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Gauge Transformation and Gravitational Potentials

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It is believed in particle physics that the velocity-dependent part of potential is, in general, ambiguous as far as it is derived from S-matrix. We consider the most general form of the graviton propagator under an arbitrary q-number gauge transformation for the graviton field. The propagator depends on twelve arbitrary functions of $\boldsymbol{k}^2$, $\boldsymbol{k}$ being the space part of the momentum of the graviton. The arbitrariness of the gauge functions can be used to remove the ambiguity of one-graviton exchange potential up to the order of $(G/r)(v/c)^4$, G being the gravitational constant. The potential thus obtained depends only on one gauge parameter, say x. The perihelion motion of two-body system is gauge independent, although the potentials in the order of $(G/r)(v/c)^2$ and G^2/r^2 depend on the gauge parameter x. The potentials are derived from those in a fixed gauge parameter x by a coordinate transformation.

§ 1. Introduction

Suppose that we are of interest to obtain one-particle exchange potential between two elementary particles with masses m_1 and m_2. Then particle physicists usually consider the diagram showing that the particles with initial four momenta p_1 and p_2 and final momenta q_1 and $q_2(p_1^2=q_1^2=-m_1^2, p_2^2=q_2^2=-m_2^2)$ exchange a boson with momentum $k\equiv(p_1-q_1)=-(p_2-q_2)$, and calculate the potential contributed from the S-matrix element corresponding to this diagram. However it is well known in particle physics that the velocity-dependent part of this potential is ambiguous. Since energy is conserved in any S-matrix element, the energy transferred between two particles, k_0 is equal to $(p_{10}-q_{10})$ and also to $(q_{20}-p_{20})$. The ambiguity comes from the k_0-dependence of the S-matrix element.

To show this explicitly, let us consider one-graviton exchange potential between two spinless particles. The graviton propagator is proportional to

$$\frac{1}{\boldsymbol{k}^2-k_0^2}. \qquad (1\cdot1)$$

Since the potential should be symmetric with respect to two particles, the factor $(1\cdot1)$ can be expressed generally as

$$\frac{1}{\boldsymbol{k}^2+x(p_{10}-q_{10})(p_{20}-q_{20})-\frac{1}{2}(1-x)[(p_{10}-q_{10})^2+(p_{20}-q_{20})^2]}, \qquad (1\cdot2)$$

where x is an arbitrary real constant. Using the propagator $(1\cdot2)$, we obtain one-graviton exchange potential between two elementary particles. Summing up this potential over all elementary particles involved in two celestial bodies with masses M_1 and M_2, we get the potential[1)]

$$V_2^{(2)}=-\frac{GM_1M_2}{r}\left\{1+\frac{1}{4}(7-x)\left(\frac{\boldsymbol{P}_1^2}{M_1^2}+\frac{\boldsymbol{P}_2^2}{M_2^2}\right)-\frac{1}{2}(8-x)\frac{(\boldsymbol{P}_1\boldsymbol{P}_2)}{M_1M_2}\right.$$
$$\left.-\frac{x}{2}\frac{(\boldsymbol{P}_1\boldsymbol{n})(\boldsymbol{P}_2\boldsymbol{n})}{M_1M_2}-\frac{1}{4}(1-x)\left[\frac{(\boldsymbol{P}_1\boldsymbol{n})^2}{M_1^2}+\frac{(\boldsymbol{P}_2\boldsymbol{n})^2}{M_2^2}\right]\right\}, \qquad (1\cdot3)$$

where $\boldsymbol{n}=\boldsymbol{r}/r$ and $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are momenta of two celestial bodies with masses M_1 and M_2, respectively. The higher order terms than $(G/r)(v/c)^2$ are neglected in (1·3). Not only the denominator but also the numerator of the graviton propagator may depend on k_0. Even in this case it is easy to show that one-graviton exchange potential can be expressed again by (1·3), if the arbitrary constant x is properly redefined. Thus we have no reason to fix the value of x to a special value,[2)] say, $x=1$. In this special case the potential (1·3) reduces to that given by Einstein, Infeld and Hoffman.[3)] This is an example to show that the velocity-dependent part of any potential is in general ambiguous, as far as the potential is obtained from S-matrix.

In the next section we consider the most general form of the graviton propagator

$$\frac{-i}{2(2\pi)^4}\int d^4k e^{ikx}\frac{X_{\alpha\beta,\gamma\delta}(k)}{k^2-i\varepsilon}$$

under the gauge transformation of the graviton field $h_{\alpha\beta}$:

$$h_{\alpha\beta}\rightarrow h_{\alpha\beta}+\partial_\alpha\rho_\beta+\partial_\beta\rho_\alpha\,,$$

where ρ_α is an arbitrary q-number vector. The numerator $X_{\alpha\beta,\gamma\delta}$ depends generally on tweleve arbitrary functions of $\boldsymbol{k}^2$. They are called gauge functions. The arbitrariness of these gauge functions can be used to remove the ambiguity of one-graviton exchange potential mentioned above. For example, in order that one-graviton exchange potential is determined unambiguously up to the order $(G/r)(v/c)^2$, three of twelve gauge functions must be fixed. Then it is shown that the x-dependence of (1·3) does not mean the existence of the ambiguity mentioned above but only shows the gauge depencence of the potential in the order $(G/r)(v/c)^2$. When we require that one-graviton exchange potential is determined unambiguously up to the order $(G/r)(v/c)^4$, eleven of twelve gauge functions are fixed. The potential thus obtained depends only on the gauge parameter x. The one-graviton exchange potentials in the order of $(G/r)(v/c)^{2n}$ $(n\geq3)$ cannot be determined unambiguously as far as the potentials are obtained from S-matrix and the arbitrariness of the gauge functions are used.

As was mentioned, the potential in the order $(G/r)(v/c)^2$ depends on gauge parameter x. Then *virial theorem* in classical mechanics suggests that the static part of two-graviton exchange potential, which is in the order of G^2/r^2, also depends on x. Section 3 is devoted to the calculation of the x-dependence of the latter potential. It is shown that the potential actually depends on x. For example, the static part of two-graviton exchange potential $V_2^{(4)}$ vanishes in the special gauge, $x=3$:

$$V_2^{(4)}=0\left(\frac{G^2}{r^2}\left(\frac{v}{c}\right)^2\right).$$

It is shown in the last section that the perihelion motion of two-body system is gauge independent, although the potentials in the order of $(G/r)(v/c)^2$ and G^2/r^2depend on x. It is also shown in the center-of mass system that the gauge dependent potentials in an arbitrary gauge parameter x are obtained from the standard ones given by Einstein, Infeld and Hoffman[3] and defined in the special gauge of $x=1$, by a coordinate transformation in the form

$$\boldsymbol{r}'=\boldsymbol{r}\left[1+\alpha\frac{G(M_1+M_2)}{r}\right],$$

where $\alpha=-\frac{1}{4}(1-x)$.

§ 2. Gauge transformations

As is well known, potential itself is not a physical observable and it depends on the gauge in the case of electromagnetic interactions. This is also true for gravitational interactions. The purpose of this section is to show how the velocity-dependent part of one-graviton exchange potential changes under gauge transformations. Before entering into gravitational interactions, let us consider electromagnetic interactions, briefly.

Since the gauge transformation of the photon field A_α is given by

$$A_\alpha \to A_\alpha+\partial_\alpha \Lambda\,,$$

the most general form of the photon propagator is[4]

$$-\frac{i}{(2\pi)^4}\int d^4k e^{ikx}\frac{D_{\alpha\beta}(k)}{k^2-i\varepsilon}\,,\qquad D_{\alpha\beta}(k)=\delta_{\alpha\beta}+k_\alpha\lambda_\beta+k_\beta\lambda_\alpha\,, \tag{2·1}$$

where λ_α is an arbitrary vector and it is a function of k_α and unit time-like vector $n_\alpha(n_\alpha^2=-1)$. We can define the vector $\bar{k}_\alpha=-k_\alpha-2n_\alpha(kn)$ and the scalar $k^2+(kn)^2$, which reduce in the special frame $n_\alpha=(0,0,0,i)$ to the space-reflected vector $\bar{k}_\alpha=(-\boldsymbol{k}, ik_0)$ and $\boldsymbol{k}^2$, respectively. In the following discussions we shall choose this frame.

The vector λ_α can be expressed as

$$\lambda_\alpha=ak_\alpha+b\bar{k}_\alpha\,. \tag{2·2}$$

Substituting this into (2·1), we obtain

$$D_{\alpha\beta}=\delta_{\alpha\beta}+2ak_\alpha k_\beta+b(\bar{k}_\alpha k_\beta+k_\alpha\bar{k}_\beta)\,. \tag{2·3}$$

Now we require that one-photon exchange potential is free from the ambiguity discussed in the previous section at least to the order of $(v/c)^2$. Then we must require that 1)

$$\frac{D_{00}}{k^2-i\varepsilon}=-\frac{1}{\boldsymbol{k}^2},$$

and 2) $D_{0\alpha}$ does not depend on k_0. From these requirements we obtain

$$(a+b)=\frac{1}{2\boldsymbol{k}^2} \quad \text{and} \quad a=0,$$

respectively. Therefore

$$D_{\alpha\beta}(k)=\delta_{\alpha\beta}+\frac{(\bar{k}_\alpha k_\beta+k_\alpha\bar{k}_\beta)}{2\boldsymbol{k}^2}. \tag{2·4}$$

The expression (2·1) with (2·4) is just the photon propagator in the Coulomb gauge. Thus we can say that the Coulomb gauge should be used in order to calculate the velocity-dependent part of one-photon exchange potential from S-matrix. Further we can show that the relastivistic correction term in the form of $(e^4/r^2)(v/c)^2$ to the two-photon exchange potential can be calculated uniquely in this gauge.

The gauge transformation of the graviton field $h_{\alpha\beta}$ is given by

$$h_{\alpha\beta}\to h_{\alpha\beta}+\partial_\alpha\rho_\beta+\partial_\beta\rho_\alpha, \tag{2·5}$$

where ρ_α is an arbitrary q-number vector. The graviton propagator can be expressed as

$$-\frac{i}{2(2\pi)^4}\int d^4k e^{ikx}\frac{X_{\alpha\beta,\gamma\delta}(k)}{k^2-i\varepsilon}, \tag{2·6}$$

where the numerator $X_{\alpha\beta,\gamma\delta}$ has the form

$$\begin{aligned} X_{\alpha\beta,\gamma\delta}(k) &= (\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\beta}\delta_{\gamma\delta}) \\ &\quad +k_\alpha k_\gamma\eta_{\beta\delta}+k_\beta k_\delta\eta_{\alpha\gamma}+k_\alpha k_\delta\eta_{\beta\gamma}+k_\beta k_\gamma\eta_{\alpha\delta} \\ &\quad +k_\gamma\Lambda_{\alpha\beta,\delta}+k_\delta\Lambda_{\alpha\beta,\gamma}+k_\alpha\Lambda_{\gamma\delta,\beta}+k_\beta\Lambda_{\gamma\delta,\alpha}. \end{aligned} \tag{2·7}$$

The tensors $\eta_{\alpha\beta}$ and $\Lambda_{\alpha\beta,\gamma}$ should be symmetric with respect to α and β, and they are functions of k_α and $\bar{k}_\alpha$. The most general form of these tensors is given by

$$\begin{aligned} \eta_{\alpha\beta} &= \frac{1}{\boldsymbol{k}^2}\{a_1\delta_{\alpha4}\delta_{\beta4}+a_2\bar{\delta}_{\alpha\beta}\}+\frac{1}{\boldsymbol{k}^4}\{b_1k_\alpha k_\beta+b_2(\bar{k}_\alpha k_\beta+k_\alpha\bar{k}_\beta)+b_3\bar{k}_\alpha\bar{k}_\beta\}, \\ \Lambda_{\alpha\beta,\gamma} &= \frac{1}{\boldsymbol{k}^2}\{\delta_{\alpha4}\delta_{\beta4}(c_1k_\gamma+c_2\bar{k}_\gamma)+\bar{\delta}_{\alpha\beta}(c_3k_\gamma+c_4\bar{k}_\gamma) \\ &\quad +\delta_{\alpha4}\delta_{\gamma4}(d_1k_\beta+d_2\bar{k}_\beta)+\bar{\delta}_{\alpha\gamma}(d_3k_\beta+d_4\bar{k}_\beta) \end{aligned} \tag{2·8}$$

$$+\delta_{\beta4}\delta_{\gamma4}(d_1k_\alpha+d_2\bar{k}_\alpha)+\bar{\delta}_{\beta\gamma}(d_3k_\alpha+d_4\bar{k}_\alpha)\}$$
$$+\frac{1}{\boldsymbol{k}^4}\{e_1k_\alpha k_\beta k_\gamma+e_2(\bar{k}_\alpha k_\beta k_\gamma+k_\alpha\bar{k}_\beta k_\gamma)+e_3k_\alpha k_\beta\bar{k}_\gamma$$
$$+e_4\bar{k}_\alpha\bar{k}_\beta k_\gamma+e_5(\bar{k}_\alpha k_\beta\bar{k}_\gamma+k_\alpha\bar{k}_\beta\bar{k}_\gamma)+e_6\bar{k}_\alpha\bar{k}_\beta\bar{k}_\gamma\},$$

which depends on nineteen functions $a_1, \cdots, e_6$ and $\bar{\delta}_{\alpha\beta}=\delta_{\alpha\beta}-\delta_{\alpha4}\delta_{\beta4}$. We shall call these arbitrary functions $a_1, \cdots, e_6$ gauge functions. Substituting these expressions for the tesors $\eta_{\alpha\beta}$ and $\Lambda_{\alpha\beta,\gamma}$ into (2·7), we find that $X_{\alpha\beta,\gamma\delta}$ depends on twelve independent gauge functions, because some of the nineteen functions appear only in some combination in it. For example, b_3 and e_5 appear only in the combination (b_3+2e_5).

Now let us impose on $X_{\alpha\beta,\gamma\delta}$ the requirement that the velocity-dependent part of one-graviton exchange potential is free from the ambiguity mentioned in the previous section up to the order of $(G/r)(v/c)^2$, then we get the conditions

$$a_1+c_1+2d_1=0\,,$$
$$c_2+2d_2=\tfrac{1}{4}\,, \tag{2·9}$$
$$b_1+2b_2+b_3+e_1+2e_2+e_3+e_4+2e_5+e_6=0\,.$$

Thus $X_{\alpha\beta,\gamma\delta}$ depends on nine gauge functions.

The S-matrix element for one-graviton exchange between two spinless particles with masses m_1 and m_2 is given by

$$S=\frac{iG}{\pi}(p_{10}p_{20}q_{10}q_{20})^{-1/2}p_{1\alpha}q_{1\beta}p_{2\gamma}q_{2\delta}\frac{X_{\alpha\beta,\gamma\delta}(k)}{k_2}\delta^{(4)}(p_1+p_2-q_1-q_2)$$
$$\simeq\frac{iG}{\pi}\frac{m_1m_2}{\boldsymbol{k}^2}\left\{1+\frac{3}{2}\left(\frac{\boldsymbol{p}_1^2}{m_1^2}+\frac{\boldsymbol{p}_2^2}{m_2^2}\right)-4\frac{(\boldsymbol{p}_1\boldsymbol{p}_2)}{m_1m_2}+4(c_1+2d_2)\frac{(\boldsymbol{p}_1\boldsymbol{k})(\boldsymbol{p}_2\boldsymbol{k})}{m_1m_2\boldsymbol{k}^2}\right.$$
$$\left.+\frac{1}{2}(1-4c_1-8d_2)\frac{1}{\boldsymbol{k}^2}\left[\frac{(\boldsymbol{p}_1\boldsymbol{k})^2}{m_1^2}+\frac{(\boldsymbol{p}_2\boldsymbol{k})^2}{m_2^2}\right]\right\}\delta^{(4)}(p_1+p_2-q_1-q_2), \tag{2·10}$$

where $k=(p_1-q_1)=-(p_2-q_2)$ and the terms which contribute to the potentials proportional to $\hbar^n(n\geq1)$ and which are proportional to $(v/c)^{2n}(n\geq2)$ are neglected. It is interesting to observe that the expression (2·10) depends only on the function (c_1+2d_2) in spite of the fact that the expression for $X_{\alpha\beta,\gamma\delta}$ depends on nine functions. From the beginning we assume that gauge functions are non-singular functions of $\boldsymbol{k}^2$. Let $(\underline{c_1}+2\underline{d_2})$ be the leading term of (c_1+2d_2) when it is expanded in the powers of $\boldsymbol{k}^2$. Then it is easy to get from (2·10) one-graviton exchange potential between two spinless particles. Summing up over all particles involved in two celestial bodies with masses M_1 and M_2 and momenta P_1 and P_2, respectively, we obtain the one-graviton exchange potential between the celestial bodies:

$$V_2^{(2)} = -\frac{GM_1M_2}{r}\Big\{1+\frac{1}{4}(7-4\underline{c_1}-8\underline{d_2})\Big(\frac{\boldsymbol{P}_1^2}{M_1^2}+\frac{\boldsymbol{P}_2^2}{M_2^2}\Big)-2(2-\underline{c_1}-2\underline{d_2})\frac{(\boldsymbol{P}_1\boldsymbol{P}_2)}{M_1M_2}$$
$$-2(\underline{c_1}+2\underline{d_2})\frac{(\boldsymbol{P}_1\boldsymbol{n})(\boldsymbol{P}_2\boldsymbol{n})}{M_1M_2}-\frac{1}{4}(1-4\underline{c_1}-8\underline{d_2})\Big[\frac{(\boldsymbol{P}_1\boldsymbol{n})^2}{M_1^2}+\frac{(\boldsymbol{P}_2\boldsymbol{n})^2}{M_2^2}\Big]\Big\}, \quad (2\cdot 11)$$

which is identical with $(1\cdot 3)$ when

$$x = 4\underline{c_1}+8\underline{d_2}. \quad (2\cdot 12)$$

Thus we have shown that the x-dependence of the potential $(1\cdot 3)$ does not mean the existence of the ambiguity mentioned in the previous section but it only shows the gauge dependence of the potential. The potential given by Einstein, Infeld and Hoffman is obtained from $(1\cdot 3)$ by putting $x=1$. As was shown explicitly in Ref. 1), if the static part of two-graviton exchange potential is independent of x, the perihelion motion, say, of the Mercury depends on the gauge parameter x. Thus the static potential should also depend on x. This x-dependence will be disscussed in § 3.

Now we want to calculate the one-graviton exchange potential which is free from the ambiguity mentioned in the previous section up to the order of $(G/r)\times(v/c)^4$. Then we must impose further eight conditions on $X_{\alpha\beta,\gamma\delta}$. They are

$$\begin{aligned} &a_2+2d_3=0, \qquad b_1+e_1=0, \\ &2b_2+2e_2+e_3=-\frac{1}{16}, \qquad c_3=0, \\ &c_4=-\frac{1}{2}, \qquad d_4=\frac{1}{2}, \\ &e_4=\frac{\bar{x}}{16}, \qquad e_6=-\frac{1}{16}, \end{aligned} \quad (2\cdot 13)$$

where $\bar{x}$ denotes the gauge function $(4c_1+8d_2)$.

Under the conditions $(2\cdot 9)$ and $(2\cdot 13)$, $X_{\alpha\beta,\gamma\delta}$ can be expressed as

$$X_{\alpha\beta,\gamma\delta}(k)=\frac{k^2}{\boldsymbol{k}^2}\Big\{\delta_{\alpha 4}\delta_{\beta 4}\delta_{\gamma 4}\delta_{\delta 4}-\delta_{\alpha 4}\delta_{\beta 4}\bar{\delta}_{\gamma\delta}-\delta_{\gamma 4}\delta_{\delta 4}\bar{\delta}_{\alpha\beta}$$
$$+\delta_{\alpha 4}\delta_{\gamma 4}\bar{\delta}_{\beta\delta}+\delta_{\beta 4}\delta_{\delta 4}\bar{\delta}_{\alpha\gamma}+\delta_{\alpha 4}\delta_{\delta 4}\bar{\delta}_{\beta\gamma}+\delta_{\beta 4}\delta_{\gamma 4}\bar{\delta}_{\alpha\delta}$$
$$-\frac{k_ik_j}{\boldsymbol{k}^2}\Big[\frac{\bar{x}}{4}(\delta_{\alpha i}\delta_{\gamma j}\delta_{\beta 4}\delta_{\delta 4}+\delta_{\beta i}\delta_{\delta j}\delta_{\alpha 4}\delta_{\gamma 4}$$
$$+\delta_{\alpha i}\delta_{\delta j}\delta_{\beta 4}\delta_{\gamma 4}+\delta_{\beta i}\delta_{\gamma j}\delta_{\alpha 4}\delta_{\delta 4})$$
$$+\frac{(1-\bar{x})}{2}(\delta_{\alpha i}\delta_{\beta j}\delta_{\gamma 4}\delta_{\delta 4}+\delta_{\gamma i}\delta_{\delta j}\delta_{\alpha 4}\delta_{\beta 4})\Big]\Big\}$$
$$+\Big\{\Big(\bar{\delta}_{\alpha\gamma}-\delta_{\alpha i}\delta_{\gamma j}\frac{k_ik_j}{\boldsymbol{k}^2}\Big)\Big(\bar{\delta}_{\beta\delta}-\delta_{\beta k}\delta_{\delta l}\frac{k_kk_l}{\boldsymbol{k}^2}\Big)$$

$$+\left(\bar{\delta}_{\alpha\delta}-\delta_{\alpha i}\delta_{\delta j}\frac{k_i k_j}{\boldsymbol{k}^2}\right)\left(\bar{\delta}_{\beta\gamma}-\delta_{\beta k}\delta_{\gamma l}\frac{k_k k_l}{\boldsymbol{k}^2}\right)$$

$$-\left(\bar{\delta}_{\alpha\beta}-\delta_{\alpha i}\delta_{\beta j}\frac{k_i k_j}{\boldsymbol{k}^2}\right)\left(\bar{\delta}_{\gamma\delta}-\delta_{\gamma k}\delta_{\delta l}\frac{k_k k_l}{\boldsymbol{k}^2}\right)\Bigg\}, \qquad (2\cdot14)$$

where the suffices i, j, k and l take the values 1, 2 and 3. It is interesting to see that $X_{\alpha\beta,\gamma\delta}$ satisfies automatically the transverse condition

$$k_i X_{ij,kl}(k)=0, \qquad (2\cdot15)$$

though it is not required in the course to get (2·14).

Now we shall rederive (2·14) from a different viewpoint. Let $T^{(i)}_{\alpha\beta}$ be the energy-momentum tensor for i-th particle. It is conserved:

$$\partial_\alpha T^{(i)}_{\alpha\beta}=0. \qquad (2\cdot16)$$

For simplicity, we shall tentatively choose the coordinate as

$$k_\alpha=(0,0,k_3,ik_0). \qquad (2\cdot17)$$

Then

$$T^{(i)}_{3\alpha}=\frac{k_0}{k_3}T^{(i)}_{0\alpha} \qquad (2\cdot18)$$

is obtained from (2·16). Feynman[5] and De Witt[6] use (2·18) and obtain

$$T^{(1)}_{\alpha\beta}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\beta}\delta_{\gamma\delta})T^{(2)}_{\gamma\delta}$$

$$=\left(\frac{k_3^2-k_0^2}{k_3^2}\right)\{T^{(1)}_{00}T^{(2)}_{00}+T^{(1)}_{00}T^{(2)}_{kk}+T^{(1)}_{kk}T^{(2)}_{00}-4T^{(1)}_{0k}T^{(2)}_{0k}$$

$$+3T^{(1)}_{03}T^{(2)}_{03}-T^{(1)}_{00}T^{(2)}_{33}-T^{(1)}_{33}T^{(2)}_{00}\}$$

$$+[2T^{(1)}_{st}T^{(2)}_{st}-T^{(1)}_{ss}T^{(2)}_{tt}], \qquad (2\cdot19)$$

where suffix k takes the values 1, 2, 3 but suffices s, t take the values 1 and 2.

However there is an ambiguity when (2·19) is written down. Using (2·18), we can rewrite (2·19) as

$$\left(\frac{k_3^2-k_0^2}{k_3^2}\right)\{T^{(1)}_{00}T^{(2)}_{00}+T^{(1)}_{00}T^{(2)}_{kk}+T^{(1)}_{kk}T^{(2)}_{00}-4T^{(1)}_{0k}T^{(2)}_{0k}$$

$$+\bar{x}T^{(1)}_{03}T^{(2)}_{03}+\tfrac{1}{2}(1-\bar{x})[T^{(1)}_{00}T^{(2)}_{33}+T^{(1)}_{33}T^{(2)}_{00}]\}$$

$$+[2T^{(1)}_{st}T^{(2)}_{st}-T^{(1)}_{ss}T^{(2)}_{tt}]$$

$$\equiv T^{(1)}_{\alpha\beta}X_{\alpha\beta,\gamma\delta}(k)T^{(2)}_{\gamma\delta}. \qquad (2\cdot20)$$

The expression (2·19) is a special case of (2·20), that is,

$$\bar{x}=3. \qquad (2\cdot21)$$

In the coordinate $(\boldsymbol{k}, ik_0)$ the expression for $X_{\alpha\beta,\gamma\delta}$ obtained from (2·20) coincides with (2·14). Thus the gauge function $\bar{x}$ is introduced as an ambiguity in this derivation.

The S-matrix element for one-graviton exchange obtained from (2·14) is given by

$$S \simeq (2\cdot 10) + \frac{iG}{\pi}\frac{m_1 m_2}{\boldsymbol{k}^2}\left\{\frac{\boldsymbol{p}_1^2 \boldsymbol{p}_2^2}{4m_1^2 m_2^2} - \frac{5}{8}\left(\frac{\boldsymbol{p}_1^4}{m_1^4}+\frac{\boldsymbol{p}_2^4}{m_2^4}\right) + \frac{2(\boldsymbol{p}_1\boldsymbol{p}_2)^2}{m_1^2 m_2^2}\right.$$
$$-\frac{4(\boldsymbol{p}_1\boldsymbol{p}_2)}{m_1^2 m_2^2}\frac{(\boldsymbol{p}_1\boldsymbol{k})(\boldsymbol{p}_2\boldsymbol{k})}{\boldsymbol{k}^2} + \frac{1}{4}\left[(1-\bar{x})\left(\frac{\boldsymbol{p}_2^2}{m_2^2}-\frac{\boldsymbol{p}_1^2}{m_1^2}\right)+\frac{4\boldsymbol{p}_2^2}{m_2^2}\right]\frac{(\boldsymbol{p}_1\boldsymbol{k})^2}{m_1^2\boldsymbol{k}^2}$$
$$\left.+\frac{1}{4}\left[(1-\bar{x})\left(\frac{\boldsymbol{p}_1^2}{m_1^2}-\frac{\boldsymbol{p}_2^2}{m_2^2}\right)+\frac{4\boldsymbol{p}_1^2}{m_1^2}\right]\frac{(\boldsymbol{p}_2\boldsymbol{k})^2}{m_2^2\boldsymbol{k}^2} + \frac{(\boldsymbol{p}_1\boldsymbol{k})^2(\boldsymbol{p}_2\boldsymbol{k})^2}{m_1^2 m_2^2 \boldsymbol{k}^4}\right\}$$
$$\times\delta^{(4)}(p_1+p_2-q_1-q_2). \tag{2·22}$$

One-graviton exchange potential between two celestial bodies up to the order $(G/r)(v/c)^4$ is given by

$$V_2^{(2)} = (2\cdot 11) - \frac{GM_1M_2}{8r}\left\{(13-2x)\frac{\boldsymbol{P}_1^2\boldsymbol{P}_2^2}{M_1^2M_2^2} + 2\frac{(\boldsymbol{P}_1\boldsymbol{P}_2)^2}{M_1^2M_2^2}\right.$$
$$-(6-x)\left[\frac{\boldsymbol{P}_1^4}{M_1^4}+\frac{\boldsymbol{P}_2^4}{M_2^4}+\frac{\boldsymbol{P}_1^2(\boldsymbol{P}_2\boldsymbol{n})^2+\boldsymbol{P}_2^2(\boldsymbol{P}_1\boldsymbol{n})^2}{M_1^2M_2^2}\right]$$
$$+12\frac{(\boldsymbol{P}_1\boldsymbol{P}_2)(\boldsymbol{P}_1\boldsymbol{n})(\boldsymbol{P}_2\boldsymbol{n})}{M_1^2M_2^2} + (1-x)\left(\frac{\boldsymbol{P}_1^2(\boldsymbol{P}_1\boldsymbol{n})^2}{M_1^4}+\frac{\boldsymbol{P}_2^2(\boldsymbol{P}_2\boldsymbol{n})^2}{M_2^4}\right)$$
$$\left.+3\frac{(\boldsymbol{P}_1\boldsymbol{n})^2(\boldsymbol{P}_2\boldsymbol{n})^2}{M_1^2M_2^2}\right\}. \tag{2·23}$$

§ 3. Gauge dependence of two-graviton exchange potential

In this section we shall calculate static two-graviton exchange potential $V_2^{(4)}$ to know the gauge dependence of it. As was shown by (2·11), one-graviton exchange potential $V_2^{(2)}$ up to the order $(G/r)(v/c)^2$ depends on the gauge para-

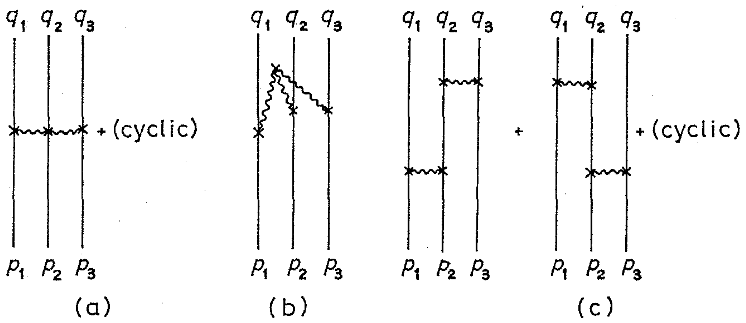

Fig. 1. Tree diagrams for the scattering of three particles. The solid and wavy lines represent a spinless particle and the graviton, respectively.

meter x. This and the *virial theorem* in classical mechanics suggest that $V_2^{(4)}$ depends also on the gauge parameter x.

In order to calculate static two-graviton exchange potential $V_2^{(4)}$ between two celestial bodies, we shall consider three-body potential $V_3^{(4)}$ among three spinless particles. The substitution law to get $V_2^{(4)}$ from $v_3^{(4)}$ was discussed by the present authors.[7] The tree diagrams which contribute to $v_3^{(4)}$ are shown in Fig. 1. The contributions from these tree diagrams to $v_3^{(4)}$ are calculated separately in the following. The gauge function $\bar{x}$ is simply written as x in the following S-matrix elements.

3. 1. *Contribntion from Fig.* 1(*a*)

Let p_i, q_i and m_i $(i=1, 2, 3)$ be the initial and the final momenta and the mass of i-th particle, respectively. The S-matrix element for the first diagram in Fig. 1(a) is given in the static limit by

$$S_a=\frac{iG^2}{2\pi^3}\frac{m_1m_2m_3}{\boldsymbol{k}_1^2\boldsymbol{k}_3^2}\{4X_{44,4\alpha}(k_1)\,X_{44,4\alpha}(k_3)$$

$$-X_{44,44}(k_1)\,X_{44,\alpha\alpha}(k_3)-X_{44,\alpha\alpha}(k_1)\,X_{44,44}(k_3)\}$$

$$\times\delta^{(4)}(p_1+p_2+p_3-q_1-q_2-q_3)\,. \tag{3·1}$$

From (2·14) we get in the static limit

$$X_{44,44}(k)=1\,,\qquad X_{44,4\alpha}(k)=\delta_{4\alpha}\,,$$

$$X_{44,\alpha\alpha}(k)=-\tfrac{1}{2}(5-x)\,.$$

Thus the S-matrix element for Fig. 1(a) is given by

$$S_a=\frac{iG^2}{2\pi^3}(9-x)\,m_1m_2m_3\left\{\frac{1}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}+\frac{1}{\boldsymbol{k}_1^2\boldsymbol{k}_3^2}+\frac{1}{\boldsymbol{k}_2^2\boldsymbol{k}_3^2}\right\}$$

$$\times\delta^{(4)}(p_1+p_2+p_3-q_1-q_2-q_3)\,, \tag{3·2}$$

where $k_i=(p_i-q_i)$. The three-body static potential obtained from (3·2) is

$$v_{3a}^{(4)}=-G^2(9-x)m_1m_2m_3\left\{\frac{1}{r_{12}r_{13}}+\frac{1}{r_{12}r_{23}}+\frac{1}{r_{13}r_{23}}\right\}, \tag{3·3}$$

r_{ij} being the distance between i-th and j-th particles.

3. 2. *Contribution from Fig.* 1(*b*)

As was shown by (3·7) in Ref. 1), the three-graviton vertex involved in Fig. 1(b) is expressed by thirteen terms. We shall call again these terms in turn as 1, 2, 3, ···, 13 following the order written in the expression (3·7).

We shall define six functions K_i by

$$K_1=k_1^2+k_2^2+k_3^2\,,$$

$$K_2=\frac{1}{4k_1^2k_2^2k_3^2}\{k_1^2k_2^2k_3^2K_1+2(k_1^4k_2^4+k_1^4k_3^4+k_2^4k_3^4)-[k_1^6(k_2^2+k_3^2)$$
$$+k_2^6(k_1^2+k_3^2)+k_3^6(k_1^2+k_2^2)]\},$$

$$K_3=\frac{1}{8k_1^2k_2^2k_3^2}\{6(k_1^4k_2^4+k_1^4k_3^4+k_2^4k_3^4)-4[k_1^6(k_2^2+k_3^2)$$
$$+k_2^6(k_1^2+k_3^2)+k_3^6(k_1^2+k_2^2)]+(k_1^8+k_2^8+k_3^8)\},$$

$$K_4=\frac{1}{4k_1^2k_2^2k_3^2}\{4k_1^2k_2^2k_3^2K_1-2(k_1^4k_2^4+k_1^4k_3^4+k_2^4k_3^4)\tag{3·4}$$
$$+(k_1^8+k_2^8+k_3^8)\},$$

$$K_5=\frac{1}{4k_1^2k_2^2k_3^2}\{2k_1^2k_2^2k_3^2K_1+2(k_1^4k_2^4+k_1^4k_3^4+k_2^4k_3^4)-2[k_1^6(k_2^2+k_3^2)$$
$$+k_2^6(k_1^2+k_3^2)+k_3^6(k_1^2+k_2^2)]+(k_1^8+k_2^8+k_3^8)\},$$

$$K_6=-\frac{1}{8k_1^2k_2^2k_3^2}\{2(k_1^4k_2^4+k_1^4k_3^4+k_2^4k_3^4)-(k_1^8+k_2^8+k_3^8)\}.$$

The S-matrix element for Fig. 1(b) has the form

$$S_b=\frac{iG^2}{2\pi^3}m_1m_2m_3\frac{1}{k_1^2k_2^2k_3^2}\sum_{i=1}^{13}N_i\delta^{(4)}(p_1+p_2+p_3-q_1-q_2-q_3).\tag{3·5}$$

Each numerator N_i in (3·5) denotes the contribution from i-th term for the three-graviton vertex. Thirteen numerators are given by

$$N_1=\tfrac{1}{2}(5-x)K_1-\tfrac{1}{2}(5-x)(1-x)K_2-\tfrac{1}{4}(1-x)^2K_3+\tfrac{1}{16}(1-x)^3K_6,$$
$$N_2=-\tfrac{1}{8}(5-x)^2K_1+\tfrac{1}{8}(5-x)^2(1-x)K_2,$$
$$N_3=-(2-x)K_1+(1-x)K_2-\tfrac{1}{4}(1-x)^2K_4-\tfrac{1}{8}(1-x)^3K_6,$$
$$N_4=\tfrac{1}{2}(5-x)(3-x)K_1+\tfrac{1}{8}(3-x)(1-x)^2K_5,$$
$$N_5=\tfrac{1}{2}(7-3x)K_1+\tfrac{1}{4}(1-x)^2K_4+\tfrac{1}{8}(1-x)^3K_6,$$
$$N_6=-\tfrac{1}{2}(5-x)^2K_1-\tfrac{1}{8}(5-x)(1-x)^2K_5,$$
$$N_7=\tfrac{1}{4}(5-x)(3-x)K_1-\tfrac{1}{2}(5-x)(1-x)K_2+\tfrac{1}{8}(5-x)(1-x)^2K_5,\tag{3·6}$$
$$N_8=\tfrac{1}{4}(5-x)(3-x)K_1-\tfrac{1}{4}(5-x)(3-x)(1-x)K_2,$$
$$N_9=-(4-2x)K_1+(3-x)(1-x)K_2+(1-x)^2K_3-\tfrac{1}{2}(1-x)^2K_5-\tfrac{1}{4}(1-x)^3K_6,$$
$$N_{10}=-\tfrac{1}{8}(5-x)^2K_1+\tfrac{1}{16}(5-x)(1-x)^2K_3,$$
$$N_{11}=\tfrac{1}{32}(5-x)^3K_1,$$
$$N_{12}=\tfrac{1}{4}(5-x)(2-x)K_1-\tfrac{1}{8}(5-x)(1-x)^2K_3,$$
$$N_{13}=-\tfrac{1}{16}(5-x)^2(3-x)K_1.$$

Substituting (3·6) into (3·5) and using (3·4), we can rewrite (3·5) in the static limit as

$$S_b = -\frac{iG^2}{\pi^3} m_1 m_2 m_3 \left\{ \left(\frac{1}{\boldsymbol{k}_1^2 \boldsymbol{k}_2^2} + \frac{1}{\boldsymbol{k}_1^2 \boldsymbol{k}_3^2} + \frac{1}{\boldsymbol{k}_2^2 \boldsymbol{k}_3^2} \right) \right.$$
$$- \frac{1}{4}(1-x) \left[\left(\frac{1}{\boldsymbol{k}_1^2} + \frac{1}{\boldsymbol{k}_2^2} \right) \frac{(\boldsymbol{k}_1 \boldsymbol{k}_2)}{\boldsymbol{k}_1^2 \boldsymbol{k}_2^2} + \left(\frac{1}{\boldsymbol{k}_1^2} + \frac{1}{\boldsymbol{k}_3^2} \right) \frac{(\boldsymbol{k}_1 \boldsymbol{k}_3)}{\boldsymbol{k}_1^2 \boldsymbol{k}_3^2} \right.$$
$$\left. \left. + \left(\frac{1}{\boldsymbol{k}_2^2} + \frac{1}{\boldsymbol{k}_3^2} \right) \frac{(\boldsymbol{k}_2 \boldsymbol{k}_3)}{\boldsymbol{k}_2^2 \boldsymbol{k}_3^2} \right] \right\} \delta^{(4)}(p_1 + p_2 + p_3 - q_1 - q_2 - q_3). \qquad (3\cdot7)$$

This matrix element leads to the three-body potential

$$v_{3b}^{(4)} = 2G^2 m_1 m_2 m_3 \left\{ \left(\frac{1}{r_{12} r_{13}} + \frac{1}{r_{12} r_{23}} + \frac{1}{r_{13} r_{23}} \right) \right.$$
$$+ \frac{1}{8}(1-x) \left[\left(\frac{1}{r_{12}^2} + \frac{1}{r_{13}^2} \right) \frac{(\boldsymbol{r}_{21} \boldsymbol{r}_{31})}{r_{12} r_{13}} + \left(\frac{1}{r_{12}^2} + \frac{1}{r_{23}^2} \right) \frac{(\boldsymbol{r}_{12} \boldsymbol{r}_{32})}{r_{12} r_{23}} \right.$$
$$\left. \left. + \left(\frac{1}{r_{13}^2} + \frac{1}{r_{23}^2} \right) \frac{(\boldsymbol{r}_{13} \boldsymbol{r}_{23})}{r_{13} r_{23}} \right] \right\}, \qquad (3\cdot8)$$

where $\boldsymbol{r}_{ij} = (\boldsymbol{x}_j - \boldsymbol{x}_i)$, and $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ denote the positions in space of i-th and j-th particles, respectively.

3. 3. *Contribution from Fig.* 1(*c*)

The first diagram in Fig. 1(c) gives the S-matrix element

$$S_{c1} = \frac{iG^2}{\pi^3} (p_{10} p_{20} p_{30} q_{10} q_{20} q_{30})^{-1/2} \frac{1}{(p_1 + p_2 - q_1)^2 + m_2^2} E(p_1, q_1; p_2, (p_1 + p_2 - q_1); k_1)$$
$$\times E(p_3, q_3; (p_1 + p_2 - q_1), q_2; k_3) \delta^{(4)}(p_1 + p_2 + p_3 - q_1 - q_2 - q_3). \qquad (3\cdot9)$$

Here the function E is defined by

$$E(A, B; C, D; k) = \frac{1}{k^2} A_\alpha B_\beta C_\gamma D_\delta X_{\alpha\beta,\gamma\delta}(k)$$
$$- \frac{1}{2}(AB + m_i^2) \frac{1}{k^2} C_\gamma D_\delta X_{\alpha\alpha,\gamma\delta}(k) - \frac{1}{2}(CD + m_j^2) \frac{1}{k^2} A_\alpha B_\beta X_{\alpha\beta,\gamma\gamma}(k)$$
$$+ \frac{1}{4}(AB + m_i^2)(CD + m_j^2) \frac{1}{k^2} X_{\alpha\alpha,\gamma\gamma}(k), \qquad (3\cdot10)$$

m_i and m_j being the masses of A and B, and C and D, respectively. The second Born term corresponding to (3·9) is

$$S_{B1} = -\frac{iG^2}{2\pi^3} (p_{10} p_{20} p_{30} q_{10} q_{20} q_{30})^{-1/2} \frac{1}{\omega_0 (p_{10} + p_{20} - q_{10} - \omega_0)}$$
$$\times E(p_1, q_1; p_2, \omega; k_1) E(p_3, q_3; \omega, q_2; k_3) \delta^{(4)}(p_1 + p_2 + p_3 - q_1 - q_2 - q_3), \quad (3\cdot11)$$

where ω is defined by

$$\omega=[(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{q}_1),\ \sqrt{(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{q}_1)^2+m_2^2}]. \tag{3·12}$$

Let us define the functions $F(A, B; C, \omega; k)$ and $F(A, B; \omega, C; k)$ by

$$\begin{aligned}
&E(A, B; C, \omega; k)-E(A, B; C, (p_1+p_2-q_1); k)\\
&\qquad\qquad=(p_{10}+p_{20}-q_{10}-\omega_0)F(A, B; C, \omega; k),\\
&E(A, B; \omega, C; k)-E(A, B; (p_1+p_2-q_1), C; k)\\
&\qquad\qquad=(p_{10}+p_{20}-q_{10}-\omega_0)F(A, B; \omega, C; k).
\end{aligned} \tag{3·13}$$

It is easy to show that these functions F's do not vanish in the limit $(p_{10}+p_{20}-q_{10}-\omega_0)\to 0$. Using the formula

$$\frac{2}{(p_1+p_2-q_1)^2+m_2^2}+\frac{1}{\omega_0(p_{10}+p_{20}-q_{10}-\omega_0)}=\frac{1}{2p_{20}^2}+0\left(k_1, k_2, \frac{1}{p_{20}^6}\right), \tag{3·14}$$

(3·9), (3·11) and (3·13), we obtain

$$\begin{aligned}
S_{c1}-S_{B1}=&\frac{iG^2}{4\pi^3}(p_{10}p_{20}p_{30}q_{10}q_{20}q_{30})^{-1/2}\delta^{(4)}(p_1+p_2+p_3-q_1-q_2-q_3)\\
&\times\left\{\frac{1}{p_{20}^2}E(p_1,q_1; p_2, (p_1+p_2-q_1); k_1)E(p_3, q_3; (p_1+p_2-q_1), q_2; k_3)\right.\\
&+\frac{2}{\omega_0}E(p_1, q_1; p_2, (p_1+p_2-q_1); k_1)F(p_3, q_3; \omega, q_2; k_3)\\
&\left.+\frac{2}{\omega_0}F(p_1, q_1; p_2, \omega; k_1)E(p_3, q_3; (p_1+p_2-q_1), q_2; k_3)\right\}\\
&+0\left(k_1, k_2; \frac{1}{p_{20}^6}\right).
\end{aligned} \tag{3·15}$$

From (2·14), (3·10), (3·13) and (3·15) we get in the static limit

$$S_{c1}-S_{B1}=-\frac{iG^2}{4\pi^3}(8-x)\frac{m_1m_2m_3}{\boldsymbol{k}_1^2\boldsymbol{k}_3^2}\delta^{(4)}(p_1+p_2+p_3-q_1-q_2-q_3). \tag{3·16}$$

Summing up all the contributions from Fig. 1(c), we finally get

$$\begin{aligned}
S_c-S_B=&-\frac{iG^2}{2\pi^3}(8-x)m_1m_2m_3\left\{\frac{1}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}+\frac{1}{\boldsymbol{k}_1^2\boldsymbol{k}_3^2}+\frac{1}{\boldsymbol{k}_2^2\boldsymbol{k}_3^2}\right\}\\
&\times\delta^{(4)}(p_1+p_2+p_3-q_1-q_2-q_3).
\end{aligned} \tag{3·17}$$

This matrix element leads to the three-body potential

$$v_{3c}^{(4)}=G^2(8-x)m_1m_2m_3\left\{\frac{1}{r_{12}r_{13}}+\frac{1}{r_{12}r_{23}}+\frac{1}{r_{13}r_{23}}\right\}. \tag{3·18}$$

3. 4. *Two-graviton exchange potential*

The three-body potentials $v_{3a}^{(4)}$, $v_{3b}^{(4)}$ and $v_{3c}^{(4)}$ obtained from Fig. 1(a), (b) and (c) are given by (3·3), (3·8) and (3·18), respectively. Using the substitution law,[7] we get the two-graviton exchange potentials $V_{2a}^{(4)}$, $V_{2b}^{(4)}$ and $V_{2c}^{(4)}$ between two celestial bodies with masses M_1 and M_2 from the three-body potentials $v_{3a}^{(4)}$, $v_{3b}^{(4)}$ and $v_{3c}^{(4)}$, respectively. They are

$$V_{2a}^{(4)} = -\frac{1}{2}(9-x)\frac{G^2M_1M_2}{r^2}(M_1+M_2),$$

$$V_{2b}^{(4)} = \frac{1}{4}(5-x)\frac{G^2M_1M_2}{r^2}(M_1+M_2), \tag{3·19}$$

$$V_{2c}^{(4)} = \frac{1}{2}(8-x)\frac{G^2M_1M_2}{r^2}(M_1+M_2),$$

which lead to the two-graviton exchange potential

$$V_2^{(4)} = \frac{1}{4}(3-x)\frac{G^2M_1M_2}{r^2}(M_1+M_2). \tag{3·20}$$

Thus $V_2^{(4)}$ is gauge dependent. It is interesting to observe that $V_2^{(4)} \simeq 0$ in the gauge $x=3$.

§ 4. Discussion

4. 1. *Perihelion motion*

As was shown by (2·11) and (3·20), both the velocity-dependent part of one-graviton exchange potential and the static part of two-graviton exchange potential are gauge-dependent. We shall show, however, that the perihelion motion of two-body system is, as it should be, gauge-independent.

The Hamiltonian for two-body system has the following form in the center-of-mass system:

$$H = \frac{1}{2}\left(\frac{1}{M_1}+\frac{1}{M_2}\right)\boldsymbol{P}^2 - \frac{1}{8}\left(\frac{1}{M_1^3}+\frac{1}{M_2^3}\right)\boldsymbol{P}^4 - \frac{GM_1M_2}{r}$$
$$-\frac{G}{r}[a\boldsymbol{P}^2 + b(\boldsymbol{Pn})^2] + c\frac{G^2M_1M_2(M_1+M_2)}{r^2}. \tag{4·1}$$

The rotation angle of the perihelion of this system for one turn is given by

$$\delta\phi = \frac{G^2M_1^2M_2^2}{\underline{M}^2}\left\{(2a+b)\frac{2M_1M_2}{(M_1+M_2)^2} + \frac{1}{M_1^3} + \frac{1}{M_2^3}\right)\left(\frac{M_1M_2}{M_1+M_2}\right)^3 - 2c\Big\}, \tag{4·2}$$

where $\underline{M}$ denotes the angular momentum of this system and a, b and c are gauge-dependent.

At a glance of this expression, one may feel that $\delta\phi$ cannot be gauge-invariant, because a and b have the mass-dependent coefficient $M_1M_2/(M_1+M_2)^2$ but c has not it. However this feeling is not right. In fact we get from (2·11) and (2·12),

$$a=\frac{1}{2}+\frac{1}{4}(7-x)\frac{(M_1+M_2)^2}{M_1M_2},$$
$$b=\frac{1}{2}-\frac{1}{4}(1-x)\frac{(M_1+M_2)^2}{M_1M_2}. \qquad (4\cdot3)$$

We see that the x-dependent parts of a and b have the mass-dependent factor $(M_1+M_2)^2/M_1M_2$. Substituting (4·3) into (4·2), we get

$$\delta\phi=\frac{G^2M_1^2M_2^2}{\underline{M}^2}\left\{\frac{3M_1M_2}{(M_1M_2)^2}+\left(\frac{1}{M_1^3}+\frac{1}{M_2^3}\right)\left(\frac{M_1M_2}{M_1+M_2}\right)^3+\frac{13}{2}-\frac{1}{2}(x+4c)\right\}. \qquad (4\cdot4)$$

We get, on the other hand, from (3·20)

$$c=\tfrac{1}{4}(3-x). \qquad (4\cdot5)$$

Thus it was shown that $\delta\phi$ is gauge-independent.

4. 2. *Coordinate transformation*[8)]

In this subsection we shall show that our Hamiltonian (4·1) with (4·3) and (4·5) is obtained from the standard one

$$H=\frac{1}{2}\left(\frac{1}{M_1}+\frac{1}{M_2}\right)\boldsymbol{P}^2-\frac{1}{8}\left(\frac{1}{M_1^3}+\frac{1}{M_2^3}\right)\boldsymbol{P}^4$$
$$-\frac{GM_1M_2}{r}\left\{1+\frac{1}{2}\left[3\left(\frac{1}{M_1^2}+\frac{1}{M_2^2}\right)+\frac{7}{M_1M_2}\right]\boldsymbol{P}^2+\frac{(\boldsymbol{Pn})^2}{2M_1M_2}\right\}+\frac{G^2M_1M_2(M_1+M_2)}{2r^2} \qquad (4\cdot6)$$

given by Einstein, Infeld and Hoffman,[3)] by a coordinate transformation in the form

$$\boldsymbol{r}'=\boldsymbol{r}\left[1+\alpha\frac{G(M_1+M_2)}{r}\right], \qquad (4\cdot7)$$

where α is a dimensionless constant.

From (4·7)

$$\boldsymbol{r}=\boldsymbol{r}'\left[1-\alpha\frac{G(M_1+M_2)}{r'}\right],$$
$$\boldsymbol{P}=\boldsymbol{P}'+\frac{\alpha G(M_1+M_2)}{r'}\left[\boldsymbol{P}'-\frac{\boldsymbol{r}'(\boldsymbol{P}'\boldsymbol{r}')}{r'^2}\right] \qquad (4\cdot8)$$

are obtained. Substituting (4·8) into (4·6), we get

$$H \simeq \frac{1}{2}\left(\frac{1}{M_1}+\frac{1}{M_2}\right)(\boldsymbol{P}')^2-\frac{1}{8}\left(\frac{1}{M_1^3}+\frac{1}{M_2^3}\right)(\boldsymbol{P}')^4-\frac{GM_1M_2}{r'}$$
$$-\frac{G}{r'}\left\{\left[\frac{1}{2}+\left(\frac{3}{2}-\alpha\right)\frac{(M_1+M_2)^2}{M_1M_2}\right](\boldsymbol{P}')^2+\left[\frac{1}{2}+\alpha\frac{(M_1+M_2)^2}{M_1M_2}\right](\boldsymbol{P}'\boldsymbol{n}')\right\}$$
$$+\left(\frac{1}{2}-\alpha\right)\frac{G^2M_1M_2(M_1+M_2)}{(r')^2}\,. \tag{4·9}$$

When

$$\alpha=-\tfrac{1}{4}(1-x), \tag{4·10}$$

the expression (4·9) coincides with (4·1) with (4·3) and (4·5).

Thus it was shown explicitly that there is no physical meaning to fix the gauge parameter x to a special value, $x=1$ or $x=3$, etc. This is, as is well known, due to the general covariance of the theory.

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