned} \tag{2.1.18}$$

Then, by integrating on τ and letting ε tend to zero, we prove:

Theorem 2.1.1. *Let r_p, $c(p)$ and a be as above. Then for any $0 < \sigma < \rho < r_p$, we have*

$$\begin{aligned}(2.1.19)\qquad &\pm\rho^{4-n}e^{\pm a\rho^2}\int_{B_\rho(p)}\phi|F_A|^2dV_g \mp \sigma^{4-n}e^{\pm a\sigma^2}\int_{B_\sigma(p)}\phi|F_A|^2dV_g\\ &\mp 4\int_{B_\rho(p)\backslash B_\sigma(p)} r^{4-n}e^{\pm\ ar^2}\phi|\frac{\partial}{\partial r}\rfloor F_A|^2dV_g\\ &\geq -4\int_\sigma^\rho \tau^{3-n}e^{\pm\ a\tau^2}d\tau\int_{B_\tau(p)}|\frac{\partial}{\partial r}\rfloor F_A||\nabla\phi\rfloor F_A|dV_g.\end{aligned}$$

This inequality is needed for establishing the existence of tangent cones of blow-up loci. Taking $\phi = 1$, we obtain:

Theorem 2.1.2 (Price). *Let r_p, $c(p)$ and a be as above. Then for any $0 < \sigma < \rho < r_p$, we have*

$$\begin{aligned}(2.1.20)\qquad &\rho^{4-n}e^{a\rho^2}\int_{B_\rho(p)}|F_A|^2dV_g - \sigma^{4-n}e^{a\sigma^2}\int_{B_\sigma(p)}|F_A|^2dV_g\\ &\geq\ 4\int_{B_\rho(p)\backslash B_\sigma(p)} r^{4-n}e^{ar^2}|\frac{\partial}{\partial r}\rfloor F_A|^2dV_g.\end{aligned}$$

Moreover, if $M = \mathbb{R}^n$ and g is flat, then the equality holds in (2.1.20) *for $\rho \in (0,\infty)$ and $a = 0$.*

Remark 3. Both (2.1.19) and (2.1.20) still hold when A is only a smooth Yang-Mills connection on $M\backslash\{p\}$ with

$$\int_M |F_A|^2dV_g < \infty.$$

To see this, we replace η in defining ξ in (2.1.16) by η_ε for sufficiently small ε, where $\eta_\varepsilon(t) = 0$ for either $t \leq \varepsilon$ or $t \geq 1+\varepsilon$, and $\eta_\varepsilon(t) = 1$ for $t \in (\varepsilon, 1-\varepsilon)$. Then we can follow the same arguments from (2.1.16) on and obtain both (2.1.19) and (2.1.20) for such a Yang-Mills connection A with isolated singularity at p.

It follows from this theorem that $\rho^{4-n}e^{a\rho^2}\int_{B_\rho(p)}|F_A|^2dV_g$ is a nondecreasing function in $(0, r_p)$. Another simple corollary of (2.1.20) is the following:

Corollary 2.1.3. *Let A be a Yang-Mills G-connection of the trivial bundle $(\mathbb{R}^n\backslash\{0\}) \times \mathbb{R}^r$, such that $\rho^{4-n}\int_{B_\rho(0)}|F_A|^2dV_{g_0}$ is independent of $\rho \in (0,\infty)$, where g_0 is a flat metric on $\mathbb{R}^n$. Then A is gauge equivalent to $d + A_s$, where $A_s : S^{n-1} \longrightarrow T^*S^{n-1} \otimes \mathrm{Lie}(G)$ is a* Lie(G)*-valued* 1*-form.*

Proof. By (2.1.20) for the flat metric g_0, we obtain

$$(2.1.21)\qquad \frac{\partial}{\partial r}\rfloor F_A \equiv 0.$$

Let D be the associated covariant derivative of A. Write $D = d + \tilde{a}dr + \tilde{A}$, where $\tilde{a} : \mathbb{R}^n\backslash\{0\} \to \mathrm{Lie}(G)$ and $\tilde{A} : \mathbb{R}^n\backslash\{0\} \to T^*S^{n-1}\otimes\mathrm{Lie}(G)$. For any gauge transformation $\sigma : \mathbb{R}^n\backslash\{0\} \to G$, we have a similar representation $d+\tilde{a}_\sigma dr+\tilde{A}_\sigma$ for $D_{\sigma(A)}$; moreover,

$$\tilde{a}_\sigma = (\sigma \cdot \tilde{a} - \frac{\partial\sigma}{\partial r}) \cdot \sigma^{-1}.$$

Choose σ by solving the ordinary differential equation on G:

$$\sigma \cdot \tilde{a} - \frac{\partial\sigma}{\partial r} = 0.$$

Then $\tilde{a}_\sigma = 0$. Together with (2.1.21), we deduce $\frac{\partial\tilde{A}_\sigma}{\partial r} = 0$; i.e., $\tilde{A}_\sigma(r,\theta) = A_s(\theta)$ for some $A_s : S^{n-1} \to T^*S^{n-1} \otimes \mathrm{Lie}(G)$. The corollary is proved. □

2.2. *Curvature estimates.* In this section, we give a basic curvature estimate for Yang-Mills connections. This estimate was first derived by K. Uhlenbeck (also see [Na]). Since it is crucial to us here, we will outline its proof for the reader's convenience.

We will adopt the notation of the last section.

Theorem 2.2.1 (K. Uhlenbeck). *Let A be any Yang-Mills connection of a G-bundle E over M. Then there are $\varepsilon = \varepsilon(n) > 0$ and $C = C(n) > 0$, which depend only on n and M, such that for any $p \in M$ and $\rho < r_p$, whenever*

$$\rho^{4-n}\int_{B_\rho(p)} |F_A|^2 dV_g \le \varepsilon,$$

then

$$|F_A|(p) \le \frac{C}{\rho^2}\left(\rho^{4-n}\int_{B_\rho(p)} |F_A|^2 dV_g\right)^{\frac{1}{2}}. \tag{2.2.1}$$

Our proof here uses R. Schoen's arguments in [Sc] for harmonic maps.

By scaling, we may assume that $\rho = 1$. Define a function

$$f(r) = (1-2r)^2 \sup_{x\in B_r(p)} |F_A|(x), \qquad r \in [0, \frac{1}{2}]. \tag{2.2.2}$$

Then $f(r)$ is continuous in $[0, \frac{1}{2}]$ with $f(\frac{1}{2}) = 0$, so that f attains its maximum at a certain r_0 in $[0, \frac{1}{2}]$.

First we claim that $f(r_0) \le 64$ if ε is sufficiently small. Assume that $f(r_0) > 64$. Put $b = \sup_{x\in B_{r_0}(p)} |F_A|(x) = |F_A|(x_0)$; then taking $\sigma = \frac{1}{4}(1-2r_0)$, we get

$$\begin{aligned} \sup_{x\in B_\sigma(x_0)} |F_A| &\le \sup_{x\in B_{r_0+\sigma}(p)} |F_A|(x) \\ &\le \frac{(1-2r_0)^2}{(1-2r_0-2\sigma)^2} \sup_{x\in B_{r_0}(p)} |F_A|(x) = 4b. \end{aligned} \tag{2.2.3}$$

Clearly, $16\sigma^2 b \geq 64$; i.e., $\sigma\sqrt{b} \geq 2$. Define a scaled metric $\tilde{g} = bg$; then with respect to $\tilde{g}$, the norm $|F_A|_{\tilde{g}}$ of F_A is equal to $b^{-1}|F_A|$. Hence,

$$\sup_{x \in B_2(x_0,\tilde{g})} |F_A|_{\tilde{g}} \leq 4, \tag{2.2.4}$$

where $B_2(x_0,\tilde{g})$ denotes the geodesic ball of $\tilde{g}$ with radius 2 and center at x_0.

Since A is a Yang-Mills connection, by the second Bianchi identity and straightforward computations, we can derive the following equation:

$$\frac{1}{2}\Delta_{\tilde{g}}|F_A|^2_{\tilde{g}} = |\tilde{\nabla} F_A|^2_{\tilde{g}} - 2F_A\#F_A\#R(\tilde{g}) - 2F_A * F_A * F_A, \tag{2.2.5}$$

where $F_A\#F_A\#R(\tilde{g})$ and $F_A * F_A * F_A$ are defined as follows: in any orthonormal basis $e_1, \ldots, e_n$ of $\tilde{g}$,

(2.2.6)
$$\begin{aligned} F_A\#F_A\#R(\tilde{g}) \quad = \quad & \sum_{l,k,i,j} \Bigg(\langle F_A(e_l,e_k), F_A(e_i,e_j)\rangle \\ & - \sum_m \langle F_A(e_l,e_m), F_A(e_i,e_m)\rangle \delta_{jk} \Bigg) R(\tilde{g})(e_l,e_j,e_k,e_i), \end{aligned}$$

and

$$F_A * F_A * F_A \sum_{i,j,k} \langle [F_A(e_i,e_j), F_A(e_j,e_k)], F_A(e_k,e_i)\rangle. \tag{2.2.7}$$

It follows from (2.2.5)–(2.2.7) that there are uniform constants c_1, c_2, depending only on n, such that

$$-\Delta_{\tilde{g}}|F_A|_{\tilde{g}} \leq c_1|F_A|_{\tilde{g}} + c_2|F_A|^2_{\tilde{g}}. \tag{2.2.8}$$

Using (2.2.4), we deduce from (2.2.8) that in $B_2(x_0,\tilde{g})$,

$$-\Delta_{\tilde{g}}|F_A|_{\tilde{g}} \leq (c_1 + 4c_2)|F_A|_{\tilde{g}}. \tag{2.2.9}$$

Then, by using either the mean-value theorem or the standard Moser iteration, we obtain

$$1 = |F_A|_{\tilde{g}}(x_0) \leq \tilde{c}\left(\int_{B_1(x_0,\tilde{g})} |F_A|^2_{\tilde{g}} dV_{\tilde{g}}\right)^{\frac{1}{2}}, \tag{2.2.10}$$

where $\tilde{c}$ is some uniform constant.

However, by the monotonicity (Theorem 2.1.1),

$$\begin{aligned} \int_{B_1(x_0,\tilde{g})} |F_A|^2_{\tilde{g}} dV_{\tilde{g}} \quad &= \quad (\sqrt{b})^{n-4} \int_{B_{\frac{1}{\sqrt{b}}}(x_0)} |F_A|^2 dV_g \\ &\leq \quad \left(\frac{1}{2}\right)^{4-n} e^{\frac{a}{4}} \int_{B_{\frac{1}{2}}(x_0)} |F_A|^2 dV_g \;\leq\; \varepsilon 2^{n-4} e^{\frac{a}{4}}. \end{aligned}$$

Combining this with (2.2.10), we obtain

$$1 \leq \tilde{c}\varepsilon 2^{n-4} e^{\frac{a}{4}}.$$

It is impossible when $\varepsilon = \varepsilon(n)$ is sufficiently small. The claim is proved.

Thus, we have

$$\sup_{x \in B_{\frac{1}{4}}(p)} |F_A|(x) \leq 4f(r_0) \leq 256. \tag{2.2.11}$$

It follows from this and (2.2.5) with $\tilde{g}$ replaced by g that for some uniform constant c',

$$-\Delta_g |F_A| \leq c' |F_A|. \tag{2.2.12}$$

Then (2.2.1) follows from (2.2.12) and a standard Moser iteration.

2.3. *Admissible Yang-Mills connections.* In order to compactify the moduli space of Yang-Mills connections, we need to use singular Yang-Mills connections of a certain type. Those singular connections behave like the usual smooth connections in many ways; for instance, one can define the first two terms of the Chern character by using their curvature forms.

An admissible Yang-Mills connection is a smooth connection A defined outside a closed subset $S(A)$ in M, such that (1) $H^{n-4}(S(A) \cap K) < \infty$ for any compact subset $K \subset M$, where $H^{n-4}(\cdot)$ stands for the $(n-4)$-dimensional Hausdorff measure (cf. [Si2]); (2) A is Yang-Mills on $M \backslash S(A)$; (3) A satisfies

$$\int_{M \backslash S(A)} |F_A|^2 \, dV_g < \infty. \tag{2.3.1}$$

Together with (2.3.1), this implies that for any smooth Lie(G)-valued 1-form u over M with compact support,

$$\int_M \langle F_A, du \rangle \, dV_g = 0. \tag{2.3.2}$$

Clearly, A is smooth on M if $S(A) = \emptyset$. We will call $S(A)$ the singular set of A. This is not invariant under gauge transformations. Even if $S(A) \neq \emptyset$, there may be a gauge transformation σ on $M \backslash S(A)$ such that $\sigma(A)$ extends to become a smooth connection on M.

Two admissible connections A_1 and A_2 are gauge equivalent if there is a gauge transformation σ of E over $M \backslash S(A_1) \cup S(A_2)$ such that $\sigma(A_1) = A_2$ outside $S(A_1) \cup S(A_2)$. This new gauge equivalence extends the previous one for smooth connections.

Similarly, by requiring that A be Ω-anti-self-dual outside $S(A)$, we can also define admissible Ω-anti-self-dual instantons.

Now let us assume that G is a unitary group. By the standard Chern-Weil theory, associated to each smooth connection A, we have closed forms

$\frac{\sqrt{-1}}{2\pi}\operatorname{tr}(F_A)$ and $\left(\frac{\sqrt{-1}}{2\pi}\right)^2\operatorname{tr}(F_A\wedge F_A)$ of degree 2 or 4. If M is compact, they represent the first two Chern characters $\mathrm{Ch}_1(E)$ or $\mathrm{Ch}_2(E)$ respectively. We now extend these to admissible Yang-Mills connections.

Let A be an admissible Yang-Mills connection with the singular set $S = S(A)$. Then $\operatorname{tr}(F_A)$ and $\operatorname{tr}(F_A\wedge F_A)$ are closed forms on $M\backslash S$. Because of (3) above, we can extend them to forms on M in the sense of distribution. Clearly, these forms are invariant under gauge transformations.

PROPOSITION 2.3.1. *The extended forms* $\frac{\sqrt{-1}}{2\pi}\operatorname{tr}(F_A)$ *and* $(\frac{\sqrt{-1}}{2\pi})^2\operatorname{tr}(F_A\wedge F_A)$ *are closed on* M. *They are denoted by* $\mathrm{Ch}_1(A)$ *and* $\mathrm{Ch}_2(A)$.

Proof. We only show the closedness of $\left(\frac{\sqrt{-1}}{2\pi}\right)^2\operatorname{tr}(F_A\wedge F_A)$ here. The other case is easier. We will always denote by C a uniform constant in this proof.

It is sufficient to show that for any smooth form φ of degree $n-5$ with compact support in M,

$$\int_M d\varphi\wedge\operatorname{tr}(F_A\wedge F_A) = 0. \tag{2.3.3}$$

Note that this is well-defined since F_A is L^2-integrable.

Without loss of generality, we may assume that M is a ball in $\mathbb{R}^n$ and E is a trivial bundle over M. Let K be a compact subset in M containing $\operatorname{supp}(\varphi)$ in its interior.

Fixing any $\varepsilon\le\varepsilon(n)$, as given in Theorem 2.2.1, we define

$$E_r = \{x\in K \mid r^{4-n}e^{ar^2}\int_{B_r(x)}|F_A|^2 dV_g \geqslant \varepsilon\}, \tag{2.3.4}$$

where a is as in Theorem 2.1.2. By Theorem 2.1.2, $E_r\supset E_{r'}$ whenever $r\geqslant r'$. We can find a finite covering $\{B_{2r}(x_k)\}_{1\leqslant k\leqslant L_r}$ of E_r such that (1) $x_k\in E_r$; (2) $B_r(x_k)\cap B_r(x_l)=\emptyset$ for $k\neq l$. Next we expand $\{B_{2r}(x_k)\}_{1\le k\le L_r}$ to a covering $\{B_{2r}(x_k)\}_{1\leqslant k\leqslant L_r'}$ ($L_r'\geqslant L_r$) of $(S\cap K)\cup E_r$, such that $x_k\in(S\cap K)\cup E_r$, $B_r(x_k)\cap B_r(x_l)=\emptyset$ for $k\neq l$. Note that for any k, the number of x_l with $B_{8r}(x_k)\cap B_{8r}(x_l)\neq\emptyset$ is bounded by a constant depending only on n and M.

For any $x\notin\bigcup_{k=1}^{L_r'}B_{2r}(x_k)$,

$$r^{4-n}\int_{B_r(x)}|F_A|^2\, dV_g < \varepsilon. \tag{2.3.5}$$

It follows from Uhlenbeck's estimate (Theorem 2.2.1) that

$$|F_A|(x)\leqslant\frac{C}{r^2}\left(r^{4-n}\int_{B_r(x)}|F_A|^2 dV_g\right)^{\frac12}\leqslant\frac{C\sqrt{\varepsilon}}{r^2}. \tag{2.3.6}$$

Then, by using Theorem 1.2.7 in [Uh1, p. 18], we can construct a gauge transformation σ_x over $B_r(x)$ for any $x \in M\backslash N_{3r}\left((S\cap K)\cup E_r\right)$, such that

$$|\sigma_x(A)|\,(y) \leqslant \frac{C}{r}\left(r^{4-n}\int_{B_r(x)}|F_A|^2 dV_g\right)^{\frac{1}{2}}, \quad \forall y\in B_r(x). \tag{2.3.7}$$

Note that for any $\delta>0$ and any subset $S'\subset M$,

$$N_\delta(S') = \{x\in M \mid d(x,S')\le\delta\},$$

where $d(\cdot,\cdot)$ denotes the distance function of the metric g.

Gluing these σ_x appropriately, we can construct a gauge transformation σ_k over each $B_{8r}(x_k)\backslash N_{3r}\left((S\cap K)\cup E_r\right)$, such that

$$|\sigma_k(A)|\,(x) \leqslant \frac{C}{r}\left(r^{4-n}\int_{B_r(x)}|F_A|^2 dV_g\right)^{\frac{1}{2}}, \tag{2.3.8}$$

whenever $x\in B_{8r}(x_k)\backslash N_{3r}((S\cap K)\cup E_r)$. One gets from (2.3.8) that on the overlap $\left(B_{8r}(x_l)\cap B_{8r}(x_k)\right)\backslash N_{3r}\left((S\cap K)\cup E_r\right)$,

$$|d\sigma_k\cdot\sigma_l^{-1}| \le \frac{2C\sqrt{\varepsilon}}{r}.$$

Hence, by modifying σ_k slightly on overlaps, we may assume that $\sigma_k\cdot\sigma_l^{-1}$ is constant on each connected component of $B_{8r}(x_k)\cap B_{8r}(x_l)\backslash N_{3r}\left((S\cap K)\cup E_r\right)$ for any $k\neq l$.

Let $\eta:\mathbb{R}^1\to\mathbb{R}^1$ be a cut-off C^∞-function satisfying: $\eta(t)=0$ for $t\leqslant 1$, $\eta(t)=1$ for $t\geqslant 2$, and $0\leqslant\eta'(t)\leqslant 1$. Then

$$\int_M d\varphi\wedge\mathrm{tr}(F_A\wedge F_A) = \lim_{r\to 0}\int_M \eta\left(\frac{d(x,(S\cap K)\cup E_r)}{3r}\right)d\varphi\wedge\mathrm{tr}(F_A\wedge F_A). \tag{2.3.9}$$

For each $k\leqslant L_r'$,

(2.3.10)

$$\begin{aligned}
\mathrm{tr}\left(F_A\wedge F_A\right)(x) &= \mathrm{tr}\left(F_{\sigma_k(A)}\wedge F_{\sigma_k(A)}\right)(x)\\
&= d\,\mathrm{tr}\left(\sigma_k(A)\wedge F_{\sigma_k(A)}+\frac{1}{3}\sigma_k(A)\wedge\sigma_k(A)\wedge\sigma_k(A)\right)(x),
\end{aligned}$$

where $x\in B_{8r}(x_k)\backslash N_{3r}\left((S\cap K)\cup E_r\right)$.

Since $\sigma_k\cdot\sigma_l^{-1}$ is piecewise constant, we have

$$\begin{aligned}
&\mathrm{tr}\left(\sigma_k(A)\wedge F_{\sigma_k(A)}+\frac{1}{3}\sigma_k(A)\wedge\sigma_k(A)\wedge\sigma_k(A)\right)\\
=\ &\mathrm{tr}\left(\sigma_l(A)\wedge F_{\sigma_l(A)}+\frac{1}{3}\sigma_l(A)\wedge\sigma_l(A)\wedge\sigma_l(A)\right)
\end{aligned}$$

on the overlap $B_{8r}(x_k) \cap B_{8r}(x_l) \backslash N_{3r}\left((S \cap K) \cup E_r\right)$. Therefore, there is a globally defined Chern-Simon transgression form Ψ outside $N_{3r}(S \cup E_r)$, such that

$$d\Psi = \operatorname{tr}(F_A \wedge F_A)$$

and

$$\Psi(x) = \operatorname{tr}\left(\sigma_k(A) \wedge F_{\sigma_k(A)} + \frac{1}{3}\sigma_k(A) \wedge \sigma_k(A) \wedge \sigma_k(A)\right)$$

whenever $x \in B_{8r}(x_k)$. For each k and any $x \in B_{6r}(x_k) \backslash B_{3r}(x_k)$,

$$|\psi(x)| \leq Cr^{-3}\left(r^{4-n}\int_{B_r(x)} |F_A|^2 dV_g\right)^{\frac{3}{2}} \leq Cr^{1-n}\int_{B_{8r}(x_k)} |F_A|^2 dV_g.$$

It follows that

$$\begin{aligned}
\left|\int_M d\varphi \wedge \operatorname{tr}(F_A \wedge F_A)\right| &= \lim_{r\to 0}\left|\int_M \eta\left(\frac{d(x,(S\cap K)\cup E_r)}{3r}\right) d\varphi \wedge d\Psi\right| \\
&\leq \lim_{r\to 0}\int_{3r\leqslant d(x,(S\cap K)\cup E_r)\leqslant 6r} \frac{1}{3r}|\Psi||d\varphi|\, dV_g \\
&\leq C\lim_{r\to 0}\left\{\sup_M |d\varphi| \sum_{k=1}^{L'_r} \int_{B_{8r}(x_k)} |F_A|^2 dV_g\right\} \\
&\leq C\sup_M |d\varphi| \lim_{r\to 0}\int_{N_{8r}(S\cup E_r)} |F_A|^2 dV_g.
\end{aligned}$$

Since $\bigcap_{r>0} N_{8r}(S \cup E_r) \subset S$ and $N_{8r}(S \cup E_r) \subset N_{8r'}(S \cup E_{r'})$ for $r \leq r'$, the last integral converges to zero as r tends to 0. Therefore, we have

$$\int_M d\varphi \wedge \operatorname{tr}(F_A \wedge F_A) = 0,$$

so that $\operatorname{tr}(F_A \wedge F_A)$ is closed in the sense of distribution. □

Let C_1 and C_2 denote the Chern-Weil polynomials defining the first two Chern classes. Then $C_1(A) = \mathrm{Ch}_1(A)$ is well-defined.

On $M\backslash S(A)$,

$$C_2(A) = \frac{1}{8\pi^2}\left(\operatorname{tr}(F_A \wedge F_A) - \operatorname{tr}(F_A) \wedge \operatorname{tr}(F_A)\right). \tag{2.3.11}$$

Then $C_2(A)$ extends to a form, still denoted by $C_2(A)$, on M in the sense of distribution.

Corollary 2.3.2. *The extended form $C_2(A)$ is closed.*

Proof. Since $\operatorname{tr}(F_A)$ is harmonic outside $S(A)$ and L^2-bounded, by the standard elliptic theory, it extends to be a smooth form on M. Then this corollary follows from the last proposition. □

3. Rectifiability of blow-up loci

We study the blow-up set of Yang-Mills connections which converge to an admissible Yang-Mills connection.

3.1. *Convergence of Yang-Mills connections.* Given any sequence of admissible Yang-Mills connections A_i, we say that the A_i converge weakly to an admissible Yang-Mills connection A (modulo gauge transformations), if $\int_M |F_{A_i}|^2 dV_g \le c$ for some uniform constant c and there are a closed subset S and gauge transformations σ_i of the G-bundle E over $M\backslash S$, such that for any compact $K \subset M\backslash S$, $\sigma_i(A_i)$ extend smoothly across K for i sufficiently large and converge to A in the C^∞-topology in K as i tends to infinity. Obviously, S contains $S(A)$. In particular, this implies that for any smooth form φ with compact support in M,

$$\lim_{i\to\infty} \int_M (F_{\sigma_i(A_i)}, d\varphi) dV_g = \int_M (F_A, d\varphi) dV_g.$$

This is exactly what the weak convergence is. Clearly, we have the next result:

LEMMA 3.1.1. *Weak limits of admissible connections $\{A_i\}$ are unique modulo gauge transformations.*

From now on, we always assume that $\{A_i\}$ is a sequence of smooth Yang-Mills connections with $YM(A_i) \le \Lambda$. All the discussions in this section also work for admissible Yang-Mills connections with slight modification.

PROPOSITION 3.1.2. *There is a subsequence $\{A_{i_j}\}$ which converges weakly to some admissible Yang-Mills connection A on M.*

Proof. Let ε be as in Theorem 2.2.1 and a be as in Theorem 2.1.2. We define a closed subset for each i and $r > 0$ sufficiently small:

$$E_{i,r} = \{x \in M \,|\, e^{ar^2} r^{4-n} \int_{B_r(x)} |F_{A_i}|^2 dV_g \ge \varepsilon\}. \tag{3.1.1}$$

It follows from the monotonicity formula (Theorem 2.1.2) that $E_{i,r} \subset E_{i,r'}$ for any $r \le r'$.

By the standard diagonal process, we can choose a subsequence $\{i_j\}$ of $\{i\}$ such that for each k, the $E_{i_j,2^{-k}}$ converge to a closed subset $E_{2^{-k}}$. Then $E_{2^{-k}} \subset E_{2^{-l}}$ for $k \ge l$. Put $S = \bigcap_k E_{2^{-k}}$.

We first claim that S is of Hausdorff codimension at least 4. Given $\delta > 0$ sufficiently small and any compact subset K of M, let $\{B_{4\delta}(x_\alpha)\}$ be any finite covering of $S \cap K$ such that (1) $x_\alpha \in S \cap K$; (2) $B_{2\delta}(x_\alpha) \cap B_{2\delta}(x_\beta) = \emptyset$ for $\alpha \ne \beta$. Take k big enough such that $2^{-k} < \delta$. Then for j sufficiently large, there are $y_\alpha \in E_{i_j,2^{-k}}$ such that $d(x_\alpha, y_\alpha) < \delta$. Then $\{B_{5\delta}(y_\alpha)\}$ is a finite covering of $S \cap K$ and $B_\delta(y_\alpha) \cap B_\delta(y_\beta) = \emptyset$ for $\alpha \neq \beta$. By Theorem 2.1.2,

$$e^{a\delta^2}\delta^{4-n}\int_{B_\delta(y_\alpha)}|F_{A_{i_j}}|^2 dV_g \geq e^{a2^{-2k}}2^{(n-4)k}\int_{B_{2^{-k}}(y_\alpha)}|F_{A_{i_j}}|^2 dV_g \geq \varepsilon.$$

Hence,

$$\sum_\alpha \delta^{n-4} \leq \frac{e^a}{\varepsilon}\sum_\alpha\int_{B_\delta(y_\alpha)}|F_{A_{i_j}}|^2 dV_g \leq \frac{e^a}{\varepsilon}\int_M |F_{A_{i_j}}|^2 dV_g \leq \frac{ce^a}{\varepsilon}.$$

It follows that $H^{n-4}(S \cap K)$, and consequently, $H^{n-4}(S)$, is no more than $\frac{5^{n-4}e^a c}{\varepsilon}$. This proves the claim.

Now we prove that A_{i_j} converges to some A outside S modulo gauge transformations. To save the notation, we assume $\{i_j\} = \{i\}$.

We notice that for any $r > 0$, there is an $i(r) > 0$, such that for any $i \geq i(r)$ and $x \in M$ with $d(x, E_{2^{-k}}) \geq r$, where $2^{-k-1} \leq r \leq 2^{-k}$,

$$e^{ar^2}r^{4-n}\int_{B_r(x)}|F_{A_i}|^2 dV_g < \varepsilon. \tag{3.1.2}$$

This is equivalent to saying that $x \in M\backslash E_{i,r}$. By Theorem 2.2.1, we deduce from (3.1.2) that for any $x \in M\backslash B_r(E_r)$,

$$|F_{A_i}|(x) < \frac{C\sqrt{\varepsilon}}{r^2}. \tag{3.1.3}$$

It follows from Theorem 3.6 in [Uh2] that there exists a subsequence $\{i'\} \subset \{i\}$ and gauge transformations $\sigma(i')$, such that $\sigma(i')(A_{i'})$ converge to a smooth connection A in C^1-topology on any compact subset outside S. Since A_i are Yang-Mills connections, by the standard elliptic theory, A is a Yang-Mills connection and $\sigma(i')(A_{i'})$ converge to A smoothly outside S. $\square$

In the following, we always assume that the sequence A_i converges to an admissible Yang-Mills connection A with $\int_M |F_{A_i}|^2 dV_g \leq \Lambda$.

LEMMA 3.1.3. *Define*

$$S_b(\{A_i\}) = \bigcap_{r>0}\{x \in M | \lim_{i\to\infty}\inf e^{ar^2}r^{4-n}\int_{B_r(x)}|F_{A_i}|^2 dV_g \geq \varepsilon\}, \tag{3.1.4}$$

where ε *is as given in Theorem* 2.2.1. *Then* (i) $S_b(\{A_i\})$ *is closed and contained in the above* S; (ii) *Its Hausdorff measure* $H^{n-4}(S_b(\{A_i\})) \leq C$ *for some constant* C *depending only on* M *and* Λ; (iii) A *extends to a smooth connection on* $M\backslash S_b(\{A_i\})$.

Proof. Suppose $x_0 \in M\backslash S_b(\{A_i\})$; then there is an $r_0 > 0$ such that

$$r_0^{4-n}\int_{B_{r_0}(x_0)}|F_{A_{n_i}}|^2 dV_g < \varepsilon$$

for some subsequence $n_i \to \infty$. By Theorem 2.2.1,

$$\sup_{n_i} \sup_{x \in B_{\frac{r_0}{2}}(x_0)} |F_{A_{n_i}}| \leq \frac{c_0\sqrt{\varepsilon}}{r_0^2}$$

for some constant $c_0 = c_0(n, M)$. In particular,

$$\sup_{n_i} \sup_{x \in B_{\frac{r_0}{4}}(x_0)} r^{4-n} \int_{B_r(x)} |F_{A_{n_i}}|^2 dV_g \leq \frac{\varepsilon}{2}$$

whenever $r \leq r_0/\sqrt[4]{c+1}$ for some constant c depending only on g. Hence, $B_{\frac{r_0}{4}}(x_0) \subset M \backslash S_b(\{A_i\})$, and consequently, $S_b(\{A_i\})$ is closed. This also implies that A is a limit of some subsequence of $\{A_{n_i}\}$ (modulo gauge transformations) in $B_{\frac{r_0}{4}}(x_0)$ in the C^∞-topology. Then (iii) follows.

For any $x_0 \in M \backslash S$, if r is sufficiently small,

$$r^{4-n} \int_{B_r(x_0)} |F_A|^2 dV_g < \varepsilon_0.$$

This implies that for i sufficiently large,

$$r^{4-n} \int_{B_r(x_0)} |F_{A_i}|^2 dV_g < \varepsilon_0.$$

Hence, $x_0 \in M \backslash S_b(\{A_i\})$. This shows that $S_b(\{A_i\}) \subset S$.

The estimate on $H^{n-4}(S_b(\{A_i\}))$ follows from the proof of Proposition 3.1.2. $\square$

Since A can be extended smoothly to $M \backslash S_b(\{A_i\})$, we may assume that $S(A) \subset S_b(\{A_i\})$. If $S_b(\{A_i\}) = \emptyset$, then there is a subsequence of $\{A_i\}$ which converges to A smoothly on M.

Consider the Radon measures $\mu_i = |F_{A_i}|^2 dV_g$ $(i = 1, 2, \cdots)$. By taking a subsequence if necessary, we may assume that $\mu_i \to \mu$ weakly on M as Radon measures; i.e., for any continuous function φ with compact support in M,

$$\lim_{i \to \infty} \int_M \varphi |F_{A_i}|^2 dV_g = \int_M \varphi d\mu. \tag{3.1.5}$$

Let us write (by Fatou's lemma)

$$\mu = |F_A|^2 dV_g + \nu$$

for some nonnegative Radon measure ν on M.

Lemma 3.1.4. *When $\nu(x) = \Theta(x) H^{n-4} \lfloor S_b(\{A_i\}), x \in M$, for H^{n-4}-a.e. $x \in S_b(\{A_i\})$, then*

$$\varepsilon \leq \Theta(x) \leq 4^{n-4} r_x^{4-n} e^{a r_x^2} \Lambda,$$

where r_x, a are as given in Theorem 2.1.2.

Proof. First we observe:

(a) For any $x \in M$, $e^{ar^2} r^{4-n}\mu(B_r(x))$ is a nondecreasing function of r sufficiently small; thus the density

$$\Theta(\mu, x) = \lim_{r\to 0+} r^{4-n}\mu(B_r(x)) \tag{3.1.6}$$

exists for every $x \in M$;

(b) $x \in S_b(\{A_i\})$ if and only if $\Theta(\mu, x) \geq \varepsilon$;

(c) For H^{n-4}-a.e. $x \in S_b(\{A_i\})$,

$$\lim_{r\to 0+} r^{4-n} \int_{B_r(x)} |F_A|^2 dV_g = 0.$$

Indeed, (a) follows from the monotonicity formula in Section 2.1. The statement (b) follows from the definition of $S_b(\{A_i\})$ and (a). To prove (c), we define

$$E_j = \{x \mid \overline{\lim}_{r\to 0+} r^{4-n} \int_{B_r(x)} |F_A|^2 dV_g > \frac{1}{j}\}. \tag{3.1.7}$$

It suffices to show that $H^{n-4}(E_j) = 0$ for each $j \geq 1$. For any $\delta > 0$, there is a covering of E_j by balls $B_{2r_\alpha}(x_\alpha)$ with $x_\alpha \in E_j$ and $2r_\alpha \leq \delta$, such that

$$r_\alpha^{4-n} \int_{B_{r_\alpha}(x_\alpha)} |F_A|^2 dF_g > \frac{1}{j},$$

and $B_{r_\alpha}(x_\alpha) \cap B_{r_\beta}(x_\beta) = \emptyset$. Then

$$H^{n-4}(E_j) - \psi(\delta) \leq \sum_\alpha (2r_\alpha)^{n-4} \leq j2^{n-4} \int_{N_\delta(S_b(\{A_i\}))} |F_A|^2 dV_g,$$

where $N_\delta(S_b(\{A_i\}))$ denotes the δ-tubular neighborhood of $S_b(\{A_i\})$, and $\psi(\delta) \to 0$ as $\delta \to 0$. It follows that $H^{n-4}(E_j) = 0$, since δ is arbitrarily small.

From the monotonicity formula we obtain

$$r^{4-n}\mu(B_r(x)) \leq C \tag{3.1.8}$$

for some constant C depending only on Λ and M. Therefore, $\mu|_{S_b(\{A_i\})}$ is absolutely continuous with respect to $H^{n-4}\lfloor S_b(\{A_i\})$; consequently, by the Radon-Nikodym theorem, we have

$$\mu|_{S_b(\{A_i\})}(x) = \Theta(x) H^{n-4}\lfloor S_b(\{A_i\}) \tag{3.1.9}$$

for H^{n-4}-a.e. $x \in S_b(\{A_i\})$. Then by (c),

$$\nu(x) = \Theta(x) H^{n-4}\lfloor S_b(\{A_i\})$$

for H^{n-4}-a.e. $x \in S_b(\{A_i\})$. We notice that μ is a Borel regular measure. The estimates of $\Theta(x)$ follow the above density estimate (b) and the fact that for H^{n-4}-a.e. $x \in S_b(\{A_i\})$,

$$2^{4-n} \leq \overline{\lim}_{r\to 0+} \frac{H^{n-4}(S_b(\{A_i\}) \cap B_r(x))}{r^{n-4}} \leq 1, \tag{3.1.10}$$

which can be easily proved (cf. [Si, Th. 3.6)]. □

Define

$$S_b = \overline{\{x \in S_b(\{A_i\}) \,|\, \Theta(\mu, x) > 0,\ \lim_{r\to 0+} r^{4-n} \int_{B_r(x)} |F_A|^2 dV_g = 0\}}. \tag{3.1.11}$$

Then $S_b(\{A_i\}) = S_b \cup S(A)$. We call (S_b, Θ) the blow-up locus of the weakly convergent sequence $\{A_i\}$. Here, S_b is the support of the blow-up locus and Θ is its multiplicity. If no confusion can occur, we may simply say that S_b is the blow-up locus.

3.2. *Tangent cones of blow-up loci.* We adopt the notation of the last section unless specified otherwise. For simplicity, we write $S = S_b$ for the blow-up locus. In this section, we study the properties of tangent cones of S.

Recall that μ is the limit Radon measure of $\mu_i = |F_{A_i}|^2 dV_g$. For any $y \in M$ and sufficiently small λ, we define the scaled measure $\mu_{y,\lambda}$ as follows: for any E in T_yM,

$$\mu_{y,\lambda}(E) = \lambda^{4-n} \mu(\exp_y(\lambda E)), \tag{3.2.1}$$

where $\exp_y : T_yM \to M$ is the exponential map of the metric g and

$$\lambda E = \{x \in T_yM \,|\, \lambda^{-1}x \in E\}. \tag{3.2.2}$$

LEMMA 3.2.1. *Let $\{\lambda_k\}$ be any sequence with $\lim_{k\to\infty} \lambda_k = 0$. Then there exist a subsequence $\{\lambda'_k\}$ and a Radon measure η on T_yM such that μ_{y,λ'_k} converges to η weakly. Moreover, $\eta_{0,\lambda} = \eta$ for each $\lambda > 0$; i.e., η is a cone measure.*

Proof. Define a connection on T_yM for each y and λ by

$$A_{i,y,\lambda} = \tau_\lambda^* \exp_y^* A_i, \tag{3.2.3}$$

where $\tau_\lambda : T_yM \mapsto T_yM$ maps v to λv. Then $A_{i,y,\lambda}$ is a Yang-Mills connection with respect to the metric $\lambda^{-2} \exp_y^* g$, which will be denoted by $g_{y,\lambda}$. Clearly, $g_{y,\lambda}$ converges to the flat metric $g_{y,0} = g|_{T_yM}$ on T_yM as $\lambda \to 0+$. Moreover, by the monotonicity (Theorem 2.1.2), for any small $r > 0$,

$$\begin{aligned} &e^{a\lambda^2 r^2} r^{4-n} \int_{B_r(0,g_{y,\lambda})} |F_{A_{i,y,\lambda}}|^2 dV_{y,\lambda} \\ &\qquad = e^{a\lambda^2 r^2} (\lambda r)^{4-n} \int_{B_{\lambda r}(y)} |F_{A_i}|^2 dV_g \leq C(M, \Lambda), \end{aligned} \tag{3.2.4}$$

where $C(M,\Lambda)$ denotes a constant depending only on M,Λ, and $B_r(0,g_{y,\lambda})$ denotes the geodesic ball of $g_{y,\lambda}$ in T_yM with radius r and center at 0. Clearly, $|F_{A_{i,y,\lambda}}|^2dV_{y,\lambda}$ converges to $\mu_{y,\lambda}$ weakly on $B_{\lambda^{-1}r_0}(0,g_{y,\lambda})$, where r_0 depends only on y. Letting i go to infinity, we obtain

$$\mu_{y,\lambda}(B_r(0,g_{y,\lambda}))\le C(M,\Lambda)r^{n-4} \tag{3.2.5}$$

for any $r\le\lambda^{-1}r_0$. Hence, we may find a subsequence $\{\lambda_k'\}\subset\{\lambda_k\}$ such that the $\mu_{y,\lambda_k'}$ converge to η weakly as Radon measures on T_yM. Then there are (by the standard diagonal process) $i_k\to\infty$ such that

$$|F_{A_{i_k,y,\lambda_k}}|^2dV_{y,\lambda}\to\eta. \tag{3.2.6}$$

Since μ is the weak limit of $\mu_i=|F_{A_i}|^2dV_g$, for $0<\sigma<\rho$ sufficiently small,

$$e^{a\sigma^2}\sigma^{4-n}\mu(B_\sigma(y))\le e^{a\rho^2}\rho^{4-n}\mu(B_\rho(y)). \tag{3.2.7}$$

This implies that $\lim_{r\to 0+}r^{4-n}\mu(B_r(y))=\Theta(\mu,y)$, and consequently, for any $r>0$,

$$\begin{aligned} r^{4-n}\eta(B_r(0,g_{y,0})) &= \lim_{\lambda_k'\to 0} r^{4-n}\mu_{y,\lambda_k'}(B_r(0,g_{y,\lambda_k'})) \\ &= \lim_{\lambda_k'\to 0}(\lambda_k'r)^{4-n}\mu(B_{\lambda_k'r}(y)). \end{aligned} \tag{3.2.8}$$

That is,

$$\eta(B_r(0,g_{y,0})) \;=\; \Theta(\mu,y)r^{n-4}.$$

This indicates that η is a cone measure. To prove it rigorously, we first observe that for any $0<\sigma<\rho<\infty$,

$$\int_{B_\rho(0,g_{y,\lambda_k'})\backslash B_\sigma(0,g_{y,\lambda_k'})} r^{4-n}|\frac{\partial}{\partial r}\rfloor F_{A_{i_k,y,\lambda_k'}}|^2dV_{y,\lambda_k'}\to 0 \text{ as } k\to\infty. \tag{3.2.9}$$

Here we have used Theorem 2.1.2 and (3.2.8).

Let $\phi(\theta)$ be any positive function on the unit sphere $S^{n-1}\subset T_yM$. It follows from Theorem 2.1.1 that for any $0<\sigma<\rho$ and λ_k' sufficiently small,

$$\begin{aligned} &\sigma^{4-n}e^{a(\lambda_k'\sigma)^2}\int_{B_\sigma(0,g_{y,\lambda_k'})}|F_{A_{i_k,y,\lambda_k'}}|^2\phi(\theta)dV_{y,\lambda_{k'}} \\ &\le \rho^{4-n}e^{a(\lambda_k'\rho)^2}\int_{B_\rho(0,g_{y,\lambda_k'})}|F_{A_{i_k,y,\lambda_k'}}|^2\phi(\theta)dV_{y,\lambda_k'} \\ &\quad -4\int_{B_\rho(0,g_{y,\lambda_k'})\backslash B_\sigma(0,g_{y,\lambda_k'})} r^{4-n}|\frac{\partial}{\partial r}\rfloor F_{A_{i_k,y,\lambda_k'}}|^2\phi dV_{y,\lambda_k'} \\ &\quad -4\int_\sigma^\rho \tau^{3-n}d\tau\int_{B_\tau(0,g_{y,\lambda_k'})}|\nabla\phi||\frac{\partial}{\partial r}\rfloor F_{A_{i_k,y,\lambda_k'}}|^2dV_{y,\lambda_k'}. \end{aligned}$$

Letting k go to infinity, by (3.2.9), we obtain

$$\sigma^{4-n}\int_{B_\sigma(0,g_{y,0})}\phi d\eta=\rho^{4-n}\int_{B_\rho(0,g_{y,0})}\phi d\eta. \tag{3.2.10}$$

Differentiating (3.2.10) on ρ, we have

$$\rho^{n-4}\int_{\partial B_\rho(0,g_{y,0})}\phi d\xi=(n-4)\int_{B_\rho(0,g_{y,0})}\phi d\eta, \tag{3.2.11}$$

where $d\eta(r,\theta)=r^{n-5}drd\xi(r,\theta)$. Combining (3.2.10) and (3.2.11), we get

$$\int_{\partial B_\rho(0,g_{y,0})}\phi(\theta)d\xi(\rho,\theta)=\int_{\partial B_\sigma(0,g_{y},0)}\phi(\theta)d\xi(\sigma,\theta), \tag{3.2.12}$$

for any $0<\sigma<\rho<\infty$. This implies

$$drd\xi(r+r_1,\theta)=drd\xi(r,\theta)$$

for any $r_1>0$. That is, $r^{5-n}d\eta(r,\theta)$ is translation invariant in r, or $d\eta(r,\theta)=r^{n-5}drd\xi(\theta)$ for some Radon measure $d\xi(\theta)$ on S^{n-1}. □

Next we study the tangent cones η with support in T_yM at H^{n-4}-a.e. $y\in S$. First we recall two elementary lemmas about the Radon measure μ given above.

Lemma 3.2.2. *The density function $\Theta(\mu,x)$ is H^{n-4}-approximately continuous at H^{n-4}-a.e. x in S. Here $\Theta(\mu,\cdot)$ is H^{n-4}-approximately continuous at $x\in S$ if for any $\varepsilon>0$,*

$$\lim_{r\to 0}\frac{H^{n-4}(\{y\in B_r(x)\cap S\,|\,|\Theta(\mu,y)-\Theta(\mu,x)|>\varepsilon\})}{r^{n-4}}=0. \tag{3.2.13}$$

Proof. The density function $\Theta(\mu,x)(x\in S)$ is upper-semi-continuous, so that $E_c=\{x\,|\,\Theta(\mu,x)<c\}$ is open, and consequently, for any $c_1<c_2$, $E_{c_2}\backslash E_{c_1}$ is a Borel set and thus measurable. Now we define

$$E_i=\{x\in S\mid \frac{(i-1)\varepsilon}{2}\le\Theta(\mu,x)<\frac{i\varepsilon}{2}\}.$$

Clearly, each E_i is contained in S and $H^{n-4}(S\backslash\bigcup_i E_i)=0$. Then for any $x\in E_i$, we have

$$\lim_{r\to 0}\frac{H^{n-4}(\{y\in B_r(x)\cap S\,|\,|\Theta(\mu,y)-\Theta(\mu,x)|>\varepsilon\})}{r^{n-4}}$$
$$=\overline{\lim}_{r\to 0}\frac{H^{n-4}(B_r(x)\cap(S\backslash E_i))}{r^{n-4}}=0.$$

Here we have used Theorem 3.5 in [Si2]. Thus the lemma follows. □

Lemma 3.2.3. *Let $x\in S$ be such that $\Theta(\mu,x)\ge\varepsilon_0>0$ and $\Theta(\mu,\cdot)$ is H^{n-4}-approximately continuous at x. Then there is a $r_x>0$, such that for each $r\in(0,r_x)$, we may find $n-4$ points $x_1,\ldots,x_{n-4}$ in $B_r(x)\cap S$ satisfying:*

(i) $\Theta(\mu, x_j) \geq \Theta(\mu, x) - \varepsilon(r)$ *for* $j = 1, 2, \cdots, n-4$, *where* $\varepsilon(r) \to 0$ *as* r *tends to zero*;

(ii) *Let* $\exp_x$ *be the exponential map of* (M, g) *at* x. *Then for some* $s \in (0, \frac{1}{2})$ *depending only on* n, $d(x_1, x) \geq sr$ *and* $d(x_k, \exp_x(V_{k-1})) \geq sr$ *for* $k \geq 2$, *where* V_{k-1} *denotes the subspace in* T_xM *spanned by* $(\exp_x|_{B_{r(0)}})^{-1}(x_1), \ldots, (\exp_x|_{B_r(0)})^{-1}(x_{k-1})$.

Proof. The arguments here are essentially due to F.H. Lin in [Li]. By the assumption, we may find a positive function $\varepsilon(r)$ for $0 < r < r_x$ such that $\lim_{r\to 0} \varepsilon(r) = 0$ and

$$\frac{H^{n-4}(\{y \in B_r(x) \cap S \mid |\Theta(\mu, y) - \Theta(\mu, x)| \geq \varepsilon(r)\})}{r^{n-4}} \leq \frac{s(n)}{2} < \frac{1}{2} \tag{3.2.14}$$

($s(n) > 0$ will be determined later).

Suppose that the lemma is false. Then there would be a sufficiently small $r > 0$ such that one cannot find $n-4$ points $x_1, \cdots, x_{n-4}$ inside the set

$$\{y \in S \cap B_r(x) \,|\, |\Theta(\mu, y) - \Theta(\mu, x)| < \varepsilon(r)\} \tag{3.2.15}$$

satisfying condition (ii) of Lemma 3.2.3. Therefore, the set in (3.2.15) is contained in an sr-neighborhood of $\exp_x(L)$ for some $(n-5)$-dimensional subspace L in T_xM. In particular, this implies

$$\begin{aligned} &\mu(\{y \in B_r(x) \cap S \,|\, |\Theta(\mu, y) - \Theta(\mu, x)| < \varepsilon(r)\}) \\ &\qquad \leq \; C(n)s(n)r^{n-4}\Theta(\mu, x), \end{aligned} \tag{3.2.16}$$

where $C(n)$ is some uniform constant independent of $s(n)$.

On the other hand, by the upper-semi-continuity of $\Theta(\mu, \cdot)$, we may assume that for any $y \in B_r(x)$,

$$\Theta(\mu, y) \leq 2\Theta(\mu, x).$$

Thus

$$\begin{aligned} &\mu(\{y \in B_r(x) \cap S \,|\, |\Theta(\mu, y) - \Theta(\mu, x)| \geq \varepsilon(r)\}) \\ &\quad \leq 2\Theta(\mu, x)H^{n-4}(\{y \in B_r(x) \cap S||\Theta(\mu, y) - \Theta(\mu, y)| \geq \varepsilon(r)\}) \\ &\quad \leq \Theta(\mu, x)s(n)r^{n-4}. \end{aligned} \tag{3.2.17}$$

Putting (3.2.16) and (3.2.17) together, we obtain

$$\begin{aligned} \mu(B_r(x) \cap S) \;&\leq\; s(n)\,(1 + C(n))\,\Theta(\mu, y)r^{n-4} \\ &<\; \frac{1}{2}\Theta(\mu, x)r^{n-4}, \end{aligned} \tag{3.2.18}$$

if we choose $s(n) < \frac{1}{2(1+C(n))}$. However, (3.2.18) is impossible for r sufficiently small, since $\lim_{r\to 0} \frac{\mu(B_r(x)\cap S)}{r^{n-4}} = \Theta(\mu, x) > 0$. $\square$

Now we can state the main result of this section, which will be used in proving the rectifiability of blow-up loci.

PROPOSITION 3.2.4. *Let μ be the Radon measure given at the beginning of this section. Then for H^{n-4}-a.e. $x \in S \subset M$, any tangent cone measure η on T_xM of μ is of the form $\Theta(\mu,x)H^{n-4}\lfloor F$ for some $(n-4)$-dimensional subspace F in T_xM.*

Note that the existence of η is assured by Lemma 3.2.1.

The rest of this section is devoted to proving this proposition. First we recall (cf. Lemma 3.1.4) that $\mu = |F_A|^2 dV_g + \Theta(\mu,\cdot)H^{n-4}\lfloor S$, where A is the weak limit of a sequence $\{A_i\}$. By Lemma 3.1.4 and the observation (c) in its proof, for H^{n-4}-a.e. $x \in S$,

$$\Theta(\mu,x) \geq \varepsilon_0 > 0, \quad \lim_{r\to 0} r^{4-n}\int_{B_r(x)} |F_A|^2 dV_g = 0. \tag{3.2.19}$$

Furthermore, it follows from Lemma 3.2.2 that $\Theta(\mu,\cdot)$ is H^{n-4}-approximately continuous at H^{n-4}-a.e. x in S.

From now on, we fix a point $x \in S$ such that (3.2.19) holds and $\Theta(\mu,\cdot)$ is H^{n-4}-approximately continuous at x.

Assume that η is the weak limit of μ_{x,r_k}, where $\lim_{k\to\infty} r_k = 0$. For k sufficiently large, by Lemma 3.2.3, we may find $n-4$ points $x_1^k, \cdots, x_{n-4}^k$ in $B_{r_k}(x) \cap S$, such that for $j = 1, 2, \cdots, n-4$,

$$\Theta(\mu, x_j^k) \geq \Theta(\mu,x) - \varepsilon(r_k), \tag{3.2.20}$$

$$d(x_j^k, \exp_x(V_{j-1}^k)) \geq s r_k, \tag{3.2.21}$$

where V_{j-1}^k denotes the 0-dimensional space $\{0\}$ if $j=1$, and the subspace in T_xM spanned by $\xi_1^k = \exp_x^{-1}(x_1^k)$,..., $\xi_{j-1}^k = \exp_x^{-1}(x_{j-1}^k)$ for $j \geq 2$.

As before, we denote by g_{x,r_k} the scaled metric $r_k^{-2}\exp_x^* g$ on T_xM, which converges to the flat one $g_{x,0}$ as k tends to ∞. Clearly, $r_k^{-1}\xi_j^k \in B_1(0, g_{x,r_k})$ for each j, so that by taking a subsequence of $\{r_k\}$ if necessary, we may assume that as k tends to ∞, $r_k^{-1}\xi_j^k \in B_1(0,g_{x,0})$ converges to ξ_j with respect to a fixed metric $g_{x,0}$. By (3.2.21), $\xi_1, \cdots, \xi_{n-4}$ span an $(n-4)$-dimensional subspace F in T_xM, which is in fact the limit of V_{n-4}^k. Moreover, $d_{g_{x,0}}(\xi_i, 0) \geq s$ and $d_{g_{x,0}}(\xi_i,\xi_j) \geq s$ for any $i \neq j$.

From (3.2.20), we can deduce that for any $r > 0$,

$$\begin{aligned} r^{4-n}\mu_{x,r_k}(B_r(\xi_j^k, g_{x,r_k})) &= (rr_k)^{4-n}\mu(B_{rr_k}(x_j^k)) \\ &\geq \Theta(\mu, x_j^k) \geq \Theta(\mu,x) - \varepsilon(r_k). \end{aligned}$$

Thus for all $r < 0$,

$$r^{4-n}\eta(B_r(\xi_j, g_{x,0})) \geq \Theta(\mu,x) = \Theta(\eta,0). \tag{3.2.22}$$

In particular,

$$\Theta(\eta,\xi_j)\ge\Theta(\eta,0). \tag{3.2.23}$$

On the other hand, for any $r,\tilde r>0$, it follows from the monotonicity,

$$\begin{aligned} r^{4-n}\eta(B_r(\xi_j,g_{x,0})) &= \lim_{k\to\infty} r^{4-n}\mu_{x,r_k}(B_r(\xi_j^k,g_{x,r_k}))\\ &= \lim_{k\to\infty} rr_k^{4-n}\mu(B_{rr_k}(x_j^k))\\ &\le \lim_{k\to\infty}\left(e^{a\tilde r^2}\tilde r^{4-n}\mu(B_{\tilde r}(x_j^k))\right)\\ &= e^{a\tilde r^2}\tilde r^{4-n}\mu(B_{\tilde r}(x)). \end{aligned}$$

Since $\tilde r$ can be arbitrarily small,

$$r^{4-n}\eta(B_r(\xi_j,g_{x,0}))=\Theta(\eta,0)$$

for any $r>0$. Then, using Theorem 2.1.1 as in the proof of Lemma 3.2.1, we can show that η is a cone measure with center at ξ_j for each $j=1,\cdots,n-4$; i.e.,

$$d\eta(r_j,\theta)=r_j^{n-5}dr_jd\xi(\theta)$$

for some Radon measure $d\xi_j(\theta)$ on the unit sphere $\{\xi\in T_xM|r_j(\xi)=1\}$, where $r_j(\xi)=|\xi-\xi_j|$. Clearly, it follows that

$$\eta(y_1,\cdots,y_{n-4},y_{n-3},\cdots,y_n)=\eta(y_{n-3},\cdots,y_n)$$

where $y_1,\cdots,y_n$ denote the euclidean coordinates of T_xM such that $y_1,\cdots,y_{n-4}$ are in F.

Finally, by the second equality in (3.2.19), we have that $\mathrm{supp}(\eta)\subset F$. Therefore, $\eta=\Theta(\mu,x)H^{n-4}\lfloor F$.

3.3. *Rectifiability.* We have shown that tangent cones exist at H^{n-4}-a.e. x in S; moreover, if (3.2.19) holds and $\Theta(\mu,\cdot)$ is H^{n-4}-approximately continuous at $x\in S$, then any tangent cones at x are $(n-4)$-subspaces in T_xM (Proposition 3.2.4). We adopt the notation of the last section. In this section, we will prove that S is rectifiable, i.e., tangent cones are unique at H^{n-4}-a.e. x in S. This in fact follows from the work of D. Priess [P], since $\Theta(\nu,\cdot)$ exists almost everywhere and ν is Borel regular. However, for the reader's convenience, we give a direct proof here by using the structure theorem of Federer (cf. [Fe], [Li]).

We may write $S=S_u\cup S_r$, where S_r is a rectifiable set and S_u is a purely unrectifiable set. We denote by $G(T_xM,n-4)$ the Grassmannian of all $(n-4)$-dimensional subspaces in T_xM.

LEMMA 3.3.1. *For any $x \in M$ and V in $G(T_xM, n-4)$,*

$$H^{n-4}(P_V(\exp_x^{-1}(B_r(x) \cap S_u))) = 0$$

where $r > 0$ is sufficiently small and P_V denotes the orthogonal projection of T_xM onto V with respect to $g_{x,0}$.

This lemma can easily be proved by modifying the arguments in the proof of [Fe, 3.3.5] or [Si2]. We omit it here.

We want to show that $H^{n-4}(S_u) = 0$. Suppose that it is not true. Then for H^{n-4}-a.e. x in S_u, $r > 0$ small and any $V \in G(T_xM, n-4)$,

$$H^{n-4}(P_V(\exp_x^{-1}(S_u \cap B_r(x)))) = 0 \tag{3.3.1}$$

and

$$\overline{\lim}_{\lambda \to 0+} \frac{H^{n-4}(S_r \cap B_\lambda(x))}{\lambda^{n-4}} = 0. \tag{3.3.2}$$

Since $H^{n-4}(S_u) > 0$, we can choose x in S_u such that (3.2.19), (3.3.1) and (3.3.2) hold, and $\Theta(\mu, \cdot)$ is H^{n-4}-approximately continuous at x. As before, we define $\mu_{x,\lambda}$ by

$$\mu_{x,\lambda}(E) = \lambda^{n-4}\mu(\exp_x(\lambda E)) \tag{3.3.3}$$

where $E \subset T_xM$. Let $\{\lambda_k\}$ be a sequence of positive numbers such that $\lim_{k\to\infty} \lambda_k = 0$ and μ_{x,λ_k} converges weakly to a tangent measure η on T_xM. By our choice of x and the proof of Proposition 3.2.4, we have that $\eta = \Theta(\mu, x)H^{n-4}\lfloor V$ for some $(n-4)$-subspace V in T_xM. We claim:

$$\overline{\lim}_{k\to\infty} \frac{H^{n-4}(P_V(\exp_x^{-1}(S \cap B_{\lambda_k}(x))))}{\lambda_k^{n-4}} > 0\,. \tag{3.3.4}$$

If this is true, then

$$\overline{\lim}_{k\to\infty} \frac{H^{n-4}(P_V(\exp_x^{-1}(S_u \cap B_{\lambda_k}(x))))}{\lambda_k^{n-4}} > 0\,, \tag{3.3.5}$$

because of (3.3.2). However, this contradicts (3.3.1).

Now we prove the claimed inequality in (3.3.4). As in the last section, we may find a sequence of Yang-Mills connections A_{i,x,λ_k} (cf. (3.2.3)) such that the $\left|F_{A_{i,x,\lambda_k}}\right|^2 dV_{x,\lambda_k}$ converge to μ_{x,λ_k} weakly as $i \to \infty$. Note that for k large enough, the A_{i,x,λ_k} are well defined in $B_4(0, g_{x,\lambda_k}) \subset T_xM$. Let us identify T_xM with $V \times V^\perp$, so that each point $z \in T_xM$ is of the form (z', z'') with $z' \in V$ and $z'' \in V^\perp$, where $V^\perp$ is the orthogonal complement of V in T_xM. Choose orthonormal coordinates $z_1, \cdots, z_n$ of T_xM with respect to $g_{x,0}$, such that $z_1, \ldots, z_{n-4}$ are coordinates of V and $z_{n-3}, \ldots, z_n$ are coordinates of $V^\perp$. We usually denote z' by $(z_1, \ldots, z_{n-4})$ and z'' by $(z_{n-3}, \ldots, z_n)$. We put

$$B_2^2(0) = \left\{z'' \in V^\perp \,|\, |z''| < 2\right\}\,.$$

Clearly, when k is sufficiently large (so that g_{x,λ_k} is sufficiently close to the flat metric $g_{x,0}$), we have that $(z',0) + \{0\} \times B_2^2(0) \subset B_4(0, g_{x,\lambda_k})$ for any $(z',0) \in V \times \{0\} \cap B_2(0, g_{x,\lambda_k})$.

Consider

$$m_{i,k}(z') = \int_{B_2^2(0)} \left| F_{A_{i,x,\lambda_k}} \right|^2 (z', z'') \phi^2(z'')\, dV_k(z''), \tag{3.3.6}$$

where $dV_k(z'')$ denotes the induced volume form on $B_2^2(0)$ by the metric g_{x,λ_k}, and $\phi \in C_0^\infty(B_2^2(0))$ with $\int_{B_2^2(0)} \phi^2 dV_{g_{x,0}} = 1$. Then $m_{i,k}$ is a smooth function of z' in $V \cap B_2(0, g_{x,\lambda_k})$.

For simplicity, we will denote by D the covariant derivative associated to each A_{i,x,λ_k} unless further specification is needed. For simplicity, we often abbreviate $\frac{\partial}{\partial z_\alpha}$ as ∂_α. One computes

(3.3.7)

$$\begin{aligned} \frac{\partial m_{i,k}(z')}{\partial z_\alpha} &= \frac{\partial}{\partial z_\alpha} \left(\int_{B_2^2(0)} \left| F_{A_{i,x,\lambda_k}} \right|^2 (z', z'') \phi^2(z'')\, dV_k(z'') \right) \\ &= 2 \int_{B_2^2(0)} \frac{\partial}{\partial z_\alpha} \left(F_{A_{i,x,\lambda_k}}, F_{A_{i,x,\lambda_k}} \right) (z', z'') \phi^2(z'')\, dV_k(z''). \end{aligned}$$

Since g_{x,λ_k} converges to the flat metric $g_{x,0}$ on $T_x M$ as $k \to \infty$,

$$g_{x,\lambda_k}(\partial_\alpha, \partial_\beta) = \delta_{\alpha\beta} + o(1), \quad \alpha, \beta = 1, 2, \cdots, n, \tag{3.3.8}$$

$$\nabla^k_{\partial_\alpha} \partial_\beta = o(1),\ \alpha, \beta = 1, 2, \cdots, n, \tag{3.3.9}$$

where ∇^k denotes the Levi-Civita connection of g_{x,λ_k}, and $o(1)$ always denotes a quantity which converges to zero as $k \to \infty$. It follows from (3.3.8) and (3.3.9) that

$$\begin{aligned} \frac{\partial}{\partial z_\alpha} &= 2 \sum_{\beta,\gamma,\beta',\gamma'=1}^n \left(\left| F_{A_{i,x,\lambda_k}} \right|^2 D_{\partial_\alpha} F_{A_{i,x,\lambda_k}} (\partial_\beta, \partial_\gamma), F_{A_{i,x,\lambda_k}} (\partial_{\beta'}, \partial_{\gamma'}) \right) \\ &\quad \cdot g_{x,\lambda_k}^{\beta\beta'} g_{x,\lambda_k}^{\gamma\gamma'} + o(1) \left| F_{A_{i,x,\lambda_k}} \right|^2, \end{aligned}$$

where $\{g_{x,\lambda_k}^{\alpha\beta}\}$ is the inverse matrix of $\{g_{x,\lambda_k}(\partial_\alpha, \partial_\beta)\}$. By the second Bianchi identity $DF_{A_{i,x,\lambda_k}} = 0$, we deduce from the above

(3.3.10)

$$\begin{aligned} \frac{\partial}{\partial z_\alpha} \left| F_{A_{i,x,\lambda_k}} \right|^2 &= 4 \sum_{\beta,\gamma,\beta',\gamma'=1}^n \left(D_{\partial_\beta} F_{A_{i,x,\lambda_k}} (\partial_\alpha, \partial_\gamma), F_{A_{i,x,\lambda_k}} (\partial_{\beta'}, \partial_{\gamma'}) \right) \\ &\quad \cdot g_{x,\lambda_k}^{\beta\beta'} g_{x,\lambda_k}^{\gamma\gamma'} + o(1) \left| F_{A_{i,x,\lambda_k}} \right|^2 \end{aligned}$$

$$
\begin{aligned}
&= 4\sum \frac{\partial}{\partial z_\beta}\left(F_{A_{i,x,\lambda_k}}(\partial_\alpha\,,\partial_\gamma)\,,\,F_{A_{i,x,\lambda_k}}(\partial_{\beta'}\,,\partial_{\gamma'})\right)g^{\beta\beta'}_{x,\lambda_k}g^{\gamma\gamma'}_{x,\lambda_k}\\
&\quad - 4\sum_{\beta,\gamma=1}^{2}\left(F_{A_{i,x,\lambda_k}}(\partial_\alpha\,,\partial_\gamma)\,,\,D_{\partial_\beta}F_{A_{i,x,\lambda_k}}(\partial_{\beta'}\,,\partial_{\gamma'})\right)\\
&\quad \cdot g^{\beta\beta'}_{x,\lambda_k}g^{\gamma\gamma'}_{x,\lambda_k} + o(1)\left|F_{A_{i,x,\lambda_k}}\right|^2 .
\end{aligned}
$$

Since A_{i,x,λ_k} is a Yang-Mills connection with respect to g_{x,λ_k},

$$
g^{\beta\beta'}_{x,\lambda_k}D_{\partial_\beta}F_{A_{i,x,\lambda_k}}(\partial_{\beta'}\,,\partial_\gamma)=0. \tag{3.3.11}
$$

Combining this with (3.3.10), we deduce for $\alpha\le n-4$,

$$
\begin{aligned}
\frac{\partial m_{i,k}(z')}{\partial z_\alpha} &= 4\sum_{\beta,\gamma=1}^{n}\int_{B_2^2(0)}\partial_\beta\left(F_{A_{i,x,\lambda_k}}(\partial_\alpha\,,\partial_\gamma)\,,\,F_{A_{i,x,\lambda_k}}(\partial_{\beta'}\,,\partial_{\gamma'})\right)\\
&\quad\cdot g^{\beta\beta'}_{x,\lambda_k}g^{\gamma\gamma'}_{x,\lambda_k}\phi^2(z'')dV_k(z'') + o(1)\int_{B_2^2(0)}\left|F_{A_{i,x,\lambda_k}}\right|^2\phi^2(z'')dV_k(z'')\\
&= -4\sum_{\beta=n-3}^{n}\int_{B_2^2(0)}\left(\partial_\alpha\rfloor F_{A_{i,x,\lambda_k}}\,,\,\partial_{\beta'}\rfloor F_{A_{i,x,\lambda_k}}\right)g^{\beta\beta'}_{x,\lambda_k}\partial_\beta\phi^2(z'')dV_k(z'')\\
&\quad + 4\sum_{\beta=1}^{n-4}\frac{\partial}{\partial z_\beta}\left(\int_{B_2^2(0)}\left(\partial_\alpha\rfloor F_{A_{i,x,\lambda_k}}\,,\,\partial_\beta\rfloor F_{A_{i,x,\lambda_k}}\right)\phi^2(z'')dV_k(z'')\right)\\
&\quad + o(1)\int_{B_2^2(0)}\left|F_{A_{i,x,\lambda_k}}\right|^2\phi^2(z'')dV_k(z'') .
\end{aligned}
$$

To estimate these derivatives, we need the following:

LEMMA 3.3.2. *Let $\{A_{i,x,\lambda,k}\}$, x etc. be defined as above. Then for any $\alpha\le n-4$,*

$$
\lim_{k\to\infty}\lim_{i\to\infty}\int_{B_4(0,g_{x,\lambda_k})}\left|\frac{\partial}{\partial z_\alpha}\rfloor F_{A_{i,x,\lambda_k}}\right|^2 dV_{x,\lambda_k}=0\,. \tag{3.3.12}
$$

Proof. By our assumption, $\left|F_{A_{i,x,\lambda_k}}\right|^2 dV_k$ converges to μ_{x,λ_k} weakly as $i\to\infty$ and $\mu_{x,\lambda_k}\to\eta$ as $k\to\infty$. Moreover, η is of the form $\Theta(\mu,x)H^{n-4}\lfloor V$. Therefore, for any $\delta>0$,

$$
\lim_{k\to\infty}\lim_{i\to\infty}\int_{B_4(0,g_{x,\lambda_k})\setminus T_\delta(V)}\left|F_{A_{i,x,\lambda_k}}\right|^2 dV_{x,\lambda_k}=0, \tag{3.3.13}
$$

where $T_\delta(V)$ denotes the δ-tubular neighborhood of the subspace V.

Let $x_1^k, \cdots, x_{n-4}^k \in B_{\lambda_k}(x) \cap S$ be chosen as in (3.2.25) and (3.2.26). We let V^k be the subspace in T_xM spanned by $\xi_1^k = \exp_x^{-1}(x_1^k), \ldots, \xi_{n-4}^k = \exp_x^{-1}(x_{n-4}^k)$. Then the V^k converge to V, and these V^k are spanned by $\xi_j = \lim_{k\to\infty} \xi_j^k$. Moreover, we may assume that $d_{g_{x,0}}(\xi_i, \xi_j) \geq s$ for $i \neq j$ and $d_{g_{x,0}}(\xi_i, 0) \geq s$, where s is as given in Lemma 3.2.3. We have shown in the proof of Lemma 3.2.3,

$$6^{4-n}\mu_{x,\lambda_k}(B_6(\xi_j^k, g_{x,\lambda_k})) \geq \Theta(\mu, x) - \varepsilon(\lambda_k). \tag{3.3.14}$$

Note that $\varepsilon(\cdot)$ is a nondecreasing function with $\lim_{r\to 0}\varepsilon(r) = 0$. Choose $i(k)$ such that for any $i \geq i(k)$,

$$6^{4-n}\int_{B_6(\xi_j^k, g_{x,\lambda_k})} |F_{A_{i,x,\lambda_k}}|^2 dV_k \geq \Theta(\mu, x) - 2\varepsilon(\lambda_k). \tag{3.3.15}$$

Since $\lim\limits_{k\to\infty} \mu_{x,\lambda_k} = \eta$ and $\Theta(\eta, \xi_j) = \Theta(\mu, x)$ (cf. (3.2.28)), by increasing $\varepsilon(r)$ if necessary, we may assume that

$$\left|6^{4-n}\mu_{x,\lambda_k}(B_6(\xi_j^k, g_{x,\lambda_k})) - \Theta(\mu, x)\right| \leq \varepsilon(\lambda_k). \tag{3.3.16}$$

By taking $i(k)$ big enough, we may further have that for $i \geq i(k)$,

$$6^{4-n}\int_{B_6(\xi_j^k, g_{x,\lambda_k})} |F_{A_{i,x,\lambda_k}}|^2 dV_k \leq \Theta(\mu, x) + 2\varepsilon(\lambda_k). \tag{3.3.17}$$

Then we deduce from this and the monotonicity (Theorem 2.1.2) that

$$\int_{B_6(\xi_j^k, g_{x,\lambda_k})\setminus B_s(\xi_j^k, g_{x,\lambda_k})} (\rho_j^k)^{4-n} \left|\frac{\partial}{\partial \rho_j^k} \rfloor F_{A_{i,x,\lambda_k}}\right|^2 dV_k \leq 2\varepsilon(\lambda_k), \tag{3.3.18}$$

where $\xi_0 = 0$, ρ_j^k is the distance from ξ_j^k of g_{x,λ_k} $(j = 0, 1, \cdots, n-4)$.

Then the lemma follows from (3.3.18) and (3.3.13) and the fact that the ξ_j span the subspace V. □

Notice that the integral

$$\int_{B_4(0, g_{x,\lambda_k})} \left|F_{A_{i,x,\lambda_k}}\right|^2 dV_{x,\lambda_k}$$

is uniformly bounded. Thus we have

$$\text{grad } m_{i,k} = f_{i,k} + \operatorname{div}(u_{i,k})\,, \tag{3.3.19}$$

where $f_{i,k} : V \cap B_2(0, g_{x,\lambda_k}) \to V$ and $u_{i,k} : V \cap B_2(0, g_{x,\lambda_k}) \to V \times V$ are functions, such that

$$\lim_{k\to\infty}\lim_{i\to\infty}\int_{V\cap B_2(0, g_{x,\lambda_k})} (|f_{i,k}| + |u_{i,k}|)\, dV_{x,0} = 0\,. \tag{3.3.20}$$

Then it follows (cf. [AL]) that there are constants $C_{i,k}$, such that

$$\lim_{k\to\infty}\lim_{i\to\infty}\|m_{i,k}-C_{i,k}\|_{L^1(V\cap B_{\frac{4}{3}}(0,g_{x,\lambda_k}))}=0. \tag{3.3.21}$$

In fact, since

$$\lim_{k\to\infty}\lim_{i\to\infty}\left|F_{A_{i,x,\lambda_k}}\right|^2 dV_{x,\lambda_k}=\eta$$

and

$$\eta=\Theta(\mu,x)H^{n-4}\lfloor V,$$

we have

$$\lim_{k\to\infty}\lim_{i\to\infty}C_{i,k}=\Theta(\mu,x)>0.$$

For k sufficiently large, the ball $B_{\frac{3}{2}}(0,g_{x,0})$ is contained in every $B_{\frac{4}{3}}(0,g_{x,\lambda_k})$. Then for any $\xi\in C_0^\infty(V\cap B_{\frac{3}{2}}(0,g_{x,0}))$,

(3.3.22)

$$\begin{aligned}
&\Theta(\mu,x)\int_{V\cap B_{\frac{3}{2}}(0,g_{x,0})}\xi(z')\,dz'\\
&\quad=\lim_{k\to\infty}\lim_{i\to\infty}\int_{V\cap B_{\frac{3}{2}}(0,g_{x,0})}\xi(z')m_{i,k}(z')\,dz'\\
&\quad=\lim_{k\to\infty}\lim_{i\to\infty}\int_{B_2(0,g_{x,\lambda_k})}\left|F_{A_{i,x,\lambda_k}}\right|^2(z',z'')\xi(z')\phi^2(z'')\,dV_{x,\lambda_k}\\
&\quad=\lim_{k\to\infty}\int_{B_2(0,g_{x,\lambda_k})}\xi(z')\phi^2(z'')\,d\mu_{x,\lambda_k}(z',z'')\,.
\end{aligned}$$

However, as a weak limit of Radon measures $|F_{A_i}|^2\,dV_g$, the measure μ is of the form $|F_A|^2\,dV_g+\nu$. After scaling, we have

$$\mu_{x,\lambda_k}=\left|F_{A_{x,\lambda_k}}\right|^2 dV_{x,\lambda_k}+\nu_{x,\lambda_k}\,, \tag{3.3.23}$$

where A_{x,λ_k} is a connection on $T_xM\backslash\lambda_k^{-1}\exp_x^{-1}(S)$ as defined in (3.2.3), and ν_{x,λ_k} is a Radon measure on T_xM of the form

$$\Theta(\mu_{x,\lambda_k},\cdot)H^{n-4}\lfloor\lambda_k^{-1}\exp_x^{-1}(S)\,. \tag{3.3.24}$$

Using the second equation in (3.2.19) which holds at x by our assumption, we see that

$$\lim_{k\to\infty}\int_{B_2(0,g_{x,\lambda_k})}\xi(z')\phi^2(z'')\left|F_{A_{x,\lambda_k}}\right|^2 dV_{x,\lambda_k}=0\,. \tag{3.3.25}$$

Hence, by (3.3.25)

$$\begin{aligned}
&\Theta(\mu,x)\int_{V\cap B_{\frac{3}{2}}(0,g_{x,0})}\xi(z')\,dz' \\
&\quad= \lim_{k\to\infty}\int_{B_{\frac{3}{2}}(0,g_{x,0})\cap\lambda_k^{-1}\exp_x^{-1}(S)}\xi(z')\Theta(\mu_{x,\lambda_k},(z',z''))\,dH^{n-4}(z',z'') \\
&\quad= \Theta(\mu,x)\lim_{k\to\infty}\int_{B_{\frac{3}{2}}(0,g_{x,0})\cap\lambda_k^{-1}\exp_x^{-1}(S)}\xi(z')\,dH^{n-4}(z',z'')\,.
\end{aligned}$$

Since $\Theta(\mu,x)>0$, this implies

$$\begin{aligned}
&\overline{\lim_{k\to\infty}}\frac{H^{n-4}(P_V(\exp_x^{-1}(S\cap B_{\lambda_k}(x))))}{\lambda_k^{n-4}} \\
&\qquad= \overline{\lim_{k\to\infty}}H^{n-4}(P_V(\lambda_k^{-1}\exp_x^{-1}(S\cap B_1(0,g_{x,\lambda_k})) \\
&\qquad\geq \mathrm{Vol}\,(V\cap B_{\frac{1}{2}}(0,g_{x,0}))>0.
\end{aligned}$$

Thus (3.3.4) is proved and we obtain a contradiction to (3.3.1). Hence, $H^{n-4}(S_u)=0$ and we have shown the following:

PROPOSITION 3.3.3. *Let (S_b,Θ) be the blow-up locus of a weakly convergent sequence $\{A_i\}$. Then its support S_b is H^{n-4}-rectifiable. In particular, for H^{n-4}-a.e. x in S_b, there is a unique tangent subspace $T_xS_b\subset T_xM$.*

4. Structure of blow-up loci

In this chapter, we study the geometry of blow-up loci.

4.1. *Bubbling Yang-Mills connections.* We assume that $\{A_i\}$ converges to an admissible Yang-Mills connection A with the blow-up locus (S,Θ) (cf. Lemma 3.1.4). It is shown in Section 3.3 that S is H^{n-4}-rectifiable. We will adopt the notation of the last chapter.

If $n=4$, S consists of finitely many points. K. Uhlenbeck further showed that when i is sufficiently large, A_i approaches a connected sum of A with certain Yang-Mills connections on the unit sphere S^4. These later connections are called bubbling connections.

In this section, we analyze the structure of A_i near S when i is sufficiently large. We will construct bubbling connections on $\mathbb{R}^n$ as A_i approaches A.

Recall that μ is the weak limit of Radon measures $|F_{A_i}|^2dV_g$ and is of the form $|F_A|^2dV_g+\Theta(\mu,\cdot)H^{n-4}\lfloor S$.

PROPOSITION 4.1.1. *Let $x \in S$ satisfy*:

(1) *The tangent plane $V = T_xS \subset T_xM$ exists uniquely*;

(2) (3.2.19) *holds for μ and A.*

Then there are linear transformations $\sigma_i : T_xM \mapsto T_xM$ such that a subsequence of $\sigma_i^* \exp_x^* A_i$ converges to a Yang-Mills connection B on T_xM such that $F_B \neq 0$ and $v \rfloor F_B \equiv 0$ for any $v \in V$. Such a connection B is called a bubbling connection at $x \in S$.

The rest of this section is devoted to the proof of Proposition 4.1.1.

Let $A_{i,x,\lambda}$ be the scaled connections on T_xM defined in (3.2.4), i.e.,

$$A_{i,x,\lambda} = \tau_\lambda^* \exp_x^* A_i, \tag{4.1.1}$$

where $\tau_\lambda(v) = \lambda v$ for any v in T_xM. Each $A_{i,x,\lambda}$ is a Yang-Mills connection with respect to the scaled metric $g_{x,\lambda}$. As i tends to infinity, $|F_{A_{i,x,\lambda}}|^2 dV_{x,\lambda}$ converges to $\mu_{x,\lambda}$ weakly. On the other hand, as λ tends to zero, $\mu_{x,\lambda}$ converges to $\Theta(\mu, x)H^{n-4}\lfloor V$ weakly. Therefore, there is a sequence λ_i such that the Radon measure $|F_{A_{i,x,\lambda_i}}|^2 dV_{x,\lambda_i}$ converges to $\Theta(\mu, x)H^{n-4}\lfloor V$ weakly. Moreover, modulo gauge transformations, A_{i,x,λ_i} converges to 0 uniformly on any compact subsets in $T_xM \backslash V$. This implies particularly that for i sufficiently large,

$$|F_{A_{i,x,\lambda_i}}|(v) \leq \frac{\varepsilon(r)}{r^2}. \tag{4.1.2}$$

We also have (cf. Lemma 3.3.2)

$$\lim_{i\to\infty} \left(\sum_{\alpha=1}^{n-4} \int_{B_2(0,g_{x,0})} |\frac{\partial}{\partial z_\alpha} \rfloor F_{A_{i,x,\lambda_i}}|^2 dV_{g_{x,\lambda_i}} \right) = 0, \tag{4.1.3}$$

where $\{z_1, \cdots, z_{n-4}\}$ is an orthogonal coordinate system of V.

As in the last section, we denote by $z = (z', z'')$ a point in T_xM with $z' \in V, z'' \in V^\perp$. We will identify V and $V^\perp$ with the subspaces $V \times \{0\}$ and $\{0\} \times V^\perp$ in T_xM.

LEMMA 4.1.2. *There are points z_i' in $V \cap B_{\frac{1}{2}}(0, g_{x,0})$ with $\lim_{i\to\infty} z_i' = 0$, such that*

(4.1.4)

$$\lim_{i\to\infty} \left(\sup_{0<r\leq\frac{1}{2}} r^{4-n} \int_{V\cap B_r(z_i',g_{x,0})} dx' \int_{V^\perp \cap B_{\frac{1}{2}}(0,g_{x,0})} \sum_{\alpha=1}^{n-4} |\frac{\partial}{\partial z_\alpha} \rfloor F_{A_{i,y,\lambda_i}}|^2 dV_{x,\lambda_i} \right) = 0.$$

Proof. We prove this by contradiction. Suppose that the lemma is false, then we can find $\delta > 0$ and $s \in (0, \frac{1}{2})$, such that for any i and $z' \in V \cap$

$B_s(0, g_{x,0})$, there is at least one $r = r(i, z')$ such that

$$r^{4-n} \int_{V\cap B_r(z',g_{x,0})} dx' \int_{V^\perp \cap B_{\frac{1}{2}}(0,g_{x,0})} \sum_{\alpha=1}^{n-4} |\frac{\partial}{\partial z_\alpha} \rfloor F_{A_{i,x,\lambda_i}}|^2 dx'' \geq \delta. \tag{4.1.5}$$

By (4.1.3), $\lim_{i\to\infty} r(i, z') = 0$ for any z'. For each i, we cover $V \cap B_{\frac{1}{2}}(0, g_{x,0})$ by finitely many disjoint balls $V \cap B_{r(i,z'_{i\alpha})}(z'_{i\alpha}, g_{x,0})$ $(\alpha = 1, 2, \cdots, m_i)$, such that

$$V \cap B_s(0, g_{x,0}) \subset \bigcup_{\alpha=1}^{m_i} V \cap B_{2r(i,z'_{i\alpha})}(z'_{i\alpha,}, g_{x,0}). \tag{4.1.6}$$

Then,

$$\begin{aligned}
\delta\left(\frac{s}{2}\right)^{n-4} &\leq \delta 2^{4-n} \sum_{\alpha=1}^{m_i} (2r(i, z'_{i\alpha}))^{n-4} = \delta \sum_{\alpha=1}^{m_i} r(i, z'_{i\alpha})^{n-4} \\
&\leq \sum_{\alpha=1}^{m_i} \int_{V\cap B_{r(i,z'_{i\alpha})}(z'_{i\alpha},g_{x,0})} dx' \int_{V^\perp\cap B_{\frac{1}{2}}(0,g_{x,0})} \sum_{\beta=1}^{n-4} \left|\frac{\partial}{\partial z_\beta} \rfloor F_{A_{i,x,\lambda_i}}\right|^2 dx'' \\
&\leq \int_{B_2(0,g_{x,0})} \left(\sum_{\beta=1}^{n-4} \left|\frac{\partial}{\partial z_\beta} \rfloor F_{A_{i,x,\lambda_i}}\right|^2\right) dV_{x,\lambda_i}.
\end{aligned}$$

This is impossible when i is sufficiently large because of (4.1.3). $\square$

Observe that for any $\delta > 0$,

$$\max_{z''\in V^\perp\cap B_{\frac{1}{2}}(0,g_{x,0})} \delta^{4-n} \int_{B_\delta(z'_i+z'',g_{x,0})} |F_{A_{i,x,\lambda_i}}|^2 dV_{x,\lambda_i} \geq \varepsilon, \tag{4.1.7}$$

where ε is as in Theorem 2.2.1. Otherwise, A_{i,x,λ_i} would converge to a smooth Yang-Mills connection on $(V \cap B_\delta(z'_i, g_{x,0})) \times (V^\perp \cap B_{\frac{1}{2}}(0, g_{x,0}))$, contradicting our assumption on $A_{i,x,\lambda}$.

Because of (4.1.7), we can find $\delta_i \in (0, \frac{1}{2})$ and $z''_i \in V^\perp \cap B_{\frac{1}{4}}(0, g_{x,0})$, such that

$$\begin{aligned}
\delta_i^{4-n} &\int_{B_{\delta_i}(z'_i+z''_i,g_{x,0})} |F_{A_{i,x,\lambda_i}}|^2 dV_{x,\lambda_i} \\
&= \max_{z''\in V^\perp\cap B_{\frac{1}{2}}(0,g_{x,0})} \delta_i^{4-n} \int_{B_{\delta_i}(z'_i+z'',g_{x,0})} |F_{A_{i,x,\lambda_i}}|^2 dV_{x,\lambda_i} = \frac{\varepsilon}{4}.
\end{aligned} \tag{4.1.8}$$

One may even take z''_i such that $\lim_{i\to\infty} z''_i = 0$. Now we define new connections

$$B_i(y) = A_{i,x,\lambda_i}(z'_i + z''_i + \delta_i y). \tag{4.1.9}$$

Each B_i is a Yang-Mills connection with respect to the scaled metric $g_i' = \delta_i^{-2} g_{x,\lambda_i}$ on $B_{4R_i}(0, g_{x,0})$, where $R_i = (4\delta_i)^{-1}$. Note that the based manifolds $(T_xM, g_i', z_i' + z_i'')$ converge to $(T_xM, g_{x,0}, 0)$ as $i \to \infty$.

Using (4.1.4) and (4.1.8), we have

$$\lim_{i\to\infty}\left(\sum_{\alpha=1}^{n-4}\int_{B_{R_i}(0,g_{x,0})}|\frac{\partial}{\partial z_\alpha}\rfloor F_{B_i}|^2 dV_{g_i'}\right) = 0, \tag{4.1.10}$$

(4.1.11)

$$\int_{B_1(0,g_{x,0})}|F_{B_i}|^2 dV_{g_i'} = \max_{y\in V^\perp\cap B_{R_i-1}(0,g_{x,0})}\int_{B_1((0,y),g_{x,0})}|F_{B_i}|^2 dV_{g_i'} = \frac{\varepsilon}{4}.$$

It follows from the monotonicity formula that

$$\sup_i\left\{\int_{B_R(0,g_{x,0})}|F_{B_i}|^2 dV_{g_i'}\right\} \le C(\Lambda)R^{n-4}, \tag{4.1.12}$$

for $0 < R < R_i$, where $C(\Lambda)$ denotes a constant depending only on Λ.

By (4.1.12), Proposition 3.1.2 and by taking a subsequence if necessary, we may assume that B_i converges to an admissible Yang-Mills connection B. It follows from (4.1.11) that B is a smooth Yang-Mills connection on $(V \cap B_1(0, g_{x,0})) \times V^\perp \subset T_xM$ with respect to $g_{x,0}$.

Moreover, (4.1.10) implies that for any $v \in V$,

$$v\rfloor F_B = 0, \tag{4.1.13}$$

whenever B is well-defined.

On the band $((V \cap B_1(0, g_{x,0}))) \times V^\perp$, we write

$$B = \sum_{\alpha=1}^{n} B^\alpha dy_\alpha,$$

where $B^\alpha \in \mathrm{Lie}(G)$ and $y_1, \cdots, y_n$ are euclidean coordinates such that $y_1, \cdots, y_{n-4}$ are tangent to V along V. Let us eliminate B^α for $\alpha \le n-4$ inductively. First, by a gauge transformation, we may assume that $B^1 = 0$; then (4.1.13) implies that all B^α are independent of y_1. Again taking a gauge transformation, we can get rid of B^2, and so on. Eventually, by finitely many gauge transformations, we arrive at a connection, still denoted by B, which is a pull-back of some connection on $V^\perp$. This implies that B extends to a smooth connection on T_xM. Proposition 4.1.1 is proved.

4.2. *Blow-up loci of anti-self-dual instantons.* Now we assume that $\{A_i\}$ is a sequence of Ω-anti-self-dual instantons which converge to an admissible Ω-anti-self-dual instanton A, where Ω is a form on M of degree $n-4$. The closedness of Ω is not needed in this section. Let $S \subset M$ be the blow-up locus of $\{A_i\}$. Here, we will show that Ω restricts to the induced volume form on S. If Ω is a calibrating form as in [HL], then S is calibrated by Ω and is particularly minimal.

First we observe that there is more information on the bubbling connection constructed in Proposition 4.1.1 in case of anti-self-dual instantons.

PROPOSITION 4.2.1. *Let M, g, Ω, $\{A_i\}$, A and S be as above. Suppose that $x \in S$ satisfies*:

(1) *The tangent cone $T_xS \subset T_xM$ exists uniquely*;

(2) (3.2.19) *holds for μ and A, where μ is the weak limit of Radon measures $|F_{A_i}|^2 dV_g$.*

Then there is an Ω_x-anti-self-dual instanton B on T_xM, where $\Omega_x = \Omega|_{T_xM}$, such that $F_B \neq 0$, $\mathrm{tr}(F_B) = 0$ and $v \rfloor F_B = 0$ for any $v \in T_xS$.

Proof. The proof is basically the same as the proof of Proposition 4.1.1.

First, we observe that $\mathrm{tr}(F_{B_i})$ converges to zero uniformly as i tends to infinity, where B_i are the scaled connections defined in (4.1.9). This is because $\mathrm{tr}(F_{A_i})$ are harmonic 2-forms with uniformly bounded L^2-norm.

Secondly, we observe that B_i are Ω_i'-anti-self-dual with respect to the metric g_i' and the closed form Ω_i' of degree $n-4$ on $B_{4R_i}(0, g_{x,0})$ defined by

$$\Omega_i' = \tau^{\delta_i *}_{(z_i', z_i'')} \exp_x^* \Omega,$$

where $\tau^{\delta_i}_{(z_i', z_i'')} : T_xM \mapsto T_xM,\ y \mapsto (z_i', z_i'') + \delta_i y$.

Since (z_i', z_i'') goes to 0 as i tends to ∞, Ω_i' converges to Ω_x. Therefore, the limit connection B is Ω_x-anti-self-dual with respect to $g_{x,0}$ and $\mathrm{tr}(F_B) = 0$.

The rest of the proof follows the same arguments as those in the proof of Proposition 4.1.1. □

COROLLARY 4.2.2. *Let $x \in S$ be as in the last proposition; then Ω_x restricts to a volume form on $T_xS \subset T_xM$ which is induced by the flat metric $g_{x,0}$.*

Proof. We identify T_xM with $\mathbb{R}^n$, where n is the dimension of M, such that $g_{x,0}$ is the standard euclidean metric g_0. Let $*$ be the Hodge operator of g_0. Then the connection B satisfies

$$F_B = - * (\Omega_x \wedge F_B). \tag{4.2.1}$$

Define a degree $n-4$, constant form $\Phi_{S,x}$ on T_xM as follows: let $x_1, \cdots, x_n$ be any euclidean coordinates of T_xM such that $x_1, \cdots, x_{n-4}$ are tangent to T_xS; then

$$\Phi_{S,x} = dx_1 \wedge \cdots \wedge dx_{n-4}.$$

Now we decompose $\Omega_x = \alpha \Phi_{S,x} + \Omega_0$, where α is a constant and $\Omega_0|_{T_xS} = 0$.

Since $v\rfloor F_B = 0$ for any $v \in T_xS$, by taking a gauge transformation if necessary, we may assume that $B = \pi_L^* B_L$ for some nontrivial connection B_L on L, where L is the orthogonal complement of T_xS and π_L is the orthogonal projection from T_xM onto L. Then (4.2.1) becomes

$$F_{B_L} = -\alpha *_L F_{B_L}, \tag{4.2.2}$$

$$0 = *(\Omega_0 \wedge F_B), \tag{4.2.3}$$

where $*_L$ is the Hodge operator of L.

Since $F_{B_L} \neq 0$, we deduce from (4.2.2) that $\alpha = \pm 1$. The corollary is proved. □

THEOREM 4.2.3. *Let (M, g) be a compact Riemannian manifold, Ω be a closed form of degree $n-4$ and $\{A_i\}$ be a sequence of Ω-anti-self-dual instantons. Then by taking a subsequence if necessary, A_i converges to an admissible Ω-anti-self-dual instanton A with the blow-up locus (S, Θ), such that* (1) *S is rectifiable and $\Omega|_S$ is one of its volume forms induced by g. In particular, S carries a natural orientation*; (2) $\frac{1}{8\pi^2}\Theta$ *is integer-valued*; (3) $C_2(S,\Theta)$ *is closed in M, where $C_2(S,\Theta)$ is an integral current defined by*

$$C_2(S,\Theta)(\varphi) = \frac{1}{8\pi^2}\int_S (\varphi, \Omega|_S)\Theta d(H^{n-4}\lfloor S), \tag{4.2.4}$$

where φ is any smooth form with compact support in M. Moreover, as currents, we have

$$\lim_{i\to\infty} C_2(A_i) = C_2(A) + C_2(S,\Theta), \tag{4.2.5}$$

where $C_2(A)$ is as defined in Corollary 2.3.2.

Remark 4. Applying (4.2.4) to the smooth form $4\pi^2\Omega$, we obtain the conservation of the action:

$$\lim_{i\to\infty}\int_M |\, F_i\,|^2\, dV_g = \int_M |\, F_A\,|^2\, dV_g + \int_S \Theta d(H^{n-4}\lfloor S).$$

The rest of this section is devoted to the proof of Theorem 4.2.3. We will adopt the notations in the proof of Proposition 4.1.1 and Corollary 4.2.2.

It is clear that (1) follows from Proposition 3.3.3, Proposition 4.2.1, Corollary 4.2.2 and results of the last chapter, so it suffices to prove (2) and (3).

First we show that the density $\frac{1}{8\pi^2}\Theta(\mu,\cdot)$ is integer-valued. Let x be any point in S such that (3.2.14) holds and there is a unique tangent space T_xS. Then (4.1.3) holds. Now,

$$\Theta(\mu, x) = \lim_{i\to\infty}\int_{B_1(0,g_{x,0})} |\, F_{A_{i,x,\lambda_i}}\,|^2\, dV_{x,\lambda_i}. \tag{4.2.6}$$

Since A_{i,x,λ_i} converges to zero uniformly on any compact subset away from $V = T_x S$, for any $z' \in V \cap B_1(x, g_{x,0})$, $A_{i,x,\lambda_i} \mid_{\{z'\}\times V^\perp \cap B_{\sqrt{1-|z'|^2}}(0,g_{x,0})}$ converges to zero uniformly away from $(z', 0)$. Then by the standard transgression arguments, we can deduce

$$\lim_{i\to\infty} \frac{1}{8\pi^2} \int_{\{z'\}\times V^\perp\cap B_{\sqrt{1-|z'|^2}}(0,g_{x,0})} \operatorname{tr}(F_{A_{i,x,\lambda_i}} \wedge F_{A_{i,x,\lambda_i}}) \in \mathbb{Z}. \tag{4.2.7}$$

Clearly, the limit on the right of (4.2.7) is a topological number and does not depend on z'.

For simplicity, we will denote by $F^V_{A_{i,x,\lambda_i}}$ the curvature of the restricted connection $A_{i,x,\lambda_i}|_{z'\times V^\perp}$. Since A_{i,x,λ_i} is $\tau^*_{\lambda_i}\exp^*\Omega$-anti-self-dual with respect to g_{x,λ_i} and $\lim_{i\to\infty} g_{x,\lambda_i} = g_{x,0}$, we obtain

(4.2.8)
$$\begin{aligned}
\frac{1}{8\pi^2}|F_{A_{i,x,\lambda_i}}|^2 dV_{x,\lambda_i} &= -\frac{1}{8\pi^2}\operatorname{tr}(F_{A_{i,x,\lambda_i}} \wedge F_{A_{i,x,\lambda_i}}) \wedge \tau^*_{\lambda_i}\exp^*\Omega \\
&= \frac{1}{8\pi^2}\Bigg(-\operatorname{tr}(F^V_{A_{i,x,\lambda_i}} \wedge F^V_{A_{i,x,\lambda_i}}) \\
&\qquad + (O(1)\sum_{\alpha=1}^{n-4} \Big| \frac{\partial}{\partial z_\alpha} \rfloor F_{A_{i,x,\lambda_i}} \Big| + o(1)|F_{A_{i,x,\lambda_i}}|)\,|F_{A_{i,x,\lambda_i}}|\Bigg) dV_{x,\lambda_i},
\end{aligned}$$

where $o(1)$ denotes a quantity which converges to zero as i tends to infinity. Together with (4.2.7) and (4.1.3), this implies

$$\begin{aligned}
\frac{1}{8\pi^2}\Theta(\mu, x)] &= \lim_{i\to\infty}\frac{1}{8\pi^2}\int_{B_1(0,g_{x,0})} |F_{A_{i,x,\lambda_i}}|^2\, dV_{x,\lambda_i} \\
&= \lim_{i\to\infty}\int_{V\cap B_1(0,g_{x,0})} d(H^{n-4}\lfloor V) \\
&\qquad \cdot \left(\frac{1}{8\pi^2}\int_{\{z'\}\times V^\perp\cap B_{\sqrt{1-|z|^2}}(0,g_{x,0})} \operatorname{tr}(F_{A_{i,x,\lambda_i}} \wedge F_{A_{i,x,\lambda_i}})\right).
\end{aligned}$$

Hence, by (4.2.6), $\frac{1}{8\pi^2}\Theta(\mu,\cdot)$ is integer-valued.

Next we show that $C_2(S,\Theta)$ is closed, i.e., for any smooth form ψ of degree $n-5$ and with compact support in M,

$$\partial C_2(S,\Theta)(\psi) = C_2(S,\Theta)(d\psi) = 0. \tag{4.2.9}$$

This will follow from (4.2.4) and Corollary 2.3.2, since

$$\int_M d\psi \wedge \operatorname{tr}(F_{A_i} \wedge F_{A_i}) = 0 \quad \text{for any } i.$$

We also have

$$\lim_{i\to\infty}\int_M \operatorname{tr}(F_{A_i})\wedge\operatorname{tr}(F_{A_i}) = \int_M \operatorname{tr}(F_A)\wedge\operatorname{tr}(F_A).$$

Therefore, it suffices to prove that by taking a subsequence if necessary, for any smooth φ of degree $n-4$,

(4.2.10)

$$\frac{1}{8\pi^2}\lim_{i\to\infty}\int_M \varphi\wedge\operatorname{tr}(F_{A_i}\wedge F_{A_i}) = \frac{1}{8\pi^2}\int_M \varphi\wedge\operatorname{tr}(F_A\wedge F_A) + C_2(S,\Theta)(\varphi).$$

Define currents T_i by

$$T_i(\varphi) = \frac{1}{8\pi^2}\int_M \varphi\wedge\left(\operatorname{tr}(F_{A_i}\wedge F_{A_i}) - \operatorname{tr}(F_A\wedge F_A)\right);$$

then by Proposition 2.3.1, $\partial T_i = 0$. Moreover, the total mass of T_i is uniformly bounded; i.e., for any φ with $||\varphi||_{C^0}\le 1$,

$$|T_i(\varphi)| \le \frac{1}{8\pi^2}\int_M \left(|F_{A_i}|^2 - |F_A|^2\right) dV_g \le \Lambda. \tag{4.2.11}$$

This implies, after taking a subsequence if necessary, that T_i converges weakly to a closed current T. Clearly, the mass of T is also bounded by Λ and we have $\partial T = 0$. Hence, by Theorem 3.2.1 in [Si2], T is rectifiable; more precisely, there is a rectifiable set S' with orientation vector $\eta : S' \to \Lambda^{n-4}T^*S'$ and a density function $\Theta'(x)$, such that

$$T(\varphi) = \frac{1}{4\pi^2}\int_{S'} (\varphi,\eta)\Theta'\, d(H^{n-4}\lfloor S').$$

Take φ to be $f\Omega$, where f is a smooth function with compact support; then

$$T(f\Omega) = \frac{1}{4\pi^2}\int_{S'} f(\Omega,\eta)\Theta'(x) d(H^{n-4}\lfloor S'). \tag{4.2.12}$$

On the other hand, since $\operatorname{tr}(F_{A_i})$ converges to $\operatorname{tr}(F_A)$ uniformly on M, we have

$$\begin{aligned}
\text{(4.2.13)}\quad T(f\Omega) &= \lim_{i\to\infty} T_i(f\Omega)\\
&= \frac{1}{8\pi^2}\lim_{i\to\infty}\int_M f\Omega\wedge\left(\operatorname{tr}(F_{A_i}\wedge F_{A_i}) - \operatorname{tr}(F_A\wedge F_A)\right)\\
&= \frac{1}{8\pi^2}\lim_{i\to\infty}\int_M f\left(|F_{A_i}|^2 - |F_A|^2\right) dV_g\\
\text{(4.1)}\quad &= \frac{1}{8\pi^2}\int_S f(x)\Theta(\mu,x) d(H^{n-4}\lfloor S).
\end{aligned}$$

Comparing this with (4.2.12), we conclude that $S' = S$ and $\Theta(\mu,\cdot) = (\Omega,\eta)\Theta'$. Finally, since Ω_S is one of the volume forms of S, we obtain that $(\Omega,\eta) = 1$ and consequently, $T = C_2(S,\Theta)$. This finishes the proof of Theorem 4.2.3.

Remark 5. We need the compactness of M and the closedness of Ω only to derive an *a priori* bound on $YM(A_i)$ in the above proof.

4.3. *Calibrated geometry and blow-up loci.* Let (M, g) be an n-dimensional Riemannian manifold and Ω be a closed form of degree $n-4$. We further assume that for any $x \in M$ and subspace F of T_xM of codimension 4, $\Omega|_F \leq dV_F$, where dV_F denotes the induced volume form on F by g. Following [HL], we say that (F, dV_F) is calibrated by Ω if $\Omega|_F = dV_F$. Moreover, if $\Phi = (S, \xi, \Theta)$ is an integral current with orientation ξ and density Θ, where S is the support of Φ and rectifiable, then we say that Φ is Ω-calibrated if $(T_xS, \xi(x))$ is calibrated by Ω for H^{n-4}-a.e. $x \in S$.

The following lemma is trivial.

Lemma 4.3.1. *Any integral current calibrated by Ω is minimizing in its homology class. In particular, its generalized mean curvature vanishes.*

Proof. Let $\Phi = (S, \xi, \Theta)$ be an integral current calibrated by Ω, and $\Psi = (S', \xi', \Theta')$ be another integral current homologous to Φ; i.e., there is a current R of degree $n-5$ such that for any smooth form φ on M,

$$\int_S (\varphi, \xi)\Theta dH^{n-4} - \int_{S'} (\varphi, \xi')\Theta' dH^{n-4} = R(d\varphi). \tag{4.3.1}$$

By our assumption, $(\Omega, \xi') \leq 1$ and $(\Omega, \xi) = 1$. Hence,

$$\int_S \Theta dH^{n-4} \leq \int_{S'} \Theta' dH^{n-4} + R(d\Omega) = \int_{S'} \Theta' dH^{n-4}, \tag{4.3.2}$$

and it follows that Φ is minimizing. □

Clearly, such a Φ is determined by S with multiplicity Θ. We will also call (S, Θ) an Ω-calibrated cycle. It is known from the geometry measure theory that for such a cycle, S is regular in an open and dense subset. In fact, it follows from [Am] that S can be decomposed as $\bigcup_a S_a$, such that each S_a is closed and smooth outside a closed subset of Hausdorff codimension at least two and Θ restricts to a positive integer on each S_a.

Theorem 4.3.2. *Let (M, g), Ω be as above, and $\{A_i\}$ be a sequence of Ω-anti-self-dual instantons. Further assume that either M is compact or the $YM(A_i)$ are uniformly bounded. Then by taking a subsequence if necessary, A_i converges to an admissible Ω-anti-self-dual instanton A with the blow-up locus (S, Θ), such that (S, Θ) is is an Ω-calibrated cycle, and*

$$\lim_{i\to\infty} C_2(A_i) = C_2(A) + C_2(S, \Theta). \tag{4.3.3}$$

This follows from Theorem 4.2.3 and the discussions above.

In the case of Hermitian-Yang-Mills connections, we have

THEOREM 4.3.3. *Let (M, g) be a complex m-dimensional compact Kähler manifold with the Kähler form ω, and $\{A_i\}$ be a sequence of Hermitian-Yang-Mills connections on a given unitary bundle E. Then by taking a subsequence if necessary, A_i converges weakly to an admissible Hermitian-Yang-Mills connection A with the blow-up locus (S, Θ), such that $S = \bigcup_\alpha S_\alpha$ and $\Theta|_{S_\alpha} = 8\pi^2 m_\alpha$, where each S_α is a holomorphic subvariety in M and m_α is a positive integer. Moreover, for any smooth φ,*

$$\lim_{i\to\infty}\int_M \varphi\wedge C_2(A_i) = \int_M \varphi\wedge C_2(A) + \sum_\alpha m_\alpha \int_{S_\alpha}\varphi. \tag{4.3.4}$$

Proof. By Theorem 4.3.2, we may assume that A_i converges to an admissible Hermitian-Yang-Mills connection A with an $\frac{\omega^{m-2}}{(m-2)!}$-calibrated cycle (S, Θ) as its blow-up locus. It suffices to show that (S, Θ) is a holomorphic cycle.

A straightforward computation shows that for any $x \in M$ and subspace $F \subset T_x M$ of codimension 4, $\frac{\omega^{m-2}}{(m-2)!}|_F \leq dV_F$ and the equality holds if and only if F is a complex subspace in $T_x M$. Therefore, $T_x S$ is a complex subspace in $T_x M$ for H^{2m-4}-a.e. $x \in S$. Since $C_2(S, \Theta)$ is a closed integral current, it follows from a result of J. King [Ki] or Harvey and Shiffman [HS] that there are holomorphic subvarieties S_α and positive integers m_α such that

$$C_2(S, \Theta)(\varphi) = \sum_\alpha m_\alpha \int_{S_\alpha}\varphi$$

for any φ. The theorem is proved. □

Remark 6. Let A be the Hermitian-Yang-Mills connection in the above theorem. It follows from a result of Bando and Siu [BS] that there is a gauge transformation σ on $M\backslash S$ such that $\sigma(A)$ extends to a smooth Hermitian-Yang-Mills connection outside a holomorphic subvariety in M of codimension at least three. In fact, the (0,1)-part of A induces a holomorphic structure on the underlying complex vector bundle. Then the induced holomorphic bundle on $M\backslash S$ extends to a coherent sheaf which is locally free outside a subvariety of codimension at least three.

4.4. *Cayley cycles and complex anti-self-dual instantons.* In this section, we assume that (M, g) is a Calabi-Yau 4-fold with the Kähler form ω and a holomorphic $(4, 0)$-form θ. We normalize

$$\theta\wedge\overline{\theta} = \frac{\omega^4}{4!}. \tag{4.4.1}$$

As in Section 1.3, we put

$$\Omega = 2(\theta+\overline{\theta}) + \frac{\omega^2}{2}. \tag{4.4.2}$$

LEMMA 4.4.1. *For any* 4*-dimensional subspace* $L \subset TM$, $\Omega\,|_L \le dV_L$.

Proof. This should be well-known. For the reader's convenience, we include an elementary proof here. Without loss of generality, we may assume that $M = \mathbb{C}^4$ and $L \subset \mathbb{C}^4$. In any euclidean coordinates $z_1, \cdots, z_4$ of $\mathbb{C}^4$,

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^4 dz_i \wedge d\bar{z}_i, \quad \theta = \frac{1}{4} dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4.$$

Let J be the standard complex structure on $\mathbb{C}^4$. Then $\dim_{\mathbb{R}} JL \cap L = 0$ or 2 or 4. Since $\pi_L \cdot (J|_L)$ is skewsymmetric, where π_L denotes the orthogonal projection onto L, one can choose an orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of L, such that

$$\begin{aligned} Ju_1 &= u_1^\perp + \lambda u_2, \quad Ju_2 = u_2^\perp - \lambda u_1, \\ Ju_3 &= u_3^\perp + \lambda' u_4, \quad Ju_4 = u_4^\perp - \lambda' u_3, \end{aligned} \tag{4.4.3}$$

where $u_1^\perp, u_2^\perp, u_3^\perp, u_4^\perp$ are in the orthogonal complement $L^\perp$.

First we assume that $\dim_{\mathbb{R}} JL \cap L = 0$, i.e., L is totally real. Then $|\lambda| < 1$, $|\lambda'| < 1$. Define

$$\begin{aligned} v_1 &= u_1\,,\ v_2 = \sqrt{1-\lambda^2} u_2 - \frac{\lambda u_1^\perp}{\sqrt{1-\lambda^2}}\,, \\ v_3 &= u_3\,,\ v_4 = \sqrt{1-\lambda'^2} u_4 - \frac{\lambda u_3^\perp}{\sqrt{1-\lambda'^2}}\,. \end{aligned} \tag{4.4.4}$$

Then $\{v_i, J_0 v_i\}_{1 \le i \le 4}$ is an orthonormal basis of $\mathbb{C}^4$, such that

$$Jv_2 = \frac{u_2^\perp}{\sqrt{1-\lambda}^2} \in L^\perp\,,\ Jv_4 = \frac{u_4^\perp}{\sqrt{1-\lambda'^2}} \in L^\perp\,. \tag{4.4.5}$$

Let $\{v_i^*, (Jv_i)^*\}$ be its the dual basis. Put $\varphi_i^* = v_i^* - \sqrt{-1}(Jv_i)^*$. Then

$$\omega = \frac{\sqrt{-1}}{2}\sum_{r=1}^4 \varphi_i^* \wedge \overline{\varphi_i^*} \tag{4.4.6}$$

$$\theta = \frac{1}{4} e^{\sqrt{-1}\gamma} \varphi_1^* \wedge \varphi_2^* \wedge \varphi_3^* \wedge \varphi_4^*\,,\ \gamma \in \mathbb{R}. \tag{4.4.7}$$

Using (4.4.5), (4.4.6) and (4.4.7), one shows

$$\theta|_L = \frac{e^{\sqrt{-1}\gamma}}{4}\sqrt{(1-\lambda^2)(1-\lambda'^2)}\, dV_L, \tag{4.4.8}$$

$$\omega^2|_L = 2\lambda\lambda'\, dV_L. \tag{4.4.9}$$

It follows that

$$\Omega|_L = \left(\cos\gamma\sqrt{(1-\lambda^2)(1-\lambda'^2)} + \lambda\lambda'\right) dV_L,$$

and so $\Omega|_L \le dV_L$.

Two other cases can be easily reduced to this case by perturbing L slightly. □

Following [HL], we see that (S,Θ) is a Cayley cycle if it is calibrated by the Ω. In this case, for H^4-a.e. $x \in S$, the tangent space T_xS is a Cayley plane in T_xM.

The following observation is of considerable interest, though simple.

Proposition 4.4.2. *Let (S,Θ) be a Cayley cycle. Then $C_2(S,\Theta)\cdot[\theta]$ is a nonnegative real number. Moreover, $C_2(S,\Theta)\cdot[\theta] = 0$ if and only if (S,Θ) is a holomorphic cycle.*

Proof. We adopt the notation in the proof of Lemma 4.4.1. If T_xS exists and is a Cayley subspace, then $e^{\sqrt{-1}\gamma} = 1$ and $\lambda = \lambda'$; this implies

$$(\theta|_L, \xi_S) = (1-\lambda^2) \ge 0.$$

The first statement is proved. If $C_2(S,\Theta)\cdot[\theta] = 0$, then $\lambda^2 = 1$; i.e., T_xS is a complex subspace. Then the proposition follows from the main result in [HS] or [Ki]. □

Remark 7. Since S is of codimension greater than 3, we may simply define $C_1(S,\Theta) = 0$. Then the above result can be rephrased as

$$\left(2C_2(S,\Theta) - \frac{r-1}{r}C_1(S,\Theta)^2\right)\cdot[\theta] \ge 0,$$

and the equality holds if and only if (S,Θ) is a holomorphic cycle. This is analogous to (1.3.8).

Remark 8. Similarly, one can easily show that for any Cayley cycle (S,Θ), $C_2(S,\Theta)\cdot[\omega^2] \ge 0$. Moreover, the equality holds if and only if S is special Langrangian.

The next result follows from Theorem 4.2.3 and the above discussions.

Theorem 4.4.3. *Let (M,g) be a compact Calabi-Yau 4-fold with Kähler form ω and a holomorphic $(4,0)$-form θ. Let $\{A_i\}$ be a sequence of complex anti-self-dual instantons. Then by taking a subsequence if necessary, the A_i converge to an admissible complex anti-self-dual instanton A with the blow-up locus (S,Θ), such that (S,Θ) is a Cayley cycle and*

$$\lim_{i\to\infty} C_2(A_i) = C_2(A) + C_2(S,\Theta).$$

Remark 9. The above theorem also holds for general Spin(7)-manifolds, which contain Calabi-Yau 4-folds as special examples of Spin(7)-manifolds.

Next we assume that M is an 8-dimensional compact manifold which admits Calabi-Yau structures. A Calabi-Yau structure on M is given by a complex structure J, a Kähler metric g compatible with J and a holomorphic $(4,0)$-form θ satisfying (4.4.1). We denote by $\mathcal{M}$ the moduli space of all Calabi-Yau structures modulo obvious equivalence relations.

Let E be a fixed $U(r)$-bundle over M. We define

(4.4.10)
$$\begin{aligned} &\mathcal{M}(E) \\ = \quad &\{(J,g,\theta)\in\mathcal{M}\mid \left(2C_2(E)-\frac{r-1}{r}C_1(E)^2\right)\cdot\varphi\geq 0,\ \text{for } \varphi=[\theta] \text{ or } [\omega_g^2]\}, \end{aligned}$$

where ω_g denotes the Kähler form of g, $C_1(E)$ and $C_2(E)$ denote the first and second Chern character of E. It is easy to show that $\mathcal{M}(E)$ is a connected, analytic variety.

For a fixed Calabi-Yau structure (J,g,θ), we denote by $\mathcal{Y}_{J,g,\theta}(E)$ the moduli space of all complex anti-self-dual connections of E on the Calabi-Yau 4-fold (M,J,g,θ) modulo gauge transformations. By Theorem 4.4.3, modulo gauge transformations, its compactification $\overline{\mathcal{Y}}_{J,g,\theta}(E)$ consists of all triples (A,S,Θ) satisfying: (1) A is an admissible complex anti-self-dual instanton on M; (2) (S,Θ) is a Cayley cycle; (3) $C_i(E)=[C_i(A)]+[C_i(S,\Theta)]$ in $H^{2*}(M,\mathbb{R})$ for $i=1,2$. The topology of $\overline{\mathcal{Y}}_{J,g,\theta}(E)$ is the one determined by the convergence property given at the beginning of Section 3.1.

We define
$$\overline{\mathcal{Y}}(E)=\bigcup_{(J,g,\theta)\in\mathcal{M}}\overline{\mathcal{Y}}_{J,g,\theta}(E).$$

By Proposition 4.4.2, there is an obvious map $f_c:\overline{\mathcal{Y}}(E)\mapsto\mathcal{M}(E)$. Denote by $\mathcal{M}_c(E)$ its image. Then the next result follows from the same arguments as those in the proof of Theorem 4.4.3.

THEOREM 4.4.4. *The set $\mathcal{M}_c(E)$ is closed in $\mathcal{M}(E)$.*

It is easy to show that $\mathcal{M}(E)$ is an analytic variety. We conjecture that *$\mathcal{M}_c(E)$ is an analytic subvariety in $\mathcal{M}(E)$.*

Remark 10. All the discussions above work as well for general anti-self-dual instantons. We single out the complex anti-self-dual case because of its plausible connection to the Hodge conjecture on holomorphic cycles on Calabi-Yau 4-folds.

4.5. *General blow-up loci.* Let $\{A_i\}$ be a sequence of smooth Yang-Mills connections which converge weakly to an admissible Yang-Mills connection A with blow-up locus (S,Θ).

THEOREM 4.5.1. *For any vector field X with compact support in M,*

$$-\int_S \operatorname{div}_S X\Theta\, dH^{n-4} = \int_M \Big(|F_A|^2 \operatorname{div} X - 4(F_A(\nabla X,\cdot), F_A)\Big)\, dV_g, \tag{4.5.1}$$

where $(F_A(\nabla X,\cdot), F_A)$ is defined in any local orthonormal basis $\{e_i\}$ of M as

$$\sum_{i,j=1}^{n} (F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j))$$

and $\operatorname{div}_S X$ denotes the divergence of X along S. That is, if T_pS exists and $\{v_i\}$ is any orthonormal basis of T_pS, $\operatorname{div}_S X(p) = \sum_{i=1}^{n-4}(\nabla_{v_i} X, v_i)(p)$.

Proof. As above, c always denotes a uniform constant. Since S is rectifiable, we can find a countable set of submanifolds $\{N_\alpha\}$ such that $S = S_0 \bigcup_\alpha S_\alpha$, where $S_\alpha = N_\alpha \cup S$ and $H^{n-4}(S_0) = 0$ (cf. [Si2]). Moreover, we may assume that $T_xS = T_xN_\alpha$ for H^{n-4}-a.e. $x \in S_\alpha$.

Fixing any $\delta > 0$, we can arrange N_α such that for some $\alpha_\delta > 0$,

$$\begin{aligned} S_\alpha \cap S_{\alpha'} &= \emptyset, \quad \text{for } \alpha, \alpha' \le \alpha_\delta; \\ H^{n-4}\Big(\bigcup_{\alpha > \alpha_\delta} S_\alpha\Big) &\le \delta. \end{aligned} \tag{4.5.2}$$

It follows that by taking a subsequence if necessary, we have

$$\lim_{\varepsilon\to 0}\lim_{i\to\infty} \int_{B_\varepsilon(\bigcup_{\alpha>\alpha_\delta} S_\alpha)} |F_{A_i}|^2 dV_g \le 2\delta. \tag{4.5.3}$$

Since δ can be arbitrarily small, it suffices to prove that for each $\alpha \le \alpha_\delta$,

(4.5.4)

$$\lim_{\varepsilon\to 0}\lim_{i\to\infty} \int_{B_\varepsilon(S_\alpha)} \left(|F_{A_i}|^2 \operatorname{div} X - 4\sum_{k,l}(F_{A_i}(\nabla_{e_k} X, e_l), F_{A_i}(e_k, e_l))\right) dV_g$$
$$= \int_{S_\alpha} \operatorname{div}_S X\Theta\, dH^{n-4}.$$

Without loss of generality, we may assume that $e_1, \cdots, e_{n-4}$ are tangent to N_α, while $e_{n-3}, \cdots, e_n$ are normal to N_α. Then it follows from Lemma 3.3.2 that (4.5.4) is the same as

(4.5.5)

$$\lim_{\varepsilon\to 0}\lim_{i\to\infty} \int_{B_\varepsilon(S_\alpha)} \left(|F_{A_i}|^2 \operatorname{div}^\perp X - 4\sum_{k,l=n-3}^{n}(F_{A_i}(\nabla_{e_k} X, e_l), F_{A_i}(e_k, e_l))\right) dV_g$$
$$= 0,$$

where $\operatorname{div}^\perp X = \sum_{k=n-3}^n g(\nabla_{e_k} X, e_k)$ is the divergence of X in normal directions of N_α.

Write $\nabla_{e_k} X = X_{i,k} e_i$, then $\operatorname{div}^\perp X = \sum_{l=n-3}^n X_{l,l}$ and (4.5.5) becomes

(4.5.6)

$$\lim_{\varepsilon\to 0}\lim_{i\to\infty}\int_{B_\varepsilon(S_\alpha)}\sum_{k,l=n-3}^n X_{k,l}$$

$$\cdot\left(|F_{A_i}|^2\delta_{kl} - 4\sum_{j=n-3}^n (F_{A_i}(e_k,e_j), F_{A_i}(e_l,e_j))\right) dV_g = 0.$$

By taking subsequences if necessary, we may assume that there are measures μ_{kl} $(k,l = n-3,\cdots,n)$, defined by

(4.5.7)

$$\mu_{kl}(h) = \lim_{i\to\infty}\int_{B_\varepsilon(N_\alpha)} h(|F_{A_i}|^2\delta_{kl} - 4\sum_{j=n-3}^n (F_{A_i}(e_k,e_j), F_{A_i}(e_l,e_j)))dV_g,$$

where h is any function with compact support in $B_\varepsilon(N_\alpha)$. It follows from the monotonicity (Theorem 2.1.2) that for any $x\in S$ and r sufficiently small,

$$\mu_{kl}(B_r(x)) \le ce^{ar^2} r^{n-4}.$$

Hence, in order to prove (4.5.6), it suffices to show that the upper-density $\overline{\Theta}(\mu_{kl},x)$ $(k,l = n-3,\cdots,n)$ vanishes for H^{n-4}-a.e. $x\in S_\alpha$, where

$$\overline{\Theta}(\mu_{kl},x) = \limsup_{r\to 0} r^{4-n}|\mu_{kl}(B_r(x))|. \tag{4.5.8}$$

We will prove (4.5.8) by contradiction. If (4.5.8) is false, there is an $S'_\alpha\subset S_\alpha$ such that $H^{n-4}(S'_\alpha) > 0$ and for some k,l, $\overline{\Theta}(\mu_{kl},x) > 0$ for any $x\in S'_\alpha$. By orthogonal transformations, we may assume that $k = l = n$. We can also have that for $x\in S'_\alpha$, the tangent space $T_xS = T_xS'_\alpha$ exists and

$$\lim_{r\to 0} r^{4-n}\int_{B_r(x)} |F_A|^2 dV_g = 0. \tag{4.5.9}$$

Then, by using the arguments in the proof of Lemma 4.1.2 and taking a subsequence if necessary, we can find $\varepsilon_i, r_i > 0$ with $\lim\varepsilon_i = 0$ and $\lim\frac{r_i}{\varepsilon_i} = 0$, $x_i\in S'_\alpha$, such that

(4.5.10)

$$r_i^{4-n}\left|\int_{B_{r_i}(x_i)}\left(|F_{A_i}|^2 - 4\sum_{j=n-3}^n (F_{A_i}(e_n,e_j), F_{A_i}(e_n,e_j))\right) dV_g\right| \ge \eta_0,$$

$$\lim_{i\to\infty}\varepsilon_i^{4-n}\int_{B_{\varepsilon_i}(x_i)}\sum_{j=1}^{n-4}|e_j\rfloor F_{A_i}|^2 dV_g = 0. \tag{4.5.11}$$

For simplicity, we assume that $M \subset \mathbb{R}^n$ and g is flat. The general case can be treated with slight modifications. Put $B_i(y) = r_i A_i(x_i + r_i y)$. Then B_i converges to zero outside a subspace $\mathbb{R}^{n-4} \times \{0\} = \lim_{i\to\infty} T_{x_i} N_\alpha$.

Let X be a vector field with compact support in $B_2(0) \subset \mathbb{R}^n$. Since B_i is Yang-Mills, we have that for any $j \le n-4$,

$$\begin{aligned}
\int_{B_2(0)} |F_{B_i}|^2 X_{j,j} dV_g &= -2\int_{B_2(0)} \sum_{k,l=1}^n (F_{B_i}(e_k,e_l), \nabla_{e_j} F_{B_i}(e_k,e_l)) X_j dV_g \\
\text{(Bianchi identity)} &= -4\int_{B_2(0)} \sum_{k,l=1}^n (F_{B_i}(e_k,e_l), \nabla_{e_l} F_{B_i}(e_k,e_j)) X_j dV_g \\
&= 4\int_{B_2(0)} \sum_{k,l=1}^n (F_{B_i}(e_k,e_l), F_{B_i}(e_k,e_j)) X_{j,l} dV_g \\
&\mapsto 0, \quad \text{as } i \to \infty.
\end{aligned}$$

Then we have

$$\begin{aligned}
0 &= \int_{B_2(0)} \left(|F_{B_i}|^2 \operatorname{div} X - 4\sum_{k,l=1}^n (F_{B_i}(\nabla_{e_k} X, e_l), F_{B_i}(e_k,e_l)) \right) dV_g \\
&= \int_{B_2(0)} \sum_{k,l=n-3}^n X_{k,l} \left(|F_{B_i}|^2 \delta_{kl} - 4\sum_j (F_{B_i}(e_k,e_j), F_{B_i}(e_l,e_j)) \right) dV_g.
\end{aligned}$$

Let η be a nonnegative function on $\mathbb{R}^1$ satisfying: $\eta(t) = 1$ for $t \le 1$ and $\eta(t) = 0$ for $t > \frac{4}{3}$. Choose

$$X = \eta(|y'|)\eta(|y''|) y_n e_n,$$

where $y' = (y_1, \cdots, y_{n-4})$, $y'' = (y_{n-3}, \cdots, y_n)$. Then the above implies

$$\lim_{i\to\infty} \int_{B_1(0)} \left(|F_{B_i}|^2 - 4\sum_j (F_{B_i}(e_n,e_j), F_{B_i}(e_n,e_j)) \right) dV_g = 0.$$

This contradicts (4.5.10) and the theorem is proved. $\square$

We say that A is stationary if the following holds for any vector field X with compact support in M:

$$\text{(4.5.12)} \qquad \int_M \left(|F_A|^2 \operatorname{div} X - 4\sum_{i,j=1}^n (F_A(\nabla_{e_i} X, e_j), F_A(e_i,e_j)) \right) dV_g = 0$$

where $\{e_i\}$ is any orthonormal basis of M. If A is a smooth Yang-Mills connection, this follows from the first variation formula for Yang-Mills action.

Remark 11. More generally, inspired by R. Schoen's notion of stationary harmonic maps, we may define a stationary Yang-Mills connection as a weak solution of the Yang-Mills equation which satisfies (4.5.12). It is interesting to develop a regularity theory for such weak solutions. But we will confine ourselves to admissible connections.

If A is stationary, then the right side of (4.5.1) vanishes for any X. This implies:

COROLLARY 4.5.2. *If A is stationary, then S is stationary, i.e., S has no boundary in M and its generalized mean curvature vanishes.*

This also provides another proof of Theorem 4.3.2 with slightly weaker conclusion.

5. Removable singularities of Yang-Mills equations

In this chapter, we investigate the extension problem of admissible Yang-Mills connections. Since the extension problem is local in nature, we may assume that M is an open subset in $\mathbb{R}^n$ with a metric g, which may be nonflat.

5.1. *Stationary properties of Yang-Mills connections.* Let A be an admissible Yang-Mills connection as in Section 2.3, and $r_p, c(p), a$ be as in Theorem 2.1.2. Then by the arguments of Section 2.1, we have:

PROPOSITION 5.1.1. *Let A be any admissible Yang-Mills connection satisfying* (4.5.2), *i.e.,*

$$\int_M \left(|F_A|^2 \operatorname{div} X - 4 \sum_{i,j=1}^n \langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j)\rangle \right) dV_g = 0 \tag{5.1.1}$$

where $\{e_i\}$ is any orthonormal basis of M. Then for any $0 < \sigma < \rho < r_p$,

$$\begin{aligned} \rho^{4-n} e^{a\rho^2} \int_{B_\rho(p)} |F_A|^2 dV_g - \sigma^{4-n} e^{a\sigma^2} \int_{B_\sigma(p)} |F_A|^2 dV_g \\ \geq\ 4 \int_{B_\rho(p)\setminus B_\sigma(p)} r^{4-n} |\frac{\partial}{\partial r} \rfloor F_A|^2 dV_g. \end{aligned} \tag{5.1.2}$$

Moreover, if $M = \mathbb{R}^n$ and g is flat, then the equality holds in (5.1.2) *for $\rho \in (0, \infty)$ and $a = 0$.*

Next we prove that any admissible Ω-anti-self-dual instantons are stationary; i.e., they satisfy (5.1.1).

Let A be an admissible Ω-anti-self-dual instanton with singular set $S = S(A)$.

Given any vector field X with compact support in M, let $\phi_t : M \to M$ be its integral curve. As in Section 2.1, we define A^t to be the connection $\phi_t^*(A)$. Then by the same arguments as those in Section 2.3, one can show that $\mathrm{Ch}_2(A^t)$ defines a closed 4-form on M in the sense of distribution.

First we claim that $\mathrm{Ch}_2(A^t)$ is independent of t, i.e., for any closed $(n-4)$-form φ,

$$\int_M \varphi \wedge \Big(\mathrm{Ch}_2(A^t) - \mathrm{Ch}_2(A)\Big) = 0. \tag{5.1.3}$$

Since ϕ_t is an identity near the boundary ∂M of M,

$$\mathrm{Ch}_2(A^t) - \mathrm{Ch}_2(A) = 0 \quad \text{near} \ \ \partial M.$$

Without loss of generality, we may assume that the bundle E under consideration is trivial over M. We constructed in Section 2.3 a Chern-Simon 3-form Ψ such that

$$d\Psi = \mathrm{Ch}_2(A) \quad \text{on } M\backslash S, \tag{5.1.4}$$

and for some uniform constant c,

$$|\Psi(x)| \leqslant \frac{c}{d(x,S)^3}, \qquad x \in M\backslash S. \tag{5.1.5}$$

Noticing that $\mathrm{Ch}_2(A^t) = \phi_t^*\mathrm{Ch}_2(A)$, we have that for $\Psi_t = \phi_t^*\Psi$,

$$d(\Psi_t - \Psi) = \mathrm{Ch}_2(A^t) - \mathrm{Ch}_2(A) \qquad \text{in } M\backslash(S \cup \phi_t(S)) \tag{5.1.6}$$

and

$$|\Psi_t - \Psi|(x) \leq \frac{2c}{d(x, S\cup\phi_t(S))^3}, \quad x \in M\backslash(S\cup\phi_t(S)). \tag{5.1.7}$$

Furthermore, $\Psi_t - \Psi = 0$ near ∂M and for H^{n-4}-a.e. $x \in S \cup \phi_t(S)$,

$$\lim_{x\to x_0} d(x, S\cup\phi_t(S))^3\,(\Psi_t - \Psi)\,(x) = 0. \tag{5.1.8}$$

Now (5.1.3) follows easily from (5.1.6)–(5.1.8) and the same arguments as in the proof of Proposition 2.3.1.

PROPOSITION 5.1.2. *Assume that Ω is a closed form of degree $n-4$. Then any admissible Ω-anti-self-dual instanton A on M is stationary.*

Proof. For simplicity, we assume that $\mathrm{tr}(F_A) = 0$. The general case follows from identical arguments because $\mathrm{tr}(F_A)$ is smooth on M.

Now we have

$$\int_M \mathrm{tr}(F_{A^t} \wedge F_{A^t}) \wedge \Omega = \int_M \mathrm{tr}(F_A \wedge F_A) \wedge \Omega. \tag{5.1.9}$$

This is the same as

$$\int_M (F_{A^t}, T(F_{A^t}))\, dV_g = \int_M (F_A, T(F_A))\, dV_g, \tag{5.1.10}$$

where T is the operator $-*\cdot\Omega\wedge$ acting on 2-forms. Then the Ω-anti-self-duality of A states $T(F_A) = F_A$. It follows that

$$\begin{aligned} YM(A^t) &= \frac{1}{4\pi^4}\int_M |F_{A^t}|^2 dV_g \\ &= \frac{1}{4\pi^2}\int_M (F_{A^t}, (Id-T)(F_{A^t}))\, dV_g + \frac{1}{4\pi^2}\int_M (F_{A^t}, T(F_{A^t}))\, dV_g \\ &= \frac{1}{4\pi^2}\int_M (F_{A^t}, (Id-T)(F_{A^t}))\, dV_g - \mathrm{Ch}_2(A). \end{aligned}$$

Since $(Id-T)(F_A) = 0$ and T is symmetric, the last integral above is at the order t^2. Therefore,

$$\frac{d}{dt} YM(A^t)\Big|_{t=0} = 0.$$

This implies that A is stationary. □

In fact, we believe that *any admissible Yang-Mills connection* (*possibly under certain mild conditions*) *is stationary.* If this is true, we can conclude from Corollary 4.5.2 that the blow-up locus of any Yang-Mills connections are stationary, in other words, it is a generalized minimal variety.

5.2. *A removable singularity theorem.* In this section, we always assume that A is an admissible Yang-Mills connection on M and stationary. Fix $p \in S = S(A)$, where $S(A)$ denotes the singular set of A. Let $r_p, c(p), a$ be as in Theorem 2.1.2. Our goal of this section is to prove a removable singularity theorem under appropriate assumptions. We assume that $S \cap B_{\frac{r_p}{2}}(p)$ satisfies the following uniform covering (UC) property: for any $y \in S \cap B_{\frac{r_p}{2}}(p)$ and $\delta \le r < \frac{r_p}{2}$, there are always balls $B_\delta(x_i)$ $(i = 1, \cdots, l)$ such that $x_i \in S$, $S \cap B_r(y) \subset \bigcup_i B_\delta(x_i)$ and $l\delta^{n-4} \le cr^{n-4}$ for some uniform constant $c > 0$. One can easily show that this (UC) property holds, if there is a measure μ with support S such that the total measure $\mu(S \cap B_{r_p}(p)) < \infty$, and for every $x \in S \cap B_{r_p}(p)$, $r^{4-n}\mu(S \cap B_r(x))$ is decreasing with r, and the density $\Theta(x) = \lim_{r\to 0} r^{4-n}\mu(S \cap B_r(x)) > 0$. In particular, if A is the limit of smooth Yang-Mills connections A_i outside S, then S has the (UC) property, since $\mu = \lim_{i\to\infty} |F_{A_i}|^2 dV_g$ satisfies the above conditions.

THEOREM 5.2.1. *Let A, S be as above. Then there is an $\varepsilon > 0$, depending only on $n = \dim M$, such that for any $p \in S$ and $0 < r < r_p$, if*

$$r^{4-n}\int_{B_r(p)} |\ F_A\ |^2\ dV_g < \varepsilon, \tag{5.2.1}$$

then there is a gauge transformation σ near p such that $\sigma(A)$ extends to be a smooth connection near p.

A direct corollary of this is the next result:

THEOREM 5.2.2. *Let A, S be as in Theorem 5.2.1. Then there is a gauge transformation σ such that $\sigma(A)$ is smooth outside a closed subset S' of H^{n-4}-measure zero.*

Proof. Let ε be given as in Theorem 5.2.1. Then for any $x \in M$, the limit

$$\lim_{r \to 0} r^{4-n} e^{ar^2} \int_{B_r(x)} |\ F_A\ |^2\ dV_g$$

exists. Define

$$S' = \{x \in M \mid \lim_{r \to 0} r^{4-n} e^{ar^2} \int_{B_r(x)} |\ F_A\ |^2\ dV_g \geq \varepsilon\}. \tag{5.2.2}$$

Then by Theorem 5.1.1, S' is closed. Moreover, by using standard arguments as those in the proof of Lemma 3.1.4 (c), we can show that $H^{n-4}(S') = 0$.

By Theorem 5.2.1, there is a countable covering $\{U_\alpha\}$ of $M \backslash S'$, so that for each α, there is a gauge transformation σ_α on $(M \backslash S') \cap U_\alpha$ such that E is trivial over U_α and $D_{\sigma_\alpha(A)} = d + A_\alpha$ for some smooth A_α. It follows that for any α, β, we have the transition function $g_{\alpha\beta} = \sigma_\alpha \cdot \sigma_\beta^{-1} : U_\alpha \cup U_\beta \backslash S' \to G$, where G is the structure group of E, such that

$$A_\alpha = g_{\alpha\beta}^{-1} d g_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\beta g_{\alpha\beta}. \tag{5.2.3}$$

Therefore, $g_{\alpha\beta}$ extends to a smooth map on $U_\alpha \cap U_\beta$, since $g_{\alpha\beta}$ takes values in a compact group G. Furthermore, $\{g_{\alpha\beta}\}$ satisfies the cocycle condition

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma.$$

Therefore, $\{g_{\alpha\beta}\}$ defines a G-bundle E' over $M \backslash S$ extending $E\ |_{M \backslash S(A)}$, and $\{A_\alpha\}$ defines a Yang-Mills connection for E'. The theorem is proved. □

The rest of this section is devoted to the proof of Theorem 5.2.1. By scaling, we may assume that $r = 5$, $M = B_5(p)$ and E is trivial over M. For simplicity, we may further assume that the metric g is flat. The general case can be proved by identical arguments.

We will always denote by c a uniform constant. As before, we write $S = S(A)$ as the singular set of A.

LEMMA 5.2.3. *There is a gauge transformation σ on $M \backslash S$ such that for any $x \in B_3(p) \backslash S$,*

$$\rho(x)^{n-2}\ |\ A^\sigma\ |^2\ (x) \quad \leq \quad c \int_{B_{\frac{1}{2}\rho(x)}(x)} |\ F_A\ |^2\ dV_g, \tag{5.2.4}$$

$$\int_{B_{\frac{2}{5}\rho(x)}(x)} \left(\frac{|\ A^\sigma\ |^2}{\rho(x)^2} + |\ \nabla A^\sigma\ |^2 \right) dV_g \quad \leq \quad c \int_{B_{\frac{1}{2}\rho(x)}(x)} |\ F_A\ |^2\ dV_g, \tag{5.2.5}$$

where $\rho(x) = d(x, S)$ and $D_{\sigma(A)} = d + A^\sigma$ with $A^\sigma \in \Omega^1\left(M \backslash S, \mathrm{Lie}(G)\right)$.

Proof. We may assume that $2^{n-4}e^a\varepsilon \le \varepsilon(n)$, where $\varepsilon(n)$ is as given in Theorem 2.2.1. Then by the monotonicity (5.1.2), for any $x \in B_3(p)\backslash S$,

$$\rho(x)^{4-n}\int_{B_{\rho(x)}(x)} |F_A|^2 dV_g < \varepsilon(n).$$

It follows from Uhlenbeck's curvature estimate (Theorem 2.2.1) that

$$|F_A|(y) \le \frac{c\sqrt{\varepsilon(n)}}{\rho(x)^2}, \quad \text{for } y \in B_{\frac{1}{2}\rho(x)}(x).$$

Note that c always denotes a uniform constant in this proof.

Next, using Theorem 1.2.7 in [Uh1, p. 18], we can have a gauge transformation σ_x over $B_{\frac{1}{2}\rho(x)}(x)$, such that $D_{\sigma_x(A)} = d + A^{\sigma_x}$ and for any $y \in B_{\frac{1}{40}\rho(x)}(x)$,

$$\rho(x)|A^{\sigma_x}|(y) + \rho(x)^2|\nabla A^{\sigma_x}|(y) \le c\left(\left(\frac{\rho(x)}{20}\right)^{4-n}\int_{B_{\frac{\rho(x)}{20}}(x)} |F_A|^2 dV_g\right)^{\frac{1}{2}}.$$

Now we outline the construction of σ from those σ_x. We cover $M\backslash S$ by balls $B_{r_i}(x_i)$ satisfying: (1) $x_i \in M\backslash S$ and $r_i = \frac{1}{40}\rho(x_i)$; (2) For any $x \in M\backslash S$, the number of those $B_{r_i}(x_i)$ containing x is uniformly finite. For each i, denote by σ_i the above σ_{x_i}. If $B_{r_i}(x_i) \cap B_{r_j}(x_j)$ is nonempty, then $r_i \le 2r_j$ and $r_j \le 2r_i$; so by the above estimate for A^{σ_i} and A^{σ_j}, we can obtain

$$r_i|d\sigma_i \cdot \sigma_j^{-1}| + r_i^2|\nabla d\sigma_i \cdot \sigma_j^{-1}| \le c\left(r_i^{4-n}\int_{B_{r_i}(x_i)\cup B_{r_j}(x_j)} |F_A|^2 dV_g\right)^{\frac{1}{2}},$$

on the overlap $B_{r_i}(x_i) \cap B_{r_j}(x_j)$. Notice that $B_{r_i}(x_i) \subset B_{\rho(x)/10}(x)$ whenever $x \in B_{r_i}(x_i)$. Thus we can glue these σ_i to get a gauge transformation σ such that

$$\rho(x)|A^{\sigma}|(x) + \rho(x)^2|\nabla A^{\sigma}|(x) \le c\left(\left(\frac{\rho(x)}{10}\right)^{4-n}\int_{B_{\frac{\rho(x)}{2}}(x)} |F_A|^2 dV_g\right)^{\frac{1}{2}}.$$

This implies that for any $y \in B_{\frac{2}{5}\rho(x)}(x)$, we have

$$\rho(x)|A^{\sigma}|(y) + \rho(x)^2|\nabla A^{\sigma}|(y) \le c\left(\left(\frac{\rho(x)}{2}\right)^{4-n}\int_{B_{\frac{\rho(x)}{2}}(x)} |F_A|^2 dV_g\right)^{\frac{1}{2}}.$$

Then the lemma follows easily. $\square$

For simplicity, we assume that σ can be taken to be Id in the above lemma.

LEMMA 5.2.4. *Let A be as above. Then*

$$\int_{B_1(x)} \left(\frac{| A |^2}{\rho(y)^2} + | \nabla A |^2 \right) dV_g \leq c \int_{B_3(x)} | F_A |^2 \; dV_g, \tag{5.2.6}$$

where $\rho(y) = d(y, S)$.

Proof. Since ρ is Lipschitz and $| \nabla\rho | \equiv 1$, by the co-area formula (cf. [Si2]), we have

$$\int_{B_1(x)} \frac{| A |^2}{\rho(y)^2} \, dV_g = \int_0^1 \frac{dr}{r^2} \int_{\rho^{-1}(r) \cap B_1(x)} | A |^2 \; d\mathcal{H}^{n-1} \tag{5.2.7}$$

where $d\mathcal{H}^{n-1}$ denotes the induced measure on the level surface $\rho^{-1}(r)$.

For any $r \leq 1$, there is a covering $\{B_{\frac{2r}{5}}(x_{ir})\}_{1 \leq i \leq N_r}$ of $\rho^{-1}([\frac{2}{3}r, \frac{4}{3}r]) \cap B_1(x)$, such that $\rho(x_{ir}) = r$ and for any $y \in \rho^{-1}([\frac{2}{3}r, \frac{4}{3}r])$, the number of balls $B_{\frac{r}{2}}(x_{ir})$ containing y is uniformly bounded. Hence,

$$\begin{aligned}
&\int_0^1 \frac{dr}{r^2} \int_{B_1(x) \cap \rho^{-1}(r)} | A |^2 \; d\mathcal{H}^{n-1} \\
&\qquad = \quad \frac{1}{ln2} \int_0^1 \frac{dr}{r^2} \int_{\frac{3}{4}r}^{\frac{3}{2}r} \frac{ds}{s} \int_{B_1(x) \cap \rho^{-1}(r)} | A |^2 \; d\mathcal{H}^{n-1} \\
&\qquad = \quad \frac{1}{ln2} \int_0^{\frac{3}{2}} \frac{ds}{s} \int_{B_1(x) \cap \rho^{-1}([\frac{2}{3}s, \frac{4}{3}s])} \frac{1}{\rho^2} \, | A |^2 \; dV_g \\
&\qquad \leq \quad \frac{1}{ln2} \int_0^{\frac{3}{2}} \frac{ds}{s} \left(\sum_i \int_{B_{\frac{2s}{5}}(x_{is})} \frac{| A |^2}{\rho^2(y)} \, dV_g \right) \\
\text{By (5.2.5)} &\qquad \leq \quad c \int_0^{\frac{3}{2}} \frac{ds}{s} \left(\sum_i \int_{B_{\frac{s}{2}}(x_{is})} | F_A |^2 \; dV_g \right) \\
&\qquad \leq \quad c \int_0^{\frac{3}{2}} \frac{ds}{s} \int_{B_3(x) \cap \rho^{-1}([\frac{s}{2}, \frac{3s}{2}])} | F_A |^2 \; dV_g \\
&\qquad \leq \quad c \int_0^3 dr \int_{\frac{2}{3}r}^{2r} \frac{ds}{s} \int_{B_3(x) \cap \rho^{-1}(r)} | F_A |^2 \; dV_g \\
&\qquad \leq \quad c \int_{B_3(x)} | F_A |^2 \; dV_g.
\end{aligned}$$

Similarly, we can derive

$$\int_{B_1(x)} | \nabla A |^2 \; dV_g \leqslant c \int_{B_3(x)} | F_A |^2 \; dV_g.$$

The lemma is proved. $\square$

Lemma 5.2.5. *Let A be as above. Then there are a function α and a 2-form β such that*

$$A = d\alpha + d^*\beta, \quad d\beta = 0 \quad \text{on } B_1(x), \tag{5.2.8}$$

$$\|\alpha\|_{H^{1,2}(B_1(x))} + \|\beta\|_{H^{1,2}(B_1(x))} \le c\|A\|_{L^2(B_2(x))}\,. \tag{5.2.9}$$

Proof. Let $\eta : B_3(x) \to \mathbb{R}^1$ be a cut-off function: $\eta(y) = 1$ for $d(x,y) \le 1$, $\eta(y) = 0$ for $d(x,y) \ge 2$ and $|\nabla\eta| \le 1$. By the extension of the classical Hodge-de Rham decomposition due to Iwaniec and Martin, we have unique α and β on $\mathbb{R}^n$ such that $\eta A = d\alpha + d^*\beta$ on $\mathbb{R}^n$, $d\beta = 0$, and

$$\|\alpha\|_{H^{1,2}(\mathbb{R}^n)} + \|\beta\|_{H^{1,2}(\mathbb{R}^n)} \le c\|\eta A\|_{H^{1,2}(R^n)}\,.$$

Then the lemma follows easily. □

Put $\tilde{A} = A - d\alpha$; then $d^*\tilde{A} = 0$. Since A is a Yang-Mills connection in the weak sense,

$$\begin{aligned} 0 &= D_A^* F_A \\ &= d^*F_A + [F_A, A] \\ &= d^*dA + d^*(A\wedge A) + [F_A, A] \\ &= d^*d\tilde{A} + d^*(A\wedge A) + [F_A, A] \\ &= (d^*d + dd^*)\tilde{A} + d^*(A\wedge A) + [F_A, A]. \end{aligned} \tag{5.2.10}$$

We decompose

$$\tilde{A} = \tilde{A}_0 + \tilde{A}_1, \tag{5.2.11}$$

such that

$$(d^*d + dd^*)\tilde{A}_0 = 0, \quad \text{in } B_1(x), \tag{5.2.12}$$

and

$$\begin{aligned} (d^*d + dd^*)\tilde{A}_1 &= -[F_A, A] - d^*(A\wedge A), \quad \text{in } B_1(x) \\ \tilde{A}_1 &= 0 \qquad \text{on } \partial B_1(x). \end{aligned} \tag{5.2.13}$$

Lemma 5.2.6. *There exists*

$$\|\tilde{A}_1\|_{H^{1,2}(B_1(x))} \le c\sqrt{\varepsilon}\,\|F_A\|_{L^2(B_3(x))}, \tag{5.2.14}$$

where ε is as given in (5.2.1).

Proof. First we have from (5.2.4),

$$| A | (y) \leq \frac{c\sqrt{\varepsilon}}{\rho(y)}, \quad \forall y \in B_3(x)\backslash S. \tag{5.2.15}$$

Multiplying (5.2.13) by $\tilde{A}_1$ and integrating by parts, we obtain

$$\begin{aligned}
&\int_{B_1(x)} | \nabla \tilde{A}_1 |^2 \, dV_g \\
&\quad = -\int_{B_1(x)} \Big((\tilde{A}_1, [F_A, A]) + (\tilde{A}_1, d^*(A \wedge A)) \Big) \, dV_g \\
&\quad = -\int_{B_1(x)} \Big((\tilde{A}_1, [F_A, A]) + (d\tilde{A}_1, A \wedge A) \Big) \, dV_g \\
&\quad \leq c\sqrt{\varepsilon} \left(\int_{B_1(x)} \frac{| \tilde{A}_1 || F_A |}{\rho(y)} \, dV_g + \int_{B_1(x)} \frac{| d\tilde{A}_1 || A |}{\rho(y)} \, dV_g \right)
\end{aligned}$$

by (5.2.6)

$$\leq c\sqrt{\varepsilon} \left(\int_{B_3(x)} | F_A |^2 \, dV_g \right)^{\frac{1}{2}} \left(\int_{B_1(x)} \left(\frac{| \tilde{A}_1 |^2}{\rho(y)^2} + | \nabla \tilde{A}_1 |^2 \right) dV_g \right)^{\frac{1}{2}}.$$

Then (5.2.14) follows from the next lemma. □

LEMMA 5.2.7. *For any function f vanishing on $\partial B_1(x)$,*

$$\int_{B_1(x)} \frac{| f |^2}{\rho(y)^2} \, dV_g \leq c \int_{B_1(x)} | \nabla f |^2 \, dV_g. \tag{5.2.16}$$

Proof. This lemma follows directly from a result of C. Fefferman and D. Phong [FP] (also see [CW, Th. 1.4], [CWW], [Fef]), once we verify the following: for any $y \in B_1(x)$ and $r \leq 1$,

$$\int_{B_r(y)} \frac{1}{\rho^3} dV_g \leq c r^{n-3} \tag{5.2.17}$$

where c is a uniform constant.

Let us check (5.2.17). If $\rho(y) \geq 2r$, then $r \leq \rho(z) \leq 3r$ for any $z \in B_r(y)$. Now

$$\int_{B_r(y)} \frac{1}{\rho^3} dV_g \leq \frac{c}{r^3} \int_{B_r(y)} dV_g \leq c r^{n-3}.$$

Next, we assume that $\rho(y) \leq 2r$. By our assumption on S, for any $\delta < 4r$, there are L_δ balls $B_\delta(x_i)$ such that $S \cap B_r(y) \subset \bigcup_i B_\delta(x_i)$ and $L_\delta \leq c \left(\frac{r}{\delta}\right)^{n-4}$.

Then by the co-area formula,

$$\begin{aligned}
\int_{B_r(y)} \frac{1}{\rho^3} dV_g &= \int_0^{4r} \frac{1}{s^3} ds \int_{\rho^{-1}(s)\cap B_r(y)} dH^{n-1} \\
&= (4r)^{-3} \int_{B_r(y)} dV_g + 3 \int_0^{4r} \frac{1}{s^4} ds \int_{\rho^{-1}([0,s])\cap B_r(y)} dV_g \\
&\leq cr^{n-3} + 3 \int_0^{4r} \frac{1}{s^4} ds \left(\sum_{i=1}^{L_s} \int_{B_{2s}(x_i)} dV_g \right) \\
&\leq cr^{n-3} + cr^{n-4} \int_0^{5r} ds \\
&\leq cr^{n-3}.
\end{aligned}$$

Thus (5.2.17) follows. □

Let $\theta \in (0,1)$ be fixed. Since $\tilde{A}_0$ is harmonic, we have, from standard elliptic estimates, that

$$\begin{aligned}
\frac{1}{\theta^{n-4}} \int_{B_\theta(x)} | d\tilde{A}_0 |^2 &\leq \theta^4 \int_{B_1(x)} | d\tilde{A}_0 |^2 dV_g \\
&\leq \theta^4 \int_{B_1(x)} | d\tilde{A} |^2 dV_g.
\end{aligned} \tag{5.2.18}$$

Then

(5.2.19)

$$\begin{aligned}
&\theta^{4-n} \int_{B_\theta(x)} | F_A |^2 \, dV_g \\
&= \theta^{4-n} \int_{B_\theta(x)} \left(| dA |^2 + 2(F_A, A\wedge A) - | A \wedge A |^2 \right) dV_g \\
&\leq \theta^{4-n} \int_{B_\theta(x)} \left(| d\tilde{A} |^2 + 2(F_A, A\wedge A) \right) dV_g \\
\text{by (5.2.4), (5.2.1)} \quad &\leq \theta^{4-n} \int_{B_\theta(x)} \left(| d\tilde{A} |^2 + \frac{c\sqrt{\varepsilon}|A||F_A|}{\rho(y)} \right) dV_g \\
\text{by (5.2.6)} \quad &\leq \theta^{4-n} \int_{B_\theta(x)} | d\tilde{A} |^2 \, dV_g + c\sqrt{\varepsilon}\theta^{4-n} \int_{B_3(x)} | F_A |^2 \, dV_g.
\end{aligned}$$

Similarly, we have

$$\int_{B_1(x)} | d\tilde{A}_0 |^2 dV_g \leq \int_{B_1(x)} | d\tilde{A} |^2 dV_g + c\sqrt{\varepsilon} \int_{B_3(x)} | F_A |^2 \, dV_g. \tag{5.2.20}$$

On the other hand, using Lemma 5.2.6 and Lemma 5.2.4, we deduce

(5.2.21)

$$\begin{aligned}\int_{B_\theta(x)} \mid d\tilde{A} \mid^2 dV_g &= \int_{B_\theta(x)} \Big(\mid d\tilde{A}_0 \mid^2 + |d\tilde{A}_1|^2 + 2(d\tilde{A}_0, d\tilde{A}_1) \Big) dV_g \\ &\le \int_{B_\theta(x)} \mid d\tilde{A}_0 \mid^2 dV_g + 2||\tilde{A}_1||_{H^{1,2}(B_1(x))} \\ &\quad \cdot \left(\int_{B_1(x)} |d\tilde{A}_0|^2 dV_g \right)^{\frac{1}{2}} + ||\tilde{A}_1||^2_{H^{1,2}(B_1(x))} \\ &\le \int_{B_\theta(x)} \mid d\tilde{A}_0 \mid^2 dV_g + c\sqrt{\varepsilon} \int_{B_3(x)} \mid F_A \mid^2 \ dV_g.\end{aligned}$$

It follows from the above four inequalities that

$$\theta^{4-n} \int_{B_\theta(x)} \mid F_A \mid^2 \ dV_g \le \theta^4 \int_{B_1(x)} \mid F_A \mid^2 \ dV_g + c\sqrt{\varepsilon}\theta^{4-n} \int_{B_3(x)} \mid F_A \mid^2 \ dV_g.$$

By scaling, we obtain that for $r \le 1$ and $y \in B_1(p)$,

(5.2.22)

$$\begin{aligned}(\theta r)^{4-n} \int_{B_{\theta r}(x)} \mid F_A \mid^2 \ dV_g &\le \theta^4 r^{4-n} \int_{B_r(x)} \mid F_A \mid^2 \ dV_g \\ &\quad + c\sqrt{\varepsilon}\theta^{4-n} r^{4-n} \int_{B_{3r}(x)} \mid F_A \mid^2 \ dV_g.\end{aligned}$$

Then, from the monotonicity for A, we have

$$(\lambda r)^{4-n} \int_{B_{\lambda r}(y)} \mid F_A \mid^2 \ dV_g \le \left(3^4 + c\sqrt{\varepsilon(r)}\lambda^{-n} \right) \lambda^4 r^{4-n} \int_{B_r(x)} \mid F_A \mid^2 \ dV_g,$$

where $\lambda = \frac{\theta}{3} < \frac{1}{3}$ and

$$\varepsilon(r) = r^{4-n} \int_{B_r(x)} \mid F_A \mid^2 \ dV_g \le 8\varepsilon.$$

A simple iteration yields

$$(5.2.23) \quad (\lambda^k r)^{4-n} \int_{B_{\lambda^k r}(y)} \mid F_A \mid^2 \ dV_g$$

$$\le \prod_{i=0}^{k-1} \left(1 + c\sqrt{\varepsilon(\lambda^i r)}\lambda^{-n} \right) (3\lambda)^{4k} r^{4-n} \int_{B_r(x)} \mid F_A \mid^2 \ dV_g,$$

where $k \ge 1$.

Choose λ and ε such that $6^4\lambda < 1$ and $8c\sqrt{\varepsilon}\lambda^{-n} < 1$. This implies that for any $i \le k-1$,

$$(1 + c\sqrt{\varepsilon(\lambda^i r)}\lambda^{-n})3^4\lambda < 1.$$

For any $r \leq 1$, we define k, $r_0 \in (\frac{1}{3}, 1]$ by $\lambda^k r_0 = r$. Then

$$\begin{aligned} r^{4-n} \int_{B_r(y)} \mid F_A \mid^2 \, dV_g &\leq \lambda^{3k} r_0^{4-n} \int_{B_{r_0}(y)} \mid F_A \mid^2 \, dV_g \\ &\leq r^3 r_0^{-3} \int_{B_1(y)} \mid F_A \mid^2 \, dV_g \\ &\leq cr^3. \end{aligned}$$

Now we replace $\varepsilon(r)$ in (5.2.22) by cr^3 and obtain

$$(\theta r)^{4-n} \int_{B_{\theta r}(x)} \mid F_A \mid^2 \, dV_g \leq \theta^4 r^{4-n} \int_{B_r(x)} \mid F_A \mid^2 \, dV_g + c\theta^{4-n} r^{\frac{9}{2}}.$$

Choose $\theta = \frac{1}{2}$ and c' such that $c(\frac{1}{2})^{4-n} + c'(\frac{1}{2})^{\frac{9}{2}} \leq c'(\frac{1}{2})^4$. Then

$$(\frac{r}{2})^{4-n} \int_{B_{\frac{r}{2}}(x)} \mid F_A \mid^2 \, dV_g + c' \left(\frac{r}{2}\right)^{\frac{9}{2}} \leq (\frac{1}{2})^4 \left(r^{4-n} \int_{B_r(x)} \mid F_A \mid^2 \, dV_g + c' r^{\frac{9}{2}} \right).$$

It follows from this and a simple iteration that

$$r^{4-n} \int_{B_r(x)} \mid F_A \mid^2 \, dV_g \leq c'' r^4,$$

where c'' is some uniform constant.

Therefore, the curvature F_A is bounded in $B_1(p)$. Using results in [Uh2], we can construct a gauge transformation σ such that $d^* A_\sigma = 0$ and $||A_\sigma||_{C^1(B_1(p))}$ is bounded. Since $D^*_{\sigma(A)} F_{\sigma(A)} = 0$, A_σ is smooth, and consequently, $\sigma(A)$ extends to a smooth connection near p. Theorem 5.2.1 is proved.

5.3. *Cone-like Yang-Mills connections.* In this section, we study the infinitesimal structure of stationary Yang-Mills connections at their singular points. Let A be a stationary Yang-Mills connection on M with L^2-bounded curvature F_A. It follows from Theorem 5.1.1 that for any $x \in S$, the limit

$$\lim_{r \to 0} r^{4-n} \int_{B_r(x)} |F_A|^2 dV_g$$

exists. Therefore, we can define

$$S([A]) = \{x \in M \mid \lim_{r \to 0} r^{4-n} \int_{B_r(x)} |F_A|^2 dV \geq \varepsilon\},$$

where ε is as given in Theorem 5.2.1. Then $S(A)$ contains $S([A])$. Denote by S the set $S([A])$. By Theorem 5.2.2, we have $H^{n-4}(S) = 0$; moreover, there is a gauge transformation σ on $M \backslash S(A)$ such that $\sigma(A)$ extends to a smooth connection on $M \backslash S$. Without loss of generality, we may assume that $S = S(A)$.

Now we explain why S is expected to be of Hausdorff codimension of at least 5.

To analyze A near x, we scale the metric g and A as follows: for any $\lambda \in (0,1)$, define

$$g_\lambda = \lambda^{-2} g, \quad A_\lambda = \tau_\lambda^* \exp_x^* A,$$

where $\tau_\lambda : T_xM \mapsto T_xM$ maps v to λv. Clearly, A_λ is a stationary Yang-Mills connection with respect to g_λ. Moreover, for any $R > 0$,

$$R^{4-n} \int_{B_R(x,g_\lambda)} |F_{A_\lambda}|^2 dV_{g_\lambda} = (\lambda R)^{4-n} \int_{B_{\lambda R}(x)} |F_A|^2 dV_g \leq c, \tag{5.3.1}$$

whenever λ is sufficiently small. Here and in the following, c always denotes a uniform constant.

Then we can deduce the following from results in Section 3.1: for any sequence $\{\lambda_i\}$ with $\lim_{i\to\infty} \lambda(i) = 0$, taking a subsequence and gauge transformations if necessary, we may assume that $A_{\lambda(i)}$ converges to a connection A^c outside $S_c \subset T_xM$. Here, A^c is Yang-Mills with respect to the flat metric g_0 on $T_xM = \mathbb{R}^n$ and $H^{n-4}(S^c \cap B_R(0, g_0)) < \infty$. Moreover, we may assume that $|F_{A_{\lambda(i)}}|^2 dV_g$ converges weakly to $|F_{A^c}|^2 dV_{g_0} + \Theta_c H^{n-4} \lfloor S_c$, where Θ_c is a function with its support in S_c.

Lemma 5.3.1. *With the above notation*, (1) $\frac{\partial}{\partial r}\Theta_c = 0$; (2) $a \cdot S_c = S_c$, *where $a \cdot S_c$ denotes the set of points az with $z \in S_c$*; (3) $\frac{\partial}{\partial r} \rfloor F_{A^c} = 0$.

Proof. By Theorem 4.5.1, for any vector field X with compact support,

$$\begin{aligned} &-\int_{S_c} \mathrm{div}_{S_c} X \Theta_c \, dH^{n-4} \\ &= \int_{T_xM} \left(|F_{A^c}|^2 \,\mathrm{div}\, X - 4 \sum_{i,j=1}^n (F_{A^c}(\frac{\partial X}{\partial x_i}, \frac{\partial}{\partial x_j}), F_A(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})) \right) dV_{g_0}, \end{aligned} \tag{5.3.2}$$

where $x_1, \cdots, x_n$ are euclidean coordinates of $T_xM = \mathbb{R}^n$. Choosing $X(x) = \xi(r) r \frac{\partial}{\partial r}$, where $r = \sqrt{\sum_i x_i^2}$ and ξ has compact support, we obtain

$$\begin{aligned} &\int_{S_c} (\xi' r + (n-4)\xi) \, \Theta_c \, dH^{n-4} + \int_{T_xM} (\xi' r + (n-4)\xi) \, |F_{A^c}|^2 dV_{g_0} \\ &\qquad = \int_{S_c} \xi' r |\nabla^\perp r|^2 \Theta_c \, dH^{n-4} + 4 \int_{T_xM} \xi' r \left| \frac{\partial}{\partial r} \rfloor F_{A^c} \right|^2 dV_{g_0}. \end{aligned} \tag{5.3.3}$$

Following the arguments in deriving (2.1.20) from (2.1.15), we can deduce from (5.3.3) that for any $\sigma < \rho$,

$$\int_{S_c\cap(B_\rho(0,g_0)\setminus B_\sigma(0,g_0))} r^{4-n}|\nabla^\perp r|^2\Theta_c\, dH^{n-4} \tag{5.3.4}$$
$$+ 4\int_{B_\rho(0,g_0)\setminus B_\sigma(0,g_0)} r^{4-n}\left|\frac{\partial}{\partial r}\rfloor F_{A^c}\right|^2 dV_{g_0}$$
$$= \rho^{4-n}\left(\int_{S_c\cap B_\rho(0,g_0)}\Theta_c\, dH^{n-4} + \int_{B_\rho(0,g_0)}|F_{A^c}|^2 dV_{g_0}\right)$$
$$- \sigma^{4-n}\left(\int_{S_c\cap B_\sigma(0,g_0)}\Theta_c\, dH^{n-4} + \int_{B_\sigma(0,g_0)}|F_{A^c}|^2 dV_{g_0}\right).$$

On the other hand, for any $s > 0$,

$$s^{4-n}\left(\int_{B_s(0,g_0)}|F_{A^c}|^2 dV_{g_0} + \int_{S_c\cap B_s(0,g_0)}\Theta_c\, dH^{n-4}\right)$$
$$= \lim_{i\to\infty}(\lambda(i)s)^{4-n}\int_{B_{\lambda(i)s}(x)}|F_A|^2 dV_g$$
$$= \lim_{s'\to 0} s'^{4-n}\int_{B_{s'}(x)}|F_A|^2 dV_g > 0.$$

Therefore,

$$\int_{S_c\cap(B_\rho(0,g_0)\setminus B_\sigma(0,g_0))} r^{4-n}|\nabla^\perp r|^2\Theta_c\, dH^{n-4} \tag{5.3.5}$$
$$+4\int_{B_\rho(0,g_0)\setminus B_\sigma(0,g_0)} r^{4-n}\left|\frac{\partial}{\partial r}\rfloor F_{A^c}\right|^2 dV_{g_0} = 0.$$

This implies that $\nabla^\perp r = 0$ on S_c and $\frac{\partial}{\partial r}\rfloor F_{A^c} = 0$; i.e., both (2) and (3) hold.

Furthermore, arguing as we did in the proof of Lemma 3.2.1, we can deduce that $|F_{A^c}|^2 dV_{g_0} + \Theta_c H^{n-4}\lfloor S_c$ is a cone measure. Now, (1) holds. $\square$

We will call such an A^c a tangent Yang-Mills connection of A at x. In general, A may have a different tangent Yang-Mills connection at x, which depends on choices of sequences $\{\lambda(i)\}$. By Corollary 2.1.3, A^c is gauge equivalent to $d+B$ for some $B : S^{n-1} \mapsto T^*S^{n-1}\otimes\mathrm{Lie}(G)$. Thus $S(A^c)$ is invariant under radial scaling and so is $S([A^c])$. If A^c is also stationary, then $H^{n-4}(S([A^c])) = 0$. Together with Uhlenbeck's removable singularity theorem in [Uh1] (also see Theorem 5.2.1), this implies that $S([A^c]) = \{0\}$ whenever $n = 5$. If the blow-up set S_c is empty, we further deduce that A has an isolated singularity at x. This leads us to

Conjecture 1. *If A is stationary, then the Hausdorff codimension of $S([A])$ is at least* 5.

We can expect stronger conclusion for Ω-anti-self-dual instantons. Now let A be an Ω-anti-self-dual instanton. Then its tangent Yang-Mills connection A^c is
Ω_x-anti-self-dual. Moreover, Ω_x is a nonvanishing constant form.

LEMMA 5.3.2. *If $v \rfloor F_{A^c} = 0$ for any v in a subspace $L \subset T_xM$ of dimension $n-5$, then modulo gauge transformations, A^c extends smoothly to a connection on T_xM.*

Proof. Write $\Omega_x = dV_L \wedge d\ell \ + \ \Omega'_x$, such that Ω'_x is perpendicular to any form $dV_L \wedge \varphi$, where dV_L is a volume form on L and φ is a 1-form. Then $\ell \neq 0$; otherwise, $F_{A^c} = 0$ by our assumption and Ω_x-anti-self-duality. Furthermore, we have

$$- * (F_{A^c} \wedge dV_L \wedge d\ell) = F_{A^c}.$$

Hence, $\frac{\partial}{\partial \ell} \rfloor F_{A^c} = 0$ and modulo a gauge transformation, A^c is the pull-back of an anti-self-dual connection on the 4-subspace perpendicular to L and $\frac{\partial}{\partial \ell}$. By the removable singularity theorem of Uhlenbeck in dimension 4 (also see Theorem 5.2.1), there is a gauge transformation σ such that $\sigma(A^c)$ extends to a smooth connection on T_xM. Hence, the lemma is proved. □

CONJECTURE 2. *If A is Ω-anti-self-dual, then its singular set $S([A])$ has Hausdorff codimension at least* 6.

Both conjectures can be affirmed if one can show that $\lim_{i\to\infty} S(A_i) \subset S(A)$ for any sequence of Yang-Mills connections A_i converging to A.

Finally, let us discuss briefly the classification of tangent Ω-anti-self-dual instantons on $\mathbb{R}^6$ with the only singularity at 0. Since Ω is a linear 2-form on $\mathbb{R}^6$, we may choose coordinates $x_1, \cdots, x_6$, such that

$$\Omega = a_1 dx_1 \wedge dx_2 + a_2 dx_3 \wedge dx_4 + a_3 dx_5 \wedge dx_6.$$

Let A^c be a nonflat tangent Ω-anti-self-dual instanton. Then

$$- *_5 (\alpha \wedge F_{A^c}) = F_{A^c}$$

on $S^5 \subset \mathbb{R}^6$, where $\alpha = \frac{\partial}{\partial r} \rfloor \Omega$ and $*_5$ is the Hodge operator on S^5. Since $F_A \neq 0$, $\alpha \neq 0$. A simple computation shows that $|\alpha|(x)$ has to be 1 for any $x \in S^5$. Hence, we may assume that $a_1 = a_2 = a_3 = 1$, and consequently, Ω is the standard symplectic form on $\mathbb{R}^6$. If J_0 denotes the complex structure on $\mathbb{R}^6$ such that the $dx_{2i-1} + \sqrt{-1} dx_{2i}$ $(i = 1, 2, 3)$ span the induced holomorphic tangent bundle, then A is Hermitian-Yang-Mills with respect to this complex structure. Moreover, we have $v \rfloor F_{A^c} = 0$ when v is either $\frac{\partial}{\partial r}$ or $J_0(\frac{\partial}{\partial r})$. This implies that modulo a gauge transformation, A^c is the pull-back of a Hermitian-Yang-Mills connection on $\mathbb{C}P^2$. Conversely, any Hermitian-Yang-Mills connections on $\mathbb{C}P^2$ give rise to a tangent Ω-anti-self-dual instanton on $\mathbb{R}^6$.

If Ω is the 3-form of Section 1.4, defining the G_2-structure on $\mathbb{R}^7$, then tangent Ω-anti-self-dual instantons are in one-to-one correspondence with Hermitian-Yang-Mills connections on S^6 with respect to the almost complex structure induced by Ω. These are all the possible tangent Ω-asd (anti-self-dual) instantons on $\mathbb{R}^7$.

It is also possible to classify all tangent Ω-anti-self-dual instantons on $\mathbb{R}^8$. Then one problem is how to show that singularities of any Ω-anti-self-dual instantons are modeled on these tangent connections on a manifold of dimension no more than 8.

6. Compactification of moduli spaces

In this chapter, we first construct a compactification of the moduli space of anti-self-dual instantons. Then we discuss briefly possible extensions of results proved in the last few chapters.

6.1. *Compactifying moduli spaces.* Let (M, g) be a compact Riemannian n-manifold and Ω be a closed differential form of degree $n-4$. Let E be a unitary vector bundle over M. Recall that $\mathcal{M}_{\Omega,E}$ consists of all equivalence classes of Ω-anti-self-dual, often abbreviated as Ω-asd, instantons on M, i.e., solutions of (1.2.2). Here, two solutions A_1 and A_2 are equivalent if and only if there is a gauge transformation σ of E such that $\sigma(A_1) = A_2$. In general, $\mathcal{M}_{\Omega,E}$ may not be compact.

We now describe in detail the compactification outlined in the introduction. A generalized Ω-asd instanton is made of (1) an admissible Ω-asd instanton A of E, which extends to become a smooth connection over $M \backslash S(A)$ for a closed subset $S(A)$ with $(n-4)$-dimensional Hausdorff measure $H^{n-4}(S(A)) = 0$; (2) a closed integral current $C = (S, \Theta)$ calibrated by Ω satisfying the energy identity

$$\frac{1}{4\pi^2}\int_M |F_A|^2 dV_g + \int_S \Theta dH^{n-4} = \int_M \mathrm{Ch}_2(E) \wedge \Omega,$$

where $\mathrm{Ch}_2(E)$ denotes the second Chern character of E.

If the co-norm $|\Omega| \leq 1$, C is an area-minimizing integral current, so that it follows from [Am] that C can be represented by $\sum_a m_a C_a$ satisfying: $m_a = \Theta|_{C_a}$ and each C_a is closed and of the form $C_a^0 \bigcup \mathrm{Sing}(C_a)$ such that C_a^0 is a smooth submanifold calibrated by Ω and $\mathrm{Sing}(C_a)$ is a closed subset of Hausdorff codimension at least two.

Remark 12. We believe that the singularity of each Ω-calibrated cycle is also of a certain geometric structure. If Ω-calibrated cycles C_a are holomorphic, then each singular set $\mathrm{Sing}(C_a)$ is a holomorphic subvariety.

Two generalized Ω-asd instantons (A, C), (A', C') are equivalent if and only if $C = C'$ and there is a gauge transformation σ on $M \backslash S(A) \cup S(A')$, such that $\sigma(A) = A'$ on $M \backslash S(A) \cup S(A')$. We denote by $[A, C]$ the equivalence class represented by (A, C). We identify $[A, 0]$ with $[A]$ in $\mathcal{M}_{\Omega,E}$ if A extends to a smooth connection of E over M modulo a gauge transformation.

We define $\overline{\mathcal{M}}_{\Omega,E}$ to be set of all equivalence classes of generalized Ω-anti-self-dual instantons of E.

Remark 13. A natural problem occurs when an Ω-calibrated cycle, or simply a submanifold, is actually the limit of a sequence of Ω-asd instantons. More generally, one may ask if a minimal submanifold of dimension $n-4$ can be the limit of a sequence of Yang-Mills connections. It is a delicate and interesting problem involving use of the implicit function theorem.

One can define the first two Chern forms of (A, C) as follows: $\mathrm{Ch}_1(A, C) = \mathrm{Ch}_1(A)$ is given by $\frac{\sqrt{-1}}{2\pi}\operatorname{tr}(F_A)$ and $\mathrm{Ch}_2(A, C) = \mathrm{Ch}_2(A) + \mathrm{PD}(C)$, where $\mathrm{Ch}_2(A)$ is given by $-\frac{1}{4\pi^2}\operatorname{tr}(F_A \wedge F_A)$ and $\mathrm{PD}(C)$ denotes the Poincaré dual of the integral current C. Since $H^{n-4}(S(A)) = 0$ for a generalized Ω-asd (A, C), both $\mathrm{Ch}_1(A)$ and $\mathrm{Ch}_2(A)$ are closed currents on M. So they give rise to cohomology classes of M. In fact, $\mathrm{Ch}_2(A, C)$ always represents $\mathrm{Ch}_2(E)$ in $H^*(M, \mathbb{Z})$ for any $[A, C]$ in $\overline{\mathcal{M}}_{\Omega,E}$.

The topology of $\overline{\mathcal{M}}_{\Omega,E}$ can be defined as follows: a sequence $[A_i, C_i]$ converges to $[A, C]$ in $\overline{\mathcal{M}}_{\Omega,E}$ if and only if (1) C can be decomposed into two closed integral currents $C' + C''$ such that C_i converges to C' in M with respect to the standard topology for currents; (2) There are gauge transformations σ_i such that $\sigma_i(A_i)$ converges to A outside $S(A)$ and the support of C, and the generalized Chern forms $\mathrm{Ch}_2(\sigma_i(A_i), C_i)$ converge to $\mathrm{Ch}_2(A, C)$ as currents. One can show that $\overline{\mathcal{M}}_{\Omega,E}$ is then a Hausdorff topological space which follows from results in Chapter 4 and 5.

THEOREM 6.1.1. *For any M, g, Ω and E as above, $\overline{\mathcal{M}}_{\Omega,E}$ is compact with respect to this topology.*

Let T be a compact family of metrics and closed $(n-4)$-forms g_t, Ω_t. Then by the above arguments, we can show:

COROLLARY 6.1.2. *For any M, $T = \{g_t, \Omega_t\}$ and E as above, $\bigcup_{t \in T} \overline{\mathcal{M}}_{\Omega_t,E}$ is compact with respect to the topology defined above.*

We end this section with a generalization of Theorem 6.1.1. in the case where Ω is not necessarily closed.

We will still call A an Ω-asd instanton whenever A satisfies the equation in Lemma 1.2.1, even if Ω is not closed. Suppose now that Ω has the decomposition $\Omega_1 + \Omega_2$, such that Ω_1 is closed and for any 2-form φ,

$$-\varphi \wedge \varphi \wedge \Omega_2 < |\varphi|^2 dV_g. \tag{6.1.1}$$

Then for any Ω-asd instanton A, we still have an *a priori* bound on $YM(A)$ as we did in (1.2.3). Following the arguments in the proof of Theorem 6.1.1, we can obtain the next result:

THEOREM 6.1.3. *Let $\Omega = \Omega_1 + \Omega_2$ be as above. Then $\overline{\mathcal{M}}_{\Omega,E}$ is compact.*

Note that an Ω-asd instanton may not be Yang-Mills if Ω is not closed.

6.2. *Final remarks.* We expect that one can define certain deformation invariants by using $\overline{\mathcal{M}}_{\Omega,E}$ as one did in the case of Donaldson, Gromov-Witten and Seiberg-Witten invariants, etc.

More precisely, let (M, g) be a compact Riemannian manifold, and Ω be a degree $n-4$ form satisfying (6.1.1) and the ellipticity condition: for any x in M, the symmetric operator $T = -*\Omega\wedge$ on 2-forms has 1 as its eigenvalue, of multiplicity exactly equal to $\frac{(n-1)(n-2)}{2}$.

We hope that $\overline{\mathcal{M}}_{\Omega,E}$ is a smooth manifold of expected dimension if g and Ω are in general position.

Let $\mathrm{ad}(E)$ be the adjoint bundle of E, i.e., the associated bundle $P(E) \times_\rho \mathrm{Lie}(G)$ with ρ being the adjoint representation of G in $\mathrm{Lie}(G)$, where $P(E)$ denotes the principal bundle of the G-bundle E. For any connection A, define a linear operator

$$\begin{aligned} L_A : \Omega^1(M, \mathrm{ad}(E)) &\mapsto \Omega^0(M, \mathrm{ad}(E)) \oplus \Omega^2_+(M, \mathrm{ad}(E)), \\ L_A(\varphi) &= (D_A^*\varphi, D_A\varphi + *(\Omega \wedge D_A\varphi)). \end{aligned} \tag{6.2.1}$$

By our assumption on Ω, each L_A is elliptic. Its index is the expected dimension of $\overline{\mathcal{M}}_{\Omega,E}$.

If M is a Calabi-Yau 4-fold with θ and ω as in (1.3.1), then by simple computations, one can show that the index of L_A is the same as half of the index of the $\overline{\partial}$-operator $D_A^{0,1}$ on $\Omega^{0,*}(M, \mathrm{End}(E))$. The index of $D_A^{0,1}$ can be computed easily by the Atiyah-Singer index theorem.

Therefore, to carry out this program, we need to prove only transversality for Ω-asd instantons. This will be studied in a future paper.

Let us end this section with a simple example of the above program. Let E be an SU(2)-bundle over a Calabi-Yau 3-fold V. Let θ_0 and ω_0 be, respectively, a holomorphic 3-form and a Kähler form on V, satisfying:

$$\theta_0 \wedge \overline{\theta}_0 = 2\sqrt{-1}\frac{\omega_0^3}{3!}.$$

Now let $M = V \times T$, where T is a torus of complex dimension one. We denote by dz the standard flat (1,0)-form on T. Put

$$\Omega = 4\mathrm{Re}(\theta_0 \wedge dz) + \frac{1}{2}\left(\omega_0 + \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}\right)^2.$$

Then T-invariant solutions of the Ω-asd equation on M reduce to the solutions of the following equation on V:

$$F_A^{0,2} = \overline{\partial}^* f, \quad F_A^{1,1} \wedge \omega_0^2 = [f, \bar{f}], \tag{6.2.2}$$

where A is a connection of E and f is a section of $\wedge^{0,3}(\mathrm{End}(E))$. Note that (6.2.2) is an elliptic system. Presumably, counting solutions of (6.2.2) leads to the so-called holomorphic Casson invariants as studied in [DT] and by R. Thomas in his thesis.

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