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Gauge transformation in Einstein–Yang–Mills theories

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We discuss the relation between space–time diffeomorphisms and gauge transformations in theories of the Yang–Mills type coupled with Einstein's general relativity. We show that local symmetries of the Hamiltonian and Lagrangian formalisms of these generally covariant gauge systems are equivalent when gauge transformations are required to induce transformations which are projectable under the Legendre map. Although pure Yang–Mills gauge transformations are projectable by themselves, diffeomorphisms are not. Instead, the projectable symmetry group arises from infinitesimal diffeomorphism-inducing transformations which must depend on the lapse function and shift vector of the space–time metric plus associated gauge transformations. Our results are generalizations of earlier results by ourselves and by Salisbury and Sundermeyer. © *2000 American Institute of Physics.* [S0022-2488(00)02308-2]

I. INTRODUCTION

In a recent paper[1] we discussed the relation between diffeomorphisms and gauge transformations in general relativity. Specifically, gauge transformations are required to be projectable under the Legendre map, and therefore they must depend on the lapse function and shift vector of the metric in a given coordinate neighborhood. Therefore, it is not the diffeomorphism group, which acts on the underlying manifold, which is the gauge group. The gauge group acts on the dynamical variables in the space of field configurations (including the metric); its structure is fixed by the dynamical model; but each element may also be interpreted as a family of space–time diffeomorphisms. More precisely, each pair consisting of an element of the gauge group and a metric on which it acts determines a space–time diffeomorphism (which affects tensors in the usual way).

Here we extend the discussion to include space–times having a Yang–Mills type field coupled to general relativity. Our work is an extension of a more formal treatment by Pons and Shepley.[2] Some of these results were obtained earlier by Salisbury and Sundermeyer,[3,4] Lee and Wald[5] (and others), but we have given them a broader foundation, namely one based on projectability under the Legendre map while retaining all the gauge variables. Our resulting expressions for the gauge generators are entirely new. The idea that coordinate transformation should be accompanied by gauge transformations dates back a rather long way. The articles by Jackiw[6] and Jackiw and Manton,[7] summarized by Jackiw,[8] discuss this idea but not from the point of view we espouse here, namely as a result of relating Lagrangian and Hamiltonian formulations of the theory. In passing, we should note that besides eliminating gauge variables through a quotienting procedure,

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the Lee and Wald[5] approach is incomplete in that it does not take into account that Lagrangian energies might not be projectable to the quotient space. We recently extended and completed their program by introducing an algorithmic procedure, which under most circumstances is equivalent to the Dirac–Bergmann algorithm.[9] Furthermore, our procedure is accomplished without quotienting out gauge variables. The Dirac–Bergmann constraint algorithm requires that evolution remain within the final constraint surface in phase space.

We find that pure Yang–Mills gauge transformations meet our requirement of projectability. Gauge transformations which act like diffeomorphisms not only have to be coupled to the metric as in the vacuum case but also require associated Yang–Mills gauge transformations.

In Sec. II we briefly recount the general treatment of diffeomorphism-invariant theories. We discuss Einstein–Yang–Mills field theory and describe (infinitesimal) gauge transformations therein. We show explicitly how these transformations must depend on the lapse function and shift vector of the space–time metric and what associated Yang–Mills gauge transformations they must have if they are to be projectable under the Legendre map. In Sec. III, we calculate the group structure functions and the canonical group generators. Section IV concludes with a general discussion of our results and future extensions. These will include the application of our procedures to the real triad formulation[10,11] and to the Ashtekar formulation[12] of general relativity.

II. YANG–MILLS THEORIES AND GENERAL RELATIVITY

As in our previous paper,[1] following the work of Batlle *et al.*,[13] we begin with a Lagrangian $L(q,\dot{q})$ which does does not depend explicitly on t. An infinitesimal transformation $\delta q^i(q,\dot{q},t)$ is a Noether Lagrangian symmetry if

$$\delta L = dF/dt,$$

which results in an equation for

$$G := \frac{\partial L}{\partial \dot{q}^i}\,\delta q^i - F, \tag{1}$$

namely

$$[L]_i \delta q^i + \frac{dG}{dt} = 0,$$

$[L]_i$ being the Euler–Lagrange functional derivative of L:

$$[L]_i = \alpha_i - W_{is}\ddot{q}^s,$$

where

$$W_{ij} := \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}, \quad \alpha_i := -\frac{\partial^2 L}{\partial \dot{q}^i \partial q^s}\dot{q}^s + \frac{\partial L}{\partial q^i}.$$

When the mass matrix or Legendre matrix $\mathbf{W} = (W_{ij})$ is singular, there exists a kernel for the pullback $\mathcal{F}L^*$ of the Legendre map $\mathcal{F}L$ from configuration-velocity space TQ (the tangent bundle TQ of the configuration space Q) to phase space T^*Q (the cotangent bundle). This kernel is spanned by vector fields whose components γ_A^i (A ranges over the number of these vectors) are a basis for the null vectors of W_{ij}. The Hamiltonian technique eases the calculation of the γ_A^i:

$$\gamma_A^i = \mathcal{F}L^*\left(\frac{\partial \phi_A}{\partial p_i}\right), \tag{2}$$

where the ϕ_A are the Hamiltonian primary first class constraints. Note that these constraints are here assumed to be effective (if not, they can be made effective; however, problems can arise when ineffective, secondary constraints, occur[9,14]).

The equation satisfied by G implies

$$\gamma_A^i \frac{\partial G}{\partial \dot{q}^i} = 0, \tag{3}$$

showing that G is projectable to a function G_H in T^*Q; that is, it is the pullback of a function (not necessarily unique) in T^*Q:

$$G = \mathcal{F}L^*(G_H) \tag{4}$$

(first pointed out by Kamimura[15]). The function G_H is determined up to the addition of linear combinations of the primary constraints. When δq^i is projectable to T^*Q, it is possible to select G_H satisfying (4) and such that

$$\delta q^i = \mathcal{F}L^* \left(\frac{\partial G_H}{\partial p_i} \right). \tag{5}$$

We will apply this result to diffeomorphisms and to Yang–Mills gauge transformations in the following.

A. Yang–Mills gauge transformations

The Yang–Mills Lagrangian density $\mathcal{L}_{\mathrm{YM}}$ is a functional of the vector potential fields A_μ^i, where the internal index i ranges over $\{1,...,n\}$, where n is the dimension of the gauge group, and μ is a space–time index ($\mu = 0,...,3$). (We will be using lower-case indices from the beginning of the alphabet, $a,b,...$, as spatial indices, $a,b = 1,2,3$.) The field tensor derived from these potential fields is

$$F^i_{\alpha\beta} = A^i_{\beta,\alpha} - A^i_{\alpha,\beta} - C^i_{jk} A^j_\alpha A^k_\beta, \tag{6}$$

where the comma denotes partial differentiation and where C^i_{jk} are the structure constants of the gauge group. The Yang–Mills Lagrangian density is given by

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{4}\sqrt{|{}^4 g|} F^i_{\mu\nu} F^j_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta} C_{ij}, \tag{7}$$

where C_{ij} is a nonsingular, symmetric group metric (its inverse is C^{ij}) and ${}^4 g$ is the determinant of the space–time metric tensor. (In a semi-simple group, C_{ij} is usually taken to be $C^s_{it} C^t_{js}$; in an Abelian group, one usually takes $C_{ij} = \delta_{ij}$.)

The derivatives of $\mathcal{L}_{\mathrm{YM}}$ with respect to the velocities of the configuration space variables, $\dot{A}^i_\alpha$ (here the dot is $\partial/\partial t$), give the tangent space functions $\hat{P}^\alpha_i$ corresponding to the phase space conjugate momenta:

$$\hat{P}^\alpha_i := \frac{\partial \mathcal{L}_{\mathrm{YM}}}{\partial \dot{A}^i_\alpha} = \sqrt{|{}^4 g|} F^j_{\mu\nu} g^{\alpha\mu} g^{0\nu} C_{ij}. \tag{8}$$

The Legendre map $\mathcal{F}L$ is defined by setting $\hat{P}^\alpha_i$ equal to P^α_i in phase space. Because of the antisymmetry of the field tensor, the primary constraints are

$$0 = \hat{P}_i := \hat{P}^0_i = \frac{\partial \mathcal{L}_{\mathrm{YM}}}{\partial \dot{A}^i_0} = \sqrt{|{}^4 g|} F^j_{\mu\nu} g^{0\mu} g^{0\nu} C_{ij}. \tag{9}$$

A generator of a projectable gauge transformation thus must be independent of $\dot{A}^i_0$.

An infinitesimal Yang–Mills gauge transformation is defined by an array of gauge descriptors Λ^i and transforms the potential by

$$\delta_R[\Lambda]A^i_\mu = -\Lambda^i_{,\mu} - C^i_{jk}\Lambda^j A^k_\mu \tag{10}$$

(we use the notation $\delta_R[\Lambda]$ for this Yang–Mills rotation variation to distinguish it from other variations defined later, and we write δ_R if the $[\Lambda]$ may be understood in context). We denote this transformation by

$$\delta_R A^i_\mu := -(\mathcal{D}_\mu \Lambda)^j, \tag{11}$$

where $\mathcal{D}_\mu$ is the Yang–Mills covariant derivative (in its action on space–time scalars and Yang–Mills vectors). Under this transformation, the field transforms as

$$\delta_R F^i_{\mu\nu} = -C^i_{jk}\Lambda^j F^k_{\mu\nu}, \tag{12}$$

where we work to first order in Λ^i and use the Jacobi identity

$$C^i_{jk}C^k_{mn} + C^i_{mk}C^k_{nj} + C^i_{nk}C^k_{jm} = 0.$$

The Yang–Mills Lagrangian $\mathcal{L}_{\rm YM}$ is invariant under this transformation provided that the group metric obeys

$$C^k_{mi}C_{kj} = -C^k_{mj}C_{ki}$$

(which it will if $C_{ij} = C^s_{it}C^t_{js}$).

The variation δ_R is clearly independent of $\dot{A}^i_0$ and so is projectable.

B. Diffeomorphisms

The configuration space variables for general relativity are the components of the metric tensor

$$ds^2 = g_{\mu\nu}dx^\mu\, dx^\nu = -N^2\, dt^2 + g_{ab}(dx^a + N^a\, dt)(dx^b + N^b\, dt), \tag{13}$$

where N is the lapse function, N^a the components of the shift vector, and g_{ab} is our notation for the spatial metric. The inverse of g_{ab} is e^{ab}:

$$e^{ac}g_{bc} = \delta^a_b.$$

We will use g for the determinant of the spatial metric; the relationship between it and the determinant of the space–time metric is

$${}^4g = -N^2 g.$$

In matrix form the metric and its inverse are

$$(g_{\mu\nu}) = \begin{pmatrix} -N^2 + N^c N^d g_{cd} & g_{ac}N^c \\ g_{bd}N^d & g_{ab} \end{pmatrix},$$

$$(g^{\mu\nu}) = \begin{pmatrix} -1/N^2 & N^a/N^2 \\ N^b/N^2 & e^{ab} - N^a N^b N^2 \end{pmatrix}.$$

The general relativity Lagrangian density is[16]

$$\mathcal{L}_{\rm GR}=N\sqrt{g}({}^3R+K_{ab}K^{ab}-(K^a_a)^2), \tag{14}$$

where ${}^3R$ is the scalar curvature computed from the three-metric (${}^3R={}^3R_{ab}e^{ab}$, where ${}^3R_{ab}$ is the three-metric Ricci tensor) and K_{ab} is the second fundamental form (extrinsic curvature; indices raised by e^{ab} or lowered by g_{ab}) for the constant-time three-surfaces:

$$K_{ab}=\frac{1}{2N}(\dot{g}_{ab}-N_{a|b}-N_{b|a}), \tag{15}$$

with the vertical bar meaning covariant differentiation with respect to the three-metric connection. Thus the total Lagrangian density is

$$\mathcal{L}=\mathcal{L}_{\rm YM}+\mathcal{L}_{\rm GR}. \tag{16}$$

Notice that the lapse N and shift N^a of the four-metric all appear, but their time derivatives (that is, their velocities) do not. This is required of any diffeomorphism invariant theory. To be projectable, therefore, a variation must be independent of these velocities as well as being independent of $\dot{A}^i_0$ in coupled Einstein–Yang–Mills theory.

Consider now an infinitesimal diffeomorphism, which changes the coordinates by

$$\delta_D[\epsilon]x^\mu=-\epsilon^\mu \tag{17}$$

(we write δ_D if the $[\epsilon]$ may be understood in context). Under this diffeomorphism, the space–time metric transforms as

$$\delta_D g_{\mu\nu}=g_{\mu\nu,\sigma}\epsilon^\sigma+g_{\sigma\nu}\epsilon^\sigma_{,\mu}+g_{\mu\sigma}\epsilon^\sigma_{,\nu}. \tag{18}$$

This is the Lie derivative equation.

We will show from this equation that δ_D is not a projectable transformation of the form of Eq. (5) unless it is made to depend on the lapse and shift variables. We will also show that δ_D is not allowed to depend on the Yang–Mills potential A^i_0. Finally, we will look at the variation of the Yang–Mills potential itself and show that if a new variation is defined to include a gauge transformation along with each diffeomorphism, the new variation will be projectable. We now proceed with these demonstrations.

Equation (18) implies that the variations of the lapse and shift due to a diffeomorphism are

$$\delta_D N=\dot{N}\epsilon^0+N_{,a}\epsilon^a+N\dot{\epsilon}^0-NN^a\epsilon^0_{,a}, \tag{19a}$$

$$\delta_D N^a=\dot{N}_a\epsilon^0+N^a_{,b}\epsilon^b+N^a\dot{\epsilon}^0-(N^2e^{ab}+N^aN^b)\epsilon^0_{,b}+\dot{\epsilon}^a-N^b\epsilon^a_{,b}. \tag{19b}$$

In order to eliminate the dependence of $\dot{N},\dot{N}^a$ from these variations, it is necessary that the ϵ^μ depend on the lapse and shift:[1]

$$\epsilon^0=\frac{\xi^0}{N},\quad \epsilon^a=\xi^a-\frac{N^a}{N}\xi^0, \tag{20}$$

where ξ^0,ξ^a are independent of N, N^a. Note that

$$\epsilon^\mu=\delta^\mu_a\xi^a+n^\mu\xi^0, \tag{21}$$

where n^μ is the unit normal to the $t=$const spacelike hypersurfaces:

$$n^0=\frac{1}{N},\quad n^a=-\frac{N^a}{N}.$$

A diffeomorphism with only the descriptor ξ^0 not zero is called a perpendicular diffeomorphism.

Furthermore, Eq. (19) shows that ϵ^μ cannot depend on A^i_0: Equation (19a) has a term $N\dot{\epsilon}^0$ which would involve $\dot{A}^i_0$ otherwise; and similarly, Eq. (19b) has a term $\dot{\epsilon}^a$ which would involve $\dot{A}^i_0$ unless such a dependence is outlawed.

Under a diffeomorphism, the Yang–Mills potential transforms as a covariant vector field under Lie differentiation:

$$\delta_D A^i_\mu = A^i_{\mu,\sigma}\epsilon^\sigma + A^i_\sigma \epsilon^\sigma_{,\mu}. \tag{22}$$

The variation of A^i_a is clearly independent of $\dot{N},\dot{N}^a,\dot{A}^i_0$ and so is projectable. However, the δ_D variation of A^i_0 is

$$\delta_D A^i_0 = \dot{A}^i_0\epsilon^0 + A^i_0\dot{\epsilon}^0 + A^i_a\dot{\epsilon}^a + A^i_{0,a}\epsilon^a. \tag{23}$$

It clearly is not projectable, nor does the dependence of ϵ^μ on the lapse and shift, Eq. (20), and the nondependence of ϵ^μ on A^i_0 help. What is needed is a combined diffeomorphism and gauge transformation.

Therefore, to δ_D we add a gauge transformation $\delta_R[M]$ defined by a gauge descriptor M^i:

$$(\delta_D + \delta_R[M])A^i_0 = \dot{A}^i_0\epsilon^0 + A^i_0\dot{\epsilon}^a + A^i_a\dot{\epsilon}^a + A^i_{0,a}\epsilon^a - \dot{M}^i - C^i_{jk}M^j A^k_0. \tag{24}$$

The most direct way of making this variation projectable, that is, to cancel the first three terms on the right-hand side, clearly is to choose M^i to be $A^i_\sigma\epsilon^\sigma$ (since the resulting addition of a term involving A^i_a is harmless). To this expression may be added an arbitrary additional gauge transformation, of course, provided it will not result in terms involving $\dot{N},\dot{N}^a,A^i_0$ in Eq. (24). The subtraction from $A^i_\sigma\epsilon^\sigma$ of the expression $A^i_a\xi^a$ represents just such a transformation; what remains will be a term proportional to n^μ, according to Eq. (21). For what comes later, therefore, we find it convenient to define $\delta_D + \delta_R[M]$ by using

$$M^i := A^i_\sigma n^\sigma \xi^0. \tag{25}$$

To this variation may be added an arbitrary pure Yang–Mills gauge transformation, and so a general projectable variation will depend on the descriptors

$$\xi^A := (\xi^0, \xi^a, \Lambda^i),$$

there being $4+n$ functions in all. In summary, a general projectable variation δ acts as a combined infinitesimal diffeomorphism and gauge transformation of the form:

$$\delta N = \dot{\xi}^0 + \xi^a N_{,a} - N^a \xi^0_{,a}, \tag{26a}$$

$$\delta N^a = \dot{\xi}^a - N e^{ab}\xi^0_{,b} + N_{,b}e^{ab}\xi^0 + N^a_{,b}\xi^b - N^b\xi^a_{,b}, \tag{26b}$$

$$\delta g_{ab} = \dot{g}_{ab}\frac{\xi^0}{N} + g_{ab,c}\left(\xi^c - \frac{N^c\xi^0}{N}\right) + g_{cb}\left(\xi^c_{,a} - \frac{N^c_{,a}\xi^0}{N}\right) + g_{ac}\left(\xi^c_{,b} - \frac{N^c_{,b}\xi^0}{N}\right), \tag{26c}$$

$$\delta A^i_0 = A^i_a\dot{\xi}^0 + A^i_{0,a}\xi^a + F^i_{0a}\frac{N^a\xi^0}{N} - \dot{\Lambda}^i - C^i_{jk}\Lambda^j A^k_0, \tag{26d}$$

$$\delta A^i_a = F^i_{0a}\frac{\xi^0}{N} + F^i_{ab}\frac{N^b\xi^0}{N} + A^i_b\xi^b_{,a} + A^i_{a,b}\xi^b - \Lambda_{,a} - C^i_{jk}\Lambda^j A^k_a. \tag{26e}$$

C. Hamiltonian dynamics

To discuss the group structure functions and the canonical group generators, we work in the Hamiltonian formulation. First, consider the Lagrangian energy for the Yang–Mills part of the action:

$$\hat{\mathcal{H}}_{\mathrm{YM}} := \dot{A}^i_\alpha \hat{P}^\alpha_i - \mathcal{L}_{\mathrm{YM}} = \frac{N}{2\sqrt{g}} C^{ij} g_{ab} \hat{P}^a_i \hat{P}^b_j + N^a \hat{P}^b_i F^i_{ab} + \frac{N\sqrt{g}}{4} C_{ij} e^{ac} e^{bd} F^i_{ab} F^i_{cd} - A^i_0 \mathcal{D}_a \hat{P}^a_i \,, \tag{27}$$

where C^{ij} is the matrix inverse of the group metric C_{ij}, and we performed an integration by parts to obtain the last term.

Similarly, we can define the Lagrangian momentum functions for the Hilbert action:

$$\hat{p}^{ab} := \frac{\partial \mathcal{L}_{\mathrm{GR}}}{\dot{g}_{ab}} = \sqrt{g}(K^{ab} - K^c_c e^{ab}), \tag{28}$$

and then compute the Lagrangian energy:

$$\hat{\mathcal{H}}_{\mathrm{GR}} := \hat{p}^{ab} \dot{g}_{ab} - \mathcal{L}_{\mathrm{GR}} = \frac{H}{\sqrt{g}} (\hat{p}_{ab} \hat{p}^{ab} - (\hat{p}^a_a)^2) - N\sqrt{g}\,{}^3R - 2N^a \hat{p}^b_{a|b}\,, \tag{29}$$

where the last term results from an integration by parts.

Thus the canonical Hamiltonian (whose pullback under the Legendre transformations is the Lagrangian energy) is of the form

$$H_c = \int d^3x N^A \mathcal{H}_A\,, \tag{30}$$

where N^A are the $3+n$ variables $N, N^a, -A^i_0$ whose conjugate momenta give the primary constraints $P_A = \{p, p_a, -P_i\} = 0$, and $\mathcal{H}_A = \{\mathcal{H}_0, \mathcal{H}_a, \mathcal{H}_i\}$. The time derivatives of the primary constraints are secondary constraints:

$$\dot{P}_A = \{P_A, H_c\} = -\mathcal{H}_A\,.$$

There are no more constraints. Explicitly,

$$\mathcal{H}_0 = \frac{1}{2\sqrt{g}} C^{ij} g_{ab} P^a_i P^b_j + \frac{\sqrt{g}}{4} C_{ij} e^{ac} e^{bd} F^i_{ab} F^j_{cd} + \frac{1}{\sqrt{g}} (p_{ab} p^{ab} - (p^c_c)^2) - \sqrt{g}\,{}^3R, \tag{31a}$$

$$\mathcal{H}_a = P^b_i F^i_{ab} - 2p^b_{a|b}\,, \tag{31b}$$

$$\mathcal{H}_i = \mathcal{D}_a P^a_i\,. \tag{31c}$$

We summarize our notation in the following list:

Configuration variables	g_{ab}	A^i_a	N		N^a		A^i_0		
Momentum variables	p^{ab}	P^a_i	p		p_a		P_i		
Primary constraints			p	$=$	p_a	$=$	P_i	$=$	0
Secondary constraints			$\mathcal{H}_0$	$=$	$\mathcal{H}_a$	$=$	$\mathcal{H}_i$	$=$	0

The equations of motion which follow from the Hamiltonian equations (30) are (these equations agree with those in Refs. 16 and 4):

$$\dot{g}_{ab}=\{g_{ab},H_c\}=\frac{2N}{\sqrt{g}}\left(p_{ab}-\frac{1}{2}p^c_c g_{ab}\right)+N_{a|b}+N_{b|a}, \tag{32a}$$

$$\dot{A}^i_a=\{A^i_\alpha,H_c\}=\frac{N}{\sqrt{g}}C^{ij}g_{ab}P^b_j-N^bF^i_{ab}+\mathcal{D}_aA^i_0, \tag{32b}$$

$$\begin{aligned}\dot{p}^{ab}=\{p^{ab},H_c\}=&-\frac{N}{\sqrt{g}}\left({}^3R^{ab}-\frac{1}{2}{}^3\mathrm{R}e^{ab}\right)+\frac{N}{2\sqrt{g}}e^{ab}\left(p^{cd}p_{cd}-\frac{1}{2}(p^c_c)^2\right)-\frac{2N}{\sqrt{g}}\left(p^{ac}p^b_c-\frac{1}{2}p^c_cp^{ab}\right)\\&+\sqrt{g}(N_{|ab}-e^{ab}N^{|c}{}_c)+(N^cp^{ab})_{|c}-2p^{c(a}N^{b)}{}_{|c}\\&+\frac{N}{2\sqrt{g}}C^{ij}\left(\frac{1}{2}e^{ab}g_{cd}P^c_iP^d_j-P^a_iP^b_j\right)\\&+\frac{N}{4}C_{ij}\sqrt{g}\left(2F^i_{cd}F^j_{ef}e^{ca}e^{eb}e^{df}-\frac{1}{2}F^i_{cd}F^j_{ef}e^{ab}e^{ce}e^{df}\right),\end{aligned} \tag{32c}$$

$$\dot{P}^a_i=\{P^a_i,H_c\}=2\mathcal{D}_b(N^{[b}P^{a]}_i)+\mathcal{D}_b(N\sqrt{g}C_{ij}e^{c[b}e^{a]d}F^j_{cd})+A^m_0C_{mj}C^j_{\ell i}C^{\ell k}P^a_k. \tag{32d}$$

Of course, Eqs. (32a) and (32b) are restatements of the definition of momenta.

We now derive the most general projectable variations of configuration and Lagrangian momentum variables. In Sec. III we construct the corresponding phase space generators of these variations.

First, we write down the most general projectable variation of the configuration variables, dependent on the descriptors ξ^0,ξ^a,Λ^i [these are the same as Eq. (26) but in our present notation; we have also used the notation of covariant differentiation with respect to the three-metric connection]:

$$\delta N=\dot{\xi}^0+\xi^aN_{,a}-N^a\xi^0_{,a}, \tag{33a}$$

$$\delta N^a=\dot{\xi}^a-Ne^{ab}\xi^0_{,b}+N_{,b}e^{ab}\xi^0+N^a_{|b}\xi^b-N^b\xi^a_{|b}, \tag{33b}$$

$$\delta g_{ab}=\frac{2\xi^0}{\sqrt{g}}\left(p_{ab}-\frac{1}{2}p^c_cg_{ab}\right)+\xi_{a|b}+\xi_{b|a}, \tag{33c}$$

$$\delta A^i_0=A^i_a\dot{\xi}^\alpha+A^i_{0,\alpha}\xi^a+\frac{N^\alpha\xi^0}{\sqrt{g}}C^{ij}g_{ab}P^b_j-\dot{\Lambda}^i-C^i_{jk}\Lambda^jA^k_0, \tag{33d}$$

$$\delta A^i_a=\frac{\xi^0}{\sqrt{g}}C^{ij}g_{ab}P^b_j+A^i_b\xi^b_{|a}+A^i_{a|b}\xi^b-\Lambda^i_{,a}-C^i_{jk}\Lambda^iA^k_a. \tag{33e}$$

Note also for future reference that the variations of A^i_μ which result from an infinitesimal spatial diffeomorphism $x'^\mu=x^\mu-\delta^\mu_a\xi^a$ plus a gauge rotation with descriptor $\Lambda^i=A^i_b\xi^b$ are

$$\delta A^i_0=-\xi^aF^i_{0a}=-\frac{\xi^a}{\sqrt{g}}NP^i_a-\xi^aN^bF^i_{ba}, \tag{34a}$$

$$\delta A^i_a=-\xi^bF^i_{ab}. \tag{34b}$$

We turn now to variations of the conjugate momenta. Observe that under infinitesimal general coordinate transformations for which $\delta x^0 \neq 0$, the variation (Lie derivative) will involve the time derivative. The projectable gauge transformations of the momentum variables from configuration-velocity space to phase space are limited to solutions of the equations of motion, since we use the equations of motion in computing the variations. Since the time derivatives of momenta always appear in their variations under general coordinate transformations which alter the evolution time, we note that this is a general feature of generally covariant systems: The full projectable diffeomorphism group is a transformation group on solution trajectories.

To find the variations of p^{ab}, we use the fact that p^{ab} appear in the four-dimensional connection coefficients $\Gamma^{\alpha}_{\beta\gamma}$. Thus p^{ab} can be calculated from the four-dimensional connection by

$$p^{ab} = \frac{1}{N} \mathcal{G}^{abcd} \Gamma^{0}_{cd}, \tag{35}$$

where

$$\mathcal{G}^{abcd} := \sqrt{g}(e^{ac}e^{bd} - e^{ab}e^{cd}). \tag{36}$$

The inverse of this object is

$$\mathcal{G}_{abcd} = \frac{1}{\sqrt{g}} \left(g_{ac}g_{bd} - \frac{1}{2} g_{ab}g_{cd} \right) \tag{37}$$

in the sense that

$$\mathcal{G}_{abcd}\mathcal{G}^{cdef} = \delta^{e}_{a}\delta^{f}_{b}. \tag{38}$$

The general variation of the connection coefficients (under an infinitesimal diffeomorphism defined by $x'^{\mu} = x^{\mu} - \epsilon^{\mu}$) is

$$\delta\Gamma^{\alpha}_{\beta\gamma} = -\Gamma^{\sigma}_{\beta\gamma}\epsilon^{\alpha}_{,\sigma} + \Gamma^{\alpha}_{\sigma\gamma}\epsilon^{\sigma}_{,\beta} + \Gamma^{\alpha}_{\beta\sigma}\epsilon^{\sigma}_{,\gamma} + \epsilon^{\alpha}_{,\beta\gamma} + \Gamma^{\alpha}_{\beta\gamma,\sigma}\epsilon^{\sigma}, \tag{39}$$

and thus

$$\delta\Gamma^{0}_{cd} = -\Gamma^{\sigma}_{cd}\epsilon^{0}_{,c} + \Gamma^{0}_{\sigma c}\epsilon^{\sigma}_{,b} + \epsilon^{0}_{b\sigma}\epsilon^{\sigma}_{,c} + \epsilon^{0}_{,bc} + \Gamma^{0}_{bc,\sigma}\epsilon^{\sigma}. \tag{40}$$

We therefore need the following relationships:

$$\Gamma^{0}_{cd} = \frac{1}{N} \mathcal{G}_{cdef} p^{ef}, \tag{41a}$$

$$\Gamma^{0}_{0d} = g^{0\mu}\Gamma_{\mu 0d} = \frac{1}{N} N_{,d} + N^{-1}N^{e}\mathcal{G}_{edgh}p^{gh}, \tag{41b}$$

$$\Gamma^{e}_{cd} = -\frac{1}{N} N^{e}\mathcal{G}_{cdfg}p^{fg} + {}^{3}\Gamma^{e}_{cd}. \tag{41c}$$

The calculation is far from trivial, but the most difficult part is made somewhat easier by defining, for any function f,

$$\delta' f := f'(x') - f(x) \Rightarrow \delta f = \delta' f + f_{,\sigma}\epsilon^{\sigma}. \tag{42}$$

By concentrating on the δ' variation for $\epsilon^{\sigma} = n^{\sigma}\xi^{0}$, using the equation of motion for the derivative term, and then adding the rather straightforward calculation for ξ^{a} (treating p^{ab} as a tensor density), we find

$$\delta p^{ab} = -\xi^0\sqrt{g}\left({}^3R^{ab} - \frac{1}{2}{}^3\,\mathrm{R}e^{ab}\right) + \frac{1}{2}\xi^0\sqrt{g}e^{ab}\left(p^{cd}p_{cd} - \frac{1}{2}(p^c_c)^2\right) - \frac{2}{\sqrt{g}}\xi^0\left(p^{ac}p^b_c - \frac{1}{2}(p^c_c)^2\right)$$
$$+\sqrt{g}(e^{ac}e^{bd}\xi^0_{|cd} - e^{ab}\xi^{0|c}{}_c) + \frac{1}{2\sqrt{g}}\xi^0 C^{ij}\left(\frac{1}{2}e^{ab}g_{cd}P^c_iP^d_j - P^a_iP^b_j\right)$$
$$+\frac{1}{4}\xi^0 C_{ij}\sqrt{g}\left(2F^i_{cd}F^j_{ef}e^{ca}e^{eb}e^{df} - \frac{1}{2}F^i_{cd}F^j_{ef}e^{ab}e^{ce}e^{df}\right) + p^{ab}\xi^c_{,c} - \xi^a_{,c}p^{cb} - \xi^b_{,c}p^{ac} + p^{ab}_{,c}\xi^c. \tag{43}$$

The ξ^0 part of the variation can be obtained from the equation of motion (32c) by replacing N by ξ^0 and setting $N^a = 0$.

To compute variations of the P^a_i, in principle uses the same method, namely by using the fact that P^a_i comes from a four-dimensional object, from Eq. (9). The result is

$$\delta P^a_i = \mathcal{D}_b(\xi^0\sqrt{g}C_{ij}e^{be}e^{ad}F^j_{cd}) + P^a_i\xi^b_{,b} - \xi^a_{,b}P^b_i + P^a_{i,b}\xi^b. \tag{44}$$

This is actually the variation $\delta_D + \delta_R[A_\mu n^\mu \xi^0]$. These results come directly from the definitions of momenta in configuration-velocity space; we will construct the generators of these equations in phase space in the following.

III. SYMMETRY GENERATORS

We now turn to the generators of the projectable variations. Generating functions G will be of the form[1]

$$G(t) = \int d^3x(\xi^A G^{(0)}_A + \dot{\xi}^A G^{(1)}_A) =: \xi^A G^{(0)}_A + \dot{\xi}^A G^{(1)}_A, \tag{45}$$

where we shall use a repeated index to include an integration over space as well as a sum. The descriptors ξ^A are arbitrary functions.

The functions in Eq. (45) are found using an extension of the techniques of Ref. 1: The simplest choice for the $G^{(1)}_A$ are the primary constraints P_A. The functions $G^{(0)}_A$ obey

$$G^{(0)}_A = -\{G^{(1)}_A, \mathcal{H}_A\} + \mathrm{pc}, \tag{46}$$

where pc represents a sum of primary constraints. The simplest solution for $G^{(0)}_A$ results in

$$G[\xi] = P_A\dot{\xi}^A + (\mathcal{H}_A + P_C N^B \mathcal{C}^C_{AB})\xi^A, \tag{47}$$

where the structure functions are defined by

$$\{\mathcal{H}_A, \mathcal{H}_B\} =: \mathcal{C}^C_{AB}\mathcal{H}_C. \tag{48}$$

We shall determine the structure functions by first examining the variations generated by the secondary constraints, Eq. (31). The emphasis throughout will be on the underlying transformation symmetry group. For this purpose we first introduce generators associated with our secondary constraints. Let

$$R[\xi] := \int d^3x\,\xi^i\mathcal{H}_i, \tag{49a}$$

$$V[\vec{\xi}] := \int d^3x\,\xi^a\mathcal{H}_a, \tag{49b}$$

$$S[\xi^0] := \int d^3x\, \xi^0 \mathcal{H}_0 . \tag{49c}$$

We find that $R[\xi]$ generates a Yang–Mills rotation, so we have, for example,

$$\{A^i_{a,} R[\xi]\} = \delta_R[\xi] A^i_a . \tag{50}$$

$V[\vec{\xi}]$ generates the spatial diffeomorphism plus the gauge rotation we employed in (34):

$$\delta_V[\vec{\xi}] A^i_a = \{A^i_a, V[\vec{\xi}]\} = \mathcal{L}_{\vec{\xi}} A^i_a + \delta_R[\xi^b A_b] A^i_a = -\xi^b F^i{}_{ab} , \tag{51}$$

where $\mathcal{L}_{\vec{\xi}}$ denotes the Lie derivative. It is convenient to define a related generator $D[\vec{\xi}]$ which generates a pure spatial diffeomorphism:

$$D[\vec{\xi}] := \int d^3x\, \xi^a \mathcal{G}_a , \tag{52}$$

where

$$\mathcal{G}_a := \mathcal{H}_a - A^i_a \mathcal{H}_i . \tag{53}$$

$S[\xi^0]$ generates a space–time diffeomorphism plus a gauge rotation (neither of which by itself is projectable). So, for example,

$$\delta_S[\xi^0] A^i_a = \delta_D[\xi^0] A^i_a + \delta_R[\xi^0 A_\mu n^\mu] A^i_a = \frac{\xi^0}{\sqrt{g}} C^{ij} g_{ab} P^b_j . \tag{54}$$

It is straightforward to calculate the complete Lie algebra from the calculable action of the infinitesimal group elements on the generators. (The only Poisson bracket we will not calculate in this manner is the bracket of $S[\xi^0]$ with $S[\eta^0]$. In principle, the entire Poisson bracket algebra can be derived from the transformation group, but this particular calculation is somewhat tedious, invoking time derivatives of the three-curvature and the extrinsic curvature. The result of the direct calculation of this bracket is given in the following.)

First, a gauge rotation of $\mathcal{H}_i$ yields

$$\{R[\xi], R[\eta]\} = -R[[\xi, \eta]]. \tag{55a}$$

The remaining brackets are

$$\{R[\xi], D[\vec{\eta}]\} = \int d^3x\, \xi^i \mathcal{L}_{\vec{\eta}} \mathcal{H}_i = -\int d^3x (\mathcal{L}_{\vec{\eta}} \xi^i) \mathcal{H}_i = -R[\mathcal{L}_{\vec{\eta}} \xi], \tag{55b}$$

$$\{D[\vec{\xi}], D[\vec{\eta}]\} = \int d^3x\, \xi^a \mathcal{L}_{\vec{\eta}} \mathcal{G}_a = -\int d^3x (\mathcal{L}_{\vec{\eta}} \xi^a) \mathcal{G}_a = -D[\mathcal{L}_{\vec{\eta}} \vec{\xi}] = D[[\vec{\xi}, \vec{\eta}]], \tag{55c}$$

$$\{S[\xi^0], D[\vec{\eta}]\} = \int d^3x\, \xi^0 \mathcal{L}_{\vec{\eta}} \mathcal{H}_0 = -\int d^3x (\mathcal{L}_{\vec{\eta}} \xi^0) \mathcal{H}_0 = -S[\mathcal{L}_{\vec{\eta}} \xi^0], \tag{55d}$$

$$\{S[\xi^0], R[\eta]\} = 0, \tag{55e}$$

$$\{V[\vec{\xi}], R[\eta]\} = 0. \tag{55f}$$

The last two brackets result from the fact that $\mathcal{H}_0$ and $\mathcal{G}_a$ are gauge scalars. Finally, a direct calculation yields

$$\{S[\xi^0],S[\eta^0]\}=V[\vec{\zeta}], \tag{55g}$$

where

$$\zeta^a:=(\xi^0\partial_b\eta^0-\eta^0\partial_b\xi^0)e^{ab}. \tag{56}$$

Using these brackets we next determine the brackets among the R, V, and S generators alone. We find

$$\begin{aligned}\{V[\vec{\xi}],V[\vec{\eta}]\}&=\{D[\vec{\xi}]+R[\xi^aA_a],D[\vec{\eta}]+R[\eta^bA_b]\}\\&=V[[\vec{\xi},\vec{\eta}]]-R[\xi^a\eta^bF_{ab}].\end{aligned} \tag{57a}$$

The remaining bracket is

$$\begin{aligned}\{S[\xi^0],V[\vec{\eta}]\}&=\{S[\xi^0],D[\vec{\eta}]+R[\xi^aA_a]\}\\&=-S[\mathcal{L}_{\vec{\eta}}\xi^0]-R[\eta^a\delta_S[\xi^0]A_a]=-S[\mathcal{L}_{\vec{\eta}}\xi^0]-R\left[\eta^a\frac{\xi^0}{\sqrt{g}}C^{ij}g_{ab}P_j^b\right].\end{aligned} \tag{57b}$$

We read off the following nonvanishing structure functions from the above brackets:

$$\mathcal{C}^a_{0'0''}=e^{ab}(-\delta^3(x-x')\partial_b''\delta^3(x-x'')+\delta^3(x-x'')\partial_b'\delta^3(x-x')), \tag{58a}$$

$$\mathcal{C}^a_{b'c''}=-\delta^3(x-x')\partial_b''\delta^3(x-x'')\delta_c^a+\delta^3(x-x'')\partial_c'\delta^3(x-x')\delta_b^a, \tag{58b}$$

$$\mathcal{C}^0_{0'a''}=\delta^3(x-x'')\partial_a'\delta^3(x-x'), \tag{58c}$$

$$\mathcal{C}^i_{j'k''}=-C^i_{jk}\delta^3(x-x')\delta^3(x-x''), \tag{58d}$$

$$\mathcal{C}^i_{0'a''}=-\frac{1}{\sqrt{g}}C^{ij}g_{ab}P_j^b\delta^3(x-x')\delta^3(x-x''), \tag{58e}$$

$$\mathcal{C}^i_{a'b''}=-F^i_{ab}\delta^3(x-x')\delta^3(x-x''). \tag{58f}$$

Referring to the above-derived structure functions, we obtain the following generators, where $G_R[\xi]$, $G_V[\vec{\eta}]$, and $G_S[\zeta^0]$ are, respectively, the gauge, spatial diffeomorphism plus associated gauge, and perpendicular diffeomorphism plus associated gauge generators:

$$G_R[\xi]=\int d^3x(-P_i\dot{\xi}^i+\mathcal{H}_i\xi^i-C^k_{ij}\xi^iA_0^jP_k), \tag{59a}$$

$$\begin{aligned}G_V[\vec{\eta}]=\int d^3x\bigg(&P_a\dot{\eta}^a-N^bF^i_{ba}P_i\eta^a-\frac{1}{\sqrt{g}}C^{ij}g_{ab}P_j^bN\eta^aP_i+N_{,a}P_0\eta^a\\&+N^a_{,b}P_a\eta^b-N^b\eta^a_{,b}P_a+\eta^a\mathcal{H}_a\bigg),\end{aligned} \tag{59b}$$

$$G_S[\zeta^0]=\int d^3x\left(P_0\dot{\zeta}^0+N_{,b}P_a\zeta^0e^{ab}-NP_a\zeta^0_{,b}e^{ab}-N^aP_0\zeta^0_{,a}+\zeta^0N^a\frac{1}{\sqrt{g}}C^{ij}g_{ab}P^b_jP_i+\zeta^0\mathcal{H}_0\right). \tag{59c}$$

These generators do indeed generate the variations of all variables.

We close this section by noting that we should recover the canonical Hamiltonian as the generator of a global time translation. Let us check to confirm that this is the case. First we seek the descriptors ξ^μ which correspond to $\epsilon^\mu=\delta^\mu_0$,

$$\epsilon^0=1=n^0\xi^0=N^{-1}\xi^0, \tag{60a}$$

$$\epsilon^a=0=\xi^a+n^a\xi^0=\xi^a-N^{-1}N^a\xi^0. \tag{60b}$$

We deduce that

$$\xi^0=N, \quad \xi^a=N^a. \tag{61}$$

We must bear in mind that $S[\xi^0]+D[\vec{\xi}]$ with ξ^μ given by (61) is not yet the generator of a global time translation because $S[N]$ generates a gauge transformation with descriptor

$$(A^i_\mu n^\mu)\xi^0=(A^i_0N^{-1}-A^i_aN^{-1}N^a)N=A^i_0-A^i_aN^a.$$

Thus the generator $R[A^i_0-A^i_aN^a]$ must be subtracted to obtain the Hamiltonian:

$$\begin{aligned}S[N]+D[N^a]-R[A^i_0-A^i_aN^a]&=\int d^3x(N\mathcal{H}_0+N^a\mathcal{G}_a-(A^i_0-N^aA^i_a)\mathcal{H}_i\\&=\int d^3x(N\mathcal{H}_0+N^a\mathcal{H}_a-A^i_0\mathcal{H}_i).\end{aligned} \tag{62}$$

This is precisely the canonical Hamiltonian, Eq. (30)!

It is important to point out that in this final expression the gauge variables N,N^a,A^i_0 are to be thought of as arbitrarily chosen but explicit functions of space and time. This object will then generate a global time translation only on those members of equivalence classes of solutions for which N,N^a,A^i_0 happen to have the same explicit functional forms. On all other solutions the corresponding variations correspond to more general diffeomorphism and gauge transformations.

In fact, every generator $G[\xi]$ in (47) with $\xi^0>0$ may be considered to be a Hamiltonian in the following sense: $G[\xi^A]=G_R[\xi]+G_V[\vec{\xi}]+G_S[\xi^0]$ generates a global time translation on those solutions which have

$$N=\xi^0, \tag{63a}$$

$$N^a=\xi^a, \tag{63b}$$

$$-A^i_0+A^i_aN^a=\xi^i. \tag{63c}$$

We have already demonstrated this fact for the nongauge variables, and it is instructive to verify the claim for the gauge variables N, N^a, and A^i_0. Substituting (63) into (33), we have

$$\delta N=\dot{N}+N^aN_{,a}-N^aN_{,a}=\dot{N}, \tag{64a}$$

$$\delta N^a=\dot{N}^a-Ne^{ab}N_{,b}+Ne^{ab}N_{,b}+N^a_{,b}N^b-N^a_{,b}N^b=\dot{N}^a, \tag{64b}$$

$$\delta A^i_0 = A^i_a \dot{N}^a + A^i_{0,a} N^a + \frac{N^a N}{\sqrt{g}} P^i_a - (-\dot{A}^i_0 + \dot{A}^i_a N^a + A^i_a \dot{N}^a) - C^i_{jk}(-A^j_0 + A^j_a N^a) A^k_0 = \dot{A}^i_0 . \tag{64c}$$

IV. CONCLUSION

We have been guided by the idea that a Lagrangian formulation of symmetries in a combined Yang–Mills theory and general relativity should be equivalent to the Hamiltonian formulation. As in a previous paper[1] we find that these formulations are indeed equivalent, as shown by the fact that a basis of the variations arising from gauge transformations is projectable under the Legendre map from configuration-velocity space (the tangent bundle) to phase space (the cotangent bundle). Finding these projectable variations is a major part of this paper.

We found that the most general projectable transformation coming from a diffeomorphism must depend on the lapse function N and shift vector N^a of the metric and must be accompanied by a Yang–Mills gauge transformation which also depends on these quantities and on the time component of the Yang–Mills field, A^i_0. These results had been obtained by Salisbury and Sundermeyer[3,4] (and others) but from other points of view. For example, Salisbury and Sundermeyer found them by a requirement on the commutator of various variations. We feel that our approach has several advantages: It is more direct, and it expressly indicates the equivalence of the Lagrangian and Hamiltonian approaches. Note that the gauge group acts on the dynamical variables, so that the diffeomorphism group, which one would naïvely think would be included, is not itself part of the gauge group. However, the diffeomorphism group provides the basis for the gauge group, and in this case, we can further say that the group acts specifically on solutions of the equations of motion (the Einstein–Yang–Mills field equations).

Since the Einstein–Yang–Mills Lagrangian does not depend on the gauge variable velocities $\dot{N}$, $\dot{N}^a$, and $\dot{A}^i_0$, under the Legendre map from configuration-velocity to phase space the submanifold coordinatized by these variables is mapped to a single point in phase space. Thus functions on configuration-velocity space can be the pull-back of functions on phase space only if they are constant on this submanifold. In particular, symmetry variation functions on the tangent space are projectable if and only if they do not depend on these velocities. In this manner we have determined the diffeomorphism and gauge variations which are projectable under the Legendre map.

Spatial diffeomorphisms are projectable, but four-dimensional diffeomorphisms which alter the time foliation are not. As in the case of pure conventional gravity the full four-dimensional gauge group must be reinterpreted as a transformation group on the space of metric solutions, and the group elements contain a compulsory dependence on the lapse and shift. We have found that in Einstein–Yang–Mills theories even this alteration is not sufficient. A Yang–Mills gauge transformation which is itself dependent on the full four-dimensional Yang–Mills connection must be added to the diffeomorphism. The resulting transformation group must therefore be interpreted as a transformation group on the space of metric and connection solutions.

It is natural to ask how one is to interpret variations of nonsolution trajectories in phase space which result from the generators we have constructed in this paper. The answer is that off-shell, on nonsolution trajectories, the pullback of the phase space variations to configuration-velocity space yields variations $\delta\dot{q}^i$ which are not equal to $(d/dt)\,\delta q^i$. Consequently, if these variations are used in determining the variation of the Lagrangian, the resulting Lagrangian variation is not a total time derivative. In other words, the original phase space variations do not correspond to Noether symmetries when applied off-shell. On the other hand, one could simply use the pullback of δq^i, and use $(d/dt)\,\delta q^i$ in the Lagrangian variation, thus ignoring the pullback of δp_i. This δq^i and its time derivative do yield a Noether Lagrangian symmetry. These issues will be discussed in detail in a forthcoming paper.[17]

It would seem straightforward to apply our ideas in other contexts, for example, in other formulations of general relativity. For example, the Ashtekar formulation[12] has many similarities to a Yang–Mills theory. However, it uses a complex Lagrangian and complex Hamiltonian, and so reality conditions must be imposed. The stability of these conditions under the evolution governed by a complex Hamiltonian makes the study of gauge transformations more difficult and more interesting. Other approaches to general relativity also rely on structures, such as a tetrad or a 3 +1 decomposition using triads for the spatial metric, which are added to the metric variables. They, too, present added difficulties—and interest—for the transformation law for the triads under diffeomorphisms must take into account the decomposition.

We anticipate that the resulting recovery, and significant enlargement, of the gauge symmetry group in Einstein–Yang–Mills theories will provide insights to efforts to quantize these models. Future work will deal with somewhat more complicated vacuum models in which auxiliary gravitational variables exhibit additional gauge symmetry. The first is a real tetrad formulation of Einstein's general relativity.[18] Then we shall explore the symmetry structure of Ashtekar's complex formulation of general relativity.[12,19] The former is actually a special case of the latter, and both are featured in recent attempts to construct a quantum theory of gravity. Since foliation altering diffeomorphisms and time evolution are in a sense identical, as we have explained in this paper, we may acquire insights into strategies for imposing the scalar constraint in quantum gravity.

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