Gauge Theory on Calabi-Yau manifolds

Richard Thomas
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Trinity Term 1997

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Abstract

In this thesis we study complex analogues on Calabi-Yau manifolds of gauge theories on low-dimensional real manifolds. A formal picture is set up in which it is shown that in many ways a Calabi-Yau is the correct complex analogue of a real oriented manifold, while a Fano variety with anticanonical divisor plays the role of a manifold with boundary. This works particularly well for transferring gauge theories over from real n-manifolds to complex n-folds, and we describe holomorphic versions of most of the gauge theories studied by mathematicians in the last fifteen years.

The technical problems in rigorously extracting invariants from these theories are described, and then tackled in the central theory of the holomorphic Casson invariant, which should count (stable) holomorphic bundles on a Calabi-Yau 3-fold. Examples of moduli spaces of bundles are calculated on the quintic 3-fold, the intersection of a quadric and quartic in $\mathbb{CP}^5$, and $K3\times T^2$, before turning to the general case.

Using the recent work of Li and Tian [LT] we apply excess intersection theory to create virtual moduli cycles in moduli spaces of sheaves on an algebraic Calabi-Yau 3-fold, with the "correct" dimension zero even though the moduli spaces themselves have too high a dimension. The main technical result required is a two step locally free resolution of the cotangent sheaf of the moduli space $\mathcal{M}$, of the right virtual dimension. This is obtained in some generality via the Quot scheme used in Simpson's construction of $\mathcal{M}$, and then applied to a Calabi-Yau 3-fold. This allows us to count sheaves, but we must subtract a correction term to count bundles. This and deformation invariance are discussed.

Since knot theory now has a gauge-theoretic description (albeit via a path integral [W1]) it has a complex analogue which is discussed in an appendix. A path integral is formally manipulated to give quantities we can study in their own right. The holomorphic linking number suggested some years ago by Atiyah [At1] is recovered, as is the Ray-Singer holomorphic torsion [RS2]. Higher invariants of Calabi-Yau 3-folds, and their embedded complex curves, are then discussed.

Contents

0.1 Introduction and acknowledgements

Gauge theories in low dimensions have been the focus of enormous mathematical interest over the past fifteen years, and have provided some of the most important recent advances in differential geometry and topology. This thesis is mainly concerned with complex analogues of these real field theories on low dimensional Calabi-Yau manifolds, and their interpretations.

Broadly speaking there are striking analogies between differential geometry on n-dimensional manifolds, and Kähler geometry on Calabi-Yau n-folds. Very naively, the trivialising $(n,0)$-form θ acts as a complex volume form, we replace x by $\bar{z}$, d by $\bar{\partial}$, and wedge everything with θ. This is particularly successful at giving us complex analogues of all the familiar gauge theories, and it is these which we study. Also to a certain extent a good complex analogue of a manifold with boundary is a Fano variety with anticanonical divisor (this is what we think of as the boundary, and is Calabi-Yau).

Chapter 1 sets up the formal picture, often somewhat imprecisely as deep technical issues to do with compactness and stability are postponed to Chapters 2 and 3. But the analogies and structure should be apparent. Formally at least, there is a holomorphic analogue on a Calabi-Yau three-fold of the Casson invariant of a three manifold, counting (stable) holomorphic bundles ("$\bar{\partial}_A^2 = 0$") instead of flat connections ("$d_A^2 = 0$"). Such integrable structures occur as the critical points of a functional CS on the space of all $\bar{\partial}$-operators on a fixed bundle, where CS is a complex analogue of the Chern-Simons functional. On a real three manifold the gradient flow lines of this functional can be used to define Floer homology (with Euler characteristic twice the Casson invariant), mimicking the usual calculation of the homology of a space by Morse theory. The complex analogue of Morse theory is the Picard-Lefschetz theory of monodromy of Lefschetz fibrations, and an infinite dimensional version of this is discussed.

Just as the gradient flow equations of the Chern-Simons functional are a special case of the anti-self-duality equations, similarly there exist analogous equations on a Calabi-Yau 4-fold. Solutions are "half-integrable" $\bar{\partial}$-operators, and are in fact integrable if and only if the bundle admits a single holomorphic structure (just as instantons are flat if and only if the bundle admits a single flat connection). They also satisfy the natural analogue of the Yang-Mills equations, which are the Euler-Lagrange equations of a Lagrangian. Similarly there is a Bogomolny equation on a Calabi-Yau 3-fold.

There is also an analogue of the definition of the Casson invariant via a Heegard splitting, where a singular Calabi-Yau is represented as the union with normal crossings of two Fano varieties along a common anticanonical divisor S. (A good example is provided by degenerating a smooth quintic in $\mathbb{CP}^4$ into the union (across a $K3$ surface) of a cubic and a quadric.) The complex symplectic form on the Calabi-Yau surface S induces a complex symplectic form on the space of holomorphic bundles on S, and Tyurin has shown the subspaces of bundles extending to the Fano varieties are complex Lagrangian. Thus, in direct analogy with the real case, we would like to interpret the intersection of these subspaces as the (holomorphic) Casson invariant, counting the bundles on S which extend to both Fanos. We also describe a striking analogue of this for counting curves in such singular Calabi-Yau manifolds. In fact the complex analogue of a manifold with boundary being a Fano variety with anticanonical divisor extends somewhat. The exact homology sequence of the pair (manifold, boundary) is replaced by the sheaf cohomology sequence defined by the anticanonical section and its divisor. In this context some other formulae for CS are given, relating it to $p_1 \wedge \theta$ on a Fano containing our Calabi-Yau as an anticanonical divisor, in parallel with those relating the Chern-Simons functional to p_1 on a bounding manifold with boundary. There is also a natural analogue of parts of the Atiyah-Floer conjecture.

Motivated by the discussions of Casson invariants in Chapter 1, Chapter 2 contains calculations of certain moduli spaces of holomorphic bundles on some Calabi-Yau 3-folds using the more powerful tools of algebraic geometry.

Firstly we study a quintic 3-fold $X \subset \mathbb{CP}^4$. The Serre construction is described, relating rank two bundles to curves, via the zero set of a section. We then study (semi-stable) bundles whose corresponding curves are degree three tori, and show the spaces of bundles and tori coincide in general. The main part of this calculation is to show that all such bundles have a section; this is done by passing down to successive hyperplane sections where calculations of bounds on cohomology groups and Riemann-Roch eventually yields sections. These pull up to sections on the blow up of X along the base-curve of a pencil, and it is then shown that these can be pushed back down to X. Finally it is shown these bundles are isolated by examining their deformation theory.

Next we study another example of both analogues of the Casson invariant. There is a beautiful description due to Mukai of the bundles of a particular type on intersections of quadrics in $\mathbb{CP}^5$. This was reinterpreted by Donaldson as showing that the bundles on the singular Calabi-Yau formed from the union of two quadric hypersurfaces in a fixed quadric (which is two Fanos glued along an anticanonical divisor) are just the restrictions of the two tautological bundles A and B on the fixed quadric (viewed as a Grassmannian). This is an example of the Casson invariant defined via Tyurin's work, and smoothing the Calabi-Yau we would like it to coincide with the other definition by showing the stable bundles on a smooth quartic in the quadric are just the restrictions of A and B. Given a stable bundle of the same topological type, we show it has sections vanishing on curves by similar methods to those of the last example. The difficult part is to show the curves we produce are the intersection of the quartic with one of the A or B planes in the Grassmannian (these are the zero sets of sections of A and B). Having done this it is easy to see the bundle is the restriction of either A or B. Lastly a little geometry of the Grassmannian shows the deformation theory of these bundles is isolated so we may count them each once.

Finally in this Chapter we study bundles on $K3 \times T^2$ motivated by some physics [VW] and an observation of Donaldson that relates numbers of these bundles to modular forms.

Chapter 3 describes the issues involved in making a rigorous general definition of the invariants described in Chapter 1. These are compactness and smoothness of moduli spaces of connections. The analytical results available are described, along with their limitations. The results we expect to be true and the structure of a natural compactification are described. However this programme seems to be out of reach at present – the results obtainable in dimensions higher than four are only useful in the integrable case at the moment, but to have enough perturbations to get a smooth moduli space of the correct dimension we would like to work with almost-complex structures too. There is no hope of a result similar to Donaldson's for Kähler surfaces (that the moduli space is generically smooth of the right dimension for large enough Chern class), because, seen in terms of the Serre construction, points on a surface are unobstructed, but curves on a 3-fold usually have deformations of the wrong dimension. There is some hope of this working on Fano 3-folds (those that are "convex" have curves with unobstructed deformations) and we discuss this.

So we resort to algebraic geometry, where we can use the compactifications of the moduli space of bundles of Maruyama and Simpson. As we cannot now hope for the moduli space to be of the right dimension we have to use excess intersection theory on the moduli space to create a virtual moduli cycle of the right dimension (zero for a Calabi-Yau). Recent work of Li, Tian and others does precisely this, if we are given a two-step locally free resolution (of the right virtual dimension) of the cotangent sheaf of the moduli space. Using Simpson's construction of the moduli space via a Quot scheme, we prove a theorem which shows under exactly which conditions such a resolution exists, on a general variety. These conditions are satisfied for a Calabi-Yau 3-fold allowing us to count stable sheaves. The deformation invariance of this number and the correction term we need to subtract to count less non-locally-free sheaves are then discussed.

A traditional piece of 3-manifold theory, knot theory, now has an analytical basis due to Witten's path integral derivation of the Jones polynomial ([W1]). This opens it up to our programme of "wedging with θ", and in an appendix we write down this Chern-Simons-type theory on a Calabi-Yau 3-fold. As yet there is no mathematical justification for the Path Integral, but invariants we extract by formal manipulation we can take as definition for mathematical purposes. On a bare Calabi-Yau 3-fold we obtain the holomorphic torsion of [RS2], then analogues of the invariants of [AS1]. There is still much work to be done here to prove finiteness of these numbers and study metric-independence. There is an alternative definition of the invariants of [AS1] on a 3-manifold due to Kontsevich [K] using a real differential geometric analogue of a construction in complex geometry, and so it seems natural to look for a version on a Calabi-Yau. We discuss this and the difficulties that arise. Finally the analogue of knots are introduced into the path integral, namely complex curves in the Calabi-Yau, and the first invariant obtained is shown to coincide with the suggestion some years ago [At1] for a holomorphic linking number of such curves. Higher invariants are then discussed.

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0.2 Notation

Complex manifolds will usually be called n-folds (often X or Y) to distinguish them from real differential manifolds, which will be called n-manifolds (M or N).

By a Calabi-Yau manifold we will mean a smooth compact Kähler n-fold X with holomorphically trivial canonical bundle $K_X = \Lambda^{n,0}$, with a Ricci-flat Kähler metric as provided by Yau's solution of the Calabi conjecture. We will fix a trivialising holomorphic $(n, 0)$-form θ, which can be taken to be parallel with respect to the Levi-Civita connection due to the Ricci-flat metric.

We do not insist that $h^{0,1} = 0$ for a Calabi-Yau 3-fold X, but will restrict to this case to define some invariants – this is analogous to only defining invariants for homology spheres ($b_1 = 0$) in the theory of real 3-manifolds.

The form θ gives us a Hodge star $\star : \Omega^{0,p} \to \Omega^{0,n-p}$ defined by $\star\, s = \bar{*}\,(s \wedge \theta)$, where $\bar{*}$ is the usual antilinear Hodge star $\bar{*} : \Lambda^{p,q} \to \Lambda^{n-p,n-q}$. This splits the (0,2)-forms on a Calabi-Yau 4-fold into ± 1-eigenspaces $\Lambda^{0,\pm}$ of $\star$, and similarly $H^{0,2}$ since $\star$ commutes with the Laplacian. This is a *real* splitting, not complex (in fact multiplication by i switches the eigenspaces); in the language of representation theory $\Lambda^{0,2}$ is a real representation of $SU(4)$, the complexification of $\Lambda^{0,+}$.

We shall use the term Fano variety loosely (even incorrectly); all we require is that its anticanonical divisor be effective, i.e. K_X^* admits a holomorphic section. (The usual definition is that K_X^* be ample.)

$\mathscr{A}$ will denote spaces of connections – usually $SU(2)$ connections on a real manifold or the space of $\bar{\partial}$-operators (often with fixed determinant) on a bundle on a complex manifold. The appropriate gauge group of bundle automorphisms will be denoted $\mathscr{G}$ or $\mathscr{G}_E$, with $\mathscr{B} = \mathscr{A}/\mathscr{G}$. Similarly $\mathcal{M}$ will denote a moduli space; flat or Yang-Mills connections on a real manifold, and (semi-) stable holomorphic bundles (or sheaves) on a complex manifold. $\mathfrak{g}_E$ (with complexification $\mathfrak{g}_E^c$) will be the adjoint bundle of a G-bundle E; namely the trace free skew endomorphisms for $G = SU(N)$, or $\mathrm{End}_0(E)$ or $\mathrm{End}\,(E)$ on a complex manifold. The curvature of A is F_A, while for a $\bar{\partial}$-operator $\bar{\partial}_A$ we let $F_A^{0,2} = \bar{\partial}_A^2$. This is the $(0, 2)$-part of the curvature of any connection compatible with $\bar{\partial}_A$. A metric on E uniquely determines a connection compatible with a given $\bar{\partial}$-operator. Often when we are working only with $\bar{\partial}$-operators we will slip into calling them connections.

We may confuse a holomorphic bundle with its locally free sheaf of sections, but usually E, F, L will denote bundles, and $\mathscr{E}$, $\mathscr{F}$, $\mathscr{L}$ their sheaves of sections. The sheaf of holomorphic functions and the trivial line bundle on a complex manifold X will be denoted $\mathcal{O}$ or $\mathcal{O}_X$, and $\mathscr{E}(t)$ is $\mathscr{E}$ twisted by t powers of the hyperplane bundle $\mathcal{O}(1)$ if X has a (fixed) embedding in $\mathbb{CP}^N$.

Chapter 1

Some Gauge Theory on Calabi-Yau Manifolds

1.1 Introduction

This chapter outlines a formal picture which will motivate some rigorous mathematics, some of which will be dispersed throughout, but most of which will form Chapters 2 and 3. So it will often be deliberately vague and imprecise, but the main ideas and analogies should be clear. For instance, we will gloss over issues of stability in talking about moduli spaces of holomorphic bundles, but these will be properly defined and studied in later chapters. And we will be occupied for the whole of Chapter 3 by the harmless looking statement "Thus with an appropriate perturbation and compactness theorem we might hope to count the number of bundles on a Calabi-Yau 3-fold" that appears in the next section.

So for now we only concern ourselves with the overall picture which is, very naively, that if the analogue of a real n-manifold is a complex n-fold, then the analogue of an oriented manifold is a Calabi-Yau. The trivialising $(n, 0)$-form θ acts as a complex volume form, we replace x by $\bar{z}$, d by $\bar{\partial}$, and wedge everything with θ. In particular this gives us complex analogues of the familiar gauge theories that have been so important in the study of low dimensional topology in the last fifteen years, and it is mainly these which we study. In terms of holonomy $SU(n)$ is the complex analogue of $SO(n)$, and some of what we discuss also has an extension to manifolds of exceptional holonomy (due to Joyce, Donaldson and Lewis – see [DT]; here we will look only at Calabi-Yau manifolds and Kähler geometry). Also to a certain extent a good complex analogue of a manifold with boundary is a Fano variety Y with anticanonical divisor X (this is what we think of as the boundary, and is Calabi-Yau), as will be seen in a few instances. Here the long exact homology sequence of a manifold and its boundary is replaced by the sheaf cohomology sequence of the sequence

$$0 \to K_Y \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

induced by the anticanonical section defining $X \subset Y$.

The gauge theories operate in dimensions 2, 3 and 4, and it is the central one, to which the other two relate, which we concentrate on rigorising in later chapters, since the technical issues involved are typical of those in the other dimensions. It is also in dimension 3 that we begin.

1.2 $\bar{\partial}$-operators on a Calabi-Yau 3-fold

In this section we begin with the case of a Calabi-Yau 3-fold X, as defined in the notation section. A standard example is a smooth quintic hypersurface in $\mathbb{CP}^4$: by the adjunction formula the canonical bundle of a divisor $X \in |\mathcal{O}(d)|$ is $K_X \cong \mathcal{O}_X(d) \otimes K_{\mathbb{CP}^4} \cong \mathcal{O}_X(d-5)$, which is trivial for $d = 5$.

The triviality of K_X has many consequences for holomorphic vector bundles E over X. Firstly, the deformation complex of E is self dual:

$$\begin{array}{ccccccccccc}
0 & \to & \Omega^0(\mathfrak{g}_E) & \xrightarrow{\bar{\partial}_E} & \Omega^{0,1}(\mathfrak{g}_E) & \xrightarrow{\bar{\partial}_E} & \Omega^{0,2}(\mathfrak{g}_E) & \xrightarrow{\bar{\partial}_E} & \Omega^{0,3}(\mathfrak{g}_E) & \to & 0 \\
 & & \otimes & & \otimes & & \otimes & & \otimes & & \\
0 & \leftarrow & \Omega^{0,3}(\mathfrak{g}_E) & \xleftarrow{\bar{\partial}_E} & \Omega^{0,2}(\mathfrak{g}_E) & \xleftarrow{\bar{\partial}_E} & \Omega^{0,1}(\mathfrak{g}_E) & \xleftarrow{\bar{\partial}_E} & \Omega^0(\mathfrak{g}_E) & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbb{C} & & \mathbb{C} & & \mathbb{C} & & \mathbb{C} & &
\end{array} \tag{1.2.1}$$

in the sense that all the pairings commute by Stokes' theorem, so "$\bar{\partial}_E^* = \bar{\partial}_E$". (Here the pairings are by trace on the Lie algebra $\mathfrak{g}_E$ of the structure group of E (which makes $\mathfrak{g}_E$ self-dual) and integration against the fixed holomorphic $(3,0)$-form θ.) Thus, in particular, the complex has zero index, its cohomology groups are Serre-dual to each other ($H^{0,i}(\mathfrak{g}_E) \cong H^{0,3-i}(\mathfrak{g}_E)$), and the virtual dimension of the moduli space is zero. Thus when the complex is acyclic we get a smooth zero dimensional space of bundles (the obstruction space really is $H^{0,2}$, and not $\operatorname{coker} \bar{\partial}_E \subset \Omega^{0,2}$, because of the Bianchi identity $\bar{\partial}_E F_E^{0,2} = 0$). Thus with an appropriate perturbation and compactness theorem we might hope to count the number of bundles on a Calabi-Yau 3-fold. We shall restrict attention to rank two bundles for the rest of this section, for convenience.

If, however, E is a line bundle, then (1.2.1) is the Dolbeault complex, and the topology of the Calabi-Yau prevents the cohomology groups vanishing. Therefore the condition $h^{0,1} = 0$ is often included in the definition of a Calabi-Yau. The analogy with flat bundles ($d_E^2 = 0$ instead of $\bar\partial_E^2 = 0$) on real 3-manifolds is clear – there we again have a self-dual complex with d_E instead of $\bar\partial_E$ and Poincaré duality replacing Serre duality, a finite number of flat bundles (twice the Casson invariant) under favourable conditions, and the restriction $b_1 = 0$ to avoid reducible connections (where (1.2.1) becomes the deRham complex). Thus the Casson invariant is usually only defined for homology spheres.

A self dual complex is often associated to the critical points of a (locally defined) function f on some (infinite dimensional) manifold Y. At a point $p \in Y$ where $df = 0$, the tangent space to the critical submanifold of f is the kernel (1st homology group) of

$$\begin{aligned} T_pY &\to T_p^*Y \\ v &\mapsto \nabla_v(df) \qquad \text{(intrinsically defined since } df = 0). \end{aligned} \tag{1.2.2}$$

The symmetry of the Hessian of a function translates into the self-duality of this map.

This is no accident; on a 3-manifold flat connections are the critical points of the Chern-Simons functional and (1.2.1) is just (1.2.2) with the action of a symmetry group. We shall find a similar picture on a Calabi-Yau 3-fold, mimicking the 3-manifold story by "wedging with θ". A good reference for the 3-manifold theory of the Chern-Simons functional and the gauge-theoretic Casson invariant is [T1]; we sketch the basic idea now.

The space $\mathscr{A}$ of connections on a bundle E over a 3-manifold M (with structure group G, Lie algebra $\mathfrak{g}$) is an affine space modelled on $\Omega^1(\mathfrak{g}_E)$. Fixing a basepoint A_0 (a flat connection $F_{A_0} = 0$ for simplicity), we have

$$\mathscr{A} = A_0 + \Omega^1(\mathfrak{g}_E), \quad T\mathscr{A} = \mathscr{A} \times \Omega^1(\mathfrak{g}_E).$$

Also let $\mathscr{B} = \mathscr{A}/\mathscr{G}_E$ denote the equivalence classes of connections under the action of the automorphism group (gauge group) of E. Now consider the one form σ on $\mathscr{A}$ defined at $A \in \mathscr{A}$ by

$$\sigma_A(a) = \int_M \operatorname{tr} F_A \wedge a, \qquad a \in \Omega^1(\mathfrak{g}_E).$$

This is zero along gauge orbits:

$$\sigma_A(d_A\phi) = \int_M \operatorname{tr} F_A \wedge d_A\phi = \int_M d\,(\operatorname{tr} F_A \wedge \phi) = 0,$$

using the Bianchi identity. It therefore descends to a one form on $\mathscr{B}$, which we also call σ. Since the rate of change of $\sigma(a)$ in the direction of $b \in \Omega^1(\mathfrak{g}_E)$,

$$\int_M \operatorname{tr}(d_A b \wedge a) = \int_M d\,(\operatorname{tr} b \wedge a) - \operatorname{tr}(b \wedge d_A a) = \int_M \operatorname{tr}(d_A a \wedge b),$$

is symmetric in a and b it follows that σ is closed.

Therefore, globally on $\mathscr{A}$ and locally on $\mathscr{B}$, $\sigma = 4\pi^2 dCS$ for some function $CS(A)$, the Chern-Simons functional. We have

$$\operatorname{grad}_A(4\pi^2 CS) = *F_A,$$

and the functional is given by

$$CS(A) = \frac{1}{4\pi^2}\int_X \operatorname{tr}\,(\frac{1}{2}d_A a \wedge a + \frac{1}{3}a \wedge a \wedge a).$$

On a Calabi-Yau 3-fold X there is an almost identical story, with a θ term added:

The space $\mathscr{A}$ of $\bar\partial$-operators (equivalently, unitary connections if E has a metric) on a bundle $E \to X$ is an affine space modelled on $\Omega^{0,1}(\mathfrak{g}_E)$, where $\mathfrak{g}_E$ will denote the endomorphisms of E, trace-free if we are fixing $\det E$. Fixing a basepoint A_0 (with $F^{0,2}_{A_0} = 0$, for simplicity), we have

$$\mathscr{A} = A_0 + \Omega^{0,1}(\mathfrak{g}_E), \quad T\mathscr{A} = \mathscr{A} \times \Omega^{0,1}(\mathfrak{g}_E).$$

Also let $\mathscr{B} = \mathscr{A}/\mathscr{G}_E$ denote the equivalence classes of $\bar\partial$-operators under the action of the automorphism group of E. Now consider the one form (in fact $(1,0)$-form) σ on $\mathscr{A}$ defined at $A \in \mathscr{A}$ by

$$\sigma_A(a) = \int_X \operatorname{tr} F_A^{0,2} \wedge a \wedge \theta, \qquad a \in \Omega^{0,1}(\mathfrak{g}_E).$$

This is zero along gauge orbits:

$$\sigma_A(\bar\partial_A \phi) = \int_X \operatorname{tr} F_A^{0,2} \wedge \bar\partial_A \phi \wedge \theta = \int_X \bar\partial\,(\operatorname{tr} F_A^{0,2} \wedge \phi) \wedge \theta = 0,$$

using the Bianchi identity. It therefore descends to a one form on $\mathscr{B}$, which we also call σ. Since the rate of change of $\sigma(a)$ in the direction of $b \in \Omega^{0,1}(\mathfrak{g}_E)$,

$$\int_X \operatorname{tr}\,(\bar\partial_A b \wedge a) \wedge \theta = \int_X \bar\partial \operatorname{tr}\,(b \wedge a) \wedge \theta - \operatorname{tr}\,(b \wedge \bar\partial_A a) \wedge \theta = \int_X \operatorname{tr}\,(\bar\partial_A a \wedge b) \wedge \theta,$$

is symmetric in a and b it follows that σ is closed.

Therefore, globally on $\mathscr{A}$ and locally on $\mathscr{B}$, $\sigma = 4\pi^2 d\,CS = 4\pi^2\partial CS$ for some holomorphic complex valued function for which we use the same notation $CS(A)$. We have

$$\mathrm{grad}_A(4\pi^2 CS) = \bar{*}\,(F_A^{0,2}\wedge\theta) = \star F_A^{0,2},$$

where we are defining the complex gradient by $df(v) = v.f = \langle v, \mathrm{grad}\ f\rangle$, we have fixed a metric on E, and $\star = \bar{*}\,(\cdot\wedge\theta)$ is our Hodge star $\Omega^{0,*}\to\Omega^{0,3-*}$.

1.3 Chern-Simons $\wedge\theta$

A short calculation shows that, for $A = A_0 + a$ and $CS(A_0)$ set to 0,

$$CS(A) = \frac{1}{4\pi^2}\int_X \mathrm{tr}\,(\frac{1}{2}\bar\partial_{A_0} a\wedge a + \frac{1}{3}a\wedge a\wedge a)\wedge\theta. \tag{1.3.1}$$

We will often confuse CS with the form of which it is the integral. Although we shall not need them, we shall spend a little time finding other interpretations of CS to make it more familiar. On $\mathscr{A}$, CS is constant on connected components of the gauge orbits $\mathscr{G}.A$ (since σ vanishes on tangents to the orbits), but we shall see below it may change under a "large" gauge transformation, as the Chern-Simons functional does on a 3-manifold.

The Chern-Simons functional on a 3-manifold is the "transgression class" associated to p_1 on a 4-manifold, in the following sense. If M is bounded by a 4-manifold N with a connection $\mathbb{A}$ on a bundle $\mathbb{E}$ restricting to A on $E\to M$ then

$$CS(A)-CS(A_0) = \frac{1}{4\pi^2}\int_M \mathrm{tr}\,(\frac{1}{2}d_{A_0}a\wedge a+\frac{1}{3}a\wedge a\wedge a) = \int_N p_1(\mathbb{A})-p_1(\mathbb{A}_0), \tag{1.3.2}$$

where $p_1(\mathbb{A}) = \frac{1}{4\pi^2}\mathrm{tr}\,F_{\mathbb{A}}\wedge F_{\mathbb{A}}$. This follows from the calculation

$$d\,\mathrm{tr}\,(\frac{1}{2}a\wedge d_{A_0}a + \frac{1}{3}a\wedge a\wedge a) = \mathrm{tr}\,(F_A\wedge F_A - F_{A_0}\wedge F_{A_0})\,. \tag{1.3.3}$$

From (1.3.2) we can see a mod $\mathbb{Z}$ ambiguity in CS: If $(N_i, \mathbb{E}_i, \mathbb{A}_i)$ are two different extensions then we get a bundle and connection glued over a small collar neighbourhood of M, and the difference in the two values of (1.3.2) will be p_1 of this bundle on $N = N_1\cup_M N_2$. In particular if A_0 is the trivial connection on a trivial bundle E we may take $\mathbb{E}$, $\mathbb{A}_0$ to be trivial and set $CS(A_0) = 0$ to recover the familiar formula

$$CS(A) = \int_N p_1(\mathbb{A}).$$

Since the same calculation (1.3.3) holds with some θ's and $\bar{\partial}$'s, CS can have a similar (though rather unnatural) topological interpretation. If X bounds a 7-manifold Y with a class $[\,\tilde{\theta}\,] \in H^3(Y;\mathbb{C})$ restricting to $[\,\theta\,]$ on X, then CS is the transgression of $p_1 \smile [\,\tilde{\theta}\,]$;

$$CS(A) - CS(A_0) = \int_Y p_1(\mathbb{A}) \wedge \tilde{\theta} - \int_Y p_1(\mathbb{A}_0) \wedge \tilde{\theta}, \tag{1.3.4}$$

where $\mathbb{A}$ is a connection extending $\bar{\partial}_A$ over a bundle $\mathbb{E}$ over Y extending E.

We will give a much more natural formula for CS in Theorem 1.3.7 below, based in complex geometry and perhaps the correct analogue of (1.3.2), with a Fano 4-fold playing the role of the 4-manifold, with "boundary" an anticanonical divisor X. There is also the formula (1.3.6) below which is another illustration of the principle "$d\,CS = p_1 \wedge \theta$", but there is one instance in which the above topological picture is useful, namely finding the gauge dependence of CS.

Given an automorphism g of E, we form the bundle $\mathbb{E}$ on $X \times S^1$ by using g to glue $E|_{X\times\{0\}}$ to $E|_{X\times\{1\}}$ in $\pi^* E \to X \times [0,1]$. Then any connection $\mathbb{A}$ on $\mathbb{E}$ restricting to A on $X \times \{0\}$ (and so to $g(A)$ on $X \times \{1\}$) will have

$$\langle p_1(\mathbb{A}) \smile \pi^*[\,\theta\,],\, [X \times S^1]\,\rangle = CS(A) - CS(g(A)), \tag{1.3.5}$$

by (1.3.4). Decomposing $p_1(\mathbb{A})$ according to the (integral) Künneth formula on $X \times S^1$, we see that CS is well defined on $\mathscr{B}$ modulo the periods

$$\left\{ \int_\sigma \theta \ :\ \sigma \in H_3(X;\mathbb{Z}) \right\}$$

of θ. Unfortunately these will usually be dense in $\mathbb{C}$, and in many ways CS is best thought of as an element of the dual Albanese variety

$$\left(H^{2,1} \oplus H^{3,0}\right)^* / \, H_3(X;\mathbb{Z}),$$

by considering all $\theta \in H^{2,1} \oplus H^{3,0}$, so that it is well defined modulo a discrete lattice of periods. This, however, requires E to be Hermitian so that we can define CS on all θ by using the metric to form a unitary connection from $\bar{\partial}_A$ and writing

$$CS(A_0 + a) = \frac{1}{4\pi^2} \int_X \mathrm{tr}\,(F_{A_0} \wedge b + \frac{1}{2} d_{A_0} b \wedge b + \frac{1}{3} b \wedge b \wedge b) \wedge \theta, \quad b = a - a^*.$$

This reduces to the previous formula (1.3.1) for $\theta \in \Omega^{3,0}$ and $F^{0,2}_{A_0} = 0$.

This is completely analogous to the case of a line bundle, topologically trivial for simplicity, on a Riemann surface Σ. Then the appropriate Chern-Simons functional is $a \mapsto \int_\Sigma a \wedge \omega$ for $\omega \in H^{1,0}(\Sigma)$. While this is not very well defined as a function of one ω (the periods of ω are likely to be dense) it is well defined modulo only a discrete lattice when considered as a function of a basis (ω_i) of $H^{1,0}$. That is, the function should correctly be thought of as lying in the torus

$$\left(H^{1,0}\right)^* / H_1(\Sigma; \mathbb{Z}).$$

In this case we also have the alternative formula [GH pp 331–332]

$$\int_\Sigma a \wedge \omega = \int_\gamma \omega \qquad \text{modulo periods,}$$

where γ is a path connecting the points that are the zeroes and poles of a section that is meromorphic with respect to the holomorphic structure $\bar\partial + a$. That is, if the holomorphic structure defined by a on the line bundle corresponds to the divisor $\sum a_i(p_i)$, with $a_i \in \mathbb{Z}$ and $p_i \in \Sigma$, then $\partial\gamma = \sum a_i(p_i)$. In a special case there is an analogous formula on a Calabi-Yau 3-fold, for a rank 2 holomorphic bundle E.

Proposition 1.3.6 *Suppose $A = A_0 + a$ is an integrable ($F_A^{0,2} = 0$) $\bar\partial$-operator on E with trivial determinant, and (E, A), (E, A_0) admit holomorphic sections s, s_0, with transverse zero sets $(s)_0$, $(s_0)_0$. Then CS defined by (1.3.1) may also be described as follows. As the zero sets are homologous, write $(s)_0 - (s_0)_0 = \partial\Delta$ for some singular 3-chain Δ. Then, modulo periods, $CS(A) = \int_\Delta \theta$.*

Proof. The idea of the proof is that, on a space of connections, $d\,CS = p_1 \wedge \theta$, and when $\det E$ is trivial, $p_1 = c_2$ represents the Euler class of E, as does the Dirac delta current Poincaré dual to the zero set of a section.

Over a small open set U away from the zero set of s we can pick a complementary, non-vanishing holomorphic section t of E, so that (s, t) trivialises $E|_U$. Rescaling t if necessary we can assume without loss of generality that $s \wedge t$ is the restriction to U of the global holomorphic trivialising section of $\Lambda^2 E$. Now define a connection extending $\bar\partial_A$ by requiring that s and t are parallel.

Fix $\delta > 0$. Patching together such connections by a partition of unity over open sets covering X minus a δ-neighbourhood $\nu_\delta(s)$ of $(s)_0$, and extending over $(s)_0$, we get a connection d_A, with $(0,1)$ part $\bar\partial_A$, such that $d_A s = 0$ outside $\nu_\delta(s)$.

We can repeat this for s_0, and then also for a section σ of $\mathbb{E} = \pi^* E \to X \times [0,1]$ that we choose to restrict to s and s_0 at the two ends, with transverse zeroes $(\sigma)_0$ in $X \times [0,1]$. Thus we obtain a connection $\mathbb{A}$ on $\mathbb{E}$ with trivial determinant, satisfying $d_{\mathbb{A}}\sigma = 0$ outside $\nu_\delta(\sigma)$, and also a parallel section τ outside $\nu_\delta(\sigma)$ such that (σ, τ) trivialises $\mathbb{A}$ and $\mathbb{E}$ on $X \times [0,1] \setminus \nu_\delta(\sigma)$.

Thus $F_{\mathbb{A}}\sigma = 0$ on $X \times [0,1] \setminus \nu_\delta(\sigma)$. Locally, therefore, with respect to a basis containing σ, $F_{\mathbb{A}}$ must be of the form $\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$, and in fact $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, since $\operatorname{tr} F_{\mathbb{A}} = 0$. This implies that $4\pi^2 p_1(\mathbb{A}) = \operatorname{tr}(F_{\mathbb{A}} \wedge F_{\mathbb{A}}) = 0$ away from ν_δ. Using (1.3.5) we have then, modulo periods,

$$CS(A) = CS(A) - CS(A_0) = \int_{X\times[0,1]} p_1(\mathbb{A}) \wedge \pi^*\theta = \int_{\nu_\delta(\sigma)} c_2(\mathbb{A}) \wedge \pi^*\theta.$$

We would like to show that we can choose $\mathbb{A}$ so that the form $c_2(\mathbb{A})$ converges to the current $[\,(\sigma)_0\,]$ representing the Poincaré dual of the zero set $(\sigma)_0$, as $\delta \to 0$. For then we would have

$$CS(A) = \int_{(\sigma)_0} \pi^*\theta = \int_\Delta \theta,$$

where $\Delta = \pi_*(\sigma)_0$ satisfies $\partial\Delta = (s)_0 - (s_0)_0$. Since the integral of θ against any other such Δ differs only by a period, this would complete the proof. This is where it gets a little messy, however, and since we do not need the proposition again, we only sketch the details.

By the tubular neighbourhood theorem, for δ sufficiently small, σ induces a diffeomorphism between $\nu_{2\delta}$ and a neighbourhood of the zero set in the total space of $\mathbb{E}|_{(\sigma)_0}$. In fact, pick a metric on $\mathbb{E}|_{(\sigma)_0}$. Then we can choose $\nu_{2\delta}$ to be the 2δ-ball bundle of $\mathbb{E}$ over $(\sigma)_0$, for δ sufficiently small. Picking an orthonormal trivialisation over a coordinate patch of $(\sigma)_0$, $\nu_{2\delta}$ looks like $\mathbb{R}^3 \times B_{2\delta}(\mathbb{C}^2)$, $\mathbb{E}$ looks like $\mathbb{C}^2$, and the section σ is the identity $(z_1, z_2) : \mathbb{C}^2 \to \mathbb{C}^2$, restricted to $B_{2\delta}(\mathbb{C}^2)$ and pulled back in the $\mathbb{R}^3$ direction.

Working in such a coordinate patch, we can now take $\mathbb{A}$ to be of a standard form on $\nu_{2\delta}$. For instance, pull back the standard one-instanton ([DK] pp 116–7), with half its energy inside a ball B_{δ^2} of radius δ^2 in $\mathbb{C}^2$, to $\mathbb{R}^3 \times B_{2\delta}$. If we pick a different orthonormal trivialisation the pull back differs by a gauge transformation g that varies only in the $\mathbb{R}^3$ direction, in which $\mathbb{A}$ is just the exterior derivative. Patching together these two connections by a patching function ψ, again only varying the $\mathbb{R}^3$ direction, the curvature changes by a bounded amount

$$F_{(1-\psi)\mathbb{A}+\psi g(\mathbb{A})} - F_{\mathbb{A}} = g^{-1} d\psi \wedge dg,$$

which therefore tends to zero as a current as $\delta \to 0$ (we can fix one ψ for all δ), since its support ν_δ shrinks. Thus we can work on such a coordinate patch, and glue together the connections at the end.

This connection does not decay at infinity in $\mathbb{C}^2 \cong \mathbb{R}^4$ as it is in a gauge which extends over $0 \in \mathbb{R}^4$, and not $\infty \in S^4$. That is, in our $\mathbb{C}^2$ model, the trivialising sections of the gauge are $(1,0)$ and $(0,1)$, instead of (z_1, z_2) and $(\frac{1}{2z_2}, -\frac{1}{2z_1})$. Changing to this second trivialisation, i.e. changing gauge by a degree one map $S^3 \to SU(2)$, the connection form decays like $\delta^4/|x|^3$. We may choose the original σ and τ to be the sections (z_1, z_2) and $(\frac{1}{2z_2}, -\frac{1}{2z_1})$ in $\nu_{2\delta} \setminus \nu_\delta$ (there was a lot of choice in σ and τ, the only problem is choosing them to be holomorphic on restriction to the ends of the product $X \times I$, but choosing the connections and sections to be pull-backs in a small collar neighbourhood of the two ends satisfies this). In this gauge we smoothly cut off the connection by a patching function ψ on $\mathbb{R}^4$ which is $O(1)$ with $d\psi = O(\delta^{-1})$, changing $\mathbb{A}$ to $\psi\mathbb{A}$ in $B_\delta \setminus B_{\delta/2}$, with curvature

$$F_{\psi\mathbb{A}} - \psi F_{\mathbb{A}} = (\psi^2 - \psi)\mathbb{A} \wedge \mathbb{A} + d\psi \wedge \mathbb{A}$$

where all the terms on the right are bounded since $\mathbb{A}$ decays as $\delta^4/|x|^3 = O(\delta)$ in this region. Thus the c_2 form of this connection converges, as $\delta \to 0$, to the current $[\,(\sigma)_0\,]$, since c_2 of the instanton tends to the Dirac delta measure at the origin of $\mathbb{R}^4$. □

Finally we have another formula for CS, the complex analogue of (1.3.2). First we rephrase (1.3.2) in a way which will make the analogy more apparent. The long exact homology and cohomology sequences of the pair $(N = N^4, M = \partial N)$ give rise to the following commutative pairings:

$$\begin{array}{ccccccc}
 & [CS(A)] & \mapsto & [p_1(\mathbb{A}) - p_1(\mathbb{A}_0)] & \\
\to & H^3(M) & \xrightarrow{d} & H^4(N,M) & \to 0 \\
 & \otimes & & \otimes & \\
\leftarrow & H_3(M) & \xleftarrow{\partial} & H_4(N,M) & \leftarrow 0. \\
 & [M] & \leftarrow\!\shortmid & [N] &
\end{array}$$

That is, since the fundamental class of M is in the image of the lower map, coming from the fundamental class of N, to find $\int_M CS(A)$ we can map the class $[CS(A)]$ into $H^4(N,M)$ (using the calculation (1.3.3)) and evaluate on $[N]$, giving (1.3.2).

We now replace the exact sequence of the pair (N, M) by the sheaf sequence of the pair (Y, X) in the case that X is an anticanonical divisor in a Fano 4-fold Y.

Theorem 1.3.7 *Suppose that the Calabi-Yau 3-fold X is a smooth effective anticanonical divisor in a 4-fold Y defined by $s \in H^0(K_Y^{-1})$. If $E \to X$ is a bundle that* *extends to a bundle* $\mathbb{E} \to Y$*, then for a* $\bar{\partial}$*-operator* A *on* E*, let* $\mathbb{A}$ *be any* $\bar{\partial}$*-operator on* $\mathbb{E}$ *extending* A*. Then we have, modulo periods,*

$$CS(A) = \frac{1}{4\pi^2} \int_Y \operatorname{tr} F_{\mathbb{A}}^{0,2} \wedge F_{\mathbb{A}}^{0,2} \wedge s^{-1}.$$

Proof. We have the sequence

$$0 \to K_Y \xrightarrow{s} \mathcal{O}_Y \to \mathcal{O}_X \to 0,$$

where $s \in H^0(K_Y^{-1})$ is the section defining X, and also $H^4(\mathcal{O}_Y) = H^0(K_Y)^* = 0$ (if $t \in H^0(K_Y)$ then $s.t$ is a holomorphic function on Y vanishing on X; thus $t = 0$). So we get the following commutative diagram of Serre duality pairings:

$$\begin{array}{ccccccc} & [CS(A)] & \mapsto & [p_1(\mathbb{A}) - p_1(\mathbb{A}_0)] \wedge s^{-1} & & \\ \to & H^3(\mathcal{O}_X) & \longrightarrow & H^4(K_Y) & \longrightarrow & 0 \\ & \otimes & & \otimes & & \\ \leftarrow & H^0(\mathcal{O}_X) & \longleftarrow & H^0(\mathcal{O}_Y) & \longleftarrow & 0, \\ & [1] & \leftarrow\!\shortmid & [1] & & \end{array} \tag{1.3.8}$$

(where the first pairing is by integrating against θ) since the upper map takes a holomorphic $(0,3)$-form on X, extends it to a C^∞ form on Y, and takes $\bar{\partial}(\,\cdot\,) \wedge s^{-1}$ of the result. Fixing constants by setting $CS(A_0) = \int_Y p_1(\mathbb{A}_0)$ gives the theorem. □

Just as the real case $CS(A) = \int_N p_1(\mathbb{A})$ can be proved directly by Stokes' theorem, the above amounts to an application of Stoke's theorem *and* the Cauchy residue theorem [Kh] (hence reducing dimensions by two). If $\nu_\delta(X)$ denotes a small tubular neighbourhood of $X \subset Y$ then $\int_Y p_1(\mathbb{A}) \wedge s^{-1} = \int_Y d\,(CS(\mathbb{A}) \wedge s^{-1}) = \lim_{\delta \to 0} \int_{\partial \nu_\delta} CS(\mathbb{A}) \wedge s^{-1}$, which can be integrated first over the fibres of the circle bundle $\nu_\delta \to X$, by Cauchy's reside formula, then along X, to give $\int_X CS(A) \wedge \theta$. (Here s^{-1} induces a canonical volume form θ on X [GH p 147].)

Diagram (1.3.8) shows we have a canonical lifting to $\mathbb{C}$ of our badly defined functional CS given by this formula. Different extensions $\mathbb{E}_i$ and Y_i of the bundle $E|_X$ will give different liftings, and the ambiguity in CS derives from the difference in these values. Given two such extensions we can glue them together across X to get a singular Calabi-Yau $Y = Y_1 \cup_X Y_2$ as will be described in a similar setting in Section 1.5. Its holomorphic (4,0)-form θ_Y is s_i^{-1} on Y_i, and the difference in the values (1.3.7) for CS is

$$\frac{1}{4\pi^2} \int_Y \operatorname{tr} F_{\mathbb{A}}^{0,2} \wedge F_{\mathbb{A}}^{0,2} \wedge \theta_Y = \langle p_1(\mathbb{E}) \smile [\,\theta_Y\,], Y \rangle.$$

Thus we see the ambiguity as periods of θ in a form similar to (1.3.5), and directly analogous to the $\mathbb{Z}$ ambiguity in the real Chern-Simons functional mentioned earlier.

1.4 The Casson invariant $\wedge\theta$

In [T1], the Casson invariant is given as a sensible definition of (one half of) the number of non-trivial flat $SU(2)$ connections on the trivial bundle on a homology 3-sphere. Flat connections are the zeroes of the gradient of the Chern-Simons functional, and Taubes perturbs this vector field on $\mathscr{B}$ to obtain a finite number of transverse zeroes, which can be counted with sign. Only the relative signs are well defined (in infinite dimensions the determinant of the Hessian does not exist) by spectral flow, but then the trivial connection is assigned a positive sign (but not counted as it is highly reducible).

The key to the success of Taubes' method is the compactness of the space of flat connections, and the choice of suitably controlled compact perturbations that do not alter the gradient field from being a Fredholm section of $T\mathscr{B}$ and do not introduce any zeroes at infinity, while breaking up the zero set into transverse points. This compactness also ensures the number of these points is finite, so can be counted.

By analogy, we would like to be able to count the holomorphic bundles on a Calabi-Yau 3-fold as the zeroes of

$$\operatorname{grad} CS = \star F_A^{0,2}$$

on $\mathscr{B}$. The analytical details are, however, much harder than in Taubes' case due to the lack of a suitable compactness theorem for moduli of Hermitian-Yang-Mills connections (which correspond to *stable* holomorphic structures on a bundle, but this is in some sense most of them). We discuss this compactness issue in Chapter 3, but to allow suitable perturbations we would like a compactness result for perturbed equations too, or for the natural generalisation (3.1.4) of the equations on a symplectic manifold, so that we could perturb the Kähler metric. If this could be done it should be possible to push Taubes' method through.

The approach we take, in Chapter 3, is an algebraic one. There is an algebraic compactification of the moduli space $\mathcal{M}$ of stable bundles, using sheaves, but we lose the benefits of perturbations, so the moduli space may well have strictly positive dimension, and be singular, etc. But algebraic geometry copes well with singularities, and we know what to do about components of $\mathcal{M}$ of positive dimension if they are smooth – take their Euler characteristic. This is because if we have a function f on a (finite dimensional) manifold Y with a smooth critical submanifold $Z = (f)_0$, it is the Euler charactersitic of Z that gives the Euler characteristic of Y (i.e. the number of critical points of a perturbation of f) since the cokernel of $\nabla(\operatorname{grad} f)$ on Z is naturally isomorphic to TZ. In fact we are interested in the zeroes of a holomorphic cotangent vector field df rather than the vector field $\operatorname{grad} f$ so our cokernel bundle will be the *cotangent* bundle of Z, as can be seen from the duality (1.2.2),

$$0 \to TZ \to TY \xrightarrow{\nabla(df)} T^*Y \to T^*Z \to 0.$$

Thus it is the Euler number of T^*Z in which we are interested. In our infinite dimensional set-up, the point is that the cokernel of the derivative of $d\,CS = F_A^{0,2}$ at a critical point will be naturally isomorphic to the cotangent space to the space of critical points $\mathcal{M}$ by the duality (1.2.1). Namely on a Calabi-Yau 3-fold we have the Serre duality

$$H^2(\operatorname{End}_0 E) \cong H^1(\operatorname{End}_0 E)^* = T_E^*\mathcal{M}$$

for a vector bundle E, and a generalisation for sheaves using Ext groups.

So if we could find a small transverse perturbation of $d\,CS$ it would break up the moduli space into a number of isolated points corresponding to the zeroes of a small section of the cokernel bundle over $\mathcal{M}$. These points would correspond to the Euler class of the cokernel bundle, i.e. $e(T^*\mathcal{M})$, giving $\chi(\mathcal{M})$ to within a sign, as claimed.

Chapter 3 will set up the machine to handle all these issues correctly to give a definition of this holomorphic Casson invariant. To show it is a sensible definition we would then like to show it is deformation invariant, as any definition using Fredholm analysis and perturbations would immediately be. While we cannot expect the moduli space to be completely independent of the complex structure as in one or two complex dimensions (where the Hermitian-Yang-Mills equations have extra symmetry and are the projectively flat equations and ASD equations respectively, independent of the complex structure), we do expect certain topological quantities to be conserved. Again, this is best seen from the PDE point of view, and we would only expect the moduli space of solutions of a *complex* PDE to change at points of real codimension two in a family, so the generic moduli spaces should be diffeomorphic.

This deformation invariance is discussed for smooth Calabi-Yau manifolds in Section 3.5; what we would really like to do is be able to deform to a Calabi-Yau with normal crossing singularities and show that the invariant there equals the one discussed in the next section.

1.5 Handlebodies and Calabi-Yaus with normal crossing singularities

The original definition of the Casson invariant used a Heegard splitting of $M = H_1 \cup_S H_2$ along a surface S. Then the space $\mathcal{M}_S$ of equivalence classes of flat G-connections on S is, at smooth points, a symplectic manifold with tangent space $H^1(\mathfrak{g}_E)$, and symplectic form

$$\Omega_A(a,b) = \int_S \operatorname{tr} a \wedge b, \qquad a, b \in \Omega^1(S; \mathfrak{g}_E),\ d_A a = 0 = d_A b, \tag{1.5.1}$$

at $[A]$, for $a, b \in T_A\mathcal{M}_S = H^1(\mathfrak{g}_E)$. (This is easily seen to be closed, and is symplectic on $\mathscr{A}$. Restricting to the flat connections (the zero set of the moment map) it is degenerate precisely along gauge orbits, so passing to the quotient $\mathcal{M}_S$ it is well defined and symplectic.) More algebraically, this is of course just the non-degenerate skew pairing $H^1_A(\mathfrak{g}_E) \times H^1_A(\mathfrak{g}_E) \to \mathbb{R}$ given by Poincaré duality.

Proposition 1.5.2 *The images $\mathcal{M}_{H_i} \subset \mathcal{M}_S$ of flat connections on the handlebodies H_i restricted to S are, at smooth points, Lagrangian submanifolds of $\mathcal{M}_S$.*

Proof. Let A be a flat connection on H_1, and $a, b \in \Omega^1(H_1; \mathfrak{g}_E)$, $d_A a = 0 = d_A b$ be tangent vectors to $\mathcal{M}_{H_1}$. Then

$$\Omega_A(a,b) = \int_S \operatorname{tr} a \wedge b = \int_{H_1} \operatorname{tr} d_A a \wedge b - \operatorname{tr} a \wedge d_A b = 0.$$

Conversely, if $\Omega(a,b) = 0\ \ \forall a \in T_A\mathcal{M}_{H_1}$ (i.e. $a \in \Omega^1(H_1; \mathfrak{g}_E)$, $d_A a = 0$) we would like to show b is tangent to $\mathcal{M}_{H_1}$. Extending b to some $\tilde{b} \in \Omega^1(H_1; \mathfrak{g}_E)$, the above equation shows that $d_A\tilde{b}$ is zero in the relative cohomology group $H^1_A(H_1, S; \mathfrak{g}_E)$. Therefore $d_A\tilde{b} = d_A c$ for some c vanishing on S. Thus $d_A(\tilde{b} - c) = 0$, so $b = (\tilde{b} - c)|_S$ is tangent to $\mathcal{M}_{H_1} \subset \mathcal{M}_S$. □

Remark. It is clear that some exact sequences and Lefschetz and Poincaré duality are lurking behind this rather hands-on proof. We will see this more clearly in the corresponding Calabi-Yau set-up (1.5.3) below, which we shall handle more algebraically.

So, in analogy with intersection theory in a smooth manifold, we would like to be able to make the $\mathcal{M}_{H_i}$ meet, after a small perturbation, in a finite number of points corresponding to flat bundles on all of $M = H_1 \cup_S H_2$. Problems with reducible connections mean we must restrict to homology spheres; then the intersection theory can be made sense of, and the resulting intersection number is twice the Casson invariant.

A beautiful observation of Donaldson and Khesin [DT, Kh] shows there is an analogue of this for counting holomorphic bundles on a Calabi-Yau, using a construction of Tyurin [Ty]. [Kh] also outlines some complex analogues of real geometry in the spirit of this work.

Tyurin considers a Fano threefold X with a section s of K_X^* with zero set a $K3$ surface $S \subset X$. As mentioned before, we will think of this as the complex analogue of a manifold with boundary. Then for the analogue of gluing along a common boundary we take two Fano 3-folds X_i, playing the role of the handlebodies, intersecting with normal crossings along a $K3$ surface S, which is the analogue of the surface S in the 3-manifold case, and is Calabi-Yau.

The resulting 3-fold

$$X = X_1 \cup_S X_2$$

is a singular Calabi-Yau. A good example of this is a degenerate Calabi-Yau quintic in $\mathbb{CP}^4$ which is the union of a cubic and a quadric. By the adjunction formula their intersection will have trivial canonical bundle and is in fact a $K3$ surface.

The construction in [Ty], using Mukai's theorem [Mu], is then the following. Here $\mathcal{M}$ denotes the moduli space of stable holomorphic bundles of fixed determinant, so $\mathfrak{g}_E = \mathrm{End}_0\,(E)$, and the Zariski tangent space to $\mathcal{M}$ is $T_E\mathcal{M} = H^{0,1}(\mathfrak{g}_E)$.

Proposition 1.5.3 *Consider a smooth 3-fold X with a smooth effective anticanonical divisor $S \subset X$. About a bundle on X which is stable on S, $\mathcal{M}_X$ injects into $\mathcal{M}_S$ by restriction. At any smooth point E of $\mathcal{M}_X$ (i.e. $H^2(X;\mathfrak{g}_E) = 0$), $\mathcal{M}_S$ is a smooth complex symplectic manifold, and $\mathcal{M}_X$ a complex Lagrangian submanifold.*

Proof. The complex symplectic pairing ([Mu]) on the tangent space $H^1(\mathfrak{g}_E)$ to $\mathcal{M}_S$ comes from Serre duality – it is its own Serre-dual since K_S is trivial. More analytically, for $a, b \in H^1(\mathfrak{g}_E)$, the pairing is

$$\Omega(a,b) = \int_S \mathrm{tr}\, a \wedge b \wedge \theta,$$

which should be compared to (1.5.1), and where θ is a fixed trivialising holomorphic (2,0)-form.

Now suppose E is a smooth point of $\mathcal{M}_X$, so that $H^2(X;\mathfrak{g}_E)=0$. Then the sequence

$$0\to K_X\to \mathcal{O}_X\to\mathcal{O}_S\to 0$$

defining S, tensored with $\mathfrak{g}_E$, yields

$$0\to H^1(\mathfrak{g}_E)\to H^1(\mathfrak{g}_E|_S)\to H^2(\mathfrak{g}_E\otimes K_X)\to 0, \tag{1.5.4}$$

since $H^1(\mathfrak{g}_E\otimes K_X)$ vanishes by Serre duality. Thus we have

$$0\to T_E\mathcal{M}_X\to T_E\mathcal{M}_S\to (T_E\mathcal{M}_X)^*\to 0,$$

and locally $\mathcal{M}_X$ injects into $\mathcal{M}_S$. In fact taking the Serre-dual of (1.5.4) gives us the commutative pairings

$$\begin{array}{ccccccccc} 0 & \to & T_E\mathcal{M}_X & \to & T_E\mathcal{M}_S & \to & (T_E\mathcal{M}_X)^* & \to & 0 \\ & & \otimes & & \otimes & & \otimes & & \\ 0 & \leftarrow & (T_E\mathcal{M}_X)^* & \leftarrow & T_E\mathcal{M}_S & \leftarrow & T_E\mathcal{M}_X & \leftarrow & 0\,. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathbb{C} & & \mathbb{C} & & \mathbb{C} & & \end{array} \tag{1.5.5}$$

Therefore, by exactness, the annihilator of $H^1(\mathfrak{g}_E)\cong T_E\mathcal{M}_X$ under the central pairing (which is the complex symplectic form Ω) is precisely itself.

Finally, higher terms in the long exact sequence show that the vanishing of $H^2(X;\mathfrak{g}_E)$ implies the vanishing of $H^2(S;\mathfrak{g}_E)$ for E stable on X, from which the smoothness statement follows. □

Remark. More generally the same proof shows that the map $\mathcal{M}_X\to\mathcal{M}_S$ always has Lagrangian *image* (the pairing defines a symplectic structure even at non-smooth points [Mu]). Also the earlier terms in the long exact sequence (1.5.4) show that for a smooth point $E\in\mathcal{M}_X$, $E|_S$ is simple (has no trace-free endomorphisms), which in many cases is equivalent to stability.

It is now obvious we would like to take the intersection number of the two $\mathcal{M}_{X_i}$ in $\mathcal{M}_S$ to define a complex Casson invariant counting holomorphic bundles on this singular Calabi-Yau 3-fold, at least when we can ensure the restriction map exists. We would also like this to be the same as the number of bundles on a smooth Calabi-Yau in a family degenerating to this manifold, as discussed above.

While $\mathcal{M}_S$ is well understood ([Mu, HL]), is smooth (so supports intersection theory), and has a natural compactification $\overline{\mathcal{M}}_S$, the problem is to assign middle dimensional homology classes to the $\mathcal{M}_{X_i}$. Maruyama's theorem gives us compactifications $\overline{\mathcal{M}}_{X_i}$ of these spaces, and Theorems 3.3.4 and 3.4.6 of Chapter 3 will give us "virtual moduli cycles" of the correct dimension (this will be explained in Chapter 3) inside the $\mathcal{M}_{X_i}$. The problem, then, is to ensure there is a map from these to $\overline{\mathcal{M}}_S$, i.e. that stable sheaves restrict to semi-stable sheaves (of the right Hilbert polynomial).

In many cases this can be shown to hold; for instance the remark following the last proposition shows the restriction map often exists at smooth points of $\mathcal{M}_X$, and Theorem 2.2.10 will show that for the generic S the restriction of ma stable bundle is semi-stable (and so stable for rank and degree coprime, for instance). The problem is to show that the map exists on *all* of $\overline{\mathcal{M}}_X$ for fixed S. Stability of the restriction is one problem, and another is that the restrictions of non-locally free sheaves may have the wrong Hilbert polynomial.

In some cases (such as the extended example of Section 2.3) $\mathcal{M}_X$ itself is compact, and all restrictions are stable, and we do not need to consider sheaves. However the intersection number obtained may not be deformation invariant since we have not considered all of $\overline{\mathcal{M}}_X$, and points may wander off "to infinity". Similarly we could consider the closure of the birational image of $\mathcal{M}_X$ in $\mathcal{M}_S$, but again deformation invariance of the resulting intersection number would be unclear.

1.6 An analogous construction to count curves

Understanding compactness etc., of moduli spaces of (pseudo-)holomorphic curves is a little easier than for bundles – the space of bundles is, roughly, made up of fibrations over different spaces of curves, by the Serre construction of Chapter 2. Different strata correspond to the different twists needed to first get a section of a particular bundle, and stability bounds these twists.

So this suggests it might be easier to carry out the method of the last section to count curves in Calabi-Yau manifolds, which also have moduli of virtual dimension zero. Gromov-Witten invariants are now defined for all algebraic varieties, and in fact [LT] gives a "virtual moduli cycle" (see Chapter 3 for this theory), of the correct dimension and with all the right properties, in the moduli space of curves on an algebraic variety. In particular we get such a cycle on a Fano (where in fact older, less sophisticated techniques often work, for instance if the Fano is "semi-positive" or "convex").

So fix a class $\beta \in H_2(X)$ in a 3-fold X and a section $s \in H^0(K_X^{-1})$ with zero set a Calabi-Yau surface S. Let n denote $\beta \,.\, [S]$, and let U be the Zariski-open subset of the moduli space of curves on X, of homology class β, corresponding to curves that have a zero dimensional intersection with S. Then intersection with S gives us a map from U to the Hilbert scheme $\mathrm{Hilb}^n S$, and we have (compare (1.5.3)):

Theorem 1.6.1 *The image of the above map is a complex Lagrangian in the complex symplectic manifold* $\mathrm{Hilb}^n S$*. If the curve represents a smooth point of the moduli space then the map is locally an injection.*

Remark. When the curve corresponds (via the Serre construction) to a stable bundle on X, and the n-tuple of points corresponds to the bundle restricted to S, this theorem is essentially the same as (1.5.3). Relating the deformation theory of the bundle to that of the points, this amounts to saying that the symplectic structure on the moduli of bundles is induced by that on S (and more generally on $\mathrm{Hilb}^n S$) – vectors tangent to $\mathcal{M}$ induce vectors on $\mathrm{Hilb}^n S$ which can be contracted with its symplectic form.

Proof. We have assumed the section s vanishes only on a set of points (more precisely a zero dimensional subscheme) Z on C, giving us a sequence

$$0 \to \nu_C \otimes K_X|_C \to \nu_C \to \nu_C|_Z \to 0,$$

on C, where ν_C is the normal bundle to C in X. Taking the long exact cohomology sequence and its dual, and using the fact that the Serre-dual bundle to $\nu_C \otimes K_X|_C$ is ν_C (since $K_C \cong K_X|_C \otimes \Lambda^2 \nu_C$), yields

$$\begin{array}{cccccccccccc}
0 \to & H^1(\nu_c)^* & \to & H^0(\nu_c) & \to & \nu_C|_Z & \to & H^0(\nu_c)^* & \to & H^1(\nu_c) & \to 0 \\
& \otimes & & \otimes & & \otimes & & \otimes & & \otimes & \\
0 \leftarrow & H^1(\nu_c) & \leftarrow & H^0(\nu_c)^* & \leftarrow & \nu_C|_Z & \leftarrow & H^0(\nu_c) & \leftarrow & H^1(\nu_c)^* & \leftarrow 0\,, \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& \mathbb{C} & & \mathbb{C} & & \mathbb{C} & & \mathbb{C} & & \mathbb{C} &
\end{array}$$

where the pairings commute, and the bottom sequence is the same as the top. The central pairing is via the trivialisation of $\Lambda^2 \nu_C|_Z \cong \Lambda^2 T_Z S$ given by the Calabi-Yau form on S, i.e. it is the natural holomorphic symplectic structure on $\mathrm{Hilb}^n S$. Thus we see that the image of $H^0(\nu_C)$ is its own annihilator under this pairing. Since this is the Zariski tangent space at C to the moduli space of curves on X, with the map being the obvious one (the derivative of the map assigning to a curve its intersection with S), the result follows.

$H^1(\nu_C)$ is the obstruction space for deformations of C, so if C is a smooth point in the moduli space of curves on X then this vanishes and the above sequences reduce from five terms to three, taking the shape of (1.5.5) and proving injectivity. □

So again it is clear what we would like to do given two such pairs (X_i, β_i) and a common anticanonical divisor S of the X_i along which we glue. The intersection of the Lagrangians in $\text{Hilb}^n S$ should be a finite number N of points corresponding to those zero-cycles which lie on curves on both of the X_i. Gluing these curves gives N nodal curves of homology class $\beta_1 + \beta_2$. On deformation these curves deform to curves in the smoothed Calabi-Yau, if, for instance, their obstruction spaces vanish. The hard part would be to show that this is all of them.

Thus for Fanos with smooth moduli spaces of curves (e.g. "convex" Fanos like projective space and the Grassmannian with the property that $H^1(TX|_C) = 0$ for all embedded curves) compactness and smoothness issues do not arise. But we still must deal with those curves that do not intersect S in points but have components in common with it. Passing through the Serre construction yet again this often corresponds to stable bundles restricting to non-stable bundles on S, as can be seen by trying to restrict the Koszul resolution (2.1.2) to S, so we see a similar problem arising to that in the last section. At least though we have more control over it – in many cases dimension counts and repeated use of the word generic will avoid such intersections.

This then looks like a very promising construction. However I have not pursued it since Paul Seidel has informed me that Tian has done some work on something very similar.

1.7 Some gauge theory $\wedge\theta$

The following sections outline a formal picture of other parts of gauge theory that have complex versions on Calabi-Yau manifolds. Having mentioned the central 3-dimensional case and passed down to two dimensions, we now pass up to four dimensions via Floer theory.

Floer theory $\wedge\theta$

A generalisation of the Casson invariant is the Floer homology of a 3-manifold. We refer to [At2] for a nice overview of the subject, but the main idea is to mimic Witten's method of computing the homology of a compact manifold on the infinite dimensional space of connections $\mathscr{B}$, finding its "middle dimensional homology" using the Chern-Simons functional as a Morse function. The Euler characteristic of the resulting Floer groups is then (twice) the Casson invariant. Thus we have a chain complex generated by the critical points, i.e. the flat connections, and boundary operator defined via the gradient flow lines between them. That is, consider the family of connections A_t on $M^3 \times \mathbb{R}$ satisfying

$$\partial_t A_t = *F_{A_t}. \tag{1.7.1}$$

Spectral flow of $\nabla(dCS)$ along such paths defines a relative grading of the critical points (only defined mod 8 due to the spectral flow around a closed loop being a multiple of 8, the index of the ASD equations (1.7.1) on $M \times S^1$). Then the boundary operator takes a critical point A to $\partial A = \sum_B n_{AB} B$, where B runs over the critical points of relative index 1, and n_{AB} is the number of gradient lines from A to B, with an appropriate sign.

The complex analogue of Morse theory is the Picard-Lefschetz theory of Lefschetz fibrations over $\mathbb{CP}^1$, i.e. meromorphic functions on the total space with isolated critical points satisfying a Morse lemma. Whereas Morse theory builds up a space's topological type by passing through critical points, here we can go round any critical point and so all the non-singular fibres are diffeomorphic. What we study instead is the monodromy around critical points and vanishing cycles – these are cycles in a smooth fibre that get transported to zero in a singular fibre – see the first chapter of [AGV] for a gentle overview of this. Intersecting a cycle a in a non-singular fibre with a vanishing cycle b associated to a critical point determines the "monodromy reflection" around the corresponding singular fibre: the image (in homology) of a under the monodromy is

$$a \mapsto a - 2(a.b)b.$$

This middle dimensional intersection theory might be hard to make sense of in infinite dimensions, but it also has a description via gradient flow lines.

Namely, consider the deformed pseudo-holomorphic curve equation

$$\frac{\partial u}{\partial t} + J\frac{\partial u}{\partial s} = \operatorname{grad}\pi,$$

for a map u of the tube $\{(s,t) \in S^1 \times \mathbb{R}\} = \mathbb{C}/\mathbb{Z}$ into the total space M of a (local) Lefschetz fibration $\pi : M \to \mathbb{C}$, with initial and final points two critical points of π. Here the gradient is a real vector defined by the complex equation

$$d\pi(X) = (X, \operatorname{grad}\pi), \quad \text{where} \quad (a,b) = \langle a,b\rangle + i\langle a, Jb\rangle,$$

and as such is in fact $\mathrm{grad}(\mathrm{Re}\,\pi) = J\,\mathrm{grad}(\mathrm{Im}\,\pi)$, since π is holomorphic. (There is only an asymmetry singling out a real direction due to the natural real t-direction on the tube.) As the Morse indices are all the same, since the function is holomorphic, we will generically have no solutions for "thin tubes" homologous to s-invariant solutions (i.e. lines between critical points), but in the one parameter family of such equations obtained by multiplying π by unit complex numbers (i.e. considering all possible real directions) we generically obtain a finite number of solutions after dividing out by the translations.

Deforming the equations so that the tube becomes thin and $\partial u/\partial s$ small, what we have is approximately the gradient flow equation

$$\frac{\partial u}{\partial t} = \mathrm{grad}\,(\mathrm{Re}\,\pi)$$

for $\mathrm{Re}\,\pi$. Projecting down to $\mathbb{C}$ we move in the direction of the real axis:

$$\frac{\partial(\pi \circ u)}{\partial t} = D_{\frac{\partial u}{\partial t}}\,\pi = \Big\langle \frac{\partial u}{\partial t}, \mathrm{grad}\,(\mathrm{Re}\,\pi) \Big\rangle = \|\,\mathrm{grad}\,(\mathrm{Re}\,\pi)\,\|^2,$$

so generically missing the second critical value. But on multiplying π by unit complex numbers we rotate the real axis and pass through the second critical value in the one parameter family.

Now the number of such solutions is equal to the intersection number of the vanishing cycles of the two points, transported to a common non-singular fibre by the gradient flow (or equivalently by the lift of the straight line in $\mathbb{C}$ between the images of the critical values). This is because the vanishing cycle is just the unstable sphere of the critical point, as a small calculation in local coordinates about the critical point shows, and intersections of these spheres correspond to gradient flow lines between the critical points, just as in Morse theory.

So we could try to mimic this picture in infinite dimensions with the holomorphic Chern-Simons functional CS on $\mathscr{B}$, or a cover on which CS is single-valued, as our complex analogue of Floer theory.

This would give us the gradient flow equations

$$\frac{\partial a}{\partial \bar{z}} = \bar{*}\,(F_A^{0,2} \wedge \theta) = \star F_A^{0,2}, \tag{1.7.2}$$

for a family $A = A_0 + a(z)$ of connections on $X \times \mathbb{C}/\mathbb{Z}$, with coordinate z on $\mathbb{C}/\mathbb{Z}$, the Calabi-Yau analogue of the real line. For this to make sense we need a metric on E to give a notion of adjoint on $\mathfrak{g}_E$, so that $\bar{*}$ is the usual $\bar{*}$ on forms tensored with * on $\mathfrak{g}_E$. Then (1.7.2) is not an equation on the space of isomorphism classes of $\bar\partial$-operators since the equation is not invariant under the full complex gauge group, just the unitary one. So we can, for instance, fix the metric on E using the standard equation

$$i\Lambda F_A^{1,1} = \lambda I,$$

where λ is a constant proportional to the degree of the bundle, and F_A is the curvature of the connection induced by the metric and the $\bar\partial$-operator. Another possibility is discussed in the next section.

Multiplying the Calabi-Yau form θ by unit complex numbers gives the S^1 family of such equations (1.7.2) which should then have a generically zero dimensional space of solutions converging to fixed holomorphic connections at either end of the tube. This follows from ellipticity of the equations modulo gauge (see the next section on ASD equations), and the fact that the spectrum of the Hessian

$$\nabla(d\,CS)$$

is symmetric about 0 as CS is holomorphic, so J switches eigenspaces of opposite signs (this is the statement that the real part of a holomorphic function is harmonic), so the spectral flow is zero. We would then like to count solutions, computing an infinite dimensional analogue of monodromy around the critical points – the holomorphic bundles. The issue is, then, the compactness and smoothness of the moduli space of gradient flow lines, i.e. the equations (1.7.2). But these are essentially just the ASD equations of the next section.

ASD $\wedge\theta$, Yang-Mills $\wedge\theta$, etc.

We can write down the following analogue of the ASD equations on a Calabi-Yau 4-fold Y with holomorphic (4,0)-form θ;

$$\star F_A^{0,2} = \bar{*}\,(F_A^{0,2}\wedge\theta) = -F_A^{0,2}. \tag{1.7.3}$$

Again this uses a metric on E, reducing the symmetry (gauge) group, so to fix the metric and make the equations elliptic we supplement them with

$$i\Lambda F_A^{1,1} = \lambda I.$$

These are now first order elliptic equations for A modulo unitary gauge transformations, with linearisation the elliptic complex

$$\begin{array}{l} 0\to\Omega^0(\mathfrak{g}_E^c)\xrightarrow{\bar\partial_A}\Omega^{0,1}(\mathfrak{g}_E^c)\xrightarrow{\bar\partial_A^+}\Omega^{0,+}(\mathfrak{g}_E)\to 0 \\ \qquad\qquad\qquad\qquad\qquad\quad {\scriptstyle\Lambda\bar\partial_A}\searrow\ \Omega^0(\mathfrak{g}_E)\to 0\,, \end{array} \tag{1.7.4}$$

where $\Omega^{0,+}$ is the $+1$ eigenspace of $\star$ on $\Omega^{0,2}$, and $\bar{\partial}_A^+ = 1/2(1+\star)\bar{\partial}_A$. Whereas the usual ASD equations are related to the intersection form $H^2 \times H^2 \to \mathbb{Z}$, $(\alpha, \beta) \mapsto \int \alpha \wedge \beta$, these equations are related to the bilinear form

$$H^{0,2} \times H^{0,2} \to \mathbb{C}, \qquad (\alpha, \beta) \mapsto \int_Y \alpha \wedge \beta \wedge \theta. \tag{1.7.5}$$

The form is real positive definite on $\mathscr{H}^{0,+}$ and negative definite on $\mathscr{H}^{0,-}$, and these two spaces are interchanged by multiplication by i. ($\mathscr{H}^{0,\pm}$ are the spaces of harmonic ± 1 eigen-2-forms of $\star$, using the fact that $\star$ commutes with the Laplacian as θ is parallel.)

Just as solutions of the ASD equations satisfy the Yang-Mills equations $d_A^* F_A = 0$, solutions of (1.7.3) satisfy a Yang-Mills $\wedge\theta$ equation

$$\bar{\partial}_A^* F_A^{0,2} = 0,$$

which is the Euler-Lagrange equation for critical points of the Lagrangian

$$\|F_A^{0,2}\|^2 = \int_Y \operatorname{tr} F_A^{0,2} \wedge \star F_A^{0,2} \wedge \theta.$$

This equals

$$4\pi^2 \langle p_1(E) \smile [\,\theta\,], Y \rangle + 2\,\|F_A^{0,+}\|^2,$$

so is clearly minimised by connections satisfying (1.7.3). This also demonstrates that on a bundle admitting a holomorphic connection (more generally those with $\langle p_1(E) \smile [\,\theta\,], Y \rangle = 0$) the "half-integrable" connections $F_A^{0,+} = 0$ are actually holomorphic, $F_A^{0,2} = 0$, just as ordinary instantons are flat on a bundle admitting a single flat connection.

Thus, with the extra term to fix the metric, these half-instanton equations on a Calabi-Yau 4-fold,

$$F_A^{0,+} = 0, \quad i\Lambda F_A^{1,1} = \lambda I,$$

reduce to the Hermitian Yang-Mills equations on bundles with $p_1.\theta = 0$. These equations, for $\lambda = 0$, in fact have a symmetry under $Spin(7)$ and can be written down on any manifold with exceptional holonomy $Spin(7)$ (see [DT] for the work of Donaldson, Joyce and Lewis on this). Calabi-Yau 4-folds are $Spin(7)$ manifolds: the splitting $\Lambda^{0,2} \cong \Lambda^{0,+} \oplus \Lambda^{0,-}$ exhibits $SU(4)$ as a double cover of $SO(6)$ (as $\Lambda^{0,+} \cong \mathbb{R}^6$), so $SU(4)$ sits inside $Spin(7)$ as $Spin(6)$. So in this particular case solutions of the Hermitian-Yang-Mills equations do not depend directly on the complex structure of the Calabi-Yau 4-fold but only its $Spin(7)$ structure (which is the 4-form $\theta + \bar{\theta} - \omega \wedge \omega$). This is again suggestive that the moduli space of stable bundles should have topology independent of the complex structure on the original manifold – see the discussion at the end of Section 1.4.

If X is a Calabi-Yau 3-fold then $X \times \mathbb{C}/\mathbb{Z}$ is canonically a Calabi-Yau 4-fold, with holomorphic (4,0)-form $\pi_1^*\theta_X \wedge \pi_2^* dz$, in the obvious notation. By a change of gauge (parallel transporting down $\mathbb{C}/\mathbb{Z}$) we may make a connection take the standard form in the $\mathbb{C}/\mathbb{Z}$ direction, so on $X \times \mathbb{C}/\mathbb{Z}$ it may be written in terms of a family $A_0 + a(z)$, $z \in \mathbb{C}/\mathbb{Z}$ of connections on X, as $\bar{\partial}_{A_0+a} \oplus \left(d\bar{z} \otimes \frac{\partial}{\partial \bar{z}}\right)$ (in "temporal gauge"). Then the equation (1.7.3) is precisely the gradient flow equation (1.7.2) for the family $a(z)$. The only difference is that there we fixed the metric using

$$\Lambda F_{A_t} = \omega_{X_t} \lrcorner F_{A_t}$$

on each 3-fold slice X_t (with Kähler form ω_{X_t}), whereas the ASD equation fixes the metric using $\Lambda F_A = \omega \lrcorner F_A$, with ω the Kähler form on all of $X \times \mathbb{C}/\mathbb{Z}$. There are obvious equations interpolating between these two, discussed in [DT], but anyway the general idea and similarities are clear.

We can also consider the equations invariant in one complex direction to obtain complex analogues of the Bogomolny equations on a Calabi-Yau 3-fold,

$$\bar{\partial}_A \Phi = \star F_A^{0,2}, \quad \Phi \in \Omega^0(\mathfrak{g}_E),$$

again supplemented by an equation on ΛF_A to fix the metric. Other dimensional reductions of the equations give non-abelian Seiberg-Witten equations and Vafa-Witten equations [VW] in four dimensions [DT].

The Atiyah-Floer conjecture $\wedge\theta$

There is a similar Floer theory [At2] to compute "middle" dimensional homology of the space of paths between two Lagrangian submanifolds in a symplectic manifold, using the symplectic action functional. This has critical points the constant paths, i.e. the points of intersection of the submanifolds, and the gradient flow traces out pseudo-holomorphic curves in the symplectic manifold, once a compatible metric is fixed. Given a Heegard splitting $M = H_1 \cup_S H_2$ of M^3 and applying this theory to the space $\mathcal{M}_S$ of flat connections on S with its two Lagrangian submanifolds $\mathcal{M}_{H_i}$ gives another Floer homology of M. The Atiyah-Floer conjecture ([At2]) is that these two homology groups should be naturally isomorphic. As described in [At2], a path in $\mathcal{M}_S$ gives a path of connections on S, and so a connection on a neck $S \times I$. The paths we consider in the gradient flow end in the $\mathcal{M}_{H_i}$ and so this connection extends over the handlebodies H_i to give a connection on $H_1 \cup_{S \times I} H_2 \cong M$.

So the gradient flow of the symplectic action functional gives a family of connections interpolating between two flat connections on M, just as the gradient flow of the Chern-Simons functional does. The two families of connections will be different, but it can be shown that their various indices, and the boundary operators, etc. agree; see [DS] for a version of this. The main idea is to study the ASD equations (which give the Chern-Simons gradient flow involved in the original Floer theory) on the neck times $\mathbb{R}$, i.e. $S\times I\times\mathbb{R}$, in the limit of scaling down the metric on S, or equivalently on stretching the length of the neck I. These tend towards flat connections on the S factor, giving a map from $I\times\mathbb{R}$ to $\mathcal{M}_S$, and the equations for these flat connections are the Cauchy-Riemann equations for a pseudo-holomorphic curve in $\mathcal{M}_S$. (In fact the curve is holomorphic – the Riemann surface is Kähler, essentially because of the isomorphism $U(1)\cong SO(2)$, and this makes the space of flat connections Kähler. Similarly everything below will not not just be complex symplectic but hyperkähler, as $Sp(1)\cong SU(2)$.)

We can speculate on analogues of this for our complex Casson invariant. Just as the symplectic structure on the Riemann surface above induced a symplectic structure on the space of flat bundles, the holomorphic symplectic structure on the $K3$ surface S makes the moduli space $\mathcal{M}_S$ of holomorphic bundles complex symplectic, and so Calabi-Yau (the top power of the symplectic form trivialises the canonical bundle). Complex curves in $\mathcal{M}_S$ correspond, at least locally, to bundles on $S\times\Delta$ (Δ a neighbourhood of $0\in\mathbb{C}$). If there is a smooth deformation of the singular Calabi-Yau $X=X_1\cup_S X_2$ that contains S then S has trivial normal bundle, so that a small neighbourhood will look like $S\times\Delta$, at least to first order. Then the ASD$\wedge\theta$ equation on $S\times\Delta_1\times\Delta_2$, with the Kähler metric on S scaled by ϵ, takes the form

$$(F_A^{0,2})_S=\epsilon\star(F_A^{0,2})_{\Delta_1\times\Delta_2},$$
$$\frac{\partial A}{\partial\bar z_1}=-\star\frac{\partial A}{\partial\bar z_2},$$

in the obvious notation. Thus, schematically, we take the limit $\epsilon\to 0$ to obtain a family of holomorphic connections on the S factor, and thus a map $\Delta\times\Delta\to\mathcal{M}_S$. Since $\star$ on $\Omega^{0,1}$ is just the quaternionic J derived from the complex symplectic structure, this map satisfies the quaternionic equation

$$\frac{\partial A}{\partial\bar z_1}+J\frac{\partial A}{\partial\bar z_2}=0,$$

an elliptic quaternionic analogue of the pseudo-holomorphic curve equation. Thus we get a quaternionic curve in the hyperkähler manifold $\mathcal{M}_S$.

So there is a complex version of this adiabatic-limit part of the story. Perhaps hoping for a complex Floer theory for Lagrangian intersections and analogies of the full Atiyah-Floer conjecture are a little too much to hope for.

For paths with boundary lying in the Lagrangians we might substitute $\mathbb{CP}^1$'s with two points (this is a Fano plus anticanonical divisor notice) lying in the two Lagrangians. This does not seem to sit well with the quaternionic mapping picture above however. Perhaps a holomorphic symplectic action functional, defined by integrating the complex symplectic form against a holomorphic 2-form on a Fano variety with suitable anticanonical divisor, is also too much to expect.

The last few sections have been deliberately vague as there are so many technical details we have glossed over. So we need to tackle these in one case, namely the holomorphic Casson invariant, which should be representative of the key issues in trying to extract invariants from many of these gauge theories, and which will now concern us for the next two chapters. Finally we will end with an appendix describing some physics which motivates some more definitions and still-to-be-done mathematics which is the complex analogue of knot theory.

Chapter 2

Holomorphic Bundles on a Calabi-Yau 3-fold

2.1 The Serre construction

In this chapter we look at some examples of counting holomorphic bundles on Calabi-Yau 3-folds. Our main technique for studying rank two bundles will be the Serre construction. This is the codimension-two analogue of the correspondence between line bundles and divisors – a section of a rank two bundle not vanishing on a divisor defines a codimension two subscheme, and Serre gives the converse construction of a bundle from the subscheme, once certain necessary and sufficient conditions are satisfied.

Since it will be used repeatedly in dimensions 2 and 3, we briefly review it here. This is a purely algebraic description, by no means comprehensible to anyone not familiar with Ext groups, who should refer to [DK], pp 389–395, for an elegant geometric treatment in the case of points on a surface.

The basic idea is that we would like an extension (2.1.2) below, which away from the codimension two subscheme Y is an extension of $\mathcal{O}$ by $\mathscr{L}$, classified by $H^1(\mathscr{L}^*)$, and we find this group gives the choices of all the extensions with fixed data on Y. On Y, this data turns out to just be a number on each component of Y, giving a locally free extension (i.e. a vector bundle) if and only if these numbers are all non-zero. So we have a sequence

$$H^1(\mathscr{L}^*) \to \{\text{Extensions}\} \to H^0(\mathcal{O}_Y),$$

and there exists a locally free extension if and only if an everywhere non-zero element of $H^0(\mathcal{O}_Y)$ is in the image of the second map. The conditions in the theorem are just a Serre-duality criterion for determining this.

Theorem 2.1.1 *Let $Y \subset X$, with components $Y = \bigcup Y_i$, be codimension 2 subvariety of a complex n-fold X, with normal bundle $\nu \to Y$. Suppose that $\Lambda^2\nu$ extends to a line bundle L on X. Then there exists a rank 2 bundle E with $c_2(E) \cap [X] = [Y]$, $\Lambda^2 E = L$ and section s vanishing precisely on Y if and only if there exist non-zero complex numbers (a_i) such that $\sum a_i \int_{Y_i} \omega = 0$ for all $\omega \in H^{n-2}(L \otimes K_X)$.*

Remarks.

1. Although we will usually consider only smooth subvarieties the results easily generalise ([OSS] pp 90–101) to local complete intersection subschemes, where locally we have a section of a rank 2 vector bundle cutting out Y scheme-theoretically. In particular this is true for zero dimensional subschemes which need not be reduced for the result to hold. So in later sections "n points" will mean a length n zero-dimensional scheme, and a curve containing these points will be tangent at any double points, etc.

2. We are using the fact that, on restriction to Y, $\omega \in H^{n-2}(\mathcal{L} \otimes K_X)$ becomes an element of $H^{n-2}(L \otimes K_X|_Y) \cong H^{n-2}(K_Y)$.

Proof. Let $\mathcal{I}_Y$ be the sheaf of ideals defining Y. We are looking for a Koszul resolution ([GH] p 688)

$$0 \to \mathcal{O} \xrightarrow{s} \mathcal{E} \xrightarrow{\wedge s} \mathcal{L} \otimes \mathcal{I}_Y \to 0, \qquad (2.1.2)$$

with $\mathcal{E}$ a rank 2 locally free sheaf, s a section defining Y, and $L = \Lambda^2 E$. Such extensions of $\mathcal{O}$ by $\mathcal{L} \otimes \mathcal{I}_Y$ are classified by the group $\mathrm{Ext}^1(\mathcal{L} \otimes \mathcal{I}_Y, \mathcal{O})$ ([GH] p 725). It is traditional at this point ([OSS] p 94, [GH] p 728) to examine the spectral sequence of the hypercohomology of $\mathcal{E}xt$ sheaves to recover $\mathrm{Ext}^1(\mathcal{L} \otimes \mathcal{I}_Y, \mathcal{O})$, but since the general Serre duality theorem ([GH] p 708) is no more difficult a concept, it seems simpler to note that

$$\mathrm{Ext}^1(\mathcal{L} \otimes \mathcal{I}_Y, \mathcal{O}) \cong H^{n-1}(\mathcal{L} \otimes \mathcal{I}_Y \otimes K_X)^*. \qquad (2.1.3)$$

Then the exact sequence $0 \to \mathcal{L} \otimes \mathcal{I}_Y \otimes K_X \to \mathcal{L} \otimes K_X \to \mathcal{L} \otimes K_X|_Y \to 0$ yields

$$\begin{array}{ccccccccc}
\to & H^{n-2}(\mathcal{L}\otimes K_X) & \to & H^{n-2}(K_Y|_Y) & \to & H^{n-1}(\mathcal{L}\otimes\mathcal{I}_Y\otimes K_X) & \to & H^{n-1}(\mathcal{L}\otimes K_X) & \to \\
 & \otimes & & \otimes & & \otimes & & \otimes & \\
\leftarrow & H^2(\mathcal{L}^*) & \leftarrow & H^0(\mathcal{O}_Y|_Y) & \leftarrow & \mathrm{Ext}^1(\mathcal{L}\otimes\mathcal{I}_Y, \mathcal{O}) & \leftarrow & H^1(\mathcal{L}^*) & \leftarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \mathbb{C} & & \mathbb{C} & & \mathbb{C} & & \mathbb{C} &
\end{array} \qquad (2.1.4)$$

where the second sequence is the Serre-dual of the first (using (2.1.3)); the pairings commute by functoriality.

Not any extension will do, of course. $0 \in \mathrm{Ext}^1$ corresponds to the extension $\mathscr{E} = \mathcal{O} \oplus \mathscr{L} \otimes \mathscr{I}_Y$ which is clearly not locally free. The necessary and sufficient condition for an element of $\mathrm{Ext}^1(\mathscr{L} \otimes \mathscr{I}_Y, \mathcal{O})$ to define a locally free extension is ([OSS] p 98) that it restrict, at each $x \in Y$, to a generator of the $\mathcal{O}_x$-module $\mathscr{E}xt^1_x(\mathscr{L} \otimes \mathscr{I}_Y, \mathcal{O}) \cong \mathrm{Ext}^1(\mathscr{I}_{Y,x}, \mathcal{O}_x) \otimes \mathscr{L}^*_x$. From the sequence $0 \to \mathscr{I}_Y \to \mathcal{O} \to \mathcal{O}_Y \to 0$ at x this group is isomorphic to $\mathrm{Ext}^2(\mathcal{O}_{Y,x}, \mathcal{O}_x) \otimes \mathscr{L}^*_x$.

Analysing the Koszul complex at x shows ([GH] p 690) that this is isomorphic to $(\Lambda^2 \nu \otimes \mathcal{O}_Y \otimes \mathscr{L}^*)_x \cong \mathcal{O}_{Y,x}$. Of course the restriction map to x factors through the map $\mathrm{Ext}^1(\mathscr{L} \otimes \mathscr{I}_Y, \mathcal{O}) \to H^0(\mathcal{O}_Y)$ in the sequence (2.1.4), so the condition that $\mathscr{E}$ be locally free is that we can choose $H^0(\mathcal{O}_Y) \ni a = (a_i) \in \bigoplus_i H^0(\mathcal{O}_{Y_i})$ in the image of this map, such that all the a_i are non-zero.

Thus we are left with showing that there are non-zero a_i such that $(a_i) \in H^0(\mathcal{O}_Y)$ maps to zero in $H^2(\mathscr{L}^*)$. But by the Serre duality of (2.1.4) this is equivalent to showing that its image is annihilated by all elements of $H^{n-2}(\mathscr{L} \otimes K_X)$, that is $\sum a_i \int_{Y_i} \omega = 0 \ \forall \omega \in H^{n-2}(\mathscr{L} \otimes K_X)$. □

Remarks.

1. If Y is connected the conditions of the theorem reduce to $\int_Y \omega = 0 \ \forall \omega \in H^{n-2}(L \otimes K_X)$.

2. If $n = 2$ then Y is a finite number of points (x_i) in a surface X, and the formula reduces to the usual residue formula (see for instance [DK] pp 392–395, where a much more attractive geometric proof is given). That is we must pick bivectors (a_i) in $(L \otimes K_X)^*|_{x_i}$ such that $\sum \langle \omega|_{x_i}, a_i \rangle = 0$ for all $\omega \in H^0(L \otimes K_X)$. To make contact with the above theorem we must identify L^* with K_X (and so $(L \otimes K_X)^*$ with $\mathbb{C}$) at the x_i via the determinant of the intrinsically defined derivative of s at its zeroes.

3. The image of $H^1(L^*)$ in Ext^1 in (2.1.4) gives the different choices of extension for a given set of (a_i). This is best illustrated in the case Y is empty; then extensions $0 \to \mathcal{O} \to \mathscr{E} \to \mathscr{L} \to 0$ are well known to be classified by $H^1(L^*)$.

Corollary 2.1.5 *If Y is a smooth curve in a projective 3-fold $X \subset \mathbb{CP}^4$ the corresponding bundle exists if and only if $TY = \mathcal{O}(a)|_Y$ for some power a of the hyperplane bundle. The bundle is unique up to isomorphism if and only if Y is connected.*

Proof. By the adjunction formula the extendability of $\Lambda^2\nu$ is equivalent to the extendability of TY. But by the Lefschetz hyperplane theorem over $\mathbb{Z}$, the line bundles on X are just those on $\mathbb{CP}^4$.

We have on $\mathbb{CP}^4$ the sequence, for $d=\deg(X)$,

$$0 \to \mathcal{O}(k-d) \to \mathcal{O}(k) \to \mathcal{O}_X(k) \to 0.$$

But as the cohomology of line bundles on $\mathbb{CP}^n$ all lies in dimensions 0 and n ([OSS] p 8) the exact cohomology sequence shows that $H^1(\mathcal{O}_X(k)) = 0 = H^2(\mathcal{O}_X(k))$. Thus (2.1.4) shows that the Ext group is just $H^0(\mathcal{O}_Y)$. The result follows. □

2.2 Bundles on a quintic 3-fold, and Mirror Symmetry

We now look at a particular example of counting bundles. It should be noted that in this example the bundles will not be strictly stable, and so will not fit into the general theory of Chapter 3 without some modification. It is an instructive example, however, clearly motivated by the holomorphic Casson invariant, and the example in the next section *will* deal with strictly stable bundles.

A bundle is slope stable if all subsheaves of its sheaf of sections have strictly smaller slope=degree/rank, and semi-stable if the non-strict inequality holds. (We will describe the similar condition of Gieseker stability in Chapter three, but shall only consider slope stability in this chapter.) For a rank two bundle we need only consider subsheaves that are line bundles (though not necessarily subbundles). This is because we need only consider rank one subsheaves, whose double duals are also subsheaves of the same slope but are line bundles. Also, twisting a bundle by a line bundle clearly does not affect stability.

So on a projective variety whose only line bundles are the $\mathcal{O}(n)$, we can normalise any rank two bundle to have determinant $\Lambda^2\mathscr{E}$ either $\mathcal{O}$ or $\mathcal{O}(-1)$. Considering subsheaves $\mathcal{O}(t) \to \mathscr{E}$ we then see that $\mathscr{E}$ is stable if and only if it has no sections, and is semi-stable if $H^0(\mathscr{E}(-1)) = 0$ in the first case (if $\Lambda^2\mathscr{E} = \mathcal{O}(-1)$ then semi-stablity is equivalent to stability).

We consider bundles on a smooth quintic Calabi-Yau 3-fold X in $\mathbb{CP}^4$ defined by degree d, genus g curves $C \subset X$. We require that TC extends to a line bundle $\mathcal{O}(a)$ on X, which will pull back to a degree $da = \chi(C) = 2-2g$ bundle on C (more generally in the l.c.i. curve case we require that the determinant of its normal bundle extends to $\mathcal{O}(-a)$). Therefore d must divide $2-2g$, and the bundle will exist and be unique by (2.1.5).

Proposition 2.2.1 *Let C be a union of smooth curves in X with $TC = \mathcal{O}(a)|_C$ (so in particular each component has degree and genus satisfying $da = \chi(C) = 2 - 2g$ for some fixed a). Then C defines a holomorphic bundle $E \to X$ with section s cutting out C, $\Lambda^2 E = \mathcal{O}(-a)$, and*

1. *If a is even then E is*

   stable *if and only if $g \geq 2$ and C lies in no degree $-\frac{5a}{2}$ surfaces in X,*

   semi-stable *if and only if $g = 1, 2$ or C lies in no degree $-5\left(\frac{a}{2}+1\right)$ surfaces in X.*

2. *If a is odd then E is (semi-)stable if and only if $g \geq 2$ and C does not lie in any degree $-5\left(\frac{a+1}{2}\right)$ surfaces in X.*

Proof. (For a even only, the odd case is similar).

$$\Lambda^2 E|_C \cong \Lambda^2 \nu \cong K_X^*|_C \otimes T^*C \cong T^*C \cong \mathcal{O}(-a)|_C$$

shows we have $\Lambda^2 E \cong \mathcal{O}(-a)$.

Thus we have $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_C(-a) \to 0$. Tensoring with $\mathcal{O}\left(\frac{a}{2}\right)$ and taking cohomology,

$$0 \to H^0\left(\mathcal{O}\left(\frac{a}{2}\right)\right) \to H^0\left(E\left(\frac{a}{2}\right)\right) \to H^0\left(\mathcal{I}_C\left(\frac{-a}{2}\right)\right) \to 0,$$

as $H^1(\mathcal{O}(k))$ vanishes on X, as shown in Corollary 2.1.5.

If $g \geq 2$ then $a < 0$, so $h^0\left(E\left(\frac{a}{2}\right)\right) = h^0\left(\mathcal{I}_C\left(-\frac{a}{2}\right)\right)$ is non-zero if and only if C lies in a divisor in $\left|\mathcal{O}\left(-\frac{a}{2}\right)\right|$, i.e. a degree $\left(-\frac{5a}{2}\right)$ surface in X (the 5 comes from the degree of $X \subset \mathbb{CP}^4$). But as $\Lambda^2\left(E\left(\frac{a}{2}\right)\right) = \mathcal{O}$, $E\left(\frac{a}{2}\right)$ is stable if and only if it has no sections.

As the stability of E is equivalent to that of any of its twists we are done for $g \geq 2$. If $g \leq 1$ then $a \geq 0$ and $\Lambda^2 E = \mathcal{O}(-a)$ is nonpositive, but E has a section, so is not stable.

The semi-stability statement is similar, tensoring with $\mathcal{O}\left(\frac{a}{2}-1\right)$ and using the fact that E is semi-stable if and only if $E\left(\frac{a}{2}-1\right)$ has no sections, as the $\mathcal{O}(k)$ are the only line bundles on X. □

It is not at all clear that, generically, bundles defined by curves of genus $g \geq 2$ should exist. The virtual dimension of the moduli space of curves in a Calabi-Yau is zero, and we must cut down this space to those whose tangent bundles extend to X. For genus 0 and 1 the tangent bundle is fixed, but for higher genus the condition on the tangent bundle should cut down the space of relevant curves to negative dimensions, unless the curves that exist in a Calabi-Yau are precisely those whose tangent bundles extend.

It may be that here the analytical approach described in Section 3.1, using (almost-) Hermitian-Yang-Mills connections on the bundles, and perhaps pseudo-holomorphic curves, is more useful, allowing perturbations. It is not clear, however, that on perturbation the condition "TC extends" does not get perturbed to some other condition precisely satisfied by the perturbed curves.

It is also quite feasible that bundles in fact correspond precisely to non-generic families of curves of positive dimension that often exist. This is what happens in the case studied in Section (2.3). For instance on a quintic 3-fold there is a large family of quintic curves given by intersection with two hyperplanes. These, however, almost by construction, correspond to unstable bundles.

In the absense of a proper understanding of $g \geq 2$, and since we have seen rational and quintic curves form unstable bundles, we concentrate on tori. Then the bundles are all semi-stable, so should still form a moduli space.

The first mathematical accounts of Mirror Symmetry made only predictions about rational curves in Calabi-Yau manifolds, and in fact much of this work started out as a (failed) attempt to relate Mirror Symmetry and counting curves to counting bundles. More recently Bershadsky, Cecotti, Ooguri and Vafa have extended these predictions to higher genus curves (see [V] for example). Since we might hope to count curves by counting the corresponding bundles, the first interesting case in [V] is that of degree 3 tori in the quintic 3-fold, of which there are predicted to be 609250.

To relate this number to counting bundles, and for us to be able to get a good hold on the moduli space of all such bundles, we would like all bundles of this same topological type to have a section vanishing on a degree 3 torus, and for compactness of the moduli space we would like the bundles to be generated by their sections after a fixed twist.

Theorem 2.2.2 *Let E be a rank 2 bundle on a smooth quintic 3-fold $X \subset \mathbb{CP}^4$, with $\Lambda^2 E$ trivial and $\langle c_2(E) \smile \omega, [X] \rangle = 3$. Then E is semi-stable, and has a section cutting out a union of tori of total degree 3 in X. Conversely, any degree 3 union of smooth tori (or, more generally, locally complete intersection curves with trivial determinant of their normal bundle) in X determines such a bundle. Also, all such bundles are generated by their sections after twisting by a fixed line bundle.*

Remark. We also analyse the dimension and smoothness of the space of such bundles at the end of this section.

The converse part of the theorem has already been dealt with by Corollary 2.1.5 and Proposition 2.2.1, since the tangent bundle to any holomorphic torus is trivial. Similarly if a bundle of this topological type has a section its zero locus C has $TC \cong K_X^* \otimes \Lambda^2 E^*|_C \cong \mathcal{O}_C$, so must be a union of tori.

To study bundles on X we follow the standard procedure of passing down to hyperplane sections and studying the simpler problem there. See, for example, [Ha2]. His method is more general in that he studies all bundles (on $\mathbb{CP}^3$), and it can't be followed through in all generality on a quintic 3-fold (though see the comments at the end of this section). Even in our simple case of degree 3 tori, the problem becomes much harder and we have to diverge from his methods at one stage and use what [OSS] call "the standard construction".

Choose a generic smooth hyperplane section $S \subset X$, a quintic surface in $\mathbb{CP}^3$ with canonical bundle $\mathcal{O}(1)$. Let F denote $E|_S$. Then $c_2(F) = 3$, $\Lambda^2 F = \mathcal{O}$ and a messy calculation gives the Riemann-Roch formula for $F(t)$ as

$$h^0(F(t)) - h^1(F(t)) + h^0(F(1-t)) = 5t^2 - 5t + 7 > 0. \tag{2.2.3}$$

Proposition 2.2.4 *$F = E|_S$ has a section.*

Proof. Substituting $t = 0$ into the Riemann-Roch formula shows that $F(1)$ must have a section. Suppose it vanishes on a divisor. Since we may choose S to be sufficiently general, we can assume that all the line bundles on S are multiples of $\mathcal{O}(1)$ (as the Noether-Lefschetz theorem [GH2] applies: since $H^{2,0} = H^0(\mathcal{O}(1)) \neq 0$, the integer lattice $H^2(S;\mathbb{Z}) \subset H^2(S;\mathbb{C})$ generically misses the part of $H^{1,1}(S)$ orthogonal to $c_1(\mathcal{O}(1))$). Therefore we can untwist by the divisor to get a section of some $F(-k)$ and so of F.

So we now suppose that F has no section and $F(1)$ has a section vanishing only in codimension two,

$$0 \to \mathcal{O} \xrightarrow{s} \mathscr{F}(1) \to \mathscr{I}_8(2) \to 0, \tag{2.2.5}$$

where $\mathscr{I}_8$ is the ideal belonging to the zero set of s (8 points (x_i) counted with multiplicities – see the first remark after Theorem 2.1.1). By Theorem 2.1.1 such an extension defines a vector bundle if and only if $\exists\, (a_i)_{i=1}^8 \in (\mathbb{C}^*)^8$ such that $\sum a_i s(x_i) = 0 \quad \forall s \in H^0(\mathcal{O}(3))$. (The identification of $\mathcal{O}(3)$ with $\mathcal{O}$ at each x_i is discussed in Remark 2 following the proof of Theorem 2.1.1.)

Therefore such extensions exist if and only if the restriction map $H^0(\mathcal{O}(3)) \to \bigoplus_{i=1}^8 \mathcal{O}_{x_i}(3)$ on S is not onto. The exact sequence $0 \to \mathcal{O}_{\mathbb{CP}^3}(-2) \to \mathcal{O}_{\mathbb{CP}^3}(3) \to \mathcal{O}_S(3) \to 0$ on $\mathbb{CP}^3$ shows that $h^0(\mathcal{O}_S(3)) = h^0(\mathcal{O}_{\mathbb{CP}^3}(3)) = 20$, so the extensions exist only when the kernel of the restriction map is ≥ 13 dimensional. That is, there must be a 12 parameter family of divisors in $|\mathcal{O}(3)|$ containing the x_i.

Lemma 2.2.6 *The x_i lie in a 12 parameter family of cubics in $|\mathcal{O}(3)|$ if and only if they either lie in the 10 point intersection of an element of $|\mathcal{O}(1)|$ and a coprime element of $|\mathcal{O}(2)|$, or if five of them lie on the intersection of two hyperplanes in $|\mathcal{O}(1)|$.*

Proof. Suppose the x_i lie on the intersection of the zero sets of $s \in H^0(\mathcal{O}(1))$ and $t \in H^0(\mathcal{O}(2))$ with s and t coprime. Then they also lie in $|s.H^0(\mathcal{O}(2))|$ and $|t.H^0(\mathcal{O}(1))|$. The intersection of these spaces is the span of $(s.t)$, so the x_i lie in an $h^0(\mathcal{O}(2)) + h^0(\mathcal{O}(1)) - 1 - 1 = 10 + 4 - 2 = 12$ dimensional space of divisors.

Similarly if five of the points lie on the intersection of $s_1,\ s_2 \in H^0(\mathcal{O}(1))$ then they lie in $|s_1.H^0(\mathcal{O}(2))|$ and $|s_2.H^0(\mathcal{O}(2))|$. The intersection of these spaces is $|s_1.s_2.H^0(\mathcal{O}(1))|$ giving a linear system of cubics of dimension $h^0(\mathcal{O}(2)) + h^0(\mathcal{O}(2)) - h^0(\mathcal{O}(1)) - 1 = 15$ containing the five points. Thus all eight points lie on at least a 12 parameter family of cubics.

The converse is a known result in enumerative geometry whose proof is quite long and too much of a digression to go into here [RV]. □

Proof of Proposition. If the x_i lie on a hyperplane we may tensor (2.2.5) with $\mathcal{O}(-1)$ and take cohomology to see that $H^0(F) \cong H^0(\mathscr{I}_8(1))$ is non-zero, a contradiction. So five of the points lie on the intersection of two hyperplanes given by sections $s_1,\ s_2 \in H^0(\mathcal{O}(1))$.

Since $H^0(F) = 0$, the Riemann-Roch formula (2.2.3) shows that $h^0(F(1)) \geq 7$, which implies, by the above sequence defining F, that $h^0(\mathscr{I}_8(2)) \geq 6$. But if $\mathscr{I}_5$ denotes the ideal corresponding to the 5 points lying on the two hyperplanes, then $h^0(\mathscr{I}_5(2)) = 7$ by the sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathscr{I}_5(2) \to 0;$$

that is the quadratics vanishing on the 5 points are

$$\{s_1.\alpha + s_2.\beta \ :\ \alpha, \beta \in H^0(\mathcal{O}(1))\}, \tag{2.2.7}$$

of dimension $2h^0(\mathcal{O}(1)) - 1 = 7$. Pick a sixth point x_6 from the eight. Since the intersection of two hyperplanes in the quintic surface S is 5 points, x_6 does not also lie in the intersection. But by taking a linear combination of s_1 and s_2 if necessary, we may assume without loss of generality that $s_1(x_6) = 0$ (and $s_2(x_6) \neq 0$). Then the quadratics vanishing on all 6 points are the quadratics of (2.2.7) with $\beta(x_6) = 0$.

Thus $h^0(\mathscr{I}_6(2)) = 6$, and yet $h^0(\mathscr{I}_8(2)) \geq 6$, so $H^0(\mathscr{I}_8(2)) = H^0(\mathscr{I}_6(2))$ and the quadratics (2.2.7) with $\beta(x_6) = 0$ must vanish on all eight points. Therefore they must all, like x_6, lie on the hyperplane $s_1 = 0$. Thus $h^0(F) = h^0(\mathscr{I}_8(1)) = 1$ and F has a section. □

We note at this point that the moduli space of such bundles $F \to X$ that are also semi-stable is therefore quite simple, at least over an open set. They all have a section vanishing only on points x_i of total multiplicity 3 where the restriction $H^0(\mathcal{O}(1)) \to \bigoplus_i \mathcal{O}_{x_i}(1)$ is not onto and so the kernel is at least 2 dimensional. Thus the x_i lie on a pencil of hyperplane sections. Two hyperplanes generically intersect in 5 points, and we have $\binom{5}{2} = 10$ choices of the 3 points from these 5. Thus an open set of the moduli space is a 10-sheeted cover of $\mathbb{P}H^0(\mathcal{O}(1)) \times \mathbb{P}H^0(\mathcal{O}(1)) = \mathbb{CP}^3 \times \mathbb{CP}^3$.

Following [Ha2] we now obtain bounds on $h^1(F(-t))$ in order to pass up to bounds on $h^1(E(-t))$ which we can use in the Riemann-Roch formula on X to get sections of $E(t)$. To do this we shall first assume F has a section not vanishing on a divisor for some smooth S; later we shall show this is a consequence of the semi-stability of E on X. (Thus $F = E|_S$ is semi-stable and the above description of the moduli space of such bundles should be useful in describing the moduli space of bundles E on X. Very few such F's come from an E; for instance those that do must have the same extension data at each of the eight points (i.e. $a_i = 1 \;\; \forall i$) since we will show that E has a section vanishing on a connected curve C whose intersection with S is the eight points, and the extension data $1 \in H^0(\mathcal{O}_C)$ on X restricts to the same at the eight points.)

Lemma 2.2.8 *For a bundle $F \to S$ with trivial determinant and a section vanishing on 3 points, $H^1(F(-t)) = 0 \;\; \forall t > 0$.*

Proof. While twisting $0 \to \mathcal{O} \to \mathscr{F} \to \mathscr{I}_3 \to 0$ by $\mathcal{O}(-t)$ and taking cohomology (as in [Ha2]) is of no help because $H^2(F(-t)) \cong H^0(F(1+t))$ is large, we may twist by $\mathcal{O}(1+t)$ and use Serre duality:

$$\begin{array}{cccccc} 0 \to & H^1(F(1+t)) & \to H^1(\mathscr{I}_3(1+t)) \to & H^2(\mathcal{O}(1+t)) \\ & \downarrow \wr & & \downarrow \wr \\ & H^1(F(-t))^* & & H^0(\mathcal{O}(-t))^* \end{array}$$

Thus, for $t > 0$, $h^1(F(-t)) = h^1(\mathscr{I}_3(1+t))$.

But $0 \to \mathscr{I}_3 \to \mathcal{O} \to \mathcal{O}_3 \to 0$ (where $\mathcal{O}_3$ is the structure sheaf of the three points) gives us

$$0 \to H^0(\mathscr{I}_3(l)) \to H^0(\mathcal{O}(l)) \to \mathcal{O}_3(l) \to H^1(\mathscr{I}_3(l)) \to 0,$$

so that

$$h^1(\mathscr{I}_3(l)) = \dim \operatorname{coker}\left[H^0(\mathcal{O}(l)) \to \mathcal{O}_3(l)\right],$$

and it is now sufficient to show this is zero $\forall l \geq 2$.

These are not any three points, however. They define a locally free extension $\mathscr{F}$ and so by Theorem 2.1.1 we know that the restriction $H^0(\mathcal{O}(1)) \to \mathcal{O}_3(1)$ is not onto; that is the three points lie on a pencil of hyperplane sections of S, i.e. on the intersection of a line in $\mathbb{CP}^3$ with S.

We can now play the game in reverse, passing up to the $\mathbb{CP}^3$ hyperplane in $\mathbb{CP}^4$ that contains S, where we consider three points lying on a line. The problem is now easy, however – sections of $\mathcal{O}_S(n)$ all come from sections on $\mathbb{CP}^3$ which restrict to the degree n polynomials in one variable on the line. We then know we can make a quadratic, and so any higher degree polynomial, take on any prescribed value at 3 points on a line, so the above cokernel is zero. □

Proposition 2.2.9 *If $E|_S = F$ has a section not vanishing on a divisor then $E \to X$ has a section.*

Proof. We have the exact sequence

$$0 \to H^1(E(-1-t)) \to H^1(E(-t)) \to H^1(E(-t)|_S) = 0$$

for $t > 0$, by Lemma 2.2.8. Thus $h^1(E(-t)) = h^1(E(-t-1))\ \ \forall t > 0$, and of course $h^1(E(-t)) = 0$ for t sufficiently large. Therefore $h^1(E(-t)) = 0\ \ \forall t > 0$.

The Riemann-Roch formula now becomes $h^0(E(t)) - h^1(E(t)) - h^0(E(-t)) = \frac{t}{3}(5t^2 + 16)$ showing that $E(1)$ has a section.

It has nothing to say about E, however, but the vanishing of $H^1(E(-1))$ implies that $H^0(E) \to H^0(E|_S)$ is onto. Thus by Proposition 2.2.4, E has a section. □

We now turn to the case of $E|_S = F$ having a section vanishing on a divisor for generic S; we would like to show it pulls up to a section of $E \to X$. However, the above calculations were on the borderline of working and will not extend to the case of $F(-k)$ having a section vanishing on points; in fact they give large values for $h^1(F(-t))$ which are of no use to us in lifting sections to X. So we need to appeal to the semi-stability of E on X. The idea is that if $E(-1)|_S$ has a section for all generic hyperplane sections S in a pencil, we can try to glue them together to give a section of $E(-1)$ on X, contradicting semi-stability. Of course this will not work in general, either because of the word "generic" above (compare the phenomenon of jumping lines on $\mathbb{CP}^2$ ([OSS] pp 26–38)) or because of the base-curve of the pencil. What this will give us is a subsheaf of the pull-back of $E(-1)$ to the blow-up of X along the base-curve, but it could be twisted over the exceptional divisor. In our case, however, the sections agree over the base-curve and we can show that they push down to X using the "standard construction" (see [OSS] pp 46–53 for the case of a point).

Theorem 2.2.10 *Suppose that for some $k > 0$ and a nonempty Zariski-open subset of the hyperplane sections S, $F(-k) = E(-k)|_S$ has a section vanishing only on points. Then $E(-k)$ has a section over X.*

Remark. As we shall need to appeal to a similar result again in the next section, notice the proof works in more generality: a rank two bundle of negative determinant on a threefold has a section if its restriction to the generic hyperplane section has a section vanishing in codimension two. Thus on a variety whose generic hyperplane sections have only $\mathcal{O}(t)$ line bundles, *a (semi-) stable bundle is semi-stable on restriction to the generic hyperplane.*

Proof. Choose a smooth hyperplane section S on which we have the sequence

$$0 \to \mathcal{O} \to \mathscr{F}(-k) \to \mathscr{I}(-2k) \to 0, \tag{2.2.11}$$

for some zero dimensional ideal $\mathscr{I}$. Now choose a smooth hyperplane section of this S (which will be a curve C) which misses the support of $\mathcal{O}/\mathscr{I}$. Let these two hyperplanes together define our pencil of surfaces in X, with base-curve C.

Then restricting (2.2.11) to C shows (since $\mathscr{I}|_C = \mathcal{O}_C$) that $h^0(F(-k)|_C) = 1$ and the restriction $H^0(F(-k)) \to H^0(E(-k)|_C)$ is an isomorphism for a generic smooth S for which we have a sequence (2.2.11). We need more than this, however, to prevent "jumping line" type phenomena from occurring.

Now for any other S in the pencil, the sequence

$$0 \to H^0(F(-t-1)) \to H^0(F(-t)) \to H^0(F(-t)|_C)$$

shows that $H^0(F(-t)) = 0 \ \ \forall t \geq k+1$ (as $H^0(F(-k-1)|_C) = 0$), and that

$$0 \to H^0(F(-k)) \to H^0(F(-k)|_C).$$

Therefore $h^0(F(-k)) \leq 1$ but by upper semicontinuity ([Ha1] p 288; using the fact that the F's form a flat family over the linear system of hyperplane sections since E is locally free), $h^0(F(-k)) \geq 1$ for all S in the pencil. Thus $H^0(F(-k)) \to H^0(E(-k)|_C)$ is an isomorphism for all S in the pencil.

Now we can follow [OSS] pp 46-53. Blowing up X along C gives part of the "standard diagram"

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{p_1} & X \\ p_2 \downarrow & & \\ \mathbb{CP}^1 & & \end{array}$$

Using the fact that the natural map $\mathscr{E} \to (p_1)_* p_1^* \mathscr{E}$ is clearly an isomorphism, we have

$$\begin{aligned} H^0(X; \mathscr{E}(-k)) \cong \;& H^0(X; (p_1)_* p_1^* \mathscr{E}(-k)) \;\cong H^0(\tilde{X}; p_1^* \mathscr{E}(-k)) \\ & \cong H^0(\mathbb{CP}^1; (p_2)_* p_1^* \mathscr{E}(-k)) \cong H^0(\mathbb{CP}^1; (p_2)_* p_1^* \mathscr{E}(-k)|_D), \end{aligned}$$

where D is the exceptional divisor $p_1^{-1} C$, and the last isomorphism follows from the isomorphism

$$H^0(E(-k)|_S) \xrightarrow{\sim} H^0(E(-k)|_C)$$

inducing

$$H^0(p_1^* E(-k)|_{\tilde{S}}) \xrightarrow{\sim} H^0(p_1^* E(-k)|_D)$$

and so

$$(p_2)_* p_1^* E(-k) \xrightarrow{\sim} (p_2)_* p_1^* E(-k)|_D.$$

But $(p_2)_* p_1^* E(-k)|_D \cong H^0(E(-k)|_C) \otimes \mathcal{O}_{\mathbb{CP}^1}$ since $D = \mathbb{CP}^1 \times C \xrightarrow{p_2} \mathbb{CP}^1$. Therefore $H^0(X; \mathscr{E}(-k)) \cong H^0(E(-k)|_C) \cong \mathbb{C}$. □

It is instructive to compare the above result with the case of jumping lines in $\mathbb{CP}^2$ ([OSS] pp 26–38). For example a bundle defined by the sequence $0 \to \mathcal{O}(-1) \to \mathscr{E} \to \mathscr{I} \to 0$, where $\mathscr{I}$ is the ideal of a number of points, has generic splitting type $\mathcal{O} \oplus \mathcal{O}(-1)$ (and so one section) on restriction to generic lines. But on a line passing through n of the points, it has type $\mathcal{O}(n) \oplus \mathcal{O}(-n-1)$ and so more than one section. However, on restriction to a basepoint of a pencil of lines the bundle would always have two sections, no matter how much it is "untwisted", and the restriction map would not be an isomorphism of sections. Therefore the above method would not give the bundle a section that it did not have on $\mathbb{CP}^2$.

Lemma 2.2.12 $H^1(E) = 0$ *if and only if the curve defining E is connected.*

Proof. The sequence $0 \to \mathcal{O} \to \mathscr{E} \to \mathscr{I}_C \to 0$ of E shows that $H^1(E) \cong H^1(\mathscr{I}_C)$. But then $0 \to \mathscr{I}_C \to \mathcal{O} \to \mathcal{O}_C \to 0$ gives

$$0 \to H^0(\mathcal{O}) \to H^0(\mathcal{O}_C) \to H^1(\mathscr{I}_C) \to 0,$$

which shows that $H^1(\mathscr{I}_C) = 0$ if and only if C is connected. □

In fact the quintic 3-fold generically contains no tori of degree 1 or 2 ([V]), so in this case we have set up a correspondence between degree 3 tori and bundles of a particular topological type. We can now use the above lemma to show that any such E is generated by its sections after a fixed twist. The best I have managed is

Proposition 2.2.13 $E(63)$ *is generated by its sections.*

Proof. Since I have nothing to add to the exposition in [Kl] pp 616–619, and could not really improve on it for this special case, I shall just quote Kleiman's results.

Since, by Lemma 2.2.8, Theorem 2.2.10, and Serre duality, $h^1(F(n)) = 0 = h^2(F(n))$ $\forall n \geq 2$, $F = E|_S$ is "m-regular", in the notation of [Kl], for all $m \geq 4$. Therefore E is m-regular, and so generated by its sections, for all $m \geq 4 + h^1(E(3))$. That our space of bundles forms a bounded family, generated by sections after a fixed twist, comes down, therefore, to the fact that we have a priori bounds on E's cohomology.

We have, therefore, to bound $h^1(E(3))$. We have the sequence

$$\begin{aligned} 0 \to H^0(E(n-1)) \to H^0(E(n)) \to H^0(F(n)) \\ \to H^1(E(n-1)) \to H^1(E(n)) \to H^1(F(n)). \end{aligned}$$

Letting $h_n^i = \dim H^i(E(n))$ and $f_n^i = \dim H^i(F(n))$, we see that

$$h_n^i - h_{n-1}^i \leq f_n^1 - f_n^0 + h_n^0 - h_{n-1}^0.$$

Adding all such inequalities over $n = 1, 2, 3$ gives

$$h_3^1 - h_0^1 \leq \sum_{n=0}^{3} \left(f_n^1 - f_n^0\right) + h_3^0 - h_0^0.$$

That is,

$$h_3^1 - h_3^0 \leq \sum_{n=0}^{3} (f_n^1 - f_n^0) - 1,$$

since $h_0^1 = 0,\ h_0^0 = 1$. But, similarly,

$$f_n^1 - f_n^0 \leq \sum_{k=0}^{n} h^1(E(k)|_C) - h^0(E(k)|_C),$$

which, by Riemann-Roch, equals $-\sum_{k=0}^{n}(10k-10) = -5(n+1)(n-2)$. So

$$h_3^1 \leq h_3^0 - 1 - 5\sum_{n=1}^{3}(n+1)(n-2) = h_3^0 - 11.$$

But $0 \to H^0(\mathcal{O}(3)) \to H^0(E(3)) \to H^0(\mathscr{I}(3)) \to 0$ shows that $h_3^0 \leq 2h^0(\mathcal{O}(3))$, which equals $2h^0(\mathcal{O}_{\mathbb{CP}^4}(3)) = 70$ by the exact sequence defining X in $\mathbb{CP}^4$.

Therefore E is generated by its sections after twisting $4 + h^1(E(3)) \leq 4 + 70 - 11 = 63$ times. $\square$

This number can surely be improved, but it does demonstrate how the size of various cohomology groups can grow very quickly, and how difficult it is to bound them usefully. This is why, at present, studying the general case of bundles on a quintic 3-fold seems out of reach. The method of [Ha2] is to relate the sizes of *all* cohomology groups of E and then enforce $h^1(E(n)) = 0$ for $n \gg 0$. A crucial step, Lemma 5.2, breaks down on a quintic 3-fold. It concerns basepoint-free linear systems on a line in $\mathbb{CP}^3$; in our case we need a similar result for basepoint-free linear systems on a quintic curve in $X \subset \mathbb{CP}^4$. While I can find a proof that works for a linear system that generates the line bundle at five distinct points on the curve, I cannot at the moment generalise this to just one point. It is possible, however, that by looking at only linear systems that do generate the fibre at five points (in our case $\mathcal{O}(n)$ for $n \geq 8$ on the quintic curve), Hartshorne's method can still produce something useful.

Theorem 2.2.14 *If E is a bundle corresponding to a (connected) torus of degree three in X, then $H^1(\mathrm{End}_0(E)) \cong H^0(\nu_C)$ and $H^2(\mathrm{End}_0(E)) \cong H^1(\nu_C)$.*

Proof. We now have $H^1(E) = 0$ and a section giving $0 \to \mathscr{E} \to \mathscr{E} \otimes \mathscr{E} \to \mathscr{E} \otimes \mathscr{I}_C \to 0$. This yields

$$0 \to H^1(\mathrm{End}\,\mathscr{E}) \xrightarrow{\sim} H^1(\mathscr{E} \otimes \mathscr{I}_C) \to 0.$$

Then $0 \to \mathscr{E} \otimes \mathscr{I}_C \to \mathscr{E} \to \mathscr{E}|_C \to 0$ gives

$$0 \to H^0(\mathscr{E} \otimes \mathscr{I}_C) \to H^0(\mathscr{E}) \to H^0(\mathscr{E}|_C) \to H^1(\mathscr{E} \otimes \mathscr{I}_C) \to 0.$$

But the first map is an isomorphism, so that

$$H^1(\mathrm{End}\,\mathscr{E}) \cong H^1(\mathscr{E} \otimes \mathscr{I}_C) \cong H^0(\mathscr{E}|_C) \cong H^0(\nu_C).$$

Since $H^1(\mathcal{O}) = 0$ this gives $H^1(\mathrm{End}_0(E))$, and taking duals gives H^2. $\square$

So for a connected torus the deformation and obstruction theory of the bundle is reduced to that of the curve, and for the generic quintic with only isolated connected degree three tori we have $H^1(\mathrm{End}_0(E)) = 0 = H^2(\mathrm{End}_0(E))$. Thus we would like to identify the number of such bundles with the number of tori. There is still some work to be done here, however, as the bundles are not simple ($s \otimes s$ is a non-scalar endomorphism of E, where s is its section) so that $H^0(\mathrm{End}_0(E))$ and H^3 are both one dimensional and should be taken into account. We should really consider triples consisting of holomorphic bundles and elements of these cohomology groups, and apply perturbations (or excess intersection theory) to the space of solutions to deduce that we can count each of our bundles above as contributing one to the total.

2.3 Bundles on a quartic in Gr $(2, 4)$

The last example, while clearly motivated by the material of Chapter 1, would need more work to fit into the general framework we will develop in Chapter 3 for the holomorphic Casson invariant, due to the bundles being unstable. Thus we study another example on a quartic hypersurface X of the Grassmannian $G = \mathrm{Gr}(2,4)$ of 2-planes in $\mathbb{C}^4$, which is embedded as a quadric in $\mathbb{CP}^5$ via the Plücker embedding.

A beautiful construction, due to Mukai [Mu] amongst others, of moduli spaces of ((semi-)stable) bundles on certain intersections of quadrics in $\mathbb{CP}^5$, has been reinterpreted by Donaldson (see [DT]) as an example of the Tyurin-Casson invariant. Smoothing the union of two quadrics in G to a quartic X, this suggests what the bundles (of the right topological type) should be on X, and would be an example of both the other holomorphic Casson invariant and a case where both were equal. Therefore in this section we describe the construction, and prove the natural conjecture about bundles on the quartic X.

We begin with some standard geometry of quadrics, see for example [GH]. All smooth even dimensional quadric hypersurfaces of projective space contain two families of linear subspaces of half the dimension, which we call A-planes and B-planes. In fact there is a complete symmetry between the two, and on degeneration to a singular quadric they "coalesce" into a single family of planes and are swapped by monodromy in the space of quadrics about the singular divisor (i.e. $A - B$ is the vanishing cycle).

Now the Plücker embedding exhibits the Grassmannian $G = \mathrm{Gr}(2,4)$ as a quadric in $\mathbb{CP}^5 = \mathbb{P}(\Lambda^2 \mathbb{C}^4)$ (in fact as any smooth quadric since any two non-degenerate quadratic forms are conjugate). On G there are two tautological 2-plane bundles, the subspace bundle (whose dual we denote A) and the quotient bundle B:

$$0 \to A^* \to \underline{\mathbb{C}}^4 \to B \to 0.$$

Sections of A are given by linear functionals $f \in (\mathbb{C}^4)^*$, whose vanishing locus is easily seen to be

$$\mathrm{Gr}\,(2, \mathrm{Ann}\, f) \cong \mathrm{Gr}\,(2, \mathbb{C}^3) \cong \mathbb{CP}^2,$$

while sections of B are given by elements v of $\mathbb{C}^4$ with zeroes

$$\{\Lambda \in \mathrm{Gr}, (2,4) : \Lambda \ni v\} \cong \mathbb{P}(\mathbb{C}^4/\,\mathbb{C}.v) \cong \mathbb{CP}^2.$$

Thus we have two families of planes in G which are of course precisely the A- and B- planes.

The Plücker embedding is given by the linear system of $\Lambda^2 A = \Lambda^2 B$, so this is the hyperplane bundle $\mathcal{O}(1)$ on G. Restricted to an A or B plane this is the standard hyperplane bundle on $\mathbb{CP}^2$, i.e. the planes have degree one. (For instance a two form $\psi \in \Lambda^2(\mathbb{C}^4)^*$ gives a section of $\mathcal{O}_G(1)$ which on restriction to the B-plane $\mathbb{P}(\mathbb{C}^4/\,\mathbb{C}.v)$ gives the section of $\mathcal{O}_{\mathbb{CP}^2}(1)$ corresponding to the linear functional $w \mapsto \psi(v \wedge w)$ on $\mathbb{C}^4/\,\mathbb{C}.v$.)

Thus a quartic hypersurface of G intersects the generic A- or B- plane in a quartic curve. So we see that on a Calabi-Yau 3-fold X that is a smooth quartic in G, the A- and B- bundles have sections vanishing on smooth curves of genus 3, which we shall use later.

The construction

Fix a smooth quadric $G = Q_0$ in $\mathbb{CP}^5$, in a $\mathbb{CP}^2$-family of quadrics spanned by Q_0, Q_1 and Q_2, say. The singular quadrics in the family lie on the sextic curve

$$C = \{[\lambda_0;\lambda_1;\lambda_2] \in \mathbb{CP}^2 : \det\,(\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2) = 0\} \subset \mathbb{CP}^2$$

where the determinant of the quadratic form defining the quadric becomes singular.

Thus for every point of $\mathbb{CP}^2 \setminus C$ we get two bundles A and B over the corresponding quadric and so over the $K3$ surface

$$S = Q_0 \cap Q_1 \cap Q_2.$$

On the other hand, points of C give us only one bundle (corresponding via its zero set to the single family of planes on a singular quadric). In fact these A and B bundles give a double cover $\mathcal{M}$ of $\mathbb{CP}^2$ branched along the sextic curve C (so $\mathcal{M}$

is another $K3$) as the moduli space of bundles of this topological type over the $K3$ surface $S = Q_0 \cap Q_1 \cap Q_2$.

The Fano $X_1 = Q_0 \cap Q_1$ lies in the pencil spanned by Q_0 and Q_1, which is a line $\mathbb{CP}^1$ in our $\mathbb{CP}^2$ family. The 2-fold branched cover $\mathcal{M}_1$ of this line induced by $\mathcal{M} \to \mathbb{CP}^2$ is the set of A and B bundles on the quadrics in this pencil, and again this actually exhibits this cover as the moduli space of bundles on the intersection X_1 of the quadrics. (The cover $\mathcal{M}_1$ is a genus 2 curve whose moduli space is well known to be, dually, the intersection of two quadrics, the duality being set up by the universal bundle on $X_1 \times \mathcal{M}_1$.) Clearly the embedding $\mathcal{M}_1 \to \mathcal{M}$ is induced by restriction of bundles from X_1 to S, and $\mathcal{M}_1$ is trivially a complex Lagrangian subspace of the complex symplectic $K3$ surface $\mathcal{M}$, as in Section 1.4.

We have a similar picture for $X_2 = Q_0 \cap Q_2$, with $\mathcal{M}_2$ the cover of the line in $\mathbb{CP}^2$ corresponding to the Q_0, Q_2 pencil. But S is an anticanonical divisor of the X_i's, as $K_{X_i} = \mathcal{O}(-2)$, and their union across S is a singular quartic (i.e. a singular Calabi-Yau) in $G = Q_0$. Just as in Section 1.4 we define the moduli space of bundles (of the right topology) on $X = X_1 \cup X_2$ to be the space of (semi-stable=stable) bundles on $X_1 \sqcup X_2$ that are isomorphic on S, that is the intersection of the $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\mathcal{M}$. But this is the double cover of the intersection of the two lines in $\mathbb{CP}^2$, i.e. the two points corresponding to the two bundles A and B on Q_0.

(Alternatively note that since X is X_1 glued to itself along S, we want the self intersection number of the genus 2 curve $\mathcal{M}_1$ in $\mathcal{M}$, which is the Euler number of its normal bundle. This is the Euler number of its cotangent bundle $-\chi(\mathcal{M}_1) = 2g - 2 = 2$, as required.)

Deforming this singular quartic in G to a smooth Calabi-Yau we would like, then, to prove the following.

Theorem 2.3.1 *The bundles A and B restrict to* stable *bundles of the same topological type on X, a smooth quartic hypersurface of $G = \mathrm{Gr}(2,4)$. These are the only (semi-)stable bundles of the same topological type, and they are isolated: $H^1(\mathrm{End}_0 A) = 0 = H^1(\mathrm{End}_0 B)$.*

Proof. We have shown in the introduction to this section that bundles $E = A$ or B have sections vanishing on genus 3 curves C in G, giving us a presentation

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_C(1) \to 0. \tag{2.3.2}$$

This shows $E(-1)$ has no sections, and since, by the Lefschetz hyperplane theorem, the $\mathcal{O}(n)$ are the only line bundles, E must be stable.

Next we would like to show that E is isolated. We will use repeatedly the facts that $\Lambda^2 E = \mathcal{O}(1)$, $E^* \cong E(-1)$, and the canonical bundle of X is trivial. Also, being a 3-dimensional complete intersection, $H^1(\mathcal{O}(n)) = 0 = H^2(\mathcal{O}(n))$ for all n (prove this by noting it is true in projective space and holds on passing down to hypersurfaces).

Lemma 2.3.3 *For E either of A or B, $H^1(E) = 0$.*

Proof. By Serre duality it is sufficient to show $H^2(E(-1)) = 0$. But the sequences (2.3.2) and $0 \to \mathscr{I}_C \to \mathcal{O} \to \mathcal{O}_C \to 0$ yield

$$\begin{array}{ccccccccccc} 0 \to H^2(E(-1)) \to & H^2(\mathscr{I}_C) & \to & H^3(\mathcal{O}(-1)) & \to & H^3(E(-1)) & \to & H^3(\mathscr{I}_C) & \to 0\ . \\ & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & \\ & H^1(\mathcal{O}_C) & & H^0(\mathcal{O}(1))^* & & H^0(E)^* & & H^3(\mathcal{O}) & \end{array}$$

Thus $h^1(E) = h^2(E(-1)) = 3 - 6 + h^0(E) - 1 = h^0(E) - 4 = h^0(\mathscr{I}_C(1)) - 3$, again by (2.3.2). Let P be an A- or B- plane in G whose intersection with the quartic is C. Then we relate $H^0(\mathscr{I}_C(1))$, i.e. the hyperplane sections of X containing C, with $H^0(\mathscr{I}_P(1))$, the hyperplane sections of G containing P. We have

$$\begin{array}{ccccccc} 0 \to & H^0(\mathscr{I}_P(1)) & \to & H^0(\mathcal{O}_G(1)) & \to & H^0(\mathcal{O}_P(1)) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & H^0(\mathscr{I}_C(1)) & \to & H^0(\mathcal{O}_X(1)) & \to & H^0(\mathcal{O}_C(1)), \end{array}$$

where the last two vertical arrows are isomorphisms by $0 \to \mathcal{O}(-3) \to \mathcal{O}(1) \to \mathcal{O}_D(1) \to 0$ sequences on G and X (here D is a quartic divisor in G or X). Therefore by the 5-lemma the first arrow is also an isomorphism and $h^1(E) = h^0(\mathscr{I}_C(1)) - 3 = h^0(\mathscr{I}_P(1)) - 3$.

Suppose that P is an A-plane, represented as in the introduction by $f \in (\mathbb{C}^4)^*$. Then the sections of $\mathcal{O}(1)$ (represented by elements $\psi \in \Lambda^2(\mathbb{C}^4)^*$) vanishing on P are

$$\{\psi : \psi \wedge f = 0\} = f \wedge (\mathbb{C}^4)^* \cong (\mathbb{C}^4)^* / \mathbb{C}.f,$$

so that $h^0(\mathscr{I}_P(1)) = 3$, and $h^1(E) = 0$ (the B-plane case is similar). $\square$

We can also show that $H^1(E(-1)) = 0$ using (2.3.2) and the fact that C is connected. Therefore tensoring (2.3.2) with $E(-1)$ and taking cohomology gives

$$0 \to H^1(\operatorname{End} \mathscr{E}) \to H^1(\mathscr{I}_C \otimes \mathscr{E}),$$

and the sequence $0 \to \mathscr{I}_C \otimes \mathscr{E} \to \mathscr{E} \to \mathscr{E}|_C \to 0$ yields

$$H^0(\mathscr{E}) \to H^0(\mathscr{E}|_C) \to H^1(\mathscr{I}_C \otimes \mathscr{E}) \to H^1(\mathscr{E}).$$

The last term vanishes, by the lemma, so to show that $H^1(\mathrm{End}_0\,\mathscr{E}) \cong H^1(\mathrm{End}\,\mathscr{E})$ is zero it is sufficient to show that the restriction map $H^0(\mathscr{E}) \to H^0(\mathscr{E}|_C)$ is onto.

Let $P = \mathbb{P}\left((\mathbb{C}^4)^*/\,\mathbb{C}.f\right) = \mathbb{P}(V)$ be the A-plane as before, so that C is a quartic curve in P (the B-plane case is similar). We first show the restriction of $H^0(A)$ from G to P is onto.

From the description of A as the universal quotient bundle of $(\mathbb{C}^4)^*$ on G ($0 \to B^* \to (\underline{\mathbb{C}^4})^* \to A \to 0$), we see that on P, A is given by

$$0 \to \mathcal{O}(-1) \to \underline{V} \to A \to 0. \tag{2.3.4}$$

Thus $V \cong H^0(P; A)$ and $H^0(G; A) \cong (\mathbb{C}^4)^* \to (\mathbb{C}^4)^*/\,\mathbb{C}.f = V$ is onto.

So we are left with showing that $H^0(P; A) \to H^0(C; A)$ is onto, for which it is sufficient that $H^1(P; A(-4))$ vanishes. But this is dual to $H^1(P; A)$ which vanishes by the sequence (2.3.4).

Therefore we have shown that A and B are isolated on X; what remains is to show that given a stable bundle E of the same Chern classes it must be one of A or B. As in the quintic example we first show it has sections, using Riemann-Roch. To do this we must control E's cohomology groups by studying E on restriction to hyperplane sections.

Proposition 2.3.5 *Let S be a generic hyperplane section of X, and let $F = E|_S$. Then $H^1(F(-t)) = 0 \quad \forall t \geq 1$.*

Proof. We proceed just as in the quintic 3-fold example in the last section. By Theorem 2.2.10 and the remark there, the stability of E implies that E is semi-stable (and so stable) on restriction to the generic S, whose only line bundles are powers of the hyperplane bundle by the Noether-Lefschetz theorem. Thus F's sections, which are guaranteed by the Riemann-Roch formula

$$h^0(F) = 4 + h^1(F)/2 \geq 4,$$

cannot vanish on divisors, and we get a sequence

$$0 \to \mathcal{O} \to \mathscr{F} \to \mathscr{I}_4(1) \to 0,$$

from a section vanishing on 4 points (more precisely, a subscheme of length 4). Taking cohomology $h^0(\mathscr{I}_4(1)) = h^0(F) - 1 \geq 3$, so the points lie on a web of hyperplanes in S (and S is the intersection of a quartic and a quadric in $\mathbb{CP}^4$). Therefore the points lie on a line in $\mathbb{CP}^4$.

The above sequence gives us, for $t \geq 2$,

$$\begin{array}{ccccccc} 0 \to & H^1(F(t-1)) & \to H^1(\mathscr{I}_4(t)) \to & H^2(\mathcal{O}(t-1)) & \to 0, \\ & \downarrow \wr & & \downarrow \wr & \\ & H^1(F(1-t))^* & & H^0(\mathcal{O}(2-t))^* & \end{array}$$

since $H^2(F(t-1)) \cong H^0(F(1-t))^*$ vanishes by stability. Thus it is sufficient to show that $h^1(\mathscr{I}_4(t))$ vanishes for $t \geq 3$ and is 1 for $t = 2$.

But this is easy since the points lie on a line in $\mathbb{CP}^4$. Given values at 4 points on a line (or 3 values and a derivative if there is a double point, etc.) there are polynomials in one variable of any degree t greater than or equal to 3 taking those values at the points, and all these polynomials come from sections of $\mathcal{O}(t)$ on $\mathbb{CP}^4$, which restrict to sections on S. Thus

$$H^0(\mathcal{O}(t)) \to H^0(\mathcal{O}_4(t))$$

is onto and $H^1(\mathscr{I}_4(t)) = 0$. By the same argument $h^1(\mathscr{I}_4(2)) = 1$, and we are done. □

Now the sequence

$$H^1(E(-t-1)) \to H^1(E(-t)) \to H^1(E(-t)|_S)$$

and the above proposition show that $H^1(E(-t)) = 0 \ \forall t \geq 1$, and in particular $H^1(E(-1)) = 0$. Using this and stability, the Riemann-Roch formula for such an E takes the simple form

$$h^0(E) - h^1(E) = 4;$$

thus E has at least four sections. By stability again these do not vanish on a divisor, and we have a sequence

$$0 \to \mathcal{O} \to \mathscr{E} \to \mathscr{I}_C(1) \to 0,$$

for some degree four curve C. But then $h^0(\mathscr{I}_C(1)) \geq 3$, and C lies in a web of hyperplanes in $X \subset G \subset \mathbb{CP}^5$. Thus it lies on a linear $\mathbb{CP}^2$ plane in $\mathbb{CP}^5$. Since C lies in the quadric G, the plane must do too, otherwise the quadric would intersect the plane in a conic curve containing the degree four curve C, which is impossible.

So the plane is one of the A- and B- planes, P say, in G, and C is the intersection of P with the quartic. P uniquely defines either the A or the B bundle on G via the Serre construction (2.1.1),

$$0 \to \mathcal{O} \to A/B \to \mathscr{I}_P(1) \to 0,$$

by the extension data $\operatorname{Ext}^1(\mathscr{I}_P(1), \mathcal{O}_G) \cong H^0(\mathcal{O}_P) \ni 1$. This extension restricts, on the quartic X, to $1 \in H^0(\mathcal{O}_C) \cong \operatorname{Ext}^1(\mathscr{I}_C(1), \mathcal{O}_X)$, defining our bundle E

$$0 \to \mathcal{O} \to \mathscr{E} \to \mathscr{I}_C(1) \to 0$$

by uniqueness, since $H^0(\mathcal{O}_C) \cong \mathbb{C}$ (C is a curve in $\mathbb{CP}^2$, so is connected). Therefore E is one of A or B restricted to X. □

2.4 Bundles on $K3 \times T^2$ and the Vafa-Witten equations

Our final example, suggested by Donaldson, touches on possible relations with physics and the fashionable topic of modular forms. To obtain firm conclusions would take us into analysis of the Vafa-Witten equations in four dimensions, so we will only sketch the ideas.

Recall our discussion of tori and Mirror Symmetry in 2.2, referring to the predictions of [V]. Physicists rarely count numbers, they deal in generating functions for sequences of numbers, and what is actually predicted in [V] is that the power series with coefficients the number of degree n tori in the quintic 3-fold should be a certain modular form (I think, although this is unclear from [V]). Donaldson pointed out that such modular forms arise in a completely different way via the work of Vafa and Witten [VW] on S-duality.

S-duality is a conjectural non-abelian generalisation of electromagnetic duality which therefore has consequences for instantons on 4-manifolds. The main point of [VW] is that the Vafa-Witten equations come from a Lagrangian similar to that in Witten's version of Donaldson theory, and one to which S-duality should also apply. But the Vafa-Witten theory is chosen to (formally) compute the Euler characteristic of moduli spaces of instantons, at least when a certain vanishing result holds, by a method quite similar to the discussion of Section 1.4. Suppose a space of instantons $\mathcal{M}$ is smooth of the correct dimension d. Then it is cut out by an infinite number of equations in "$\infty + d$" unknowns, in the sense of the

Fredholm theory of the ASD equations giving a section of an infinite dimensional vector bundle over the infinite dimensional space of connections modulo gauge. If we add in d more equations which in fact constitute a section of $T\mathcal{M}$ we should get $\chi(\mathcal{M})$ solutions (counted with signs). What Vafa and Witten in fact do is add in infinitely more unknowns with "$\infty + d$" more equations, with kernel isomorphic to the cokernel of the original equations. This ensures that the original moduli space $\mathcal{M}$ sits in the space of solutions (by setting the other variables to zero), and that also, if these are all of the solutions, on perturbation the moduli space breaks up into $\chi(\mathcal{M})$ solutions when counted with sign.

So, when the vanishing result (that all solutions lie in $\mathcal{M}$) holds, and ignoring compactness problems, we expect the partition function of "topologically twisted $N=4$ supersymmetric Yang-Mills theory" to compute $\chi(\mathcal{M})$, and then a consequence of the S-duality conjecture is that the generating function

$$Z_X(\tau) = \frac{q^{-s}}{\#Z(G)} \sum_n \chi(\mathcal{M}_n) q^n, \quad q = e^{2\pi i \tau}, \tag{2.4.1}$$

should be a modular form of τ with respect to the standard action of (a large subgroup of) $SL(2;\mathbb{Z})$ – the group of the S-duality. Here s is some constant, $\#Z(G)$ is the number of elements in the centre of the structure group G, and $\mathcal{M}_n$ is the moduli space of G-instantons of instanton number n (this is c_2 for $G = SU(2)$). The Euler characteristics have to be properly interpreted keeping in mind singularities and compactifications of the $\mathcal{M}_n$.

In particular for $X = K3$, where a vanishing theorem does hold, using results of Mukai, Nakashima and Qin, [VW] show that (2.4.1) is indeed a modular form.

We will now relate this to our discussion using an observation of Donaldson. The moduli space of stable holomorphic bundles on the Calabi-Yau $K3\times T^2$ contains $\mathcal{M}_{K3}$ (by pulling up). The generic number of bundles on $K3\times T^2$ that any theory of counting bundles would produce ought to be the Euler characteristic $\chi(\mathcal{M}_{K3\times T^2})$ of the moduli space of ((semi-)stable) bundles on $K3\times T^2$, as discussed in Section 1.4. So, accounting for the bundles on $K3\times T^2$ that are not pull-ups, the generating function of the Euler characteristics χ_n of the moduli spaces of bundles of the type

$$\Lambda^2 E = \mathcal{O}, \quad c_2(E) = \pi^*[n],$$

should be a modular form. (Here $[n]$ denotes n times the generator $1 \in H^4(K3)$, and π the projection to $K3$.)

So let $\mathcal{M}_{K3}$ be the moduli space of stable holomorphic bundles F with trivial determinant and fixed second Chern class n. Letting $\mathcal{M}_{K3\times T^2}$ be the corresponding space of bundles with $c_2 = \pi^* n$, we have four copies of $\mathcal{M}_{K3}$ in $\mathcal{M}_{K3\times T^2}$, corresponding to the four line bundles L on T^2 with square zero: if p denotes the projection to T^2 then $E \in \mathcal{M}_{K3} \Rightarrow \pi^*E \otimes p^*L \in \mathcal{M}_{K3\times T^2}$ since it has determinant $\pi^*\Lambda^2 E \otimes p^*L^2$, which is trivial for $L^2 = \mathcal{O}$. That is,

$$4\,\mathcal{M}_{K3} \hookrightarrow \mathcal{M}_{K3\times T^2}.$$

At least infinitesimally we can see that this is all of $\mathcal{M}_{K3\times T^2}$, i.e. that it is an open subset. Deformations of such a bundle are given, to first order, by

$$H^1(\mathrm{End}_0\,(\pi^*E \otimes p^*L)) \cong H^1(\pi^*\mathrm{End}_0(E)),$$

which, by the Künneth formula, equals $H^1(\mathrm{End}_0\,(E))$ since $H^0(\mathrm{End}_0\,(E)) = 0$ by stability.

There may, however, be other components in $\mathcal{M}_{K3\times T^2}$, as we outline very briefly. We are free to choose the polarisation (Kähler form) to make the T^2 fibres very big and so the $K3$ fibres of the projection p very small, along the lines of Friedman's "n-suitable" ample divisors in [Fr] (where $\pi^*[n] = c_2(F)$).

Suppose firstly that the torsion free (and so locally free) sheaf

$$p_*(F) \neq 0.$$

Taking a line bundle subsheaf $L^{-1} \to p_*(F)$ and using the map $p^*p_*(F) \to F$, we obtain a rank one subsheaf

$$0 \to p^*L^{-1} \to \mathscr{F}.$$

Thus there is an effective divisor D with a section $p^*L^{-1} \otimes \mathcal{O}(D) \to \mathscr{F}$ vanishing only in codimension two. By stability D must be contained in the fibres of p (otherwise, by the choice of polarisation, $[D] \otimes p^*L^{-1} \,.\, \omega$ would be positive) and so is pulled up from T^2 also. Thus we may write, without loss of generality,

$$0 \to p^*L^{-1} \to \mathscr{F} \to p^*L \otimes \mathscr{I}_C \to 0.$$

By calculating c_2 of such an F we see that the homology class of the curve C is pulled up from $K3$, so $\pi(C)$ is a zero dimensional subscheme Z of $K3$, and $C = \pi^*(Z)$. Locally free sheaves of this sort do exist, as the conditions of Serre's construction (2.1.1) are easily satisfied ($H^2(L)$ vanishes by the Künneth formula), and they are stable for L not too positive. In fact F can only be destabilised by line bundles of the form $p^*L \otimes \mathcal{O}(-\pi^*D)$ where D is a divisor on $K3$ containing Z. Stability forces this line bundle to have negative degree with respect to our polarisation and so bounds the degree of L in terms of that of D and n.

Tensoring the above sequence with $p^*(L\otimes L_1)$, for L_1 any other line bundle on T^2, shows that L is the unique line bundle such that $F\otimes p^*(L)$ has sections not vanishing on T^2 fibres. So the subscheme $Z\subset K3$ is also unique. Therefore $\mathcal{M}_{K3\times T^2}$ contains some copies of products of Hilbert schemes of points in $K3$ and Jacobian tori. These have zero Euler characteristic and so do not affect our argument.

So now we consider $p_*(F)=0$. In this case the restriction of F to the generic $K3$ fibre is stable, and so we get a rational (and therefore regular) map

$$T^2 \xrightarrow{f} \mathcal{M}_{K3}.$$

Suppose for simplicity there is a universal bundle $\mathbb{E}$ over $K3\times\mathcal{M}_{K3}$ (this is true for odd c_2), or atleast over $f(T^2)$, and consider the sheaf of homomorphisms

$$L=p_*(\mathscr{F}^*\otimes f^*\mathbb{E}).$$

On the generic $K3$ fibre, where F is stable, the maps from $F|_{K3}$ to $f^*\mathbb{E}|_{K3}$ are all multiples of a fixed isomorphism. So L is rank one, and thus a line bundle (as could be guessed from the notation) since it is torsion-free.

This gives us the sequence

$$0\to\mathscr{F}\otimes p^*L\to f^*\mathbb{E}\to Q\to 0,$$

for some torsion sheaf Q supported on the finite number of fibres, $K3_i$ say, where the second arrow is not an isomorphism. Since it has a two-step locally free resolution, $Q|_{K3_i}$ must be locally free. Fixing i, if it is rank two then the above map vanishes identically on $K3_i$ and so we may remove this fibre by adding the divisor $[K3_i]$ to L.

So we may assume that $Q|_{K3_i}$ is a line bundle. Taking second Chern classes of the above sequence shows that $c_2(f^*\mathbb{E})\,.\,\pi^*\omega_{K3}$ equals $c_2(\mathscr{F}\otimes p^*L)\,.\,\pi^*\omega_{K3}+c_2(Q)\,.\,\pi^*\omega_{K3}=c_2(\mathscr{F})\,.\,\pi^*\omega_{K3}-\sum_i c_1(Q|_{K3_i})\,.\,\omega_{K3_i}=-\sum_i c_1(Q|_{K3_i})\,.\,\omega_{K3_i}$. But $f^*\mathbb{E}|_{K3_i}\to Q|_{K3_i}\to 0$ implies, by stability, that $c_1(Q|_{K3_i})\,.\,\omega_{K3_i}>0$ and the above quantity is nonpositive. But $c_2(f^*\mathbb{E})\,.\,\pi^*\omega_{K3}$ must be nonnegative by the Bogomolov inequality ([HL] p 71), so it must vanish, which can only happen if there are no such $K3_i$ fibres. This gives us

$$0\to\mathscr{F}\otimes p^*L\xrightarrow{\sim} f^*\mathbb{E}\to 0,\quad \text{i.e. } F\cong f^*\mathbb{E}\otimes p^*L^{-1}.$$

Since $c_2(F)$ is pulled up from $K3$, the degree of the map f must be zero so that f is constant, and the induced map on determinants implies that L has square zero. Thus F is a pull-up from $K3$ twisted by a degree 2 line bundle on T^2, as required.

Chapter 3

Compactness and Virtual Moduli Cycles

3.1 The compactness and smoothness problem

By now it is clear we need some sort of compactness result for holomorphic bundles on, say, a 3-fold, or at least a Calabi-Yau 3-fold, and some control or understanding of smoothness or perturbations.

We could take a purely algebraic approach. Bundles generated by their sections are quotients of trivial bundles, and as such are classified by holomorphic maps $X \to Gr$ to some Grassmannian Gr. Thus, roughly speaking, compactness fails when we cannot find a fixed twist $E(n)$ of E such that all the $E(n)$ of one particular topological type are generated by their sections. So firstly this approach only really works for projective algebraic varieties, where we have a positive line bundle, and secondly we need this "boundedness" theorem (for instance we proved such a boundedness theorem for the particular example of bundles we considered in Section 2.2).

There is a general compactness theorem for "bounded families" of bundles (in fact we have to consider all semi-stable torsion-free sheaves to compactify the moduli space) on a projective variety in [M1,2]. What this boundedness comes down to, in effect, is that on restriction to curves the bundles have a bounded number of sections. Thus on repeatedly passing down to hyperplane sections of a variety Maruyama shows inductively that in this case the bundle is generated by its sections after a fixed twist (as we did in Section 2.2). Of course it is crucial here that the variety be projective, so we have such hyperplane divisors. In [M3] it is then proved that all families of semi-stable rank 2 bundles on projective varieties are in fact bounded.

So for a projective algebraic variety we have a compactness result. But we have lost the benefits of most perturbations, and we could only expect to get a moduli space of the correct dimension if we could prove a "generic smoothness" result along the lines of Donaldson's result for projective surfaces, namely that

$$H^2(\operatorname{End}_0 E) = 0$$

for a Zariski-open dense subset of the moduli space, for sufficiently large c_2. Even then, in the 3-fold case say, to get $H^3(\operatorname{End}_0 E) \cong H^0(\operatorname{End}_0 E \otimes K_X)^*$ to vanish we have to restrict to Fanos and Calabi-Yau manifolds. We will see several instances below, in both the algebraic and analytic approaches, where the Fano case works best while the Calabi-Yau case is borderline; the analytic approach may not work (perturbations move the border) but the algebraic machinery does (though only just).

Consider, for simplicity, $SL(2,\mathbb{C})$ bundles on complete intersection 3-folds (so that $H^1(\mathcal{O}(l)) = 0 = H^2(\mathcal{O}(l)) \quad \forall l$) whose only line bundles are powers of the hyperplane bundle. For bundles in the stratum of the moduli space consisting of bundles with a section only after twisting by $\mathcal{O}(t)$, we have the exact sequence of sheaves

$$0 \to \mathcal{O}(-t) \to \mathscr{E} \to \mathscr{I}_C(t) \to 0, \tag{3.1.1}$$

for some curve (more precisely codimension-two l.c.i. subscheme) C lying in no divisor in $|\mathcal{O}(t-1)|$.

Then tensoring (3.1.1) by $\mathscr{E} \otimes K_X$ shows that $H^2(\operatorname{End}_0 \mathscr{E})$ fits into the exact sequence

$$H^1(\mathscr{E}(-t) \otimes K_X) \to H^2(\operatorname{End}_0 \mathscr{E})^* \to H^1(\mathscr{E} \otimes \mathscr{I}_C(t) \otimes K_X)).$$

The first term is easily dealt with – by (3.1.1) it is isomorphic to $H^1(\mathscr{I}_C \otimes K_X)$, which by the sequence $0 \to \mathscr{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ is the cokernel of the restriction map $H^0(K_X) \to H^0(K_X|_C)$. Thus for a Calabi-Yau or Fano this vanishes.

The second term fits into the sequence

$$\begin{array}{ccccc} H^0((\mathscr{E}(t) \otimes K_X)|_C) & \longrightarrow & H^1(\mathscr{E} \otimes \mathscr{I}_C(t) \otimes K_X) & \longrightarrow & H^1(\mathscr{E}(t) \otimes K_X) \\ \downarrow \wr & & & & \downarrow \wr \\ H^1(\nu_C)^* & & & & H^1(\mathscr{I}_C(2t) \otimes K_X) \end{array}$$

where ν_C is the normal bundle to C (which exists since C is a local complete intersection; in fact $\nu_C = E|_C$). Thus for smoothness it would be sufficient to show that

$$H^1(C; \nu_C) = 0;$$

i.e. deformations of the curve C are unobstructed and of the correct dimension, and

$$H^1(\mathscr{I}_C(2t) \otimes K_X) = 0;$$

that is the map

$$H^0(K_X(2t)) \to H^0(K_X(2t)|_C) \cong H^{1,0}(C)$$

is onto.

The first condition is an added complication not present in the complex surface theory where deformations of codimension-two subschemes, i.e. points, are clearly unobstructed. It can often be satisfied if X is Fano, however – this is where, for instance, Gromov-Witten curve-counting invariants are most easily defined, and it is conjectured to be true in the pseudo-holomorphic curve literature for all Fano varieties and sufficiently high degree C. It is also trivially true on so-called *convex* varieties – by definition these satisfy $H^1(T_X|_C) = 0$ for every curve $C \subset X$, and $H^1(\nu_C)$ is a quotient of $H^1(T_X|_C)$. Also many examples of moduli of curves on Calabi-Yau 3-folds are known with isolated and unobstructed curves, so that $H^1(\nu_C) = 0$.

The second condition, however, is more tricky. Problems arise because of the possible non-connectedness of the curve C, as is seen most clearly on Calabi-Yau manifolds. For a complete intersection 3-fold the sequence (2.1.4) in the last chapter shows that the isomorphism classes of bundles corresponding to a fixed curve C (with extendible determinant of its normal bundle) are parameterised by

$$\mathbb{P}\left(\mathrm{Ext}^1(\mathscr{I}_C(2t), \mathcal{O})\right) \cong \mathbb{P}\left(H^0(C)\right),$$

which is zero dimensional if and only if C is connected. So on a Calabi-Yau, for instance, an isolated curve will still give a component of the moduli space of bundles $\mathcal{M}$ of too high a dimension if it is not connected. Also, of course, the "boundary" of the Maruyama compactification, involving ideal sheaves corresponding to non-connected curves and sheaves with singularities in codimension three, will have components of too high a dimension.

However, it is often still clear what contribution a particular component of $\mathcal{M}$ should contribute to the number of bundles (e.g. the Euler number of its cotangent bundle in the case it is smooth, two if it is a scheme-theoretic double point, etc.) even when we cannot show smoothness. Excess intersection theory deals with just this problem, and is the method we will adopt in Section 3.3. First, however, we discuss possible analytical approaches to the compactness problem.

The Analytical compactness problem

We describe the analytical approach to the compactness of moduli spaces, based on the result of Uhlenbeck and Yau that stable bundles correspond to Hermitian-Yang-Mills connections [UY]. While the classical compactness theorems for Yang-Mills connections apply only in dimensions four and below (see [U1]), Uhlenbeck and Nakajima have partially extended the results to higher dimensions. We work for now on a Kähler manifold of arbitrary dimension. The case relevant to us is the following.

Theorem 3.1.2 [U2, Nak] *Let A be a Hermitian-Yang-Mills connection on a compact Kähler manifold X of dimension n. There exist constants $\epsilon, \delta, C > 0$ depending only on X such that for all balls $B_r \subset X$ of radius $r < \delta$ such that $r^{2-n}\|F_A\|_{L^2(B_r)} < \epsilon$, we have $\|F_A\|_{L^n(B_r)} < C\, r^{2-n}\|F_A\|_{L^2(B_r)}$.*

As mentioned in [UY], this gives the following compactness theorem, also valid for suitably perturbed Hermitian-Yang-Mills connections (small $\sup|i\Lambda F_A^{1,1}|$ will do).

Theorem 3.1.3 *Given a sequence of Hermitian-Yang-Mills connections there exists a set $S \subset X$ of Hausdorff dimension $2n-4$ such that there exists a subsequence (A_i) convergent on $X \setminus S$ in the following sense: there exists a smooth Hermitian-Yang-Mills connection A on $X \setminus S$, and gauge transformations g_i, such that for all compact subsets $K \subset X \setminus S$, $g_i(A_i)|_K$ converges to $A|_K$ in C^∞.*

Proof. The characteristic class formula

$$p_1(E) = \frac{1}{4\pi^2}\int_X \operatorname{tr} F_A \wedge F_A = \frac{1}{4\pi^2}\left(2\,\|F_A^{0,2}\|^2 + \|\Lambda F_A\|^2 - \|F_A^\perp\|^2\right),$$

(where we decompose $\Lambda^2 \otimes \mathbb{C} = \Lambda^{0,2} \oplus \Lambda^{2,0} \oplus \mathbb{C}\langle\omega\rangle \oplus \Lambda^\perp$), shows that for a Hermitian-Yang-Mills connection,

$$\|F_A\|^2 = \|\Lambda F_A\|^2 + \|F_A^\perp\|^2 = 2\,\|\Lambda F_A\|^2 - 4\pi^2 p_1(E) = 4\pi^2(2(\deg E)^2 - p_1(E))$$

is bounded. Thus $|F_{A_i}|$ is (after passing to a subsequence) convergent as a measure μ say.

Now let S be the set of "bad" points

$$\left\{ x \in X \,:\, \forall r < \delta,\, r^{2-n}\Big(\int_{B_r(x)} d\mu\Big)^{1/2} \geq \epsilon/2 \right\},$$

and fix $r < \delta$. We may choose a finite number of points $(x_i)_{i=0}^N \subset S$ such that $x_j \notin B_{r/2}(x_i)\ \forall i \neq j$, but $\forall x \in S$, $\exists\, i$ such that $B_{r/2}(x) \cap B_{r/2}(x_i) \neq \emptyset$. Thus,

1. the $B_{r/2}(x_i)$ are disjoint, but,

2. the $B_r(x_i)$ cover S.

Take K such that $\mathrm{Vol}(B_r(y)) \leq Kr^{2n}$ for all r and y, and k such that $A = A_k$ has $r^{2-n}\|F_A\|_{L^2(B_r(x_i))} \geq \epsilon/4$ for all i. Then by 1,

$$\|F_A\|^2 \geq \sum_i \|F_A\|^2_{L^2(B_{r/2}(x_i))} \geq \frac{N\epsilon^2}{16}(r/2)^{2n-4},$$

so $N \leq \mathrm{const.}\, r^{4-2n}$ using the above bound on $\|F_A\|^2$. So, by 2, S can be covered by const. r^{4-2n} balls of radius r for all $r < \delta$. But this is precisely the statement that S has Hausdorff dimension $\leq 2n-4$.

The proof is now routine, and just as in the four dimensional case. Theorem (3.1.2) allows us to pass from estimates on $r^{2-n}\|F_A\|_{L^2(B_r)}$ to estimates on $\|F_A\|_{L^n(B_r)}$ in small balls about the good points $X \setminus S$. Then the earlier results of [U1] give us a Coulomb gauge in which we can pass to $\|A\|_{L^n_1}$ bounds and so a convergent subsequence in L^n. The patching constructions for such gauges, and the diagonal argument to get the convergent subsequence on $X \setminus S$, are quite long and tedious but the same as in four dimensions ([DK] pp 158–160). Elliptic regularity then gives C^∞ convergence to a C^∞ Hermitian-Yang-Mills connection A. □

Remark. There is also a version of this for a perturbed Hermitian-Yang-Mills equation, so we can indeed perturb the almost complex structure and metric. An appropriate elliptic system

$$F_A^{0,2} = \bar{\partial}^* u, \qquad i\Lambda F_A^{1,1} = \lambda I, \tag{3.1.4}$$

can be set up which makes sense on any almost-Kähler manifold, and reduces to the same equations when the structure is integrable, by the Bianchi identity. A Weitzenböck formula shows that for the L^2-norm of the Nijenhius tensor suitably small compared to the minimum of the scalar curvature s we still get $\|F_A\|_{L^2}$ bounds on solutions. So in the Fano case, where s is positive, we still have the beginnings of a compactness result.

The obvious hope, then, is that the "bad" set S should be, in the 3-fold case, an embedded holomorphic curve $C \subset X$. The forms $c_2(A_i) - c_2(A_\infty)$ (where A_∞ is the limiting connection of [BS], discussed below, say) converge as currents to a (2,2)-current T on X, which we would like to show is the current dual to a complex curve C. Then the results of [Siu] are very suggestive of the problem here. He shows that positive (p,p)-currents have singular sets (where they have positive Lelong number – see [GH] pp 366–392) which are holomorphic. However, although

$$-c_2(A_i)\wedge\omega = C\,\|F^{\perp}_{A_i}\|^2\omega^3 \;\longrightarrow\; -T\wedge\omega$$

is positive, we have no such result for T. So Siu's results would need extending to include currents that are somehow positive "on average" to be of use here.

Alternatively, we could try to follow similar lines to those of [T2], who shows that the zero set S of a section of a line bundle satisfying a perturbed Seiberg-Witten equation is a pseudo-holomorphic curve. He does this by giving a "positive cohomology assignment" to all 4-discs transverse to S, which turns out to be the intersection number. Continuity of this assignment translates, eventually, into S being pseudo-holomorphic. We could give hyperplane sections (in the algebraic case) a similar positive cohomology assignment in our threefold X, by looking at the holomorphic structures defined by the A_i's restricted to the hyperplanes, and tending $i\to\infty$: four dimensional theory will now give a number of singular points, at least if the holomorphic structures are stable (and thus admit HYM connections); this is our assignment.

We would like then to be able to remove the singularity to obtain a Hermitian-Yang-Mills connection (on a bundle with smaller $c_2.\omega$) on all of X. Bando and Siu [BS] prove that the singularity can indeed be removed to produce at least a reflexive sheaf with HYM metric on its locally free part. It should be clear from the proof that c_3 of this sheaf vanishes so that it is locally free. This is of limited use, however, as we have now lost the advantages of perturbations, the method of proof being so reliant on the holomorphic structure; we would like a version which would apply to the equations (3.1.4) and perhaps pseudo-holomorphic curves.

So we still require a purely analytic approach. The local result we need is that an L^2 Hermitian-Yang-Mills connection on $\Delta^n\setminus\Delta^{n-2}$ (where Δ^n is a neighbourhood of $0\in\mathbb{C}^n$) extends smoothly to one on Δ^n. This cannot, however, be true in full generality, since the pull-up of a stable bundle with $c_2\neq 0$ from $\mathbb{CP}^2$ to $\mathbb{C}^3\setminus\{0\}$ admits a (pulled-up) HYM connection which turns out to be L^2, but clearly does not extend over the origin.

Based on the results of Uhlenbeck and Nakajima, however, we could make the following conjecture, the conditions of which are not satisfied by the above example, but *are* satisfied by the model case of the pull-up of any L^2 Hermitian-Yang-Mills connection from $\Delta^2\setminus\{0\}$ to $\Delta^n\setminus\Delta^{n-2}\cong\Delta^{n-2}\times(\Delta^2\setminus\{0\})$ (using the removal of singularities result in four dimensions and some elementary integration).

Conjecture 3.1.5 *If A is a Hermitian-Yang-Mills connection over $\Delta^n\setminus\Delta^{n-2}$ such* *that* $r^{2-n}\|F_A\|_{L^2(B(r))} < \epsilon$ *for all* r *sufficiently small (where* ϵ *is a constant smaller than that in (3.1.2)), then it extends smoothly to a Hermitian-Yang-Mills connection on* Δ^n*.*

I will not record my failed attempts at proving this, as probably it should really be tackled as if it were the pull-up of a four dimensional problem, using Sobolev norms with only derivatives in the $\mathbb{R}^4$ directions.

It is clear, though, that if results such as these could be pushed through, there would be a natural analogue of the Donaldson-Uhlenbeck compactification of ASD moduli spaces in four dimensions. The boundary would consist of "ideal connections" with Dirac-delta c_2-singularities, each represented by a connection on a bundle with lower $c_2.\omega$ (this is the limiting connection with its singularity removed), and a complex curve along which the singularity lies. Perhaps this could even work in the almost-Kähler case with pseudo-holomorphic curves, allowing more perturbations to ensure a smooth moduli space.

Again, this might really only be of use on Fano 3-folds; here not only do we still have L^2 bounds on curvature if we perturb the equations, and hope that the space of curves will be smooth, but also only on Fanos are the expected dimensions of the boundary components of the above compactification smaller than that of the original moduli space. This can be seen from the formulae which can be computed for the virtual dimension

$$-\chi(\operatorname{End}_0 E) = c_1(X)(2c_2(E) - \frac{1}{8}c_2(X))$$

of the moduli spaces, and the index $\chi(\nu_C)$ of the deformation problem of a curve C,

$$\chi(\nu_C) = \langle c_1(X), [C]\rangle,$$

since $\operatorname{PD}[C]$ is the change in $c_2(E)$ when a connection bubbles off along $[C]$. So, either by algebraic or analytic means, it may be possible to introduce invariants of Fanos via Donaldson's μ-map, as a first step towards the harder problem of counting bundles on Calabi-Yau 3-folds.

The approach we take, however, is algebraic, and does not try to control the non-smoothness of the moduli space, but takes account of it.

3.2 Moduli of sheaves

We work on a smooth projective variety X of dimension n. For instance a Calabi-Yau 3-fold with $h^{0,1} = 0$ (a condition which is often included in the definition of a Calabi-Yau) is projective algebraic by the Kodaira embedding theorem: $h^{0,2} = 0 = h^{2,0}$ so all 2-cohomology classes are $(1,1)$ and we may find a rational Hodge class approximating the Kähler form. We start by fixing our notation and recalling some fundamentals of sheaf moduli theory; an excellent reference is [HL].

A coherent sheaf is of pure dimension d if the supports of all non-zero subsheaves are of dimension d. In particular it is pure of dimension $n = \dim X$ if and only if it is torsion-free. The Hilbert polynomial of a sheaf $\mathscr{E}$ is

$$P(t) = \chi(\mathscr{E}(t)) = h^0(\mathscr{E}(t)) \quad \text{for } t \gg 0,$$

and the normalised Hilbert polynomial is $P(t)/rk(E)$. A sheaf is stable (in the sense of Gieseker; this is the notion we shall use from now on) if and only if all coherent subsheaves have strictly smaller normalised Hilbert polynomial for $t \gg 0$, and semi-stable if the inequality is not strict. Importantly stable sheaves are simple – their only global endomorphisms are scalars: $H^0(\operatorname{End}_0 \mathscr{E}) = 0$.

The family of pure dimensional semi-stable sheaves of a fixed Hilbert Polynomial P form a bounded family (see [HL] for the definition and the proof, which we will not need), and Simpson [S] has constructed their moduli space $\mathcal{M}$ as a projective scheme – in particular it is compact. The locally free sheaves (vector bundles) form a Zariski open subset, which may of course be empty for some choices of P, and stability is also an open condition. We will review Simpson's construction (which is slightly different from Maruyama's) when we need it below.

The deformation theory of sheaves shows that the Zariski tangent space of $\mathcal{M}$ at a *stable* point $\mathscr{E}$ is $\operatorname{Ext}^1(\mathscr{E},\mathscr{E})$ (there is an identification of semi-stable points in the moduli space, so their tangent spaces are more complicated to describe). At locally free points this reduces to the more familiar $H^1(\operatorname{End}(\mathscr{E}))$, and more generally there is a spectral sequence relating the two with $H^0(\mathscr{E}xt^1(\mathscr{E},\mathscr{E}))$, which parameterises local deformations of the singularities (non-locally free points) of $\mathscr{E}$. There is a generalised Serre duality theorem that for arbitrary coherent sheaves $\mathscr{F}$ and $\mathscr{G}$ on X,

$$\operatorname{Ext}^i(\mathscr{F},\mathscr{G}) \cong \operatorname{Ext}^{n-i}(\mathscr{G},\mathscr{F} \otimes K_X)^*,$$

easily proved from normal Serre duality by induction on the length of a locally free resolution of $\mathscr{G}$. Thus the cotangent space to $\mathcal{M}$ is $\operatorname{Ext}^{n-1}(\mathscr{E},\mathscr{E} \otimes K_X)$.

There is a trace map on any $\operatorname{Ext}(E,E)$ group or sheaf, generalising that on the $\operatorname{Hom}(E,E)$. Taking the trace-free part Ext_0 of everything in this section corresponds to fixing the determinant of $\mathscr{E}$.

In algebraic geometry tangent spaces do not glue well to form a sheaf, and the more natural object is the cotangent sheaf Ω of a scheme. Lehn [L] has shown that about stable points,

$$\Omega_{\mathcal{M}} \cong \mathscr{E}xt^{n-1}_{X\times\mathcal{M}/\mathcal{M}}(\mathscr{E},\mathscr{E}\otimes K_X)_0,$$

where $\mathscr{E}$ is a universal sheaf on $X\times\mathcal{M}$ (one exists locally and the results patch together independently of choices), and $\mathscr{E}xt^i_{X\times\mathcal{M}/\mathcal{M}}$ are the relative Ext sheaves of the projection $X\times\mathcal{M}\to\mathcal{M}$. (Technically these are the right derived functors of $\mathscr{H}om_{X\times\mathcal{M}/\mathcal{M}}(\mathscr{F},\mathscr{G})$ which associates to sheaves $\mathscr{F}$ and $\mathscr{G}$ on $\mathcal{M}$ a sheaf whose sections over an open set $U\subset\mathcal{M}$ are global homomorphisms from $\mathscr{F}$ to $\mathscr{G}$ over $X\times U$.) So $\Omega_{\mathcal{M}}$ has fibre the Zariski cotangent space $\mathrm{Ext}^{n-1}_0(\mathscr{E},\mathscr{E}\otimes K_X)$ (base-change holds [BPS] as Ext^n_0 vanishes by stability – it is dual to $H^0(\mathrm{End}_0\,\mathscr{E})$).

The obstruction space to the deformations of $\mathscr{E}$ is $\mathrm{Ext}^2_0(\mathscr{E},\mathscr{E})$. For a torsion-free sheaf (whose determinant is therefore a line bundle) we take the trace-free part even if we do not fix determinants, since no obstruction lives in the $H^2(\mathcal{O}_X)$ part of $\mathrm{Ext}(\mathscr{E},\mathscr{E})$; the extensions of line bundles (i.e. the determinant) are unobstructed, or, in the language of gauge theory, abelian gauge theory is linear.

3.3 Excess intersection theory

The moduli space of sheaves of Hilbert Polynomial P and fixed determinant has a natural virtual dimension given by the topological invariant

$$vd(\mathcal{M}) = \sum_{i=0}^{n}(-1)^{i+1}\dim\,\mathrm{Ext}^i_0(\mathscr{E},\mathscr{E}). \tag{3.3.1}$$

However, as discussed earlier, we cannot expect the actual dimension of all of its components to coincide with this. When the higher groups $\mathrm{Ext}^i_0(\mathscr{E},\mathscr{E})$, $i\geq 3$, vanish, its dimension will be greater than or equal to vd, and what we would like to do is find a smaller cycle inside $\mathcal{M}$, of dimension vd, which carries the "correct" information of the moduli problem (it should have the right properties under deformations of X, for instance).

In simple cases in differential geometry this is well understood. Suppose a manifold M is cut out by a set of r equations inside an $(n+r)$-dimensional ambient space Z, i.e. M is the zero set of a section s of a rank r vector bundle E over Z.

Suppose also that $(r-d)$ of the equations are linearly independent and form a transverse section of a rank $(r-d)$ subbundle E' of E, while the remaining d equations are dependent on the first $(r-d)$, that is they lie in E' also. Then M will be smooth but of too high a dimension, namely $n+d$ instead of the "virtual dimension" n.

Perturbing the section, however, we can get a smooth zero set of dimension n. In fact we can get this to lie in M by choosing a smooth splitting

$$E \cong E' \oplus (E/E')$$

of E, and choosing the perturbation to be the addition of a small transverse section of E/E', whose zeroes will form the n-cycle in M that is the global zero set.

In practice our moduli space $\mathcal{M}$ will not be globally cut out, in any natural way, by a section of a vector bundle over an ambient space of the correct dimension, because it arises as a G.I.T. quotient of a space by a group action. But notice that in the above simple situation we can recover the zero set of the "correct" dimension from purely local data on M itself, without knowing the ambient space Z – it is the Euler class

$$e(E/E') \cap [M] \in H_n(M)$$

of the cokernel bundle E/E', which maps to the Euler class of E itself in Z:

$$\begin{array}{rl} e(E) \cap [Z] \in & H_n(Z) \\ & \uparrow \\ e(E/E') \cap [M] \in & H_n(M). \end{array} \tag{3.3.2}$$

So we might hope the deformation theory of the moduli problem would give us the local data on $\mathcal{M}$ necessary to define a virtual cycle, knowing the tangent spaces (or more generally the cotangent sheaf) and obstruction spaces (the generalisation of the cokernel bundle). For instance if the moduli space were smooth and the obstruction spaces all of the same dimension they would glue together to form a locally free sheaf (a relative Ext sheaf) whose top Chern class would be our virtual cycle.

The general algebraic case is more difficult – $\mathcal{M}$ may not be smooth (the local sections are not transverse) and if it were there may be no algebraic splitting of E as above, so the zeroes of a perturbed section would move off $\mathcal{M}$ into some non-existent global ambient space. But Li and Tian [LT] (see also [BF]) show we can recover a virtual moduli cycle from the deformation theory, based on Fulton and MacPherson's intersection theory [Fu] – a refined version of intersection theory which does not require a "moving lemma" and so produces intersection cycles within the original intersection, which is precisely what we require.

So we start by summarising Fulton's theory in the case of a holomorphic section s of a vector bundle E over a smooth variety Z, with scheme-theoretic zero set M (this is the theory applied to the intersection of the zero set of E with the image $s(Z)$ in the total space of E).

The section s induces a "cone" in the total space of E which, in the case that s is a transverse section of a subbundle E', is just E'. More generally it is the linearisation of s but reflecting the possibly nilpotent scheme structure of M, and can be thought of as the image of s "made vertical" in $E|_M$, i.e. the limit of λs as $\lambda \to \infty$.

Precisely, the definition is as follows. In an affine patch of Z over which E is trivial, let Γ be the graph

$$\Gamma = \{(x, s(x)) \in E : x \in Z\}$$

of s in the total space of E, and let $\mathscr{I}$ be the ideal of the zero set M in Γ. Then the cone to M in Γ is defined as

$$C_M\Gamma = \operatorname{Spec} \bigoplus_{t \geq 0} \mathscr{I}^t/\mathscr{I}^{t+1},$$

which has a canonical projection to M, because $\mathscr{I}^t/\mathscr{I}^{t+1}$ has an obvious $\mathcal{O}_M = \mathcal{O}_\Gamma/\mathscr{I}$ structure. It also has an embedding in $E|_M$,

$$E|_M = \operatorname{Spec}\operatorname{Sym}\left(\mathscr{I}_{[0]}/\mathscr{I}_{[0]}^2\right) = \operatorname{Spec}\operatorname{Sym}\mathcal{O}(E^*)$$

(where $\mathscr{I}_{[0]} = \mathcal{O}_M(E^*)$ is the ideal of the zero set of E in E), because of the quotient map $\mathscr{I}_{[0]} \to \mathscr{I}$.

Thus we have a cone $C_M\Gamma \subset E|_M$ in a vector bundle over M. The Chow groups $A_*(E)$ of a vector bundle E are isomorphic to the Chow groups of the base (shifted in degree by $rk(E)$) by pulling up the cycles from the base to the total space – an algebro-geometric Thom isomorphism theorem. Thus the cycle defined by $C_M\Gamma$ is rationally equivalent to the pull back of a cycle $c_r(E) \in A_n(M)$ in the Chow group of M, whose image in the Chow group of the ambient space Z is, as Fulton shows, the top Chern class of E,

$$\begin{array}{rl} c_r(E) \in & A_n(Z) \\ & \uparrow \\ [C_M\Gamma] \in A_{n+r}(E|_M) \cong & A_n(M), \end{array}$$

the generalisation of (3.3.2). Equivalently, we intersect the cone with the zero section M of $E \to M$.

What Li and Tian do is show that the cone $C_M\Gamma$ can be recovered from purely infinitesimal data on M, namely the derivative of the section s, as a sheaf map

$$\mathcal{O}_M(TZ) \xrightarrow{ds} \mathcal{O}_M(E).$$

In particular, by the second exact sequence of Kähler differentials ([Ha1] p 173), the dual map gives a two step locally free resolution of the cotangent sheaf Ω_M of M:

$$\mathcal{O}_M(E^*) \xrightarrow{ds^*} \mathcal{O}_M(T^*Z) \to \Omega_M \to 0,$$

and they show that from just this a unique cone can be constructed in $E \to M$. That is, given a two step locally free resolution of the cotangent sheaf of a scheme M (of finite type over $\mathbb{C}$),

$$\mathcal{O}_M(E_2^*) \to \mathcal{O}_M(E_1^*) \to \Omega_M \to 0, \qquad (3.3.3)$$

we can locally represent M as the zero set of a section of E_2 over an affine space of dimension $rk(E_1)$ (at least after stabilising by adding a trivial factor to both E_i's, with the identity map between them) inducing the above sequence. Then Fulton's construction gives a cone in E_2 over M; these local cones patch together independently of choices to give a global cone

$$C \subset E_2 \to M$$

which we intersect with the zero section to give a cycle in $A_{rk(E_1)-rk(E_2)}(M)$ – our virtual cycle of the "correct dimension".

Theorem 3.3.4 [LT] *Let M be a scheme of finite type over a field of characteristic zero, with a complex (3.3.3) with the E_i locally free, of virtual dimension n. Then there is a functorially defined cycle in $A_n(M)$ that is independent of the choice of resolution (3.3.3), which is $[M]$ if M is smooth of dimension n. Similarly if M is smooth and the kernel of (3.3.3) is a vector bundle (whose dual we denote E_2/E_1, the "obstruction bundle") then the cycle is the top Chern class of E_2/E_1.*

That this is the right virtual cycle is assured by its behaviour under deformation invariance, which we will discuss in the next section.

As $\mathcal{M}$ is certainly a scheme of finite type over $\mathbb{C}$, all we need is such a complex, of virtual dimension (3.3.1). This will certainly require the dimension of $\mathcal{M}$ to be at least vd, which for a 3-fold is equivalent to the vanishing of

$$\mathrm{Ext}^3_0(\mathscr{E},\mathscr{E}) = H^0(\mathrm{End}_0\,\mathscr{E} \otimes K_X)^*$$

which holds for simple (e.g. stable) sheaves $\mathscr{E}$ on Calabi-Yau or Fano 3-folds. We would then like the deformation theory to give us a sequence (3.3.3).

Li and Tian give such a resolution of $\Omega_{\mathcal{M}}$ for X a surface. Here $\mathcal{M}$ is the moduli space of *stable* sheaves, which are what we shall deal with from now on as deformation theory about semi-stable sheaves is more complicated – when we want $\mathcal{M}$ to be compact (i.e. projective, not just quasi-projective) we will look at sheaves with rank and degree coprime since then semi-stability implies stability. However the method of [LT] will not generalise to three dimensions and we take a different approach which first deals with the Quot scheme involved in Simpson's construction of $\mathcal{M}$, and which works in all dimensions.

A guiding principle is that, about locally free points of $\mathcal{M}$, gauge theory exhibits $\mathcal{M}$ as the zero set of a holomorphic Fredholm section $F_A^{0,2} = \bar{\partial}_A^2$ of an infinite dimensional holomorphic bundle over the infinite dimensional Kähler manifold of isomorphism classes of $\bar{\partial}$-operators. Locally, then, we should be able to take the zero set of all but a finite number of these equations to leave $\mathcal{M}$ as being cut out by a finite number of equations on a finite dimensional variety (with virtual dimension the Fredholm index of the section, which is precisely $vd(\mathcal{M})$) giving us the resolution we require.

3.4 Resolving $\Omega_{\mathcal{M}}$

At this point we require Simpson's construction of $\mathcal{M}$, described in [HL, S]. We sketch the details; for now $\dim X$ is arbitrary and $\det \mathscr{E}$ is *not* fixed.

Fixing a Hilbert polynomial P, by boundedness there is a fixed integer n_1 such that, for all $\mathscr{E}$ with Hilbert polynomial P, $\mathscr{E}(n_1)$ is generated by its sections and all higher cohomology groups $H^i(\mathscr{E}(n_1)),\ i > 0$, vanish. Thus $h^0(\mathscr{E}(n_1)) = P(n_1)$, and we have an exact sequence

$$0 \to \mathscr{K} \to \underline{H} \otimes \mathcal{O}(-n_1) \to \mathscr{E} \to 0, \tag{3.4.1}$$

for some kernel $\mathscr{K}$, and $H = H^0(\mathscr{E}(n_1))$. Thus the sheaves $\mathscr{E}$ are all quotients, with Hilbert polynomial P, of a fixed (locally free) sheaf $\mathbb{C}^{P(n_1)} \otimes \mathcal{O}(-n_1) = \mathcal{O}(-n_1)^{\oplus P(n_1)}$, parameterised by $\mathrm{Quot}\,(\mathcal{O}^{P(n_1)}, P)$. $\mathcal{M}$ is thus the quotient (in the sense of geometric invariant theory) of the subset of Quot of semi-stable sheaves by the action of $SL(H)$ that permutes the different identifications of $H = H^0(\mathscr{E}(n_1))$ with $\mathbb{C}^{P(n_1)}$. (The definition of Gieseker stability of sheaves is, by construction, the G.I.T. notion of stability for this group action [HL p 98].)

In turn the Quot scheme embeds in a Grassmannian. Tensoring the sequence (3.4.1) with $\mathcal{O}(n_2)$ and taking cohomology yields

$$0 \to H^0(\mathscr{K}(n_2)) \to H \otimes V \to H^0(\mathscr{E}(n_2)) \to 0, \tag{3.4.2}$$

for n_2 sufficiently large that $\mathscr{K}(n_2)$ and $\mathscr{E}(n_2)$ are generated by sections and have no higher cohomology. V denotes $H^0(\mathcal{O}(n_2-n_1))$.

Conversely, since the subspace $H^0(\mathscr{K}(n_2))$ of $H\otimes V$ determines the sheaf $\mathscr{K}(n_2)\hookrightarrow \underline{H}\otimes\mathcal{O}(n_2-n_1)$ by taking sections, it also determines the quotient $\mathscr{E}(n_2)$ and so $\mathscr{E}$.

Thus we get an injection of Quot into the Grassmannian $\mathrm{Gr}\,(H\otimes V, P(n_2))$ of $P(n_2)$-dimensional quotients of $H\otimes V$ which, for $n_2\gg 0$, can be shown to be an embedding.

What we would like is for Quot to be cut out in Gr by a section of a vector bundle of rank $\dim(\mathrm{Gr})-vd(\mathcal{M})+\dim SL(H)$. For instance, a point of Quot

$$0\to\mathscr{A}\to\underline{H}\otimes\mathcal{O}(n_2-n_1)\to\mathscr{B}\to 0,$$

maps to the point

$$0\to A\to H\otimes V\to B\to 0, \qquad\qquad A=H^0(\mathscr{A}),\ B=H^0(\mathscr{B}),$$

of Gr, where A generates, on taking sections inside $\underline{H}\otimes\mathcal{O}(n_2-n_1)$, the sheaf $\mathscr{A}$ whose global sections are precisely A, i.e. they have no component in B. So, conversely, we might hope that Quot$\subset$Gr is exactly the zeroes of the map

$$H^0(\mathscr{A})\hookrightarrow H\otimes V\to B,$$

where $\mathscr{A}$ is the sheaf generated by taking sections in A. While I cannot prove this, the following will be an infinitesimal version on the Quot scheme, which is all we shall eventually need to resolve Ω_{Quot}.

Recall that the Zariski tangent space to the Quot scheme of quotients (3.4.1) is $\mathrm{Hom}\,(\mathscr{K},\mathscr{E})$ [HL].

Theorem 3.4.3 *The Zariski tangent space* $\mathrm{Hom}\,(\mathscr{K},\mathscr{E})$ *to the Quot scheme of quotients (3.4.1), on an arbitrary smooth n-dimensional variety X, fits into the exact sequence*

$$0\to T_{\mathscr{E}}\,\mathrm{Quot}\to T_{\mathscr{E}}\,\mathrm{Gr}\to E_2\to\mathrm{Ext}^2(\mathscr{E},\mathscr{E})\to 0,$$

where E_2 will be defined in the proof, and has constant dimension for all $\mathscr{E}$ if $\dim\mathrm{Ext}^i(\mathscr{E},\mathscr{E})$ *is constant for all $i\geq 3$.*

Remark. This result is essentially equivalent to the resolution of the cotangent sheaf $\Omega_{\mathcal{M}}$ that we require. But since the general case is technical and messy, involving relative Ext sheaves and the like, we do this as a warm-up. The proof contains all the main ingredients for the stronger result for which we will simply dualise, set everything relative to a base and use nastier notation.

Then the duals of all of the above groups will glue together to form sheaves over $\mathcal{M}$ and the constancy of $\dim E_2$ will translate into the local freeness of the sheaf $\mathcal{O}(E_2^*)$, i.e. E_2 will form a vector bundle on $\mathcal{M}$, as will $T\,\mathrm{Gr}$.

Proof. On Quot define the sheaf $\mathscr{K}'$ to be the kernel of the map generating $\mathscr{K}(n_2)$ by its sections,

$$0 \to \mathscr{K}' \to \underline{H^0(\mathscr{K}(n_2))} \to \mathscr{K}(n_2) \to 0,$$

and take $\mathrm{Hom}\,(\,\cdot\,,\mathscr{E}(n_2))$:

$$0 \to \mathrm{Hom}\,(\mathscr{K},\mathscr{E}) \to H^0(\mathscr{K}(n_2))^* \otimes H^0(\mathscr{E}(n_2)) \qquad (3.4.4)$$

$$\to \mathrm{Hom}\,(\mathscr{K}',\mathscr{E}(n_2)) \to \mathrm{Ext}^1(\mathscr{K},\mathscr{E}) \to 0,$$

where the final zero comes from the local freeness of $\underline{H^0(\mathscr{K}(n_2))}$, and the choice of $n_2 \gg 0$. Now the first two terms are $T_{\mathscr{E}}$ Quot and $T_{\mathscr{E}}\,\mathrm{Gr}$, with the map between them induced by the embedding Quot $\subset$ Gr. We simply define

$$E_2 := \mathrm{Hom}\,(\mathscr{K}',\mathscr{E}(n_2)),$$

leaving only the last term and $\dim E_2$ to be dealt with. The higher terms in the above long exact sequence give

$$0 \to \mathrm{Ext}^i(\mathscr{K}',\mathscr{E}(n_2)) \to \mathrm{Ext}^{i+1}(\mathscr{K},\mathscr{E}) \to 0.$$

But the long exact sequence of $\mathrm{Hom}\,(\,\cdot\,,\mathscr{E})$ applied to

$$0 \to \mathscr{K} \to \underline{H}(-n_1) \to \mathscr{E} \to 0,$$

gives $0 \to \mathrm{Ext}^j(\mathscr{K},\mathscr{E}) \to \mathrm{Ext}^{j+1}(\mathscr{E},\mathscr{E}) \to 0$, by the choice of $n_1 \gg 0$. Thus

$$\mathrm{Ext}^i(\mathscr{K}',\mathscr{E}(n_2)) \cong \mathrm{Ext}^{i+1}(\mathscr{K},\mathscr{E}) \cong \mathrm{Ext}^{i+2}(\mathscr{E},\mathscr{E})$$

for all $i \geq 1$. This gives us the last term of (3.4.4) as in the statement of the theorem, and the constancy of $\dim E_2$ if $\dim \mathrm{Ext}^j(\mathscr{E},\mathscr{E})$ is constant for $j \geq 3$, since

$$\dim \mathrm{Hom}\,(\mathscr{K}',\mathscr{E}(n_2)) + \sum_{i=1}^{n} (-1)^i \dim \mathrm{Ext}^i(\mathscr{K}',\mathscr{E}(n_2))$$

is a fixed topological quantity [HL, BPS]. □

Corollary 3.4.5 *We have an exact sequence*

$$0 \to T_{\mathscr{E}}\mathcal{M} \to E_1 \to E_2 \to \operatorname{Ext}^2(\mathscr{E},\mathscr{E}) \to 0$$

where the E_i's have constant dimension for all $\mathscr{E} \in \mathcal{M}$ if the same is true of $\operatorname{Ext}^j(\mathscr{E},\mathscr{E})$ *for $j \geq 3$. Here $E_1 = T_{\mathscr{E}}(\operatorname{Gr}/SL(H))$, where $SL(H)$ has the obvious action on $\operatorname{Gr}(H \otimes V, P(n_2))$ inducing an action of $PSL(H)$ which is free at points representing simple (e.g. stable) bundles.*

Remark. The above sequence is independent of the point of Quot $\subset$ Gr we choose to represent our sheaf $\mathscr{E}$, i.e. it descends under the $SL(H)$ action.

Proof. The statement about the action of $SL(H)$ follows from the corresponding statement about its action on Quot $\subset$ Gr, which linearises to give the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & T_{\mathscr{E}}\operatorname{Quot} & \to & T_{\mathscr{E}}\operatorname{Gr} \to E_2 \to \operatorname{Ext}^2(\mathscr{E},\mathscr{E}) \to 0. \\ & \uparrow & \nearrow & \\ & \mathfrak{sl}(H) & & \end{array}$$

Taking the quotient of the first two terms by the injection (due to stability) of $\mathfrak{sl}(H)$ gives the desired sequence, if we can show that

$$T_{\mathscr{E}}\operatorname{Quot}/\operatorname{End}_0(H) \cong T_{\mathscr{E}}\mathcal{M}.$$

But this follows from applying $\operatorname{Hom}(\,\cdot\,,\mathscr{E})$ to the sequence (3.4.1):

$$0 \to \operatorname{Hom}(\mathscr{E},\mathscr{E}) \to H^* \otimes \operatorname{Hom}(\mathcal{O}(-n_1),\mathscr{E}) \to \operatorname{Hom}(\mathscr{K},\mathscr{E}) \to \operatorname{Ext}^1(\mathscr{E},\mathscr{E}) \to 0.$$

The first term is $\mathbb{C}$.id mapping to the identity in the second term End H. Thus we have the required sequence

$$0 \to \operatorname{End}_0(H) \to T_{\mathscr{E}}\operatorname{Quot} \to T_{\mathscr{E}}\mathcal{M} \to 0.$$

(This again shows that $\operatorname{End}_0(H)$ injects into $T_{\mathscr{E}}$ Quot and $T_{\mathscr{E}}$ Gr.) □

We now tackle the general case, dualising and setting everything relative to a base (i.e. $\mathcal{M}$), working on

$$p: \ X \times \mathcal{M} \to \mathcal{M}.$$

Thus $H^i(\mathscr{F})$ is replaced by the $\mathcal{M}$-sheaf $R^ip_*(\mathscr{F})$ or the obvious candidate for its dual $\mathscr{E}xt^{n-i}_{X\times\mathcal{M}/\mathcal{M}}(\mathscr{F},K_X)$ (we will discuss presently the sense in which relative duality holds). Here K_X is the canonical bundle of X, or more generally the dualising sheaf of p, and we will abbreviate $\mathscr{E}xt^j_{X\times\mathcal{M}/\mathcal{M}}$ to $\mathscr{E}xt^j_p$. Recall the discussion of relative Ext sheaves and the cotangent sheaf of $\mathcal{M}$ in Section (3.2), but otherwise notice the proof is much the same as for the last theorem.

We will change notation a little and denote points of $\mathcal{M}$ by $\mathscr{F}$, reserving $\mathscr{E}$ for a universal sheaf on $X\times\mathcal{M}$. Such an $\mathscr{E}$ exists locally at least, which is all we shall require; all results will be independent of choices. Alternatively we could work on Quot instead of $\mathcal{M}$, and all groups and sheaves will descend under the $SL(H)$ group action. The problem is the scalar endomorphisms of stable sheaves $\mathscr{F}$, but these act trivially on $\mathrm{Ext}^i(\mathscr{F},\mathscr{F})$. Eventually the case that will interest us will be rank and degree coprime sheaves, where $\mathcal{M}$ admits a global universal sheaf.

We will need to use a little Ext sheaf theory from [BPS], familiar at least for relative cohomology ([Ha1], III Section 12). Namely *for sheaves flat over the base*, if the dimensions of Ext^i groups are constant, the relevant relative $\mathscr{E}xt^i$ sheaf will be locally free with fibre Ext^i. Conversely if $\mathscr{E}xt^i$ is locally free its fibre at any point is just the Ext^i group. In the cases we use these will follow from the corresponding statements in cohomology since, due to ampleness of certain line bundles, the Ext groups and sheaves will just be cohomology groups and sheaves.

Theorem 3.4.6 *Fix a smooth n-dimensional algebraic variety X, and let $\mathcal{M}$ denote the moduli space of* stable *sheaves on X of a fixed Hilbert polynomial P. Let $\mathscr{E}$ denote a universal sheaf on $X\times\mathcal{M}$ (see the discussion above), and suppose that* $\dim \mathrm{Ext}^i(\mathscr{F},\mathscr{F})$, $i\geq 3$ *is constant for all $\mathscr{F}\in\mathcal{M}$. Then there is a resolution*

$$0\to \mathscr{E}xt^{n-2}_p(\mathscr{E},\mathscr{E}\otimes K_X)\to \mathcal{O}(E_2^*)\to \Omega_{\mathrm{Gr}}\to\Omega_{\mathrm{Quot}}\to 0,$$

where E_2 is a vector bundle on $\mathcal{M}$. Moreover this gives us a two step resolution of $\Omega_{\mathcal{M}}$:

$$0\to \mathscr{E}xt^{n-2}_p(\mathscr{E},\mathscr{E}\otimes K_X)\to \mathcal{O}(E_2^*)\to \mathcal{O}(E_1^*)\to\Omega_{\mathcal{M}}\to 0,$$

where E_1 is the vector bundle $T\,(\mathrm{Gr}/SL(H))$ over $\mathcal{M}$. (Again this descends to $\mathcal{M}$ independently of choices of points of Quot representing points of $\mathcal{M}$.)

Proof. We denote by $\mathcal{O}(1)$ a very ample line bundle on $X\times\mathcal{M}$. Choose $n_1\gg 0$ such that $\mathscr{E}(n_1)$ is generated by its fibrewise sections, i.e. we have a sequence

$$0\to\mathscr{K}\to p^*\left(p_*\mathscr{E}(n_1)\right)(-n_1)\to\mathscr{E}\to 0, \tag{3.4.7}$$

and $R^ip_*(\mathscr{E}(n_1))=0\ \forall i\geq 1$. Denote $p_*(\mathscr{E}(n_1))$ by $\mathscr{H}$, which is locally free as it has fibres of fixed dimension $P(n_1)$.

Now twist with n_2 sufficiently large such that $\mathscr{K}(n_2)$ and $\mathscr{E}(n_2)$ are generated by fibrewise sections with no $R^i p_*$, and take cohomology:

$$0 \to p_*(\mathscr{K}(n_2)) \to \mathscr{H} \otimes \mathcal{V} \to p_*(\mathscr{E}(n_2)) \to 0. \tag{3.4.8}$$

$\mathcal{V}$ denotes $p_*(\mathcal{O}(n_2 - n_1))$, and to pull the sheaf $\mathscr{H}$ through $p_* p^*$ we have used its local freeness.

Defining $\mathscr{K}'$ by

$$0 \to \mathscr{K}' \to p^* p_* \mathscr{K}(n_2) \to \mathscr{K}(n_2) \to 0,$$

and taking $\mathscr{H}om_p(\mathscr{E}(n_2), \,\cdot\, \otimes K_X)$ yields

$$\begin{aligned} 0 \to \mathscr{E}xt_p^{n-1}(\mathscr{E}, \mathscr{K} \otimes K_X) \to \mathscr{E}xt_p^n(\mathscr{E}(n_2), \mathscr{K}' \otimes K_X) \quad & (3.4.9) \\ \to \mathscr{E}xt_p^n(\mathscr{E}(n_2), p^* p_* \mathscr{K}(n_2) \otimes K_X) \to \mathscr{E}xt_p^n(\mathscr{E}, \mathscr{K} \otimes K_X) \to 0, & \end{aligned}$$

where the first zero comes from the choice of $n_2 \gg 0$.

Now the third term is dual to

$$\mathscr{H}om_p(p^* p_*(\mathscr{K}(n_2)), \mathscr{E}(n_2)) \cong p_*(\mathscr{K}(n_2))^* \otimes p_*(\mathscr{E}(n_2))$$

by relative duality and local freeness of $p_*(\mathscr{K}(n_2))$. Relative duality does not quite give the duality between relative Ext sheaves that one might naively guess from Serre duality, but an equality of right derived functors [Ha3]. What this amounts to in terms of sheaves is a spectral sequence, but for us this collapses since all the lower relative Ext sheaves $\mathscr{E}xt_p^i$, $i < n$, vanish by choice of n_2. Thus we get the duality claimed.

The lower terms of the long exact sequence (3.4.9) give

$$0 \to \mathscr{E}xt_p^{i-1}(\mathscr{E}, \mathscr{K} \otimes K_X) \to \mathscr{E}xt_p^i(\mathscr{E}(n_2), \mathscr{K}' \otimes K_X) \to 0, \quad i \leq n-1,$$

while the long exact $\mathscr{H}om_p(\mathscr{E}, \,\cdot\,)$ sequence of (3.4.7) yields

$$0 \to \mathscr{E}xt_p^{i-2}(\mathscr{E}, \mathscr{E} \otimes K_X) \to \mathscr{E}xt_p^{i-1}(\mathscr{E}, \mathscr{K} \otimes K_X) \to 0, \quad i \leq n,$$

so that

$$\mathscr{E}xt_p^i(\mathscr{E}(n_2), \mathscr{K}' \otimes K_X) \cong \mathscr{E}xt_p^{i-1}(\mathscr{E}, \mathscr{K} \otimes K_X) \cong \mathscr{E}xt_p^{i-2}(\mathscr{E}, \mathscr{E} \otimes K_X)$$

for all $i \leq n-1$. The last term is locally free, by the constancy of $\dim \operatorname{Ext}^i(\mathscr{F}, \mathscr{F} \otimes K_X)$, $i \geq 3$, for all $\mathscr{F} \in \mathcal{M}$. Thus so is the first term, giving the constancy of the dimensions of the fibres $\mathrm{Ext}^i(\mathscr{F}(n_2), \mathscr{K}' \otimes K_X)$, $i \leq n-1$. So $\mathrm{Ext}^n(\mathscr{F}(n_2), \mathscr{K}' \otimes K_X)$ has constant dimension also (as the alternating sum of these dimensions is fixed by the Hilbert polynomial of the sheaf [HL, BPS]), and $\mathscr{E}xt^n_p(\mathscr{E}(n_2), \mathscr{K}' \otimes K_X)$ is locally free. (Notice we have used here the flatness of the sheaves concerned over $\mathcal{M}$. This follows by the constancy of their Hilbert polynomials ([Ha1] III 9.9).) Denote the dual of its associated vector bundle by E_2.

So finally (3.4.9) has become

$$0 \to \mathscr{E}xt^{n-2}_p(\mathscr{E}, \mathscr{E} \otimes K_X) \to \mathcal{O}(E_2^*) \to p_*(\mathscr{E}(n_2))^* \otimes p_*(\mathscr{K}(n_2)) \to \Omega_{\mathrm{Quot}} \to 0,$$

using Lehn's description [L] of the cotangent sheaf of Quot as $\mathscr{E}xt^n_p(\mathscr{E}, \mathscr{K} \otimes K_X)$, which is completely analogous to the description of $\Omega_{\mathcal{M}}$.

We now pass from Quot to $\mathcal{M}$ to deduce the second part of the theorem. Applying $\mathscr{H}om_p(\mathscr{E}, \ \cdot\ )$ to (3.4.7) yields

$$0 \to \mathscr{E}xt^{n-1}_p(\mathscr{E}, \mathscr{E} \otimes K_X) \to \mathscr{E}xt^n_p(\mathscr{E}, \mathscr{K} \otimes K_X) \to$$
$$p_*(\mathscr{E}(n_1)) \otimes \mathscr{E}xt^n_p(\mathscr{E}, K_X(-n_1)) \to \mathscr{E}xt^n_p(\mathscr{E}, \mathscr{E} \otimes K_X) \to 0,$$

by the usual arguments. The last two terms are $p_*(\mathscr{E}(n_1)) \otimes p_*(\mathscr{E}(n_1))^*$ and $\mathcal{O}_{\mathcal{M}}$, with the trace map between them (the adjoint of the identity map). Thus we have

$$0 \to \Omega_{\mathcal{M}} \to \Omega_{\mathrm{Quot}} \to \mathscr{E}nd_0(p_*\mathscr{E}(n_1)) \to 0,$$

induced of course by the action of $SL(H^0(\mathscr{E}(n_1)))$ on Quot.

Fit this into the sequence resolving Ω_{Quot}:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \Omega_{\mathcal{M}} & & \\ & & & & & & \downarrow & & \\ 0 \to & \mathscr{E}xt^{n-2}_p(\mathscr{E}, \mathscr{E} \otimes K_X) & \to & \mathcal{O}(E_2^*) & \to & \Omega_{\mathrm{Gr}} & \to \ \Omega_{\mathrm{Quot}} & \to 0, & \\ & & & & & & \searrow \ \downarrow & & \\ & & & & & & \mathscr{E}nd_0 & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

where $\mathscr{E}nd_0$ is $\mathscr{E}nd_0\,(p_*\mathscr{E}(n_1))$. Taking the kernels of the two surjections to $\mathscr{E}nd_0$ gives the result; the kernel of $\Omega_{\mathrm{Gr}} \to \mathscr{E}nd_0$ is locally free as it is the kernel subbundle of a vector bundle surjection. $\square$

3.5 Counting sheaves

So finally we can make a definition of the "number of sheaves" on a Calabi-Yau 3-fold X with $h^{0,1}(X) = 0$.

Definition 3.5.1 *Fix a Calabi-Yau 3-fold X with $h^{0,1}(X) = 0$. The "number of sheaves" on X, of a fixed Hilbert polynomial P and with rank and degree coprime, is defined as follows. X is algebraic and the moduli space $\mathcal{M}_P$ of such* stable *sheaves is a projective scheme. The Hilbert polynomial fixes c_1 and so $\det \mathscr{E}$, since $h^{0,1}(X) = 0 = h^{0,2}(X)$. Thus also $\operatorname{Ext}^i(\mathscr{E}, \mathscr{E})$, $i = 1, 2$, do not differ from their trace-free counterparts. $\operatorname{Ext}^3(\mathscr{E}, \mathscr{E}) \cong \mathbb{C}\ \forall \mathscr{E}$ by stability, so we may apply Theorems 3.4.6 and 3.3.4 to obtain a virtual moduli cycle $Z \in A_0(\mathcal{M}_P)$. Then our number $\lambda(\mathcal{M}_P)$ is the length of the zero-dimensional projective scheme Z.*

Remark. The cycle Z has the correct dimension (i.e. the virtual dimension, zero) by adding up the ranks of the groups in the sequence of (3.4.5).

To define a holomorphic Casson invariant we would like to count less non-locally-free sheaves, and this will be tackled presently, but first we discuss deformation invariance (without proving anything).

Simpson's construction extends to the case of a flat family $X \to S$ of smooth varieties X_s over a base S, giving a relative moduli space $\mathcal{M}_S \to S$ with a morphism to the base and fibres $\mathcal{M}_s$ the individual moduli spaces associated to the varieties X_s. Thus as we vary the complex structure on X_s the moduli spaces also vary in an algebraic family.

We can study the deformation theory of $\mathcal{M}_S$, giving, at the level of tangent spaces (there is also a version for cotangent sheaves), a sequence

$$0 \to T_{\mathscr{E}}\mathcal{M}_s \to T_{\mathscr{E}}\mathcal{M}_S \to T_s S \to \operatorname{Ext}^2_{X_s}(\mathscr{E}, \mathscr{E}), \tag{3.5.2}$$

where $\mathscr{E}$ is a stable sheaf on X_s, defining a point in $\mathcal{M}_s \subset \mathcal{M}_S$. The cokernel of the last map is the obstruction space of $\mathcal{M}_S$.

Again due to stability the group action is easily dealt with (as in the last section the PSL action is free) and so we may as well work with the relative Quot scheme over S, embedded in a relative Grassmannian $\mathrm{Gr} \times S \xrightarrow{\pi} S$. We would expect a two step resolution of the cotangent sheaf of Quot of virtual dimension $vd(\mathcal{M}_S) = \dim S + vd(\mathcal{M}_s)$ to give a virtual moduli cycle which is an algebraic family of the virtual moduli cycles on the fibres. That is, in the case of sheaves on a Calabi-Yau 3-fold, we expect a cycle of dimension $\dim S$ with a proper morphism

π to the base S. This would then have finite fibres *all of the same length as schemes* giving the same "number of sheaves" on each fibre.

Li and Tian [LT] show this is precisely what happens if we can find such a resolution (3.3.3) (for the relative Quot scheme in our case), with a surjective map from E_1 to TS, or more correctly an injection

$$0 \to \pi^*\Omega_S \to \mathcal{O}(E_1^*). \tag{3.5.3}$$

This condition, put crudely, ensures the moduli space sits over the base and not in a fibre, and that the fibrewise virtual moduli cycles are the correct algebraic intersection of the global virtual cycle with the fibres. If such a resolution exists then it gives a resolution on each fibre by replacing E_1 by $\ker(E_1 \to TS)$, and it is proved in [LT] that the resulting virtual moduli cycles are compatible.

In our case it is natural to again try to cut out the Quot scheme in $\mathrm{Gr}\times S$, thus taking $E_1 = T\,\mathrm{Gr} \oplus TS$ and the same E_2. This clearly gives us the required surjection $E_1 \to TS$; all we need is to be able to extend the map $T\,\mathrm{Gr} \to E_2$ of Theorem 3.4.3 by a map $TS \to E_2$ that gives the right resolution.

We already have the map $TS \to \mathrm{Ext}^2_{X_s}(\mathcal{E},\mathcal{E})$ in the sequence (3.5.2) above, giving the obstruction to extending a sheaf $\mathcal{E}$ on X_s to one on an infinitesimally close fibre. This is obtained by taking the cup product

$$H^1(TX_s) \times \mathrm{Ext}^1_{X_s}(\mathcal{E},\mathcal{E}\otimes\Omega_{X_s}) \to \mathrm{Ext}^2_{X_s}(\mathcal{E},\mathcal{E})$$

of the Kodaira-Spencer element in $H^1(TX_s)$, representing the deformation of X_s along a particular tangent vector in T_sS, with the Atiyah class [HL] of the sheaf $\mathcal{E}$ in $\mathrm{Ext}^1_{X_s}(\mathcal{E},\mathcal{E}\otimes\Omega_{X_s})$ (for a locally free sheaf this is just the class of the curvature $[F_A^{1,1}] \in H^1(\mathrm{End}\,E \otimes T^*X_s)$ of any compatible connection on the vector bundle E).

By Theorem (3.4.3), E_2 surjects onto $\mathrm{Ext}^2_{X_s}(\mathcal{E},\mathcal{E})$, so what we require is the correct lift of $T_sS \to \mathrm{Ext}^2_{X_s}(\mathcal{E},\mathcal{E})$ to $T_sS \to E_2$. So far I have been unable to find this, however.

An alternative procedure would be to consider the following moduli space. We can consider a sheaf $\mathcal{E}$ on a fibre X_s to be a torsion sheaf on X_S by pushing it forward by the inclusion $i = i_s$ to $i_*\mathcal{E}$. The moduli space $\mathcal{M}_S$ of stable sheaves on X_S of the same Hilbert polynomial contains each moduli space $\mathcal{M}_s$ in this way, and deformation theory shows that this is actually all of (a component of) $\mathcal{M}_S$. If we could control $\mathrm{Ext}^3(i_*\mathcal{E}, i_*\mathcal{E})$ and $\mathrm{Ext}^4(i_*\mathcal{E}, i_*\mathcal{E})$ of such torsion sheaves we could apply all of the same machinery as before to get a resolution of $\Omega_{\mathcal{M}_S}$ and a virtual moduli cycle. The map from (the component of) $\mathcal{M}_S$ to S (sending a sheaf to the fibre it is supported on) induces an injection (3.5.3) which would prove deformation of the cycles.

Removing the boundary contribution

In this section we will assume we have the right definition of counting sheaves, i.e. that we can prove deformation invariance. What we would now like to do, to give a definition of a holomorphic Casson invariant of a Calabi-Yau 3-fold X, is subtract a correction term that gives the number of sheaves which are not locally free.

We make a start by showing what the contribution is from *rank two* sheaves with only certain codimension three singularities (coming back to singularities along curves later). If a rank two torsion-free sheaf contains an ideal sheaf $\mathscr{I}_p$ of points (or zero dimensional subschemes) p we say it has a singularity of type A at p, which is removed on passing to the double dual (or reflexive hull) $\widehat{\mathscr{E}}$ of $\mathscr{E}$, so that locally $\mathscr{E}$ is the kernel of an evaluation of a locally free sheaf $\widehat{\mathscr{E}}$ at p:

$$\text{(A)} \qquad\qquad 0 \to \mathscr{E} \to \widehat{\mathscr{E}} \to \mathcal{O}_p \to 0.$$

A singularity of type B is, locally, the cokernel of a section, vanishing only at p, of a rank 3 bundle F,

$$\text{(B)} \qquad\qquad 0 \to \mathcal{O} \to \mathscr{F} \to \mathscr{E} \to 0,$$

and at such a point the sheaf is still reflexive. Each A-singularity contributes -2 to $c_3(\mathscr{E})$, while B contributes $+1$. In particular a *reflexive* sheaf with only these singularities has positive third Chern class and is locally free if and only if $c_3 = 0$.

Denote by $\mathcal{M}(a, b)$ the moduli of sheaves with a singularities of type A, b of type B and Hilbert polynomial modified by changing c_3 from zero (which it is for locally free sheaves of rank two) to $b - 2a$. Then we would like to subtract from our total $\lambda(\mathcal{M})$ of all sheaves the sum of some contributions $\lambda(\mathcal{M}(a, b))$ from those $\mathcal{M}(a, b) \subset \mathcal{M}$ with $c_3 = b - 2a = 0$.

Our procedure for studying $\mathcal{M}(a, b)$ and defining the number $\lambda(\mathcal{M}(a, b))$ is to pass to double duals, giving a morphism

$$\mathcal{M}(a, b) \longrightarrow \mathcal{M}(0, b). \qquad (3.5.4)$$

(For rank and degree coprime sheaves Gieseker stability is equivalent to slope stability. Modifying a sheaf in codimension greater than one does not affect slope stability since we can also modify any subsheaves in codimension greater than one without changing their slope.) Now $\mathcal{M}(0, b)$ is a component of another moduli space (of sheaves with $c_3 = b$) which has other components with $a \neq 0$. So assuming we know how to pass from a number for $\mathcal{M}(0, b)$ back to one for $\mathcal{M}(a, b)$, we would like to be able to define $\lambda(\mathcal{M}(0, b))$ by using the "number of sheaves" in this higher moduli space and again subtracting a correction term. So we have to repeat the process, which fortunately terminates since the third Chern class c_3 of a stable sheaf is bounded by its c_1 and c_2, essentially by the boundedness of spaces of stable sheaves.

So we have an inductive process. Consider first $\mathcal{M}(0,B)$, where B is the maximum value of c_3 for our fixed c_1 and c_2. This is the whole of the moduli space of stable sheaves of this Hilbert polynomial, because if there were any sheaves with type A singularities their double duals would have higher c_3, a contradiction. Thus Definition 3.5.1 assigns a number $\lambda(\mathcal{M}(0,B))$. Similarly for $\mathcal{M}(0,B-1)$ (recall that removing an A-singularity by passing to the double dual adds 2 to c_3). Next the moduli space $\mathcal{M}_{B-2}$ for $c_3 = B-2$ consists of $\mathcal{M}(0,B-2)$ and $\mathcal{M}(1,B)$. Passing to the double dual gives us a map $\mathcal{M}(1,B) \to \mathcal{M}(0,B)$ so if we can understand the fibres of this map, and so how to get a number $\lambda(\mathcal{M}(1,B))$ from $\lambda(\mathcal{M}(0,B))$, this will also allow us to define

$$\lambda(\mathcal{M}(0,B-2)) = \lambda(\mathcal{M}_{B-2}) - \lambda(\mathcal{M}(1,B)).$$

In this way we can work down to $c_3 = 0$ and $\mathcal{M}(0,0)$, which is what we are interested in (although we have simplified things by ignoring codimension-two singularities for now). The contribution, discussed below, of the fibres of the maps (3.5.4) is deformation invariant (in fact it will be topological), so assuming deformation invariance of Definition (3.5.1) we obtain a deformation invariant quantity counting sheaves in $\mathcal{M}(0,0)$.

Therefore we now turn to the fibres of (3.5.4). For simplicity we take $a = 1 = b$ – there will be an obvious generalisation of what follows to higher numbers of singularities, replacing copies of X by $\mathrm{Hilb}^a(X)$, etc. So we would like to compare $\mathcal{M}(1,1)$ with $\mathcal{M}(0,1)$ via the double-dual map between them.

When the A-singularity (at p say) is away from the B-singularity, we have a sequence

$$0 \to \mathscr{E} \to \widehat{\mathscr{E}} \to \mathcal{O}_p \to 0, \tag{3.5.5}$$

where $\widehat{\mathscr{E}}$ is free around p. Applying $\mathrm{Hom}\,(\,\cdot\,,\mathscr{E})$ yields

$$0 \to \mathrm{Hom}\,(\mathscr{E},\mathscr{E}) \to H^0(\mathscr{E}xt^1(\mathcal{O}_p,\mathscr{E})) \to \mathrm{Ext}^1(\widehat{\mathscr{E}},\mathscr{E})$$
$$\to \mathrm{Ext}^1(\mathscr{E},\mathscr{E}) \to H^0(\mathscr{E}xt^2(\mathcal{O}_p,\mathscr{E})) \to \mathrm{Ext}^2(\widehat{\mathscr{E}},\mathscr{E}),$$

while applying $\mathscr{H}om(\mathcal{O}_p,\,\cdot\,)$ gives

$$\mathcal{O}_p \cong \mathscr{E}xt^1(\mathcal{O}_p,\mathscr{E}),$$

by the freeness of $\widehat{\mathscr{E}}$ at p, and

$$T_pX \cong \mathscr{E}xt^1(\mathcal{O}_p, \mathcal{O}_p) \cong \mathscr{E}xt^2(\mathcal{O}_p, \mathscr{E}).$$

(Here we have computed $\mathscr{E}xt^1(\mathcal{O}_p, \mathcal{O}_p)$ explicitly by its Koszul resolution; the result is that the deformations of $\mathcal{O}_p$ simply come from moving p.) By similar arguments $\mathrm{Ext}^2(\widehat{\mathscr{E}}, \mathscr{E}) \cong \mathrm{Ext}^2(\widehat{\mathscr{E}}, \widehat{\mathscr{E}})$ and we obtain

$$0 \to \mathrm{Ext}^1(\widehat{\mathscr{E}}, \mathscr{E}) \to \mathrm{Ext}^1(\mathscr{E}, \mathscr{E}) \to T_pX \to \mathrm{Ext}^2(\mathscr{E}, \mathscr{E}).$$

It is quite easy to see that the obstruction map $T_pX \to \mathrm{Ext}^2(\mathscr{E}, \mathscr{E})$ is zero, and that there is a map $T_pX \to \mathrm{Ext}^1(\mathscr{E}, \mathscr{E})$ splitting the sequence, as moving p gives deformations of E.

Applying $\mathrm{Hom}\,(\widehat{\mathscr{E}}, \cdot\,)$ to (3.5.5) gives

$$0 \to \mathrm{Hom}\,(\widehat{\mathscr{E}}, \widehat{\mathscr{E}}\,) \to \mathrm{Hom}\,(\widehat{\mathscr{E}}, \mathcal{O}_p) \to \mathrm{Ext}^1(\widehat{\mathscr{E}}, \mathscr{E}) \to \mathrm{Ext}^1(\widehat{\mathscr{E}}, \widehat{\mathscr{E}}\,) \to 0.$$

Using the stability of $\widehat{\mathscr{E}}$ and analysing the first map we see that its cokernel is the tangent space $T\,\mathbb{P}\,(\widehat{\mathscr{E}}_p^*)$ to the space of quotients $\widehat{\mathscr{E}} \to \mathcal{O}_p$ of the fibre of $\widehat{\mathscr{E}}$ at p. So we are left with

$$\mathrm{Ext}^1(\mathscr{E}, \mathscr{E}) \cong \mathrm{Ext}^1(\widehat{\mathscr{E}}, \widehat{\mathscr{E}}\,) \oplus T_pX \oplus T\,\mathbb{P}\,(\widehat{\mathscr{E}}_p^*).$$

Thus deformations of E come from those of $\widehat{\mathscr{E}}$, those of p, and those of the quotient of a *fixed* fibre $\widehat{\mathscr{E}}_p$. Also they are all unobstructed, deforming to the other kernels of quotients

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathcal{O}_p \to 0$$

of a fixed bundle $\mathscr{F} = \widehat{\mathscr{E}}$, parameterised by the projective bundle $\mathbb{P}\,(\mathscr{F}^*)$ over X away from the singularities of $\mathscr{F}$.

We now want to deal with the singular points of $\mathscr{F}$. So, changing notation, we would like to know what the $\mathcal{O}_p$-quotients of a reflexive sheaf $\mathscr{E}$ look like as we approach a singular point q of $\mathscr{E}$, where it has the presentation

$$0 \to \mathcal{O} \xrightarrow{s} \mathscr{F} \to \mathscr{E} \to 0,$$

with s a section of a rank 3 vector bundle F with zero locus q. We have the sequence

$$0 \to \mathscr{E}_p^* \to \mathscr{F}_p^* \to \mathcal{O}_p^*,$$

with the last map the transpose s^t of the section at p. Away from q this gives us the picture as before with a $\mathbb{CP}^1$ of quotients, but at q we get a $\mathbb{CP}^2$ of quotients because s^t vanishes. The local model for the resulting space of $\mathscr{E}$'s is

$$\{ (x, f) \in \mathbb{C}^3 \times (\mathbb{CP}^2)^* \, : \, f(x) = 0 \},$$

with its projection to $\mathbb{C}^3$. This is the total space of the tautological 2-plane bundle over the Grassmannian $\mathrm{Gr}\,(2,3) = (\mathbb{CP}^2)^*$ of 2-planes in $\mathbb{C}^3$. Topologically this is what a neighbourhood of the fibre over $q \in X$ of the space of such sheaves looks like. (This is analogous to a neighbourhood of the exceptional divisor in a blow-up looking like the tautological bundle over the divisor.)

Thus we get a $\mathbb{P}\,(T_q^*X)$ fibre over q (where ds canonically identifies F_q with T_qX) glued into the $\mathbb{CP}^1$-bundle $\mathbb{P}\,(E^*)$ over $X \setminus \{q\}$ by this construction. The resulting 4-fold is the fibre of the map $g = \widehat{\ } : \mathcal{M}(1,1) \to \mathcal{M}(0,1)$ and it is smooth with topological type independent of the point of $\mathcal{M}$. The deformation theory of the $\mathscr{E}$'s in $\mathcal{M}(1,1)$ is that of their double duals $\widehat{\mathscr{E}} \in \mathcal{M}(0,1)$ but with the tangents TF to the fibres added to E_1 and T^*F to E_2, in the language of the last section. Thus the cone C in $E_2 \to \mathcal{M}(0,1)$ that Li and Tian construct is replaced by $g^*C \subset E_2 \oplus T^*F \to \mathcal{M}(1,1)$, with T^*F the relative cotangent bundle of g.

To intersect g^*C with the zero section $\mathcal{M}(1,1)$ of $E_2 \oplus T^*F$ we may first intersect inside E_2, then intersect with the zero section of T^*F. The first operation simply gives g^*Z, where Z is the virtual moduli cycle of $\mathcal{M}(0,1)$, i.e. a finite number $\lambda(\mathcal{M}(0,1))$ of points. So we are left with intersecting $\lambda(\mathcal{M}(0,1))$ fibres F with the zero section in T^*F, giving finally

$$\lambda(\mathcal{M}(1,1)) = \lambda(\mathcal{M}(0,1))\,.\,\chi(F),$$

where F is a topological model of any 4-fold fibre.

It is clear there is some more work to be done here, not least to take account of sheaves $\mathscr{E}$ with codimension-two singularities. For such sheaves we can carry over the above method, modifying in codimension three as before (but leaving the curve singularity in place) and studying the fibres of the induced maps $\mathscr{E} \mapsto \widehat{\mathscr{E}}$ on moduli spaces. These will now have a modification (in fact a blow-up) along the curve (where there are more $\mathcal{O}_p$-quotients), but again we have a fixed topological model for fixed $\widehat{\mathscr{E}}$ and fixed singularity C. We then have to sum over all curves C, but relating the deformation theories of $\mathscr{E}$ and C we see we get a Gromov-Witten theory count of curves arising, and this too is deformation invariant.

What is unlikely, however, is that we could remove the contribution from sheaves with codimension-two singularities. Studying these as above would correspond to looking at sequences such as

$$0 \to \mathscr{E} \to \widehat{\mathscr{E}} \to \mathscr{L} \to 0,$$

where $\mathscr{L}$ is some torsion free (and so locally free) sheaf supported on a curve. While numbers of curves (in the sense of Gromov-Witten theory) are deformation invariant, the space of line bundles $\mathscr{L}$ on a curve, and the space of quotients $\widehat{\mathscr{E}} \to \mathscr{L}$, seem unlikely to be deformation invariant.

This is perhaps unsurprising – we know from Donaldson theory that in two complex dimensions codimension-two singularities of sheaves (namely ideal sheaves of points) correspond to limits of stable bundles (in the sense of bubbling in the ASD moduli space). Thus they should not be removed from the moduli space, and under deformation of the underlying complex structure we can expect bundles to wander off to infinity and acquire such codimension-two singularities.

Appendix A

Some Path Integral Invariants

A.1 The path integral

In this appendix some physics is outlined that motivates some mathematical definitions and further work. Due to the appearance of [FKT] I have not pursued the mathematics to completion, so little of what is here is rigorous.

Another traditional piece of low dimensional topology is knot theory. While it might seem out of reach of our naive "wedge with θ" programme, Witten's derivation of the Jones polynomial ([W1, At4]) gives a more analytical approach. We already have the analogue (1.3.1) of the Chern-Simons functional, so by analogy with [W1] we can try to define invariants of a bare Calabi-Yau 3-fold by evaluating the path integral

$$Z_k = \int_{\mathscr{B}} \mathscr{D}A \; e^{ikCS(A)}, \tag{A.1.1}$$

over the space $\mathscr{B}$ of $\bar{\partial}$-operators on a bundle $E \to X$. This computes the "partition function" Z_k of the gauge field theory associated to the Lagrangian kCS, in fact already suggested in [W2].

In [W1], Wilson loops (the product of traces, in some representation, of the holonomy around non-intersecting loops γ_i) are included in the path integral as "observables" to give a "partition function" that, if mathematical sense could be made of it, should be a topological invariant. Isotoping the loops without crossings gives a system $(M, \cup\gamma_i)$ diffeomorphic to the original one, so with Wilson loops the path integral ought to give knot and link invariants. Below, in Section 3.5, we shall add in to the path integral (A.1.1) the Calabi-Yau analogues of knots using complex tori with trivialising $(1,0)$-forms or, more generally, any complex curve with a holomorphic one-form.

To make rigorous mathematical sense of this, however, requires another formulation, either in terms of cut and paste operations (as Witten gave), quantum groups, or the perturbative, Feynman diagram evaluation about $k \to \infty$. This is the approach we shall take, formally manipulating the integral as if it were over a domain in $\mathbb{R}^N$ (as $N \to \infty$) instead of over the infinite dimensional space of connections $\mathscr{B}$, and using the result to define our invariants. We shall ignore the problems of the ambiguities in CS (which are avoided in the 3-manifold case by taking $k \in \mathbb{Z}$, since the Chern-Simons functional is well defined mod $\mathbb{Z}$, but cannot be overcome by any choice of k if the periods of θ are dense), since our final invariants will be free of this ambiguity. Similarly we shall ignore the fact that we have a complex valued Lagrangian, something unusual in physics, since the formal regularisation that makes sense of integrals such as

$$\int_{\mathbb{R}} e^{ix^2} dx = e^{i\frac{\pi}{4}}\sqrt{\pi},$$

also makes sense of the wildly divergent

$$\int_{\mathbb{R}} e^{x^2} dx = -i\sqrt{\pi},$$

all be it as a complex number! We shall also give the same value to integrals over a complex parameter z, that is

$$\int_{\mathbb{C}} e^{\lambda z^2} dz := \sqrt{\frac{\pi}{\lambda}},$$

without worrying about its physical meaning as we are just interested in extracting complex invariants of the Calabi-Yau. Here dz is meant in the physicists' sense, i.e. as $dx\,dy$, and performing the above integral over x for any fixed y gives this answer (since there are no residues); the infinite constant arising from integrating over y can then be renormalised to 1. I do not know what Witten intended when he mentioned such a theory in [W2], perhaps not this.

Thus we should, in the first instance, get invariants of the complex structure of a Calabi-Yau 3-fold X (and of the bundle E) from (A.1.1). There are however complications in the 3-manifold case that reappear here. To integrate over $\mathscr{B}$ the usual procedure is to gauge fix and integrate over $\mathscr{A}$. The choice of gauge involves picking a metric, and the evaluation can then pick up an anomalous metric dependence, which is removed in [W1] by a local counter-term.

The chapter will be brief as this work has not yet reached any satisfactory conclusion; the only part that has so far given a concrete result – the holomorphic linking number of Section A.5 – has now been written up more professionally elsewhere [FKT]. So what follows can be thought of as an overview of possibilities for future research.

A.2 Perturbation theory

To begin with we shall consider the bare path integral without any analogues of Wilson loops (links), which will be introduced later. We will formally evaluate the path integral perturbatively by the stationary phase approximation (see AGV for the result in finite dimensions) about the critical points $\{A_0 : F^{0,2}_{A_0} = 0\}$ of CS, which we will have to assume are isolated (we will comment on this later). Good references for perturbation theory are [Ram, Ax, BN, FG]. Taylor expanding about a critical point and integrating gives a non-convergent asymptotic expansion in powers of k^{-1}, which stationary phase tells us is in fact *not* asymptotic to the integral; we must add the approximations to the integral from *all* critical points to get the correct asymptotic expansion. The resulting coefficients will be our invariants.

To integrate (A.1.1) over $\mathscr{A}/\mathscr{G}$ the standard physics procedure is to integrate over a section of the $\mathscr{A} \to \mathscr{A}/\mathscr{G}$ bundle, multiplying by a Jacobian factor to take account of the "angle" at which the section meets each $\mathscr{G}$ orbit. Although there is no global section of $\mathscr{A}$ ("the Gribov ambiguity") we work locally about some holomorphic connection A_0 since any two sections should give the same result. That the result does sometimes depend on the gauge fixing method (e.g. the metric) is known as an anomaly, although in our case the metric dependence may not be so critical as an invariant of the Kähler metric and complex structure could still be valuable.

The idea, then, is to convert the integral over $\mathscr{B}$ to one over the affine space $\mathscr{A}$ which is much easier to deal with. It should be noted that all of what follows would be rigorously correct for an integral on the smooth quotient of a finite dimensional manifold by a group action; we simply "regularise" the obvious generalisation to infinite dimensions and take this as a definition. Anyone unfamiliar with functional integrals might want to skip straight to the answer.

So we choose a metric on E which with the Kähler metric on X allows us to work in Coulomb gauge $\bar\partial^*_{A_0} a = \bar\partial^*_{A_0}(A - A_0) = 0$, at least for small a so that a version of Uhlenbeck's theorem applies. (Remember that by the stationary phase approximation we only need the Taylor expansion of CS about A_0; we sum over all critical points A_0 later.) Then

$$\int_{\mathscr{B}=\mathscr{A}/\mathscr{G}} \mathscr{D}A\ e^{ikCS(A)} =$$

$$\int_{[A]\in\mathscr{A}/\mathscr{G}} e^{ikCS(A)} \left[\int_{g\in\mathscr{G}} \delta\Big(\frac{k}{8\pi^2}\bar{\partial}^*_{A_0}(g(A)-A_0)\Big)\,\mathrm{Det}\Big(\frac{k}{8\pi^2}\bar{\partial}^*_{A_0}\bar{\partial}_A\Big)\right],$$

by integrating the Dirac delta, and making some sort of regularised sense of the determinant of $\bar{\partial}^*_{A_0}\bar{\partial}_A$, via [RS1]. Here A is the representative $A = A_0 + a$ of $[A]$ with $\bar{\partial}^*_{A_0}a = 0$. Giving $\delta\Big(\frac{k}{8\pi^2}\bar{\partial}^*_{A_0}(g(A)-A_0)\Big)$ its Fourier expansion

$$\int_{\phi\in\Omega^{0,3}(\mathfrak{g}_E)} \mathscr{D}\phi\ \exp\left(i\frac{k}{8\pi^2}\int_X \mathrm{tr}\,\phi\,\bar{\partial}^*_{A_0}(g(A)-A_0)\wedge\theta\right),$$

gives the value

$$\int_{\mathscr{G}} \mathscr{D}g \int_{A\in A_0+\ker\bar{\partial}^*_{A_0}} \mathscr{D}A\ e^{ikCS(A)}\,\mathrm{Det}\Big(\frac{k}{8\pi^2}\bar{\partial}^*_{A_0}\bar{\partial}_A\Big)\int_{\phi\in\Omega^{0,3}(\mathfrak{g}_E)} \mathscr{D}\phi\, e^{i\frac{k}{8\pi^2}\int_X \mathrm{tr}\,\phi\,\bar{\partial}^*_{A_0}(g(A)-A_0)\wedge\theta}$$

$$= \int_{A\in\mathscr{A}} \mathscr{D}A\ e^{ikCS(A)+i\frac{k}{8\pi^2}\int_X \mathrm{tr}\,\phi\,\bar{\partial}^*_{A_0}a\wedge\theta}\ \mathrm{Det}\Big(\frac{k}{8\pi^2}\bar{\partial}^*_{A_0}\bar{\partial}_A\Big),$$

to $\int_{\mathscr{B}}\mathscr{D}A\ e^{ikCS(A)}$. We can lift the determinant to the exponent via noncommutative, or Fermionic, integration, of which there is a very brief and elegant account in [MQ] pp 86–88. (This is just a slick way of Taylor expanding the determinant in powers of a too, and for the first order term used in Proposition A.3.2 we do not need it and can approximate $\mathrm{Det}(\bar{\partial}^*_{A_0}\bar{\partial}_A)$ by $\mathrm{Det}(\bar{\partial}^*_A\bar{\partial}_A)$.) This yields

$$\mathrm{Det}\Big(\frac{k}{8\pi^2}\bar{\partial}^*_{A_0}\bar{\partial}_A\Big) = \int \mathscr{D}c\mathscr{D}\bar{c}\ e^{i\frac{k}{8\pi^2}\int_X \mathrm{tr}\,\bar{c}\,\bar{\partial}^*_{A_0}\bar{\partial}_A c\wedge\theta}.$$

A short calculation shows that, for $A = A_0 + a$ and A_0 holomorphic,

$$CS(A) = CS(A_0) + \frac{1}{4\pi^2}\int_X \mathrm{tr}\left(\frac{1}{2}a\wedge\bar{\partial}_{A_0}a + \frac{1}{3}a\wedge a\wedge a\right)\wedge\theta,$$

leaving us with

$$\int_{\mathscr{B}} \mathscr{D}A\ e^{ikCS(A)} = e^{ikCS(A_0)}\int \mathscr{D}a\mathscr{D}c\mathscr{D}\bar{c}\mathscr{D}\phi\ e^{iL},$$

where

$$L = \frac{k}{8\pi^2}\int_X \mathrm{tr}\left(\frac{1}{2}a\wedge\bar{\partial}_{A_0}a + \frac{1}{3}a\wedge a\wedge a + \phi\,\bar{\partial}^*_{A_0}a + \bar{c}\,\bar{\partial}^*_{A_0}\bar{\partial}_A c\right)\wedge\theta,$$

for $c \in \Omega^0(\mathfrak{g}_E),\ a \in \Omega^{0,1}(\mathfrak{g}_E),\ \bar{c}, \phi \in \Omega^{0,3}(\mathfrak{g}_E)$.

We now Taylor expand in powers of a about A_0, sum over the (isolated) critical points A_0, and rescale all variables by $\frac{\sqrt{8}\pi}{\sqrt{k}}$, giving the asymptotic expansion

$$Z_k = \sum_{A_0 : F^{0,2}_{A_0}=0} e^{ikCS(A_0)} \int \mathscr{D}(a, \phi, c, \bar{c}) \left[e^{i\int_X \operatorname{tr}\left(a \wedge \bar{\partial}_{A_0} a + \phi\, \bar{\partial}^*_{A_0} a + \bar{c}\, \bar{\partial}^*_{A_0} \bar{\partial}_{A_0} c\right) \wedge \theta} \right.$$
$$\left. \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(2k)^n} \left(\int_X \operatorname{tr}\left(\frac{1}{3} a \wedge a \wedge a + \bar{c} \wedge [a, c] \right) \wedge \theta \right)^{2n} \right] \quad \text{(A.2.1)}$$

in powers of k^{-1}. We have ignored odd powers of the cubic term which should, by symmetry, integrate to zero, and we have omitted the normalised constant $(\frac{1}{\sqrt{k}})^\infty$ which arises from the rescaling.

A.3 Holomorphic torsion and the semiclassical approximation

We are now left with an integral over an affine space, so we can proceed by analogy with finite dimensional integrals. Just as $\int e^{-\lambda x^2}\, dx = \sqrt{\frac{\pi}{\lambda}}$, and

$$\int e^{-\sum \lambda_i x_i^2}\, dx_1 \ldots dx_N = \frac{\pi^{N/2}}{\left(\prod_j \lambda_j\right)^{1/2}},$$

we can analytically continue to define

$$\int e^{ix^t \Lambda x}\, d^N x = (\det i\Lambda)^{-1/2}, \quad \text{(A.3.1)}$$

where $d^N x$ is the normalised measure $\pi^{-N/2} dx_1 \ldots dx_N$. Thus for a similar integral over an infinite dimensional space, with Λ say an elliptic operator on sections of vector bundles (the space of which we integrate over) we simply *define* the integral to have absolute value $\operatorname{Det}(\Lambda^*\Lambda)^{-1/4}$, where the determinant Det is regularised by the zeta function methods of [RS1]. (The phase can be trickier, and is discussed below.)

Proposition A.3.2 *The semiclassical contribution of an* acyclic *holomorphic connection A_0 to the partition function Z_k has absolute value $e^{ikCS(A_0)}\sqrt{\tau(A_0)}$, where $\tau(A_0)$ is the Ray-Singer holomorphic torsion [RS2] of the holomorphic bundle $(\mathfrak{g}_E, A_0)$.*

Remarks.

1. The semiclassical approximation is the term of order k^0 in the expansion (A.2.1), so called because it is concentrated about the classical solutions of the field theory, i.e. the critical points of the Lagrangian, the holomorphic connections in this case.

2. This is our first path integral invariant of a Calabi-Yau. It is, of course, entirely analogous to the R-torsion [RS1] that arises as the semiclassical approximation to Chern-Simons theory on a 3-manifold ([W1]).

3. "Acyclic" refers to the Dolbeault complex of the induced holomorphic structure on the adjoint bundle $\mathfrak{g}_E$. It is clear why we want to consider this case only, to avoid having to make sense of integrating an integrand which is constant (to second order at least) in $h^{0,1}(\mathfrak{g}_E)$ directions, leading formally to zero determinants which cannot be inverted. It is a natural condition, however, being equivalent to $[A_0]$ representing a smooth point of the moduli space of bundles with discrete isotropy group (we earlier tacitly assumed there was a *unique* Coulomb gauge; if $H^0(\mathfrak{g}_E) \neq 0$ we would have to further divide by a large isotropy group, as it is we have only a finite ambiguity). Some progress has been made in the non-acyclic case on 3-manifolds in [Ax, FG].

4. If e^{ikCS} is single-valued for some k, or if we can regularise the sum over $H_3(X;\mathbb{Z})$ of all the different values of e^{ikCS}, we get a manifold invariant by summing the contributions from all holomorphic connections (assuming they are isolated and finite). This gives us the leading order term in (A.2.1), and such a procedure applies to the higher order invariants too. Otherwise we can ignore the $e^{ikCS(A_0)}$ term to obtain an invariant of X and the holomorphic structure A_0 on E.

Proof. We wish to regularise the Gaussian integral

$$\int \mathscr{D}(a,\phi,c,\bar{c}) \;\; \exp\left(i\int_X \operatorname{tr}\left(a\wedge\bar{\partial}_{A_0}a + \phi\,\bar{\partial}^*_{A_0}\bar{\partial}_{A_0}c\right)\wedge\theta\right).$$

By the definition of the Fermionic integral, the integration over $(c,\bar{c})$ produces $\operatorname{Det}\left(i\,\bar{\partial}^*_{A_0}\bar{\partial}_{A_0}|_{\Omega^0(\mathfrak{g}_E)}\right)$, with absolute value $\operatorname{Det}\Delta^{0,0}_{A_0}$ (where $\Delta^{i,j}_{A_0} = \Delta_{A_0}|_{\Omega^{i,j}(\mathfrak{g}_E)}$). In fact we need not have bothered, at this stage, with Fermionic integration – all we have done is expanded the $\operatorname{Det} i\bar{\partial}^*_{A_0}\bar{\partial}_A$ term in powers of $a = A - A_0$ and taken the first term.

The other two terms may be written as

$$\int \mathscr{D}(a,\phi)\ \exp i\,\langle\, (a,\phi), \Lambda(a,\phi)\,\rangle, \tag{A.3.3}$$

where $\Lambda = \star\bar\partial_{A_0} + \bar\partial_{A_0}\star$ on $\Omega^{0,1}(\mathfrak{g}_E)\oplus\Omega^{0,3}(\mathfrak{g}_E)$ (and $\star = \bar{*}(\,\cdot\,\wedge\theta)$). Thus $\Lambda^*\Lambda = \Delta^{0,1}_{A_0}\oplus\Delta^{0,3}_{A_0}$, and the absolute value of (A.3.3) is

$$(\mathrm{Det}\,\Delta^{0,1}_{A_0}\,.\,\mathrm{Det}\,\Delta^{0,3}_{A_0})^{1/4}.$$

This gives the semiclassical approximation as

$$\frac{\mathrm{Det}\,\Delta^{0,0}_{A_0}}{(\mathrm{Det}\,\Delta^{0,1}_{A_0}\,.\,\mathrm{Det}\,\Delta^{0,3}_{A_0})^{1/4}},$$

which by Serre duality equals

$$\left(\frac{(\mathrm{Det}\,\Delta^{0,1}_{A_0})\,(\mathrm{Det}\,\Delta^{0,3}_{A_0})^3}{(\mathrm{Det}\,\Delta^{0,2}_{A_0})^2}\right)^{1/4},$$

the square root of the holomorphic torsion of [RS2]. □

Phase of the semiclassical approximation

The phase of the integral (A.3.1) is a little more delicate, but for Λ self-adjoint (which it is, notice, in our case) we can set it to be $e^{i\pi/4\,\sum_i \mathrm{sign}\,(\lambda_i)}$, where λ_i are the eigenvalues of Λ. This can be regularised in an infinite dimensional setting to $e^{i\pi/4\,\eta(\Lambda)}$, where $\eta(\Lambda)$ is the η-invariant of [APS]. In the 3-manifold theory this gives an anomalous metric dependence, since the η invariant of a connection varies with the η-invariant of the tangent bundle, whose mod $\mathbb{Z}$ reduction is essentially the Chern-Simons functional of the Levi-Civita connection. Thus (modulo a small problem of framing circumvented in [At3]) this dependency can be removed by adding a local counter-term $CS(\nabla_{LC})$ to the Lagrangian.

In fact,

$$\eta_{M=\partial N} = \sigma(N,M) - \frac{1}{12\pi^2}\int_N \mathrm{tr}\,F_{LC}\wedge F_{LC}, \tag{A.3.4}$$

where F_{LC} is the curvature of the Levi-Civita connection (i.e. the Riemannian curvature), and $\sigma(N,M)$ is the (integer-valued) signature of the intersection form on $H^2(N,M)$. (Also the metric is required to be a product in a neighbourhood of the boundary M.) Thus, on the one hand 3η reduces mod $\mathbb{Z}$ to $CS(\nabla_{LC})$, while on the other hand for a closed 4-manifold N,

$$\frac{1}{3}p_1(N) = \frac{1}{12\pi^2}\int_N \operatorname{tr} F_{LC} \wedge F_{LC} \tag{A.3.5}$$

is the signature $\sigma(N)$ of the intersection form on $H^2(N)$. This is also the index of the signature operator

$$L = d + d^* = d - *d* : \ \Omega^+ \to \Omega^-,$$

with kernel and cokernel maximal positive and negative definite subspaces of the intersection form.

In many ways a complex analogue of all of this holds, it is just that it is trivial since everything vanishes.

The kernel and cokernel of the "complex signature operator"

$$L = \bar\partial + \bar\partial^* = \bar\partial - \star\bar\partial\star : \ \Omega^{0,+} \to \Omega^{0,-}$$

are maximal positive and negative definite subspaces $\mathscr{H}^{\mp}$ of the "holomorphic intersection form"

$$\alpha, \beta \mapsto \int_Y \alpha \wedge \beta \wedge \theta$$

on $H^{0,2}$ of a Calabi-Yau 4-fold (Y, θ).

It even has signature given by the analogue of (A.3.5)

$$\frac{1}{3}p_1(Y) \,.\, \theta_Y = \frac{1}{12\pi^2}\int_Y \operatorname{tr} F_{LC}^{0,2} \wedge F_{LC}^{0,2} \wedge \theta_Y,$$

but only by virtue of both being trivially zero.

The complex analogue of a 4-manifold with boundary (N, M), a 4-fold Y with anticanonical section s defining $X \subset Y$, has a bilinear form analogous to that on $H_2(N, M)$:

$$\alpha, \beta \mapsto \int_Y \alpha \wedge \beta \wedge s^{-1},$$

for $\alpha, \beta \in \ker(H^2(\mathcal{O}_Y) \to H^2(\mathcal{O}_X))$, i.e. (0,2)-forms which restrict to zero on X (so we can divide by s) – just as we only consider forms vanishing on M in $H^2(N, M)$.

There is also the lifting (1.3.7) of the holomorphic Chern-Simons functional of a connection A on a bundle on X,

$$CS(A) = \frac{1}{4\pi^2}\int_Y \operatorname{tr} F_{\mathbb{A}}^{0,2} \wedge F_{\mathbb{A}}^{0,2} \wedge s^{-1},$$

but for the Levi-Civita connection on X and Y this formula, the analogue of (A.3.4), becomes trivial since X and Y have integrable complex structures with $F^{0,2}_{LC} = 0$.

So in fact the relevant η invariant is zero, and we have just described a very complicated way of seeing this. More simply, the spectrum of our operator $\Lambda = \star\bar\partial_{A_0} + \bar\partial_{A_0}\star$ is symmetric about zero since i switches eigenspaces of opposite eigenvalue, swapping $\mathscr{H}^+$ and $\mathscr{H}^-$. Thus $\eta(\Lambda) = 0$, and the phase of the semiclassical approximation to the path integral should be zero. As such it has no metric dependence, and therefore, by analogy with the 3-manifold theory [W1, AS2] (which is metric-independent once the anomalous metric-dependence of the phase of the semiclassical approximation is removed) this might suggest that the theory, were it finite to all orders, may need no corrections, and may have no metric dependence.

A.4 Manifold invariants

We now consider the higher order terms in the asymptotic expansion (A.2.1). The integral defining the coefficient of k^{-n} is an infinite dimensional version of

$$\int e^{ix^t\Lambda x}(\lambda_{ijk}x_ix_jx_k)^{2n}d^Nx \qquad \text{(using the summation convention)}$$

$$= \sum_I c_I\lambda^I \left(\text{coeff. of } J^I \text{ in } \int e^{ix^t\Lambda x + iJ^tx}d^Nx\right),$$

as can be seen by expanding e^{iJ^tx} as a power series. Here I is a set $\{(i_aj_ak_a)\}$ of $2n$ triples (ijk) of numbers between 1 and N, $\lambda^I = \prod_a \lambda_{i_aj_ak_a}$, $J^I = \prod_a J_{i_a}J_{j_a}J_{k_a}$, and c_I is a constant which we will come to presently.

Now completing the square in the exponent and integrating gives

$$\sum_I C_I\lambda^I \left(\text{coeff. of } J^I \text{ in } (\det i\Lambda)^{-1/2}e^{-\frac{i}{4}J^t\Lambda^{-1}J}\right).$$

The $(\det i\Lambda)^{-1/2}$ term is just the semiclassical approximation, so disregarding this we should get a new invariant by considering the infinite dimensional regularisation of

$$\sum_I C_I\lambda^I \left(\text{coeff. of } J^I \text{ in } (J^t\Lambda^{-1}J)^{3n}\right),$$

where we have expanded the exponential and taken out the relevant term.

Coefficients of J^I in $(J^t\Lambda^{-1}J)^{3n}$ are sums of products of Λ^{-1}_{ij}'s over partitions of the union of the elements of I into pairs. This leads quickly to the theory of Feynman diagrams [Ram, BN, Ax] with a graph for each index I with $2n$ trivalent vertices, one for each of the triples in I, and $3n$ edges connecting them corresponding to the partition into pairs. We then form the product of the λ_{ijk}'s labelling the vertices (the "cubic interaction") with the Λ_{ij}^{-1}'s labelling the edges. The constants then work out so that we take the sum over all such trivalent graphs weighted by the reciprocal of the number of automorphisms of the graph.

In infinite dimensions the vectors x and J are replaced by vectors in the infinite dimensional space $\Omega^{0,*}(\mathfrak{g}_E)$, and the coordinates x_i become the values of these forms at points of X. So the sums over indices I become integrals over $(X)^{2n}$ of $3n$ powers of the Green's function of the relevant operator $\Lambda = \star\bar\partial_{A_0} + \bar\partial_{A_0}\star$ contracted at the $2n$ points by the cubic interaction.

Following [AS1] we can simplify to the case of the cubic interaction being a $\operatorname{tr}\frac{1}{3}A\wedge A\wedge A\wedge\theta$ term, with quadratic term $\Lambda = \bar\partial_{A_0}\circ(\,\cdot\,\wedge\theta)$, by substituting $\Phi = \star\bar\partial_{A_0}\phi$, $\bar C = \star\bar\partial_{A_0}\bar c$, incorporating all the variables into one supervariable A and reducing the path integral to one over $\ker\bar\partial^*_{A_0}$. We do not have space to go into supermanifolds and BRST transformations so do not repeat the full details (which are explained in [AS1] in the entirely analogous 3-manifold theory), but taking the above finite dimensional analysis as motivation we can simply define invariants as below. First, though, we need some notation.

Let $\pi_i : X^{2n} = \prod_{k=1}^{2n} X_k \to X_i$ be the ith projection map, and $\pi_{ij} : X^{2n} \to X_i{\times}X_j$ be defined similarly. Let $\theta_i = \pi_i^*\theta \in \Omega^{3,0}(X^{2n})$, and give X^{2n} the Calabi-Yau form $\Theta = \bigwedge_{i=1}^{2n}\theta_i$.

$\mathfrak{g}_E\boxtimes\mathfrak{g}_E$ denotes $\pi_1^*\mathfrak{g}_E\otimes\pi_2^*\mathfrak{g}_E$ over X^2. An integrable $\bar\partial$-operator A_0 on E, acyclic on $\mathfrak{g}_E$, induces $\bar\partial_{A_0}$ on $\mathfrak{g}_E\boxtimes\mathfrak{g}_E$, also acyclic by the Künneth formula. $\omega\in\Omega^{0,2}(X_1\times X_2, \mathfrak{g}_E\boxtimes\mathfrak{g}_E)$ is the kernel for the propagator $\Lambda^{-1} = (\bar\partial_{A_0}(\,\cdot\,\wedge\theta))^{-1} = \theta\lrcorner(\bar\partial^*_{A_0}\Delta^{-1}_{A_0})$ on $\Omega^{3,*}(\mathfrak{g}_E)$, restricted to $\ker\bar\partial_{A_0}\to\ker\bar\partial^*_{A_0}$, in the sense that

$$\theta\lrcorner(\bar\partial^*_{A_0}\Delta^{-1}_{A_0})\phi\,(x_2) = \int_{X_1\times\{x_2\}}\omega(x_1,x_2)\wedge\phi(x_1)$$

using the trace to contract $\mathfrak{g}_E\otimes\mathfrak{g}_E\otimes\mathfrak{g}_E\to\mathfrak{g}_E$. Thus ω is the (unique, due to acyclicity) solution of

$$\bar\partial^*_{A_0}\omega = 0, \quad \bar\partial_{A_0}\omega = \theta_1\lrcorner[\Delta]\otimes I,$$

with $[\Delta]$ the current Poincaré dual to the diagonal $\Delta\subset X{\times}X$, and $I\in\mathfrak{g}_E\boxtimes\mathfrak{g}_E|_\Delta\cong\mathfrak{g}_E\otimes\mathfrak{g}_E\cong\operatorname{End}(\mathfrak{g}_E)$ the identity. Since θ_1 is parallel, $\theta_1\lrcorner[\Delta]$ is holomorphic, so a solution exists.

Definition A.4.1 *Let A_0 be an integrable $\bar\partial$-operator on a bundle $E\to X$, acyclic on $\mathfrak{g}_E$, with kernel $\omega\in\Omega^{0,2}(X^2,\mathfrak{g}_E\boxtimes\mathfrak{g}_E)$ as above. Let $\Omega = \sum_{i,j=1}^{2n}\pi_{ij}^*\omega$ be the* *total form in* $\Omega^{0,2}(X^{2n}, \oplus_{i,j=1}^{2n} \pi_{ij}^*(\mathfrak{g}_E \boxtimes \mathfrak{g}_E))$. *Then, for any integer* n, *we define an "invariant"*

$$I_n(A_0) = \int_{X^{2n}} \mathrm{TR}\, \Omega^{3n} \wedge \Theta,$$

where TR takes the $\otimes_{i=1}^n A_i B_i C_i$ *factors* $(A_i, B_i, C_i \in \pi_i^* \mathfrak{g}_E^{\otimes 3})$ *to* $\prod_{i=1}^n \mathrm{tr}\,(A_i B_i C_i - A_i C_i B_i)$, *and all other factors of the tensor algebra of the* $\pi_{ij}^*(\mathfrak{g}_E \boxtimes \mathfrak{g}_E) = \pi_i^* \mathfrak{g}_E \otimes \pi_j^* \mathfrak{g}_E$ *bundles to zero.*

Remarks.

1. Ω^{3n} is singular on all of the "diagonals" where an x_i equals an x_j, so for this to be a sensible definition we need to show the integral is finite. This is discussed below.

2. It is easy to see that I_n can be alternatively described in the following way (which is closer to the Feynman diagram interpretation):

   Let G be a (not necessarily connected) graph with $2n$ trivalent vertices and $3n$ edges. Use this to identify $(\mathfrak{g}_E^{\otimes 2n})^{\otimes 3n}$ with $(\mathfrak{g}_E^{\otimes 3n})^{\otimes 2n}$, and let $\widetilde{\mathrm{TR}}$ on $\mathfrak{g}_E^{\otimes 3}$ be given by $\widetilde{\mathrm{TR}}(A \otimes B \otimes C) = \mathrm{tr}\,(ABC - ACB)$. Letting TR_G be the tensor product of all $2n$ of the $\widetilde{\mathrm{TR}}$'s, we have

   $$I_n(A_0) = \sum_G \int_{X^{2n}} \mathrm{TR}_G \bigwedge_G \pi_{ij}^* \omega \wedge \Theta \qquad \text{(A.4.2)}$$

   summed over all such graphs G, where $\bigwedge_G$ denotes the exterior product over all vertices i, j connected by an edge in G.

   Alternatively it can be shown that $\log Z_k$ can be expanded in powers of k^{-1} with coefficients as in (A.4.2), but summed only over *connected* graphs G. (This is easy to see by exponentiating such a series.) So (A.4.2) gives another invariant summed over only connected graphs.

3. The first invariant is a litle simpler to write down; it is

   $$\int_{X^2} \mathrm{TR}\,(\omega \wedge \omega \wedge \omega) \wedge \theta_1 \wedge \theta_2, \qquad \text{(A.4.3)}$$

   and corresponds to the "theta" Feynman diagram (a circle with a diameter across it). (The other 2-vertex diagram, the dumbbell, does not contribute due to the antisymmetry in the cubic interaction which excludes diagrams with self-interacting vertices.)

4. If there are only finitely many holomorphic structures A_0 on E, all isolated, we may sum over them to get an invariant $I_n(X) = \sum I_n(A_0)$ of X, irrespective of whether or not $k\,CS$ is single-valued.

5. As in [AS1] we can get an invariant for X directly, if $h^{0,1}(X) = 0$, by considering the trivial holomorphic bundle. While $\mathfrak{g}_E$ is not acyclic we can still define a Green's function G for Λ such that $G\Lambda = \Lambda G$ is the projection along the harmonic space of $\Omega^{0,3}$.

6. Naively we expect the I_n to be metric independent. The metric dependence is the integral of a divergence, which due to singularities has a contribution from the "boundary" diagonals. Analysing this as in [AS2] we could find the metric dependence and remove it. As discussed in the last section, it may well be zero in our case.

7. We could then speculate on independence of the complex structure. Since we are summing over holomorphic bundles these could "go off to infinity" (e.g. become sheaves) as we vary the Calabi-Yau, if we do not work on a compactification of the space of bundles. We can see this is related to the holomorphic Casson invariant of Chapter 3, and in fact the first Axelrod-Singer invariant of a 3-manifold is the Casson invariant. This is more complicated than each flat connection contributing one to the total, however, and comes from the surgery rules of the two invariants.

In [AS1] finiteness of the analogous 3-manifold invariants is proved. ω is modelled, via the Hadamard parametrix method ([Ho] p 30), on the kernel

$$\sum_{i=1}^{3}(-1)^i\frac{x_i}{\|x\|^3}dx_{\{1,2,3\}\setminus\{i\}}.$$

The idea is that although this is singular as we approach the diagonal, its square is zero, so does not contribute. Rewriting in terms of $\hat{x}_i = x_i/\|x_i\|$, where the x_i are local geodesic coordinates perpendicular to the diagonal $\Delta \subset X \times X$, the exact kernel is

$$\omega = I \otimes \sum_{i=1}^{3}(-1)^i\left(\hat{x}_i d\hat{x}_j \wedge d\hat{x}_k + \hat{x}_i d\hat{x}_j \wedge \gamma_k\right) + A, \tag{A.4.4}$$

where $\{ijk\} = \{123\}$, γ_k and A are bounded forms, and I is the identity in $\mathfrak{g} \otimes \mathfrak{g}$. Now the singular forms $d\hat{x}_i$ only "point" along the unit sphere bundle of the normal bundle to the diagonal $\Delta \subset X^2$, so annihilate vectors pointing along Δ or towards

Δ. Axelrod and Singer make this precise by writing down four linearly independent vector fields that contract with ω to give zero. Roughly speaking, they then notice the only parts of the integrand which are non-integrable when n vertices come together are (by looking at the powers of $1/\|x\|$ in (A.4.4)) ones involving at least $3n-3$ of the $d\hat{x}_i$'s. Over the $3n$-dimensional space of these 3 vertices, the resulting $3n-3$ form must be zero, since it is annihilated by interior product by 4 vector fields.

In our case the relevant kernel is the Bochner-Martinelli kernel ([GH] p 383) divided by the Calabi-Yau form $\theta = d\bar{z}_1\, d\bar{z}_2\, d\bar{z}_3$;

$$\sum_{i=1}^{3}(-1)^i \frac{\bar{z}_i}{\|z\|^6} d\bar{z}_{\{1,2,3\}\setminus\{i\}}.$$

Applying the Hadamard parametrix method to find the actual kernel about a point on a diagonal we end up with a formula analogous to (A.4.4), but with worse singularities. At first sight the method of [AS2] will not work, even for the first invariant, but apparently there are reasons in string theory for believing that Z_k can be normalised [W2]. These should come down to massive cancellations in the integral due to the Ricci-flatness of the Calabi-Yau. Basically a messy local calculation needs to be performed, on which I intend collaborating with Axelrod. As a small start we have worked out the right holomorphic analogue of geodesic coordinates for calculating the Hadamard parametrix.

Holomorphic geodesic coordinates

Given a complex manifold X with Hermitian metric g we want to construct *holomorphic* coordinates which are as canonical as possible, and with respect to which the $g_{i\bar{j}}$'s take the simplest form about $z=0$.

We denote by ∇ the unique connection compatible with the complex and Hermitian structures; this will be the Levi-Civita connection (i.e. torsion free) if and only if X is Kähler. Pick local holomorphic coordinates (z^i) about a point $z=0$, and set $\partial_i = \partial/\partial z^i$, $\nabla_i = \nabla_{\partial_i}$, $\Gamma^k_{ij} = dz^k(\nabla_i \partial_j)$ and $g_{i\bar{j}} = g(\partial_i, \partial_j)$.

First of all we may assume that $g_{i\bar{j}}(0) = \delta_{i\bar{j}}$ by a constant (first order) change of coordinates $\hat{z}^i = a^i_j z^j$, i.e. $\hat{\partial}_i = a^j_i \partial_j$, taking the matrix $A = (a^i_j)$ to satisfy $AA^* = g$, with a solution unique up to a unitary transformation of the z^i's.

We now perform a second order coordinate change,

$$\hat{z}^i = z^i + \frac{1}{2} a^i_{jk} z^j z^k \qquad (\text{with } a^i_{jk} = a^i_{kj} \text{ constants}),$$

i.e.,

$$\hat{\partial}_i = (\delta_i^j + a_{ik}^j z^k)\partial_j = h_i^j \partial_j.$$

The new Christoffel symbols are then

$$\hat{\Gamma}_{ij}^k = h_i^a h_j^b h_c^k \Gamma_{ab}^c + h_i^a h_b^k (\partial_a h_j^b). \tag{A.4.5}$$

So, to top order, we have

$$\hat{\Gamma}_{ij}^k = \Gamma_{ij}^k + \partial_i h_j^k = \Gamma_{ij}^k + a_{ji}^k.$$

Thus the Γ_{ij}^k's can be chosen to vanish at $z = 0$ if and only if they are symmetric in i and j, that is if and only if ∇ is torsion-free, i.e. X is Kähler. In this case the coordinate transformation is *uniquely* determined by $a_{ij}^k = -\Gamma_{ij}^k$.

We now continue in the same way with coordinate changes of higher orders to obtain

Proposition A.4.6 *Given a Kähler manifold X and $n > 0$, there exist holomorphic coordinates about any point in which Γ_{ij}^k and $\partial_{i_1} \ldots \partial_{i_m} \Gamma_{ij}^k$ are zero for all $m < n-1$. The coordinates are unique, modulo a constant unitary transformation, up to order $(n+1)$ coordinate changes of the form $\hat{z}^i = z^i + f(z)$ where f and its partial derivatives of order $\leq n$ all vanish at $z = 0$.*

Remark. Of course we cannot hope for the $\bar{\partial}$-derivatives of Γ to vanish as these determine the curvature of X. What the proposition shows is that we have (essentially unique) coordinates in which the Taylor series of the metric has coefficients of $z^{i_1} \ldots z^{i_m}$ all zero. In fact we also have $g_{i\bar{j}} = \delta_{i\bar{j}} + O(|z|^2)$ about 0, since $\partial_i g_{j\bar{k}}(0) = \Gamma_{ij}^l g_{l\bar{k}}(0) = 0$ and $\partial_{\bar{i}} g_{j\bar{k}}(0) = \overline{\Gamma_{ij}^l} g_{l\bar{k}}(0) = 0$.

Thus, for instance, the standard coordinate z on an affine chart $\mathbb{C} \subset \mathbb{CP}^1$ is holomorphic geodesic for the Fubini-Study metric $\frac{dz\, d\bar{z}}{(1+z\bar{z})^2}$ since $g_{1\bar{1}} = 1 - 2z\bar{z} + 3(z\bar{z})^2 - \ldots$ only contains terms in $z\bar{z}$.

Proof. We suppose inductively that we have obtained coordinates as in the proposition up to the n-th stage, and attempt to further remove the $(n-1)$-th order holomorphic parts $\partial_{i_1} \ldots \partial_{i_{n-1}} \Gamma_{ij}^k(0)$ of Γ.

So we let

$$\hat{z}^i = z^i + \frac{1}{(n+1)!} a_{j_1 \cdots j_{n+1}}^i z^{j_1} \ldots z^{j_{n+1}}$$

where the coefficients $a_{\ldots}^i$ are constants symmetric in the lower indices. Thus,

$$\hat{\partial}_j = h_j^i \partial_i,$$

where

$$h^i_k = \delta^i_k + \frac{1}{n!}\, a^i_{j_1\cdots j_n k}\, z^{j_1}\dots z^{j_n}.$$

The formula (A.4.5) for the transformation of the Christoffel symbols shows that to top order in the z^i's we have

$$\hat{\Gamma}^i_{kl} = \Gamma^i_{kl} + \frac{1}{(n-1)!}\, a^i_{j_1\cdots j_{n-1}kl}\, z^{j_1}\dots z^{j_{n-1}},$$

so that, at $z=0$,

$$\partial_{j_1}\dots\partial_{j_{n-1}}\Gamma^i_{kl} = a^i_{j_1\cdots j_{n-1}kl}.$$

Thus we can remove the *symmetric* (in $j_1\dots j_{n+1}$) part of $\partial_{j_1}\dots\partial_{j_{n-1}}\Gamma^i_{j_n j_{n+1}}$, and doing so uniquely determines the coordinate change. So it remains to show that $\partial_{j_1}\dots\partial_{j_{n-1}}\Gamma^i_{j_n j_{n+1}}(0)$ is symmetric in $j_1\dots j_{n+1}$.

The integrability of the complex structure, and unitarity, imply the vanishing of the (2,0)-part of the curvature, which is

$$dz^l(\nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k) = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}.$$

Taking the $z^{j_1}\dots z^{j_{n-2}}$ coefficient of this (equivalently applying $\partial_{j_1}\dots\partial_{j_{n-2}}|_{z=0}$ to this formula) shows that $\partial_{j_1}\dots\partial_{j_{n-1}}\Gamma^i_{j_n j_{n+1}}$ is symmetric in j_{n-1} and j_n. But it is symmetric in j_n and j_{n+1} (∇ is torsion free) and in the j_i, $1\le i\le n-1$ by symmetry of partial derivatives. Therefore it has the required symmetry. □

The Kontsevich approach

Kontsevich's alternative definition [K] of Axelrod and Singer's invariants used a real-differential geometric analogue of a complex geometric construction for resolving the singularities of configuration spaces. Thus it seems natural to hope there should be a simple analogue of his definition for a Calabi-Yau 3-fold.

We concentrate on the definition of the first invariant (A.4.3), as this illustrates the difficulties involved. Firstly, by analogy with [K], we would like ω to be a harmonic (0,2)-form on the blow up $Z = \widetilde{X\times X}$ of X^2 along the diagonal Δ, which restricts to some generator on each $\mathbb{CP}^2$ fibre over the diagonal (tensored with the identity in $\mathfrak{g}_E\otimes\mathfrak{g}_E^*$). But of course $\mathbb{CP}^2$ has no (0,2)-cohomology, and so we have to consider twisting everything by a line bundle.

So let E denote the exceptional divisor $\pi^{-1}(\Delta)$. Then on restriction to each fibre the line bundle $[3E]$ is $\mathcal{O}(-3)\to\mathbb{CP}^2$, which has one dimensional $H^{0,2}$. For

$\mathfrak{g}_E$ acyclic we can find a closed form in $\Omega^{0,2}(Z;[3E]\otimes\mathfrak{g}_E\boxtimes\mathfrak{g}_E)$ restricting to a generator of $H^{0,2}(\mathbb{CP}^2;\mathcal{O}(-3))$ in each fibre, using the exact sequence

$$0\to[2E]\to[3E]\to[3E]|_{\mathbb{P}T\Delta}\to 0$$

on Z, and $K_Z=[2E]$. Then we can form $\mathrm{TR}\,\omega^3\wedge\pi^*\Theta$ as before, an element of $\Omega^{0,6}([9E])\cong\Omega^{6,6}([7E])$ (as $K_Z=[2E]$). Using the above exact sequence and Serre duality we can see this is zero in cohomology, but an analytic finiteness result as discussed earlier would presumably show it vanishes to at least seventh order on E, so that we can consider it as an element of $\Omega^{6,6}(\mathcal{O}_Z)$ and integrate it over Z.

This is clearly unsatisfactory, and it is disappointing that I cannot yet find a natural analogue of Kontsevich's construction.

A.5 Knot-type invariants and linked tori

Finally we briefly sketch the analogue of Wilson loops (knots and links) in 3-manifolds. We fix complex curves $C_i\subset X$ and holomorphic 1-forms σ_i on the C_i. So we will want C_i to have genus at least one, and tori seem the most natural curves to consider – they are the Calabi-Yau manifolds of their dimension, the 1-form is more or less unique.

We can then supplement the path integral Z_k with a term analogous to the holonomy considered in [W1]. In the abelian (line bundle) case, a $\bar\partial$-operator A on X gives the line bundle a holomorphic structure on restriction to C_i, and defines an element of the Jacobian $H^{0,1}/H_1(X;\mathbb{Z})$, which can be paired against σ_i and exponentiated to give a number which we (sloppily) denote

$$\exp\Big(ik\int_{C_i}A\wedge\sigma_i\Big),\tag{A.5.1}$$

once we have chosen a basepoint in the Jacobian; again there are problems with the periods being dense which do not affect the final answer. We then consider the path integral

$$Z_k=\int_{\mathscr{B}}\mathscr{D}A\;e^{ikCS(A)}\prod_i e^{ik\int_{C_i}A\wedge\sigma_i}\,.$$

For the abelian theory $G=U(1)$ the evaluation is quite simple by completing the square in A (there are no cubic terms anymore). That is, using

$$\int e^{-x^t\Lambda x+J^tx}\,d^Nx=(\mathrm{Det}\,\Lambda)^{-1/2}e^{\frac14 J^t\Lambda^{-1}J},$$

with $ik\bar{\partial}$ essentially playing the role of Λ, and J being the linear operator $A \mapsto ik\sum\int_{C_i} A\wedge\sigma_i$, the following holomorphic analogue of the linking number between two of the tori arises:

$$\int_{C_i} \theta \lrcorner \Big[\bar{\partial}^{-1}(\mathrm{PD}\,[C_j]\wedge\sigma_j)\Big]\wedge\sigma_i.$$

$\bar{\partial}^{-1}$ operates on $\ker\bar{\partial}^*$, so is $\Delta^{-1}\bar{\partial}^*$, which exists if $h^{0,1}=0$ (so that $h^{3,2}$ vanishes also). Compare this formula to that for the Gauss linking number of two circles (which must be in a homology three sphere, the analogue of insisting on $h^{0,1}=0$, for the linking number to make sense) where the kernel is simply that for d instead of $\bar{\partial}$. This is the Gauss kernel on flat $\mathbb{R}^3$ which would be replaced here by the Bochner-Martinelli kernel if we were in $\mathbb{C}^3$. To evaluate the whole path integral would require regularising the self-linking number of a complex curve; this is more difficult and has only recently been solved for knots in S^3.

A complex geometric interpretation of the linking number can be made as follows. Suppose C_j is a torus whose canonical bundle extends to the trivial bundle K_{C_j} on all of X with a trivialisation fixed by σ_j. Then $\bar{\partial}^{-1}(\mathrm{PD}\,[C_j])$ represents the cohomology class in $H^1(X\setminus C_j; K_X\otimes K^*_{C_j})$ (or more precisely $\mathrm{Ext}^1(K_{C_j}\otimes\mathscr{I}_{C_j}, K_X)$) corresponding to the bundle defined by the curve C_j via the Serre construction (2.1.1). Restricting the Ext group to the other (disjoint) curve C_i gives a class in $H^1(C_i; K_X\otimes K^*_{C_j})$ which is isomorphic to $H^1(C_i;K_i)$ using the Calabi-Yau forms θ, σ_i and σ_j. This may be evaluated on C_i to give a measure of how non-trivial the extension

$$0\to K_X\to E\to K_{C_j}\otimes\mathscr{I}_{C_j}\to 0$$

is over C_i. This is precisely the holomorphic linking number proposed by Atiyah in [At1] for $\mathbb{CP}^1$ fibres in a twistor space, and he showed the Serre class in Ext^1 is the Green's current solving the $\bar{\partial}$-problem for the current defined by the curve C_j. Thus his linking number agrees with ours.

While studying this I received the much more thorough preprint [FKT], and so have not pursued it further. It fits nicely into the overall theme of this thesis, however.

Higher order invariants

Considering the non-abelian theory gives higher order invariants of knots and links (in fact the whole Jones polynomial) in 3-manifolds. To mimic this we need an analogue of holonomy for a complex curve. On a circle (or knot) the functional determinant of a connection d_A defines the holonomy of A, so it is natural in our holomorphic set-up to consider the regularised determinant $\mathrm{Det}\,\bar{\partial}_A|_C$ for a complex curve, with one-form σ, in a Calabi-Yau 3-fold, where σ is used to map $\Omega^{0,1}$ to Ω^0. This can then be formally expanded in powers of A and used perturbatively in the path integral to give the linking number at first order, then higher order invariants. Their finiteness, metric dependence, and complex interpretation may be more difficult to study.

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