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Gauging 1-form center symmetries in simple SU(N) gauge theories

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Abstract: Consequences of gauging exact $\mathbb{Z}_k^C$ center symmetries in several simple SU(N) gauge theories, where k is a divisor of N, are investigated. Models discussed include: the SU(N) gauge theory with N_f copies of Weyl fermions in self-adjoint single-column antisymmetric representation, the well-discussed adjoint QCD, QCD-like theories in which the quarks are in a two-index representation of SU(N), and a chiral SU(N) theory with fermions in the symmetric as well as in anti-antisymmetric representations but without fundamentals. Mixed 't Hooft anomalies between the 1-form $\mathbb{Z}_k^C$ symmetry and some 0-form (standard) discrete symmetry provide us with useful information about the infrared dynamics of the system. In some cases they give decisive indication to select only few possiblities for the infrared phase of the theory.

Keywords: Confinement, Global Symmetries, Nonperturbative Effects, Spontaneous Symmetry Breaking

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Contents

1 Introduction

Recently the ideas of generalized symmetries and higher-form gauging have been applied to gain deeper insights on the infrared dynamics of some strongly-coupled 4D gauge theories [1]–[20]. One of the key issues, which lead to many interesting consequences, is the so-called $\mathbb{Z}_N^C$ center symmetry[1] in SU(N) theories. Although the idea itself is a familiar one,[2] it becomes more powerful when combined with the idea of "gauging" such a discrete center symmetry [1]–[20]. In some cases, this leads to mixed ([0-form]–[1-form]) 't Hooft anomalies; they carry nontrivial information on possible infrared dynamics of the system.

[1]Throughout the paper, a 1-form center symmetry will be indicated with a suffix C, to distinguish it from various 0-form (conventional) discrete chiral symmetries.

[2]A precursor of the ideas is indeed the center symmetry $\mathbb{Z}_N^C$ in Euclidean SU(N) Yang-Mills theory at finite temperature, which acts on the Polyakov loop. The unbroken (or broken) center symmetry by the VEV of the Polyakov loop, is a criterion of confinement (or de-confinement) phase [21].

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The concept of gauging a discrete symmetry might sound a bit peculiar from the point of view of conventional idea of gauging a global flavor symmetry, i.e., that of taking the transformation parameters as functions of spacetime and turning it to a local gauge symmetry. Here the gauging of a 1-form discrete symmetry means identifying the field configurations related by it, and eliminating the associated redundancies. In the case of the $\mathbb{Z}_N^C$ center symmetry in SU(N) gauge theory, gauging it effectively reduces the theory to SU$(N)/\mathbb{Z}_N$ theory [1]–[20]. We review below (section 2) how this procedure works for the case of a subgroup, $\mathbb{Z}_k^C \subset \mathbb{Z}_N^C$ discrete center symmetry.

The aim of the present paper is to apply these new ideas to several simple SU(N) gauge theories, which possess exact $\mathbb{Z}_k^C$ color center symmetries (k being a divisor of N), and to examine the implications of gauging these discrete $\mathbb{Z}_k^C$ center symmetries on their infrared dynamics. In some cases our discussion is a simple extension of (or comments on) the results already found in the literature; in most others the results presented here are new, to the best of our knowledge. Here we discuss the following models: in section 3 the SU(N) gauge theory with N_f copies of Weyl fermions in self-adjoint single column antisymmetric representation; in section 4 the adjoint QCD discussed extensively in the literature; in section 5 QCD-like theories with quarks in two-index representations of SU(N), and in section 6 some chiral SU(N) theories with fermions in the symmetric as well as in anti-antisymmetric representations but without those in the fundamental representation. We conclude in section 7 with some general discussion. Notes on Dynkin indices for some representations in SU(N) group can be found in appendix A.

2 Gauging a discrete 1-form symmetry

As the gauging of a discrete center symmetry and the calculation of anomalies under such gauging are the basic tools of this paper and will be used repeatedly below, let us briefly review the procedure here. The procedure was formulated in [4] and used in [5] for SU(N) Yang-Mills theory at $\theta = \pi$, based on and building upon some earlier results [1]–[3], and then applied to other systems and further developed: see [6]–[10], and [13]–[19]. The details and good reviews can be found in these references and will not be repeated here, except for a few basics reviewed below.

We recall that in order to gauge a $\mathbb{Z}_k^C$ discrete center symmetry in an SU(N) gauge theory (k being a divisor of N), one introduces a pair of U(1) 2-form and 1-form $\mathbb{Z}_k^C$ gauge fields $(B_c^{(2)}, B_c^{(1)})$ satisfying the constraint [4]

$$k B_{\rm c}^{(2)} = dB_{\rm c}^{(1)} \,. \tag{2.1}$$

This constraint satisfies the invariance under the U(1) 1-form gauge transformation,

$$B_{\rm c}^{(2)} \mapsto B_{\rm c}^{(2)} + d\lambda_{\rm c}, \qquad B_{\rm c}^{(1)} \mapsto B_{\rm c}^{(1)} + k\lambda_{\rm c}, \tag{2.2}$$

where $\lambda_{\rm c}$ is the 1-form gauge function, satisfying the quantized flux

$$\frac{1}{2\pi} \int_{\Sigma_2} d\lambda_{\rm c} \in \mathbb{Z} \,. \tag{2.3}$$

The SU(N) dynamical gauge field a is embedded into a U(N) gauge field,

$$\widetilde{a} = a + \frac{1}{k} B_{\mathrm{c}}^{(1)}, \tag{2.4}$$

and one requires invariance under U(N) gauge transformation. The gauge field tensor $F(a)$ is replaced by

$$F(a) \to \tilde{F}(\tilde{a}) - B_{\mathrm{c}}^{(2)} \,. \tag{2.5}$$

This determines the way these $\mathbb{Z}_k^C$ gauge fields are coupled to the standard gauge fields a; the matter fields must also be coupled to the U(N) gauge fields, such that the 1-form gauge invariance, (2.2), is satisfied. For a Weyl fermion ψ this is achieved by writing the fermion kinetic term as

$$\bar{\psi}\,\gamma^{\mu}\left(\partial + R(\tilde{a}) - \frac{n(R)}{k} B_c^{(1)}\right)_{\mu} P_L\,\psi\,, \tag{2.6}$$

with $R(\tilde{a})$ appropriate for the representation to which ψ belongs, and $n(R)$ is the N-ality of the representation R. P_L is the projection operator on the left-handed component of the Dirac spinor. This whole procedure effectively eliminates the $\mathbb{Z}_k^C$ redundancy and defines a SU(N)/$\mathbb{Z}_k$ theory.

Also, in order to study the anomaly of $\mathrm{U}_{\psi}(1)$ symmetry (or of a discrete subgroup of it), $\psi \to e^{i\alpha}\psi$, we introduce an external $\mathrm{U}_{\psi}(1)$ gauge field A_{ψ}, and couple it to the fermion as

$$\bar{\psi}\,\gamma^{\mu}\left(\partial + R(\tilde{a}) - \frac{n(R)}{k} B_c^{(1)} + A_{\psi}\right)_{\mu} P_L\,\psi\,. \tag{2.7}$$

It is easy now to compute the anomalies following the standard Stora-Zumino descent procedure [42, 43], see also [16]. For simplicity we write the expressions for a single fermion and for an Abelian symmetry, but these can be readily generalized. A good recent review of this renowned constructions can be found in [44]. According to this procedure, the anomaly can be evaluated starting from a 6D (six-dimensional) Abelian anomaly [42, 43][3]

$$\begin{aligned}
&\frac{1}{24\pi^2}\mathrm{tr}_R\big(\tilde{F} - B_c^{(2)} + dA_{\psi}\big)^3 \\
&= \frac{D(R)}{24\pi^2}\mathrm{tr}\big(\tilde{F} - B_c^{(2)}\big)^3 + \frac{2\,T(R)}{8\pi^2}\mathrm{tr}\big(\tilde{F} - B_c^{(2)}\big)^2 \wedge dA_{\psi} + \dots\,.
\end{aligned} \tag{2.8}$$

Let us recall that, in the standard quantization (i.e., in the absence of the 1-form discrete symmetry gauging),

$$B_{\mathrm{c}}^{(1)}, B_{\mathrm{c}}^{(2)} \to 0\,, \qquad \tilde{F}(\tilde{a}) \to F(a)\,, \tag{2.9}$$

and the above reduces to

$$\frac{D(R)}{24\pi^2}\mathrm{tr}F^3 + \frac{2T(R)}{8\pi^2}\mathrm{tr}F^2 \wedge dA_{\psi} + \dots\,. \tag{2.10}$$

[3] $D(R)$ is the value of the symmetric trace of the product of three generators normalized to the one evaluated in the fundamental representation; $T(R)$ is the Dynkin index of the representation R, see appendix A. Throughout, the simplified differential form notation is used, e.g., $F^2 \equiv F \wedge F = \frac{1}{2}F^{\mu\nu}F^{\rho\sigma}dx_{\mu}dx_{\nu}dx_{\rho}dx_{\sigma} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}d^4x$, etc.

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By using the identity [42, 43]

$$\mathrm{tr}F^3 = d\left\{\mathrm{tr}\left(a(da)^2 + \frac{3}{5}(a)^5 + \frac{3}{2}a^3 da\right)\right\} \tag{2.11}$$

(also $\mathrm{tr}F^2 = d\,\{\mathrm{tr}(ada + \frac{2}{3}a^3)\}$) the first term leads to the SU(N) gauge anomalies. The second term gives the boundary term

$$\frac{2T(R)}{8\pi^2}\int_{\Sigma_5} \mathrm{tr}F^2 \wedge A_\psi \tag{2.12}$$

which, after variations

$$A_\psi \equiv dA_\psi^{(0)}\,, \qquad A_\psi^{(0)} \to A_\psi^{(0)} + \delta\alpha \tag{2.13}$$

yields, by anomaly inflow, the well-known $4D$ anomaly,

$$\delta S_{\delta A_\psi^{(0)}} = \frac{2T(R)}{8\pi^2}\int \mathrm{tr}F^2\,\delta\alpha = 2\,T(R)\,\mathbb{Z}\,\delta\alpha\,, \tag{2.14}$$

where $\mathbb{Z}$ represents the integer instanton number, leading to the well-known result that the discrete subgroup

$$\mathbb{Z}_{2T(R)} \subset \mathrm{U}_\psi(1) \tag{2.15}$$

remains unbroken by instantons.

With the 1-form gauging in place, i.e., with $(B_c^{(2)}, B_c^{(1)})$ fields present in eq. (2.8), $\mathrm{U}_\psi(1)$ symmetry could be further broken to a smaller discrete subgroup, due to the replacement,

$$\mathrm{tr}F^2 \to \mathrm{tr}\big(\tilde{F} - B_c^{(2)}\big)^2\,. \tag{2.16}$$

In fact,[4]

$$\frac{1}{8\pi^2}\int_{\Sigma_4} \mathrm{tr}\big(\tilde{F} - B_c^{(2)}\big)^2 = \frac{1}{8\pi^2}\int_{\Sigma_4}\{\mathrm{tr}\tilde{F}^2 - N(B_c^{(2)})^2\}\ : \tag{2.17}$$

the first term is an integer;[5] the second term is

$$-\frac{N}{8\pi^2}\int_{\Sigma_4}\big(B_c^{(2)}\big)^2 = -\frac{N}{8\pi^2 k^2}\int_{\Sigma_4}\big(dB_c^{(1)} \wedge dB_c^{(1)}\big) = \frac{N}{k^2}\,\mathbb{Z}\,, \tag{2.18}$$

which is in general fractional.

3 Models with self-adjoint chiral fermions

We first consider a class of SU(N) gauge theories (N even) with left-handed fermions in the $\frac{N}{2}$ fully antisymmetric representation. This representation is equivalent to its complex conjugate (as can be seen by acting on it with the epsilon tensor) and so does not contribute to the gauge anomaly. In these models there is a 1-form $\mathbb{Z}^C_{\frac{N}{2}}$ center symmetry and we are particularly interested in understanding how this mixes in the 't Hooft anomalies with the other 0-form symmetries present.

[4]Observe that $B_c^{(2)}$ is Abelian, $\propto \mathbb{1}_N$, and that $\mathrm{tr}\tilde{F} = N\,B_c^{(2)}$.

[5]The combination

$$\frac{1}{8\pi^2}\int_{\Sigma_4}\{\mathrm{tr}\tilde{F}^2 - \mathrm{tr}\tilde{F}\wedge\mathrm{tr}\tilde{F}\}$$

is the second Chern number of U(N) and is an integer. The second term of the above is also an integer as $\left(\frac{N}{k}\right)^2$ is.

3.1 SU(6) models

Let first examine in detail the case $N = 6$ with N_f flavors of Weyl fermions in the representation

$$\underline{20} = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \,. \tag{3.1}$$

As will be seen below, this (SU(6)) is the simplest nontrivial case of interest. The first coefficient of the beta function is

$$b_0 = \frac{11N - 6N_f}{3} = 22 - 2N_f \,, \tag{3.2}$$

so up to $N_f = 10$ flavors are allowed for the theory to be asymptotically free. In all these models, as will be explained in the following, there is a $U(1)_\psi$ global symmetry broken by the usual ABJ anomaly and instantons to a global discrete $\mathbb{Z}^{\psi}_{6N_f}$ which is then further broken by the 1-form gauging to $\mathbb{Z}^{\psi}_{2N_f}$. Note that the latter breaking should be understood in the sense of a mixed 't Hooft anomaly: there is an obstruction to gauging such a $\mathbb{Z}^{C}_{\frac{N}{2}}$ discrete center symmetry, while trying to maintain the global $\mathbb{Z}^{\psi}_{6N_f}$ symmetry.

3.1.1 $N_f = 1$

Let us further restrict ourselves to SU(6) theory with a *single* left-handed fermion in the representation, $\underline{20}$. This model was considered recently in [18]. A good part of the analysis below indeed overlaps with [18]; nevertheless, we discuss this simplest model with certain care, in order to fix the ideas, to recall the basic techniques and notations, and to discuss physic questions involved.

There are no continuous nonanomalous symmetries in this model. There is an anomalous $U(1)_\psi$ symmetry whose nonanomalous subgroup is the $\mathbb{Z}^{\psi}_6$ symmetry given by

$$\mathbb{Z}^{\psi}_6 : \quad \psi \to e^{\frac{2\pi i}{6} j} \psi \,, \qquad j = 1, 2, \ldots, 6 \,. \tag{3.3}$$

The system possesses also an exact center symmetry which acts on Wilson loops as

$$\mathbb{Z}^{C}_3 : \quad e^{i \oint A} \to e^{\frac{2\pi i}{6} k} e^{i \oint A} \,, \qquad k = 2, 4, 6 \,, \tag{3.4}$$

and which does not act on ψ.

This is an example of generalized symmetries (in this case, a 1-form symmetry), which have received a considerable (renewed) attention in the last several years. In particular, the central idea is that of gauging a discrete symmetry (such as $\mathbb{Z}^{C}_3$ here), i.e., that of identifying field configurations related by those symmetries, and effectively modifying the path-integral sum over them. If a center $\mathbb{Z}^{C}_k$ symmetry is gauged, $SU(N)$ gauge theory is replaced by $SU(N)/\mathbb{Z}_k$ theory. The basic aspects of such a procedure were reviewed in section 2.

Let us now apply this method to our simple SU(6) toy model, to study the fate of the unbroken $\mathbb{Z}^{\psi}_6$ symmetry (3.3), in the presence of the $\mathbb{Z}^{C}_3$ gauge fields. The Abelian $6D$

anomaly takes the form[6]

$$
\begin{aligned}
&\frac{1}{24\pi^2}\,\mathrm{tr}_{\underline{20}}\big(\tilde{F}-B_c^{(2)}-dA_\psi\big)^3\\
&=\frac{6}{8\pi^2}\mathrm{tr}\big(\tilde{F}-B_c^{(2)}\big)^2\wedge dA_\psi+\dots\\
&=\frac{6}{8\pi^2}\mathrm{tr}\tilde{F}^2\wedge dA_\psi-\frac{6N}{8\pi^2}(B_c^{(2)})^2\wedge dA_\psi+\dots
\end{aligned}
\tag{3.5}
$$

where $\mathbb{Z}_3^C$ gauge fields satisfy

$$3B_{\mathrm{c}}^{(2)}=dB_{\mathrm{c}}^{(1)}\,,\tag{3.6}$$

invariant under the U(1) 1-form gauge transformation,

$$B_{\mathrm{c}}^{(2)}\mapsto B_{\mathrm{c}}^{(2)}+d\lambda_{\mathrm{c}}\,,\quad B_{\mathrm{c}}^{(1)}\mapsto B_{\mathrm{c}}^{(1)}+3\lambda_{\mathrm{c}}\,,\tag{3.7}$$

$$\frac{1}{2\pi}\int_{\Sigma_2}d\lambda_{\mathrm{c}}\in\mathbb{Z}\,.\tag{3.8}$$

The factor 6 in (3.5) is twice the Dynkin index of $\underline{20}$ (see appendix A). A_ψ is a U(1) gauge field, formally introduced to describe the $\mathbb{Z}_6^\psi$ discrete symmetry transformations.

The first term in (3.5) is clearly trivial, as

$$\frac{1}{8\pi^2}\int\mathrm{tr}\tilde{F}^2\in\mathbb{Z}\,,\qquad A_\psi=dA_\psi^{(0)}\,,\quad\delta A_\psi^{(0)}=\frac{2\pi\mathbb{Z}_6^\psi}{6}\,.\tag{3.9}$$

This corresponds to the standard gauge anomaly that breaks $\mathrm{U}(1)_\psi\longrightarrow\mathbb{Z}_6^\psi$.

The second term in (3.5) shows that $\delta A_\psi^{(0)}$ gets multiplied by

$$-\frac{6N}{8\pi^2}\int(B_c^{(2)})^2=-6N\left(\frac{1}{3}\right)^2\mathbb{Z}=-6\,\frac{2}{3}\mathbb{Z}\,.\tag{3.10}$$

The crucial step used here is the flux quantization of the $B_c^{(2)}$ field

$$\frac{1}{8\pi^2}\int(B_c^{(2)})^2=\left(\frac{1}{3}\right)^2\mathbb{Z}\,,\tag{3.11}$$

which follows from (3.6)–(3.8).[7] The global chiral $\mathbb{Z}_6^\psi$ symmetry

$$\delta A_\psi^{(0)}=\frac{2\pi\ell}{6}\,,\qquad\ell=1,2,\dots,6\tag{3.12}$$

is therefore reduced to a $\mathbb{Z}_2^\psi$ invariance obtained restricting to the elements $\ell=3,6$,

$$\mathbb{Z}_6^\psi\longrightarrow\mathbb{Z}_2^\psi\,.\tag{3.13}$$

This agrees with what was found by [18]. This implies that a confining vacuum with mass gap, with no condensate formation and with unbroken $\mathbb{Z}_6^\psi$, is not consistent.

[6]The trace tr without specification of the representation means that it is taken on the fundamental, $\underline{N}$.

[7]The factor 3 is replaced by k in the case of $\mathbb{Z}_k^C$ discrete center gauging considered below for other systems.

Strictly speaking, it is not quite correct to say that the flavor symmetry of the model is $\mathbb{Z}_6^\psi$, eq. (3.3), since $\mathbb{Z}_2^\psi \subset \mathbb{Z}_6^\psi$ (i.e., $\psi \to -\psi$) is shared with the color $\mathbb{Z}_2^C \subset \mathbb{Z}_6^C$. The correct symmetry is

$$\frac{\mathbb{Z}_6^\psi}{\mathbb{Z}_2} \sim \mathbb{Z}_3^\psi \,. \tag{3.14}$$

However our conclusion is not modified: (3.13) is actually equivalent to

$$\mathbb{Z}_3^\psi \longrightarrow \mathbf{1}\,, \tag{3.15}$$

in the vacuum with unbroken $\mathbb{Z}_2^{\text{color-flavor}}$. This feature must be kept in mind in all our analysis below: the crucial point is that in this paper we gauge only a subgroup of discrete color center group, which does not act on the fermions.[8]

The breaking $\mathbb{Z}_6^\psi \to \mathbb{Z}_2^\psi$ implies a threefold vacuum degeneracy, if the system confines (with mass gap) and if in IR there are no massless fermionic degrees of freedom on which $\mathbb{Z}_6^\psi/\mathbb{Z}_2^\psi$ can act.[9] A possible explanation naturally presents itself. As the interactions become strong in the infrared, it is reasonable to assume that bifermion condensate

$$\langle \psi\psi \rangle \sim \Lambda^3 \neq 0 \tag{3.16}$$

forms. As the field ψ is in $\underline{20}$ of the gauge group SU(6), a Lorentz invariant bifermion composite can be in one of the irreducible representations of SU(6), appearing in the decomposition

$$\yng(1,1,1) \otimes \yng(1,1,1) = \yng(1,1,1,1,1,1) \oplus \yng(2,1,1,1,1) \oplus + \dots\,. \tag{3.17}$$

The most natural candidate would be the first, $\underline{1}$, but it can be readily verified that such a condensate vanishes by the Fermi-Dirac statistics. Another possibility is that $\psi\psi$ in the adjoint representation gets a VEV, signaling a sort of dynamical Higgs mechanism [24–26]. Even though such a condensate should necessarily be regarded as a gauge dependent expression of some gauge invariant VEV (see below), it unambiguously signals[10] the breaking of global, discrete chiral symmetry as

$$\mathbb{Z}_6^\psi \to \mathbb{Z}_2^\psi\,, \tag{3.18}$$

with broken $\mathbb{Z}_6^\psi/\mathbb{Z}_2^\psi$ acting on the degenerate vacua. Four-fermion, gauge-invariant condensates such as

$$\langle \psi\psi\psi\psi \rangle \neq 0\,, \qquad \text{or} \qquad \langle \bar\psi\bar\psi\psi\psi \rangle \neq 0\,, \tag{3.19}$$

[8] The situation is subtler if one tries to gauge the full color center group, see [20].

[9] Here, as in the rest of the paper, we do not consider the more "exotic" possibility that discrete anomaly matching may be achieved with a topological field theory or by a CFT in the IR.

[10] Note that the global symmetry group $\mathbb{Z}_6^\psi$ commutes with the color SU(6): there is no way a gauge transformation eliminates the nontrivial properties of the condensate under $\mathbb{Z}_6^\psi$.

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might also form, first of which also breaks $\mathbb{Z}_6^\psi$ in the same way. The condensate (3.16) thus leads to threefold vacuum degeneracy, consistently with (3.13) implied by the $\mathbb{Z}_6^\psi - \mathbb{Z}_3^C$ mixed anomaly.

Let us pause briefly to make a few comments on dynamically induced Higgs phase. As in any system where the Higgs mechanism is at work, some (elementary or composite) scalar field gets a nonvanishing, gauge noninvariant (and gauge dependent) vacuum expectation value (VEV). In a weakly coupled Higgs type model, there is a potential having degenerate minima, and the vacuum, which necessarily breaks the gauge invariance, induces the Higgs phase, with some gauge bosons becoming massive. Also, in such a model, apparently gauge-dependent phenomena can be naturally re-interpreted in a gauge-invariant fashion.[11]

Here the situation is basically the same. One is indeed assuming that an effective composite scalar $\sim \psi\psi$ forms by strong interactions, which then condenses. It corresponds to the non-gauge-invariant VEV of a scalar field in a potential model. In contrast to a weakly coupled Higgs models, however, the effective scalar composite particle is still strongly coupled and is not described by a simple potential. Therefore, a gauge-invariant rephrasing of the phenomenon may not be straightforward. Apart from this, there is nothing unphysical about assuming gauge non-invariant bifermion condensate:[12] it is analogue of the Higgs VEV $\langle\phi\rangle$ in the standard electroweak theory.

As a final remark, it may help to remember also that the Higgs mechanism itself was first discovered in the context of superconductivity ([27, 28], see also [29]): the Cooper pair condenses due to the interactions between the electrons and the lattice phonons. The Cooper pair, having charge 2, is not a gauge invariant object. It is the first example in a physical theory of what we call dynamical Higgs mechanism, in the sense that the effective Higgs scalar (the Cooper pair) is a composite, gauge noninvariant field.[13]

The infrared system depends also on the kind of bi-fermion $\psi\psi$ condensates which actually form. The "MAC" (most attractive channel) criterion [24] suggests condensation of a $\psi\psi$ composite scalar in the adjoint representation. It is then possible that the infrared physics is described by full dynamical Abelianization [25, 26]: the low-energy theory is an Abelian $\mathrm{U}(1)^5$ theory. Although the infrared theory looks trivial, the only massless infrared degrees of freedom being five types of non-interacting photons, the system might be richer actually. There is a remnant of the $\mathbb{Z}_6$ symmetry of the UV theory, which is a threefold vacuum degeneracy. Domain walls would exist which connect the three vacua, and on which nontrivial infrared 3D physics can appear (we shall not elaborate on them here).

[11] For instance, the Higgs VEV of the form $\langle\phi\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$ found in any textbook about the standard electroweak theory, is just a gauge dependent way of describing a minimum of the potential $V(\phi^\dagger\phi)$, so is the statement such as the left hand fermion being equal to $\psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$. A similar reinterpretation of the W and Z bosons is also straightforward.

[12] In this respect we differ from the interpretation given in [18]. Indeed there is a long history of studies in strongly interacting chiral gauge theories based on such ideas, starting from [24]. See also [25, 26] and references cited therein.

[13] This brief comment is meant only to remind the reader that there is nothing unusual in having a composite, gauge noninvariant field getting a VEV, to break the gauge (and/or flavor) symmetry of a given system. Of course, conventional superconductivity is not a good model for strongly interacting gauge theories as the ones we are interested in here.

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It is interesting to check also the (conventional) $[\mathbb{Z}_6^\psi]^3$ anomaly matching constraint, following [22, 23]. The matching condition for a $[\mathbb{Z}_N]^3$ discrete symmetry is

$$\begin{aligned} A_{\text{IR}} &= A_{\text{UV}} + mN\,, && \text{for odd } N\,, \\ A_{\text{IR}} &= A_{\text{UV}} + mN + \frac{nN^3}{8}\,, && \text{for even } N\,, \end{aligned} \tag{3.20}$$

where $m, n \in \mathbb{Z}$.[14] In our case $N = 6$, $\frac{N^3}{8} = 27$ and it must be that

$$A_{\text{IR}} - A_{\text{UV}} = 0 \mod 3 \tag{3.21}$$

if $\mathbb{Z}_6^\psi$ is to remain unbroken. However, $A_{\text{UV}}\big([\mathbb{Z}_6^\psi]^3\big) = 20 = 2 \mod 3 \neq 0$ in our system, where 20 is the color multiplicity. Therefore a confining vacuum with no condensates with mass gap, and with unbroken $\mathbb{Z}_6^\psi$ ($A_{\text{IR}} = 0$), would not be consistent. On the other hand, the condensate formation (3.16), a spontaneous breaking $\mathbb{Z}_6^\psi \longrightarrow \mathbb{Z}_2^\psi$ and associated threefold vacuum degeneracy, is perfectly consistent with the $[\mathbb{Z}_2^\psi]^3$ anomaly matching condition. Consideration of $\mathbb{Z}_6^\psi\,[\text{grav}]^2$ anomaly leads to the same conclusion.

We find thus that the consideration of the 1-form $\mathbb{Z}_3^C$ center symmetry gauging (3.13) and that of the conventional $[\mathbb{Z}_6^\psi]^3$ or $\mathbb{Z}_6^\psi\,[\text{grav}]^2$ anomaly matching requirement, give a consistent indication about the infrared dynamics of our system.

3.1.2 $N_f = 2$

We now move to discuss SU(6) theory with more than one Weyl fermions in $\underline{20}$. For two flavors the global symmetry is

$$G_f = \text{SU}(2) \times \mathbb{Z}_{12}^\psi\,. \tag{3.22}$$

As before there is a 0-form and 1-form mixed anomaly

$$\mathbb{Z}_{12}^\psi\,[\mathbb{Z}_3^C]^2\,, \tag{3.23}$$

and the discrete chiral symmetry is broken by the 1-form gauging as

$$\mathbb{Z}_{12}^\psi \longrightarrow \mathbb{Z}_4^\psi\,. \tag{3.24}$$

In this case the bi-fermion scalar condensate

$$\langle \psi^{[A}\psi^{B]} \rangle \neq 0\,, \tag{3.25}$$

can be formed which is gauge-invariant, and leaves SU(2) invariant. Let us assume that such a condensate indeed is formed. The bi-fermion condensate (3.25) breaks the discrete chiral symmetry

$$\mathbb{Z}_{12}^\psi \longrightarrow \mathbb{Z}_2^\psi\,, \tag{3.26}$$

[14]The difference between the even N and odd N cases is due to the possibility that a single fermion ψ with charge $N/2$ can get a Majorana mass term for even N (which does not break the Z_N symmetry). Such a fermion provides a $\frac{N^3}{8}$ contribution to the $[\mathbb{Z}_N]^3$, therefore the anomaly matching conditions for massless fermions is weakened. For further details see [23].

implying a *six-fold* vacuum degeneracy. The latter is stronger than (3.24) but is consistent. The SU(2) triangle anomaly vanishes so there are no associated matching constraints: neither massless baryons nor NG bosons are required, and expected to occur. The Witten SU(2) anomaly is also matched between the UV (6 doublets) and the IR (0 doublet).

Strictly speaking the symmetry of the $N_f = 2$ system is not (3.22), but

$$G_f = \frac{SU(2) \times \mathbb{Z}_{12}^{\psi}}{\mathbb{Z}_2 \times \mathbb{Z}_2} \sim \frac{SU(2)}{\mathbb{Z}_2} \times \mathbb{Z}_6^{\psi} \,, \tag{3.27}$$

where one of the factors in the denominators is due to the overlap with the $\mathbb{Z}_2^{\rm color}$, the other being a $\mathbb{Z}_2$ shared between SU(2) and $\mathbb{Z}_{12}^{\psi}$. A similar observation was made in the $N_f = 1$ case, in section 3.1.1. As discussed in the previous case, and as will be in all other cases discussed below, none of our conclusions is modified by this more careful consideration of the symmetry group, as eq. (3.24) and eq. (3.26) are equivalent to

$$\mathbb{Z}_6^{\psi} \longrightarrow \mathbb{Z}_2^{\psi} \,, \tag{3.28}$$

and

$$\mathbb{Z}_6^{\psi} \longrightarrow \mathbf{1} \,, \tag{3.29}$$

respectively, in the presence of unbroken SU(2) and $\mathbb{Z}_2^{\rm color}$ (confinement).

As for the conventional discrete anomaly matching, in the UV there is a discrete 0-form

$$\mathbb{Z}_{12}^{\psi}\, [SU(2)]^2 \tag{3.30}$$

anomaly. This is due to the SU(2) instanton effects, which gives the phase variation of the partition function,

$$20\, \frac{2\pi k}{12} = 2\pi \frac{5}{3} k \,, \qquad k = 1, 2, \ldots, 12 \,, \tag{3.31}$$

breaking the chiral symmetry as

$$\mathbb{Z}_{12}^{\psi} \longrightarrow \mathbb{Z}_4^{\psi} \tag{3.32}$$

since only the transformations with $k = 3, 6, 9, 12$ leave the partition function invariant. One may wonder how such a breaking is described in the IR, where no massless fermions are present, and SU(2) is unbroken. The answer is that the condensate (3.25) breaks the discrete chiral symmetry as (3.26), which is stronger than (3.32).

We consider also the constraints following from the $[\mathbb{Z}_{12}^{\psi}]^3$ anomaly. This time an unbroken $\mathbb{Z}_{12}^{\psi}$ in the IR requires the anomaly matching (3.20)

$$A_{\rm UV} = A_{\rm IR} + 12\, m + 6^3\, n \,, \qquad m, n \in \mathbb{Z} \,, \tag{3.33}$$

that is,

$$A_{\rm UV} - A_{\rm IR} = 0 \mod 12 \,. \tag{3.34}$$

The UV anomaly is

$$A_{\rm UV} = 4 \mod 12 \,, \tag{3.35}$$

therefore the $[\mathbb{Z}_{12}^{\psi}]^3$ anomaly consideration is consistent with the assumption of the breaking $\mathbb{Z}_{12}^{\psi} \longrightarrow \mathbb{Z}_4^{\psi}$ and with consequent threefold degeneracy of the vacuum. This is also consistent with result of the 1-form gauging of the center $\mathbb{Z}_3^C$ symmetry and the mixed anomaly (3.23).

To conclude, it is possible that actually a bi-fermion gauge-invariant condensate (3.25) forms, breaking the discrete symmetry, as $\mathbb{Z}_{12}^{\psi} \longrightarrow \mathbb{Z}_2^{\psi}$. It is however also possible that the bi-fermion condensate vanishes, and e.g., some four-fermion condensates are formed. In that case, the discrete symmetry breaking pattern would coincide with what is implied by the conventional and mixed anomalies associated with the discrete $\mathbb{Z}_{12}^{\psi}$ symmetry (3.30) and (3.23).

3.1.3 $N_f = 3$

For three fermions in $\underline{20}$ the symmetry is

$$G_f = \mathrm{SU}(3) \times \mathbb{Z}_{18}^{\psi} \,. \tag{3.36}$$

The 1-form gauging of the $\mathbb{Z}_3^C$ center symmetry yields the discrete symmetry breaking

$$\mathbb{Z}_{18}^{\psi} \longrightarrow \mathbb{Z}_6^{\psi} \,. \tag{3.37}$$

In this case, a gauge-invariant bi-fermion condensate,

$$\langle \psi^{[A} \psi^{B]} \rangle \,, \qquad A, B = 1, 2 \,, \tag{3.38}$$

would break the continuous symmetry as

$$\mathrm{SU}(3) \longrightarrow \mathrm{SU}(2) \,. \tag{3.39}$$

There are $8 - 3 = 5$ NG bosons, which saturate the anomalies of the spontaneously broken $\mathrm{SU}(3)/\,\mathrm{SU}(2)$ symmetries. There are no triangle $\mathrm{SU}(2)^3$ anomalies. No massless baryons are required and expected to occur in the infrared theory. The discrete $\mathbb{Z}_{18}^{\psi}$ symmetry would be broken by the condensate (3.38) as

$$\mathbb{Z}_{18}^{\psi} \longrightarrow \mathbb{Z}_2^{\psi} \,, \tag{3.40}$$

implying a nine-fold vacuum degeneracy.

Again, let us check $\mathbb{Z}_{18}^{\psi}\,[\mathrm{SU}(3)]^2$ (to be matched mod $N = 18$) and $[\mathbb{Z}_{18}^{\psi}]^3$ anomalies. Actually, since SU(3) is broken to SU(2) by bi-fermion condensate, we shall study the $\mathbb{Z}_{18}^{\psi}\,[\mathrm{SU}(2)]^2$. As

$$A_{\mathrm{UV}}\big(\mathbb{Z}_{18}^{\psi}\,[\mathrm{SU}(2)]^2\big) = 20 = 2 \mod 18 \,, \tag{3.41}$$

the $\mathbb{Z}_{18}^{\psi}\,[\mathrm{SU}(2)]^2$ anomaly matching is consistent with what is implied by the ψ^2 condensate formation (3.38).

As for $[\mathbb{Z}_{18}^{\psi}]^3$, as N is even we must have the equality

$$A_{\mathrm{IR}} = A_{\mathrm{UV}} + 18\,m + 9^3\,n = A_{\mathrm{UV}} + 9\,k \,, \qquad m, n, k \in \mathbb{Z} \,, \tag{3.42}$$

i.e., an equality *modulo* 9, if $\mathbb{Z}_{18}^{\psi}$ is to remain unbroken in the infrared. But

$$A_{\mathrm{UV}} = 2 \cdot 20 = 4 \mod 9 \,. \tag{3.43}$$

The consideration of $\mathbb{Z}_{18}^{\psi}\,[\mathrm{SU}(2)]^2$ and $[\mathbb{Z}_{18}^{\psi}]^3$ anomaly matching is consistent with the assumption of the bi-fermion condensate (3.38) (the reduction $\mathbb{Z}_{18} \longrightarrow \mathbb{Z}_2$) and with consequent nine-fold degeneracy of the vacua. The 1-form center symmetry gauging and the mixed anomaly alone, give instead a weaker condition (3.37).

3.1.4 $N_f = 4$

The symmetry of the $N_f = 4$ model is

$$G_f = \mathrm{SU}(4) \times \mathbb{Z}_{24}^{\psi} \,. \tag{3.44}$$

The 1-form gauging of $\mathbb{Z}_2^C$ breaks the discrete chiral symmetry as

$$\mathbb{Z}_{24}^{\psi} \longrightarrow \mathbb{Z}_8^{\psi} \,. \tag{3.45}$$

In this case, the bi-fermion condensate (3.25) of the form,

$$\langle \psi^{[A} \psi^{B]} \rangle \neq 0 \,, \qquad A, B = 1, 2 \tag{3.46}$$

or

$$\langle \psi^{[A} \psi^{B]} \rangle \neq 0 \,, \quad (A, B = 1, 2) \,, \qquad \langle \psi^{[C} \psi^{D]} \rangle \neq 0 \,, \quad (C, D = 3, 4) \,, \tag{3.47}$$

if it occurs, break the symmetry as

$$\mathrm{SU}(4) \times \mathbb{Z}_{24}^{\psi} \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{Z}_2^{\psi} \,. \tag{3.48}$$

There are $15 - 6 = 9$ NG bosons. Again no massless baryons are required.

Thus for $N_f = 4$, the discrete $\mathbb{Z}_{24}^{\psi}$ would be broken to $\mathbb{Z}_8^{\psi}$ due to the 1-form center gauging, whereas it would be broken more strongly to $\mathbb{Z}_2^{\psi}$, if the condensate (3.46) or (3.47) is to form.

As for the conventional discrete anomaly is concerned, it has an anomaly at UV:

$$A_{\mathrm{UV}}\big(\mathbb{Z}_{24}^{\psi}\,[\mathrm{SU}(2)]^2\big) = 20 = -4 \mod 24 \,, \tag{3.49}$$

implying a discrete symmetry breaking to $\mathbb{Z}_4^{\psi}$ and a six-fold vacuum degeneracy, at least. This does not exclude that it is broken more strongly, as expected from the bi-fermion condensate formation.

As for the $[\mathbb{Z}_{24}^{\psi}]^3$, there is an UV anomaly

$$A_{\mathrm{UV}}\big([\mathbb{Z}_{24}^{\psi}]^3\big) = 4 \cdot 20 \,. \tag{3.50}$$

This must match to the IR modulo $\gcd\left(24, \frac{24^3}{8}\right) = 24$, a requirement satisfied by the reduction of the symmetry to $\mathbb{Z}_4^{\psi}$.

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3.1.5 $5 \leq N_f \leq 10$

For larger N_f, the symmetry of the system is

$$G_f = \mathrm{SU}(N_f) \times \mathbb{Z}^{\psi}_{6N_f}\,. \tag{3.51}$$

For illustration let consider the case $N_f = 5$. The discrete $\mathbb{Z}^{\psi}_{30}$ symmetry is broken by the 1-form gauging to $\mathbb{Z}^{\psi}_{10}$, implying some condensates to occur in the infrared. A condensate of the form, (3.38), would break the global symmetry as,

$$\mathrm{SU}(5) \times \mathbb{Z}^{\psi}_{30} \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathbb{Z}^{\psi}_{2}\,. \tag{3.52}$$

It would be a hard problem to find a set of massless baryons saturating the anomaly triangles associated with this low-energy symmetries.

It is possible, however, that the system instead chooses to produce condensates of the form,

$$\langle \psi^{[A}\psi^{B]} \rangle \neq 0\,, \quad A, B = 1, 2\,; \qquad \langle \psi^{[C}\psi^{D]} \rangle \neq 0\,, \quad C, D = 3, 4\,. \tag{3.53}$$

In this case the symmetry breaking pattern is

$$\mathrm{SU}(5) \times \mathbb{Z}^{\psi}_{30} \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{Z}^{\psi}_{2}\,, \tag{3.54}$$

and the low-energy theory is described by $24 - 6 = 18$ massless NG bosons. No massless baryons are required, and are expected to appear.

Finally, let us check the conventional anomalies involving $\mathbb{Z}^{\psi}_{30}$. In the UV, the anomalies are

$$\begin{aligned} A_{\mathrm{UV}}\big(\mathbb{Z}^{\psi}_{30}\,[\mathrm{grav}]^2\big) &= 5 \cdot 20 = 100 \neq 0 \quad \mathrm{mod}\ 30\,, \\ A_{\mathrm{UV}}\big([\mathbb{Z}^{\psi}_{30}]^3\big) &= 5 \cdot 20 = 100 \neq 0 \quad \mathrm{mod}\ 30\,, \end{aligned} \tag{3.55}$$

and would not match in the UV and IR, implying a (at least) partial breaking of $\mathbb{Z}^{\psi}_{30}$. However, if the discrete symmetry is reduced to $\mathbb{Z}^{\psi}_{2}$ by the bi-fermion condensates, then the UV $\mathbb{Z}^{\psi}_{2}$ anomalies vanish

$$\begin{aligned} A_{\mathrm{UV}}\big(\mathbb{Z}^{\psi}_{2}\,[\mathrm{grav}]^2\big) &= 5 \cdot 20 = 100 = 0 \quad \mathrm{mod}\ 2\,, \\ A_{\mathrm{UV}}\big([\mathbb{Z}^{\psi}_{2}]^3\big) &= 5 \cdot 20 = 100 = 0 \quad \mathrm{mod}\ 2\,, \end{aligned} \tag{3.56}$$

and no contradiction arises.

3.2 SU(N) generalizations

We now consider the general case of SU(N) (N even) theory with N_f left-handed fermions ψ in the self-adjoint, totally antisymmetric, one-column (of height $n = \frac{N}{2}$) representation. The first coefficient of the beta function is

$$b_0 = \frac{11N - 2N_f T_R}{3}\,. \tag{3.57}$$

For simplicity, we shall limit ourselves to the single flavor ($N_f = 1$) case below. The generalization to general N_f is quite straightforward. For SU(N) one finds that the twice Dynkin index (see appendix A) is given by

$$2\,T_R = \binom{N-2}{(N-2)/2} \;: \tag{3.58}$$

$2T_R$ and $d(R)$ are given for some even values of N in table (3.59).

N	4	6	8	10	12
$2\,T_R$	2	6	20	70	252
$d(R)$	6	20	70	252	924

(3.59)

Thus SU(4), SU(6), SU(8), SU(10) models with $N_f = 1$ are asymptotically free, SU(12) and higher are not. We shall limit ourselves to some of the asymptotically free theories.

ψ is neutral with respect to the $\mathbb{Z}^C_{\frac{N}{2}}$ symmetry, therefore the system possesses an exact 1-form:

$$\mathbb{Z}^C_{\frac{N}{2}}: \quad e^{i\oint A} \to e^{\frac{2\pi i}{N}k}\, e^{i\oint A}\,, \qquad k = 2, 4, \ldots N\,. \tag{3.60}$$

At the same time, the anomaly-free global discrete symmetry is:

$$\mathbb{Z}_{2T_R}: \quad \psi \to e^{\frac{2\pi i}{2T_R}j}\,\psi\,, \qquad j = 1, 2, \ldots 2T_R\,. \tag{3.61}$$

We are interested to find out how this discrete symmetry is realized in the infrared, and what the 1-form gauging of the center symmetry has to tell about it.

We introduce a 1-form gauge fields $(B_c^{(2)}, B_c^{(1)})$ such that

$$\frac{N}{2}B_c^{(2)} = dB_c^{(1)}\,. \tag{3.62}$$

The anomaly can be evaluated from the Stora-Zumino 6D Abelian anomaly [43]

$$\begin{aligned}
&\frac{1}{24\pi^2}\mathrm{tr}_R\big(\tilde F - B_c^{(2)} - dA_\psi\big)^3 \\
&= \frac{2T_R}{8\pi^2}\mathrm{tr}\big(\tilde F - B_c^{(2)}\big)^2 \wedge dA_\psi + \ldots \\
&= \frac{2T_R}{8\pi^2}\mathrm{tr}\tilde F^2 \wedge dA_\psi - \frac{6N}{8\pi^2}(B_c^{(2)})^2 \wedge dA_\psi + \ldots
\end{aligned} \tag{3.63}$$

so that the $5D$ effective action reads,

$$S^{5D} = \frac{2T_R}{8\pi^2}\mathrm{tr}\tilde F^2 \wedge A_\psi - \frac{6N}{8\pi^2}(B_c^{(2)})^2 \wedge A_\psi + \ldots\,. \tag{3.64}$$

The first term is clearly trivial, as

$$\frac{1}{8\pi^2}\int \mathrm{tr}\tilde F^2 \in \mathbb{Z}, \qquad A_\psi = dA_\psi^{(0)}\,, \tag{3.65}$$

and

$$\delta A_\psi^{(0)} = \frac{2\pi k}{2T_R}\,, \qquad k = 1, 2, \ldots, 2T_R\,, \tag{3.66}$$

which corresponds to the $\mathbb{Z}_{2T_R}$ transformation of the ψ field in four dimensional action. As for the second term of (3.64), as

$$-\frac{2T_R N}{8\pi^2}\int (B_c^{(2)})^2 \in -2T_R N\left(\frac{2}{N}\right)^2 \mathbb{Z} = -2T_R\frac{4}{N}\mathbb{Z} \tag{3.67}$$

the phase of the partition function is transformed by

$$-\frac{2\pi k}{2T_R}2T_R\frac{4}{N}\mathbb{Z} = -2\pi k\frac{4}{N}\mathbb{Z}\,, \qquad k = 1, 2, \ldots, 2T_R\,, \tag{3.68}$$

under $\mathbb{Z}_{2T_R}$: one sees that the 1-form gauging of the center $\mathbb{Z}^C_{N/2}$ has the effect of making $\mathbb{Z}_{2T_R}$ anomalous, in general. Stated differently, there is a mixed 't Hooft anomaly between the 1-form $\mathbb{Z}^C_{\frac{N}{2}}$ gauging and 0-form $\mathbb{Z}_{2T_R}$ symmetry. Its consequence depends on N in a nontrivial fashion:

(i) For $N = 4$, the mixed anomaly disappears, as

$$\frac{4}{N} = 1\,. \tag{3.69}$$

(ii) For $N = 4\ell$, $\ell \geq 2$,

$$\frac{4}{N} = \frac{1}{\ell}\,, \tag{3.70}$$

therefore the discrete symmetry is broken to

$$\mathbb{Z}^{\psi}_{2T_R} \longrightarrow \mathbb{Z}^{\psi}_{\frac{2T_R}{\ell}} \tag{3.71}$$

generated by

$$\psi \to e^{\frac{2\pi i}{2T_R}k}\psi\,, \qquad k = \ell, 2\ell, 3\ell, \ldots, 2T_R\,. \tag{3.72}$$

Note that for $N = 4\ell$, $2T_R$ is an integer multiple of ℓ (see appendix A).

(iii) For $N = 4\ell + 2$,

$$2T_R \cdot \frac{4}{N} = 2T_R \cdot \frac{2}{2\ell+1}\,, \tag{3.73}$$

therefore the breaking of the discrete symmetry is

$$\mathbb{Z}^{\psi}_{2T_R} \longrightarrow \mathbb{Z}^{\psi}_{\frac{2T_R}{2\ell+1}}\,; \tag{3.74}$$

only the transformations

$$A_\psi = dA^{(0)}_\psi\,, \quad \delta A^{(0)}_\psi = \frac{2\pi k}{2T_R}\,, \quad k = 2\ell+1, 4\ell+2, \ldots, 2T_R \tag{3.75}$$

remain invariant. Note that for N of the form, $N = 4\ell + 2$, $2\ell + 1$ is a divisor of $2T_R$ (see appendix A).

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Concretely, for SU(4) there are no mixed anomalies. SU(6) case has been studied in detail above: the discrete symmetry (for $N_f = 1$) is broken as (3.13) by 1-form gauging. For SU(8) the effect of the 1-form center symmetry gauging is

$$\mathbb{Z}_{20}^{\psi} \longrightarrow \mathbb{Z}_{10}^{\psi} , \tag{3.76}$$

and for SU(10) is

$$\mathbb{Z}_{70}^{\psi} \longrightarrow \mathbb{Z}_{14}^{\psi} . \tag{3.77}$$

The Fermi statistics allows, for N multiple of 4, (e.g., $N = 4$ or $N = 8$), a bi-fermion condensate

$$\langle \psi\psi \rangle , \tag{3.78}$$

which is gauge invariant. If such a condensate indeed forms the discrete symmetry is broken more strongly, as

$$\mathbb{Z}_{20}^{\psi} \longrightarrow \mathbb{Z}_{2}^{\psi} . \tag{3.79}$$

For N of the form, $N = 4\ell+2$, $\ell \in \mathbb{Z}$, a bi-fermion Lorentz invariant condensate cannot be gauge invariant. As discussed in SU(6) case, it is possible that in these cases dynamical Higgs phenomenon occurs, with gauge noninvariant bi-fermion condensate in the adjoint representation of SU(N). The system can dynamically Abelianize. The discrete symmetry is again broken to $\mathbb{Z}_2^{\psi}$.

Finally, let us check the conventional anomalies involving $\mathbb{Z}_{2T_R}^{\psi}$. In the UV, the anomalies are

$$\begin{aligned} A_{\rm UV}\big(\mathbb{Z}_{2T_R}^{\psi}\,[\text{grav}]^2\big) &= 1 \cdot d(R) \neq 0 \mod 2T_R \,, \\ A_{\rm UV}\big([\mathbb{Z}_{2T_R}^{\psi}]^3\big) &= 1 \cdot d(R) \neq 0 \mod 2T_R \,, \end{aligned} \tag{3.80}$$

except for $N = 4$ (see table (3.59)). Thus the conventional discrete anomaly matching requirement implies that some condensate forms in the infrared, breaking $\mathbb{Z}_{2T_R}^{\psi}$ spontaneously. The assumption of bi-fermion condensate and consequent spontaneous breaking of $\mathbb{Z}_{2T_R}^{\psi}$, (3.79), is compatible with the discrete anomaly matching condition.

Let us discuss these results from the point of view of the fractional instantons. If no matter fields are present in the system, one can compactify the $\mathbb{R}^4$ space on a 4-torus $\mathbb{T}^4$ and insert one unit of 't Hooft flux in the first 2-torus (x_1, x_2) and another unit in the second 2-torus (x_3, x_4). This object [45]–[47], sometimes called "toron", has topological charge equal to $\frac{1}{N}$ that of an ordinary instanton. In general we can insert n_{12} units of 't Hooft flux in the first 2-torus and n_{34} units in the second and this object has topological charge $\frac{n_{12}n_{34}}{N}$. If the instanton breaks a certain chiral symmetry to a discrete subgroup $\mathrm{U}(1) \longrightarrow \mathbb{Z}_M$ this is due to the presence of M fermion zero modes in the instanton background. A toron has a smaller amount of zero modes, precisely $\frac{M}{N}$ due to the index theorem and thus the discrete symmetry is broken further to $\mathbb{Z}_{\frac{M}{N}}$. It is known [9, 13] that gauging the 1-form center symmetry is equivalent to putting the theory on a nontrivial background with fractional instanton number.

Here we are interested in theories with matter fields, but with some residual center symmetry. This means that fractional instantons can be constructed, but not of the one of the minimal charge. For the SU(N) (N even) theory with N_f left-handed fermions ψ in the self-adjoint representation the remaining center symmetry is $\mathbb{Z}^C_{\frac{N}{2}}$ which means that only *even* numbers of fluxes are allowed on each 2-torus. With $n_{12} = n_{34} = 2$ units of fluxes we have a toron with charge $\frac{4}{N}$. This can be combined with any integer number of instanton charge to construct the minimal possible instanton charge $\frac{\gcd(4,N)}{N}$ and this is $\frac{4}{N} = \frac{1}{\ell}$ for $N = 4\ell$ and $\frac{2}{N} = \frac{1}{2\ell+1}$ when $N = 4\ell+2$. The symmetry is then broken as

$$\mathrm{U}(1)_\psi \longrightarrow \mathbb{Z}^\psi_{2T_R N_f} \longrightarrow \mathbb{Z}^\psi_{2T_R N_f \frac{\gcd(4,N)}{N}} \tag{3.81}$$

first by the ABJ anomaly and instantons, and then by the gauging of the 1-form symmetry. This result agrees with what was found above by use of the $\mathbb{Z}^C_{\frac{N}{2}}$ gauge fields $(B_c^{(2)}, B_c^{(1)})$. More about these issues at the end, see Discussion (section 7).

4 Adjoint QCD

SU(N) theories with N_f Weyl fermions λ in the adjoint representation have been the object of intense study, and our comments here will be brief. In this model, the color center 1-form $\mathbb{Z}^C_N$ symmetry is exact, therefore can be entirely gauged. The system possesses also a nonanomalous 0-form discrete chiral symmetry,

$$\mathbb{Z}^\lambda_{2N_f N}: \quad \lambda \to e^{\frac{2\pi i}{2N_f N}k}\lambda\,, \qquad k = 1, 2, \ldots, 2N_f N\,. \tag{4.1}$$

We introduce a set of gauge fields

- A_λ: $\mathbb{Z}^\lambda_{2N_f N}$ 1-form gauge field, to formally describe (4.1);
- $B_c^{(2)}$: $\mathbb{Z}^C_N$ 2-form gauge field.

The Abelian 6D anomaly is

$$\begin{aligned}
&\frac{1}{24\pi^2}\int \mathrm{tr}_{\mathrm{adj}}\big(\tilde{F} - B_c^{(2)} - dA_\lambda\big)^3 \\
&= \frac{2NN_f}{8\pi^2}\int \mathrm{tr}(\tilde{F} - B_c^{(2)})^2 \wedge dA_\lambda + \ldots \\
&= \frac{2NN_f}{8\pi^2}\int \mathrm{tr}\tilde{F}^2 \wedge dA_\lambda - \frac{2N^2N_f}{8\pi^2}\int (B_c^{(2)})^2 \wedge dA_\lambda \ldots\,.
\end{aligned} \tag{4.2}$$

As

$$A_\lambda = dA_\lambda^{(0)}\,, \qquad \delta A_\lambda^{(0)} \in \frac{2\pi i}{2NN_f}\mathbb{Z} \tag{4.3}$$

the first term is trivial (conserves $\mathbb{Z}^\lambda_{2NN_f}$); the second term gives

$$\Delta S(\delta A_\lambda^{(0)}) \in \frac{2\pi i}{N}\mathbb{Z}\,, \tag{4.4}$$

breaking the chiral discrete symmetry as

$$\mathbb{Z}^\lambda_{2NN_f} \longrightarrow \mathbb{Z}^\lambda_{2N_f} \tag{4.5}$$

in agreement with [5, 7, 10]. In this case the matter fields have no charge under the center of the gauge group and so the torons can have the minimal topological charge $\frac{1}{N}$ of that of the instanton, hence the breaking (4.5).

Let us briefly discuss the case of SU(2), $N_f = 2$ theory. The discrete chiral symmetry $\mathbb{Z}_8^\lambda$ is in this case broken by the 1-form $\mathbb{Z}_2^C$ gauging to as

$$\mathbb{Z}_8^\lambda \longrightarrow \mathbb{Z}_4^\lambda \,, \tag{4.6}$$

with

$$\mathbb{Z}_4^\lambda: \quad \lambda \to e^{\frac{2\pi i}{8}k}\lambda\,, \qquad k = 2,4,6,8\,. \tag{4.7}$$

In particular it means that the discrete chiral transformations

$$\lambda \to e^{\pm\frac{2\pi i}{8}}\lambda\,, \tag{4.8}$$

which is an invariance of the standard SU(2) theory, becomes anomalous under the gauging of $\mathbb{Z}_2^C$, i.e., in the quantum SU(2)/$\mathbb{Z}_2$ theory.

A familiar assumption about the infrared dynamics of this system is [11] that a condensate

$$\langle \lambda^{\{I}\lambda^{J\}}\rangle \neq 0\,, \qquad \mathrm{SU}_f(2) \longrightarrow \mathrm{SO}_f(2) \tag{4.9}$$

($I, J = 1, 2$ being the flavor $SU_f(2)$ indices) forms in the infrared. It would break the discrete chiral symmetry as $\mathbb{Z}_8^\lambda \to \mathbb{Z}_2^\lambda$, which leaves four-fold degenerate vacua. Note that this symmetry breaking is stronger than that would follow from the 1-form gauging, (4.6). The fact that the vacuum breaks the symmetry further with respect to what is expected from (4.6) is a common feature, seen also in the previous section 3, and is perfectly acceptable.

Anber and Poppitz (AP) [10] however propose that the system instead develops a four-fermion condensate of the form

$$\langle \lambda\lambda\lambda\lambda\rangle \neq 0\,, \quad \text{with} \quad \langle \lambda\lambda\rangle = 0\,, \tag{4.10}$$

in the infrared, which breaks $\mathbb{Z}_8^\lambda$ spontaneously to $\mathbb{Z}_4^\lambda$, leaving only doubly degenerate vacua with unbroken $SU_f(2)$ symmetry. They assume that massless baryons of spin $\frac{1}{2}$

$$B \sim \lambda\lambda\lambda \tag{4.11}$$

which is necessarily a doublet of unbroken $\mathrm{SU}_f(2)$, appear in the infrared spectrum. It is shown that all the conventional 't Hooft and Witten anomaly matching conditions are met by such a vacuum and with such low-energy degrees of freedom. The action of the broken $\mathbb{Z}_8^\lambda/\mathbb{Z}_4^\lambda$ is seen to be realized in the infrared as a transformation between the two degenerate vacua,

$$\langle \lambda\lambda\lambda\lambda\rangle \to -\langle \lambda\lambda\lambda\lambda\rangle\,. \tag{4.12}$$

Most significantly Anber and Poppitz note that the above hypothesis is consistent with what is expected from the gauging of the 1-form $\mathbb{Z}_2^C$ center symmetry, (4.6), this time just with the minimal amount of breaking necessary.

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Let us check the conventional anomalies associated with the discrete symmetry in the Anber-Poppitz scenario. Assuming the condensates of the form (4.10), the anomalies associated with the unbroken $\mathbb{Z}_4^\lambda$ symmetries must be considered. The anomalies $\mathbb{Z}_4^\lambda\,[\mathrm{SU}_f(2)]^2$ and $\mathbb{Z}_4^\lambda\,[\mathrm{grav}]^2$ have been already verified in [10] to match in the UV and in the IR, therefore only the $[\mathbb{Z}_4^\lambda]^3$ anomaly remains to be checked. In the UV λ contributes

$$A_{\mathrm{UV}}\big([\mathbb{Z}_4^\lambda]^3\big) = N_f \cdot d(\mathrm{adj}) = 2 \cdot 3 = 2 \mod 4\,, \tag{4.13}$$

whereas in the IR the baryons B gives

$$A_{\mathrm{IR}}\big([\mathbb{Z}_4^\lambda]^3\big) = N_f \cdot 3^3 = 2 \cdot 27 = 2 \mod 4\,, \tag{4.14}$$

therefore the matching works.

As in any anomaly-matching argument, these considerations only tell that a particular dynamical scenario (in this case, (4.10)–(4.12)) is consistent, but not that such a vacuum is necessarily realized. It would be important to establish which between the familiar $\mathrm{SO}_f(2)$ symmetric vacuum and the proposed $\mathrm{SU}_f(2)$ symmetric one, is actually realized in the infrared, e.g., by using the lattice simulations.

The adjoint QCD has been discussed extensively in the literature, by compactifying one space direction to S^1 and by using controlled semi-classical analysis [30], by direct lattice simulations [31], and more recently, by applying the 1-form center symmetry gauging and using mixed anomalies [5, 7, 10]. For more general approach, see [11], and for more recent work on adjoint QCD, see [13, 14].

For $N_f = 1$ the system reduces to $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, where a great number of nonperturbative results are available [35–38]. Note that for $N_f = 1$ $\mathrm{SU}(N)$ theory, the breaking of the discrete symmetry (4.5) due to the 1-form gauging implies an N fold vacuum degeneracy, in agreement with the well-known result, i.e., the Witten index of pure $\mathcal{N} = 1$ $\mathrm{SU}(N)$ Yang-Mills.

Another possibility is to start from the $\mathcal{N} = 2$ supersymmetric $\mathrm{SU}(2)$ Yang-Mills theory, where many exact results for the infrared effective theory are known [32–34]. It can be deformed to $\mathcal{N} = 1$ theory by a mass perturbation, yielding a confining, chiral symmetry breaking vacua. For the exact calculation of gauge fermion condensates $\langle\lambda\lambda\rangle$ from this viewpoint, see [40, 41]. The pure $\mathcal{N} = 2$ theory can also be deformed directly to $\mathcal{N} = 0$ [12], to give indications about $N_f = 2$ adjoint QCD.

5 QCD with quarks in a two-index representation

Consider now $\mathrm{SU}(N)$, N even, with N_f pairs of "quarks" in symmetric (or antisymmetric) representations. Namely the left-handed matter fermions are either

$$\psi, \tilde{\psi} = \square\!\square \oplus \overline{\square\!\square} \tag{5.1}$$

or

$$\psi, \tilde{\psi} = \begin{array}{c}\square\\[-4pt]\square\end{array} \oplus \overline{\begin{array}{c}\square\\[-4pt]\square\end{array}} \tag{5.2}$$

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(the quarks in standard QCD are in $\square \oplus \bar{\square}$). The first beta function coefficients are

$$b_0 = \frac{11N - 2N_f(N \pm 2)}{3} \,. \tag{5.3}$$

The $k = \frac{N}{2}$ element of the center $\mathbb{Z}_N$ does not act on ψ's, i.e., there is an exact

$$\mathbb{Z}_2^C \subset \mathbb{Z}_N^C \tag{5.4}$$

center symmetry.[15] On the other hand, there is a discrete axial symmetry

$$\mathbb{Z}^{\psi}_{2N_f(N\pm 2)} : \quad \psi \to e^{\frac{2\pi i}{2N_f(N\pm 2)}} \psi \,, \quad \tilde{\psi} \to e^{\frac{2\pi i}{2N_f(N\pm 2)}} \tilde{\psi} \,, \tag{5.5}$$

preserved by instantons. The $\pm$ signs above refer to two cases eq. (5.1) and eq. (5.2), respectively.

Let us consider for simplicity $N_f = 1$ and consider a 1-form gauging of the exact $\mathbb{Z}_2^C$. The external background fields are

- A_ψ: $\mathbb{Z}^{\psi}_{2(N\pm 2)}$ 1-form gauge field,
- $B_c^{(2)}$: $\mathbb{Z}_2^C$ 2-form gauge field.

The last satisfies

$$2B_c^{(2)} = dB_c^{(1)} \,, \qquad B_c^{(1)} \to B_c^{(1)} + 2\lambda \,, \qquad B_c^{(2)} \to B_c^{(2)} + d\lambda \,. \tag{5.6}$$

The 6D anomaly is

$$\begin{aligned}
&\frac{1}{24\pi^2} \mathrm{tr}_{R_\psi} \big(\tilde{F} - B_c^{(2)} - dA_\psi\big)^3 + \frac{1}{24\pi^2} \mathrm{tr}_{R_{\tilde{\psi}}} \big(\tilde{F} - B_c^{(2)} + dA_\psi\big)^3 + \dots \\
&= -\frac{2(N\pm 2)}{8\pi^2} \mathrm{tr}(\tilde{F} - B_c^{(2)})^2 \wedge dA_\psi + \dots \\
&= -\frac{2(N\pm 2)}{8\pi^2} \mathrm{tr}\tilde{F}^2 \wedge dA_\psi + \frac{2N(N\pm 2)}{8\pi^2} (B_c^{(2)})^2 \wedge dA_\psi + \dots
\end{aligned} \tag{5.7}$$

Now

$$\frac{2(N\pm 2)}{8\pi^2} \int \mathrm{tr}\tilde{F}^2 \in 2(N\pm 2)\mathbb{Z} \,, \tag{5.8}$$

$$A_\psi = dA_\psi^{(0)} \,, \qquad \delta A_\psi^{(0)} \in \frac{2\pi}{2(N\pm 2)} \mathbb{Z}_{2(N\pm 2)} \,, \tag{5.9}$$

so the first term is trivial. By using

$$\frac{1}{8\pi^2} \int (B_c^{(2)})^2 = \frac{1}{4}\mathbb{Z} \,, \tag{5.10}$$

[15] This aspect has been considered by Cohen [48], in particular in relation with the possible existence of an order parameter for confinement.

the second term gives an anomaly

$$A = 2\pi \frac{N}{4} \mathbb{Z} \,. \tag{5.11}$$

This means that for $N = 4\ell$ there is no anomaly, whereas for $N = 4\ell+2$ the 1-form gauging breaks the discrete symmetry as

$$\mathbb{Z}^{\psi}_{2(N\pm2)} \longrightarrow \mathbb{Z}^{\psi}_{N\pm2} \tag{5.12}$$

with the subgroup

$$\mathbb{Z}^{\psi}_{N\pm2}: \quad \psi \to e^{\frac{2\pi i}{2(N\pm2)}\ell}\,\psi\,, \quad \tilde{\psi} \to e^{\frac{2\pi i}{2(N\pm2)}\ell}\,\tilde{\psi}\,, \qquad \ell = 2, 4, \dots 2(N\pm2) \tag{5.13}$$

that remains nonanomalous.

We can construct a toron with $n_{12} = n_{34} = \frac{N}{2}$ units of fluxes and thus topological charge $\frac{N}{4}$. For N multiple of 4 this is not fractional and thus we have no mixed anomaly; for $N = 4\ell + 2$, it can be combined with a suitable number of instantons to obtain the minimal possible fractional charge $\frac{1}{2}$ and this explains the breaking (5.12).

These results are consistent with the assumption that in the IR the condensate

$$\langle \psi \tilde{\psi} \rangle \neq 0 \tag{5.14}$$

forms, even though the bi-fermion condensate itself (5.14) breaks the discrete symmetry more strongly,

$$\mathbb{Z}^{\psi}_{2(N\pm2)} \longrightarrow \mathbb{Z}^{\psi}_{2} \,. \tag{5.15}$$

6 Chiral models with $\frac{N-4}{k}$ $\psi^{\{ij\}}$'s and $\frac{N+4}{k}$ $\bar{\chi}_{[ij]}$'s

Let us consider now SU(N) gauge theories with Weyl fermions in the complex representation, $\frac{N-4}{k}$ $\psi^{\{ij\}}$'s and $\frac{N+4}{k}$ $\bar{\chi}_{[ij]}$,

$$\frac{N-4}{k}\, \yng(2) \oplus \frac{N+4}{k}\, \overline{\yng(1,1)}\,, \tag{6.1}$$

where k is a common divisor of $(N-4, N+4)$ and $N \geq 5$. With this matter content the gauge anomaly cancels. Asymptotic freedom requirement

$$11N - \frac{2}{k}(N^2 - 8) > 0\,, \tag{6.2}$$

leaves a plenty of possibilities for (N, k). Two particularly simple models which we analyze in the following are:

(i) $(N, k) = (6, 2)$: SU(6) theory with

$$\yng(2) \oplus 5\, \overline{\yng(1,1)}\,; \tag{6.3}$$

(ii) $(N, k) = (8, 4)$: SU(8) model with

$$\yng(2) \oplus 3\, \overline{\yng(1,1)}\,. \tag{6.4}$$

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6.1 SU(6) theory with $\underline{\mathbf{21}} \oplus 5 \times \underline{\mathbf{15}}^*$

Classical continuous flavor symmetry group is

$$\mathrm{SU}(5) \times \mathrm{U}(1)_\psi \times \mathrm{U}(1)_\chi \,. \tag{6.5}$$

The chiral anomalies are:

$$\begin{aligned} \mathrm{U}(1)_\psi \, [\mathrm{SU}(6)]^2 &= \frac{T_{\square\square}}{T_{\square}} = N + 2 = 8 \,, \\ \mathrm{U}(1)_\chi \, [\mathrm{SU}(6)]^2 &= \frac{5 T_{\bar{\begin{smallmatrix}\square\\\square\end{smallmatrix}}}}{T_{\square}} = 5(N-2) = 20 \,, \end{aligned} \tag{6.6}$$

meaning that the charges with respect to the unbroken $\mathrm{U}(1)_{\psi\chi} \subset \mathrm{U}(1)_\psi \times \mathrm{U}(1)_\chi$ symmetry are

$$(Q_\psi, Q_\chi) = (5, -2) \,. \tag{6.7}$$

The system has unbroken discrete groups also:

$$\mathrm{U}(1)_\psi \longrightarrow \mathbb{Z}_8^\psi \,, \qquad \mathrm{U}(1)_\chi \longrightarrow \mathbb{Z}_{20}^\chi \,. \tag{6.8}$$

One might wonder if a subgroup of $\mathbb{Z}_8^\psi \times \mathbb{Z}_{20}^\chi$ is contained in $\mathrm{U}(1)_{\psi\chi}$. In fact, $\mathrm{U}(1)_{\psi\chi}$ transformations

$$\psi \to e^{5i\alpha}\psi \,, \qquad \chi \to e^{-2i\alpha}\chi \,, \tag{6.9}$$

with

$$\alpha = \frac{2\pi k}{40} \,, \qquad k = 1, 2, \ldots, 40 \,, \tag{6.10}$$

generate the subgroup

$$\psi \to e^{\frac{2\pi i}{8}k}\psi \,, \qquad \chi \to e^{-\frac{2\pi i}{20}k}\chi \tag{6.11}$$

of $\mathbb{Z}_8^\psi \times \mathbb{Z}_{20}^\chi$. The anomaly-free symmetry subgroup of $\mathrm{U}(1)_\psi \times \mathrm{U}(1)_\chi$ is

$$\frac{\mathrm{U}(1)_{\psi\chi} \times \mathbb{Z}_8^\psi \times \mathbb{Z}_{20}^\chi}{\mathbb{Z}_{40}} \sim \mathrm{U}(1)_{\psi\chi} \times \mathbb{Z}_4 \tag{6.12}$$

(see (6.26) and (6.27) below). Actually, by considering the overlap with color center and $\mathrm{SU}_f(5)$ center, the correct anomaly-free symmetry group is:[16]

$$\frac{\mathrm{SU}(5) \times \mathrm{U}(1)_\psi \times \mathrm{U}(1)_\chi}{\mathbb{Z}_6^C \times \mathbb{Z}_5^f} \longrightarrow \frac{\mathrm{SU}(5) \times \mathrm{U}(1)_{\psi\chi} \times \mathbb{Z}_4}{\mathbb{Z}_6^C \times \mathbb{Z}_5^f} \,. \tag{6.13}$$

Let us first check the 't Hooft anomaly matching condition with respect to the continuous global symmetries, assuming that the vacuum possesses the full symmetry, (6.13). The anomaly coefficients in the UV are

repr	dim	$T_F(r)$	$[\mathrm{SU}(5)]^3$	$\mathrm{U}(1)_{\psi\chi}\,[\mathrm{SU}(5)]^2$	$[\mathrm{U}(1)_{\psi\chi}]^3$
$\square\square$	21	0	0	0	2625
$\bar{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$15 \cdot 5$	$\frac{1}{2}$	15	$\frac{1}{2} \cdot (-2) \cdot 15 = -15$	-600

[16] Here we do not gauge the full denominator of the global group, but only the exact subgroup of the center symmetry. This will be done, for a simpler chiral gauge theory, in [20].

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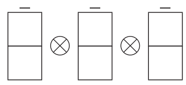
and so in total:

$$\begin{aligned}
A_{\mathrm{UV}}\big([\mathrm{SU}(5)]^3\big) &= 15\,, \\
A_{\mathrm{UV}}\big(\,\mathrm{U}(1)_{\psi\chi}\,[\mathrm{SU}(5)]^2\big) &= \frac{1}{2}\cdot(-2)\cdot 15 = -15\,, \\
A_{\mathrm{UV}}\big([\mathrm{U}(1)_{\psi\chi}]^3\big) &= 21\cdot 5^3 - 15\cdot 2^3\cdot 5 = 2025\,.
\end{aligned} \tag{6.14}$$

Let us investigate whether the system can confine without any condensates forming. We ask if color-singlet massless "baryon" states can be formed which would saturate the above anomalies. The only (simple) possibility is to contract the color as

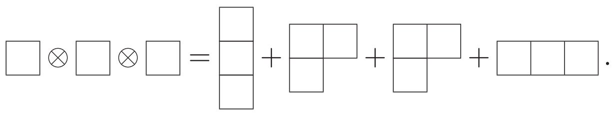


$$= (\cdot) + \dots \tag{6.15}$$

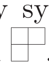
that is,

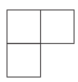
$$B^{\gamma,I,J,K} = \epsilon_{ijlmno}\chi^{\alpha,I}_{ij}\,\chi^{\beta,J}_{lm}\,\chi^{\gamma,K}_{no}\,\epsilon_{\alpha\beta} \tag{6.16}$$

where Greek letters are spin, small Latin are color, big Latin are flavor. A priori these can belong to different SU(5) flavor representations,



$$\tag{6.17}$$

Actually the first (completely antisymmetric) and the last (completely symmetric) are both excluded by the statistics. We are left with a mixed representation . Its anomaly content is

repr	dim	$T_F(r)$	$[\mathrm{SU}(5)]^3$	$\mathrm{U}(1)_{\psi\chi}\,[\mathrm{SU}(5)]^2$	$[\mathrm{U}(1)_{\psi\chi}]^3$
	40	11	16	$11\cdot(-6) = -66$	$40\cdot(-6)^3 = -8640$

Clearly these baryons cannot reproduce the anomalies (6.14) due to the fermions in the UV theory. We conclude that if the system confines in the IR, with a gapped vacuum (vacua), the global symmetry (6.12) must be broken spontaneously, at least partially.

Let us check whether the 1-form gauging of a center symmetry can give any useful information. In our system the surviving exact center symmetry is $\mathbb{Z}_2^C \subset \mathbb{Z}_6^C$:

$$\mathbb{Z}_2^C: \quad e^{i\oint A} \to e^{\frac{2\pi i}{2}k}\, e^{i\oint A}\,, \qquad k = 1, 2\,. \tag{6.18}$$

We gauge this 1-form symmetry and study its effects on the discrete symmetries, by introducing the fields

- A_ψ: $\mathbb{Z}_8^\psi$ 1-form gauge field,
- A_χ: $\mathbb{Z}_{20}^\chi$ 1-form gauge field,
- $B_c^{(2)}$: $\mathbb{Z}_2^C$ 2-form gauge field.

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The last satisfies

$$2B_c^{(2)} = dB_c^{(1)}\,, \qquad B_c^{(1)} \to B_c^{(1)} + 2\lambda\,, \qquad B_c^{(2)} \to B_c^{(2)} + d\lambda\,. \tag{6.19}$$

The 6D anomaly functional is

$$\begin{aligned} &\frac{1}{24\pi^2}\mathrm{tr}_{\square\!\square}\big(\tilde F - B_c^{(2)} - dA_\psi\big)^3 + \frac{1}{24\pi^2}\mathrm{tr}_{\substack{\square\\\square}}\big(\tilde F - B_c^{(2)} - dA_\chi\big)^3 \\ &= -\frac{8}{8\pi^2}\mathrm{tr}\big(\tilde F - B_c^{(2)}\big)^2 \wedge dA_\psi - \frac{20}{8\pi^2}\mathrm{tr}\big(\tilde F - B_c^{(2)}\big)^2 \wedge dA_\chi + \dots\,. \end{aligned} \tag{6.20}$$

The relevant part of the 5D WZW action is

$$-\frac{8}{8\pi^2}\left[\int \mathrm{tr}\tilde F^2 - 6\,(B_c^{(2)})^2\right] A_\psi - \frac{20}{8\pi^2}\left[\int \mathrm{tr}\tilde F^2 - 6\,(B_c^{(2)})^2\right] A_\chi\,. \tag{6.21}$$

$\mathbb{Z}_8^\psi$ and $\mathbb{Z}_{20}^\chi$ transformations are expressed by the variations

$$A_\psi = dA_\psi^{(0)}\,, \quad \delta A_\psi^{(0)} = \frac{2\pi k}{8}\,; \qquad A_\chi = dA_\chi^{(0)}\,, \quad \delta A_\chi^{(0)} = -\frac{2\pi \ell}{20}\,, \tag{6.22}$$

($k \in \mathbb{Z}_8$, $\ell \in \mathbb{Z}_{20}$). The integration over closed 4 cycles give

$$\frac{1}{8\pi^2}\int \mathrm{tr}\tilde F^2 \in \mathbb{Z}\,, \qquad \frac{1}{8\pi^2}\int (B_c^{(2)})^2 \in \frac{\mathbb{Z}}{4}\,. \tag{6.23}$$

Therefore one finds that both $\mathbb{Z}_8^\psi$ and $\mathbb{Z}_{20}^\chi$ are broken by the 1-form $\mathbb{Z}_2^C$ gauging:

$$\mathbb{Z}_8^\psi \longrightarrow \mathbb{Z}_4^\psi\,, \qquad \mathbb{Z}_{20}^\chi \longrightarrow \mathbb{Z}_{10}^\chi\,. \tag{6.24}$$

$$\delta A_\psi^{(0)} = \frac{2\pi k}{8}\,, \qquad k = 2, 4, \dots, 8\,, \qquad \delta A_\chi^{(0)} = -\frac{2\pi \ell}{20}\,, \qquad \ell = 2, 4, \dots, 20\,. \tag{6.25}$$

Such a result suggests a nonvanishing condensate of some sort to form in the infrared, and breaks the global symmetry at least partially.

Actually, a more careful analysis is needed to see which bifemion condensates may occur in the infrared, in order to satisfy the mixed-anomaly-matching condition. The division by $\mathbb{Z}_{40}$ in the global symmetry group, (6.12), is due to the fact that the subgroup (6.11) is inside the nonanomalous $U_{\psi\chi}(1)$. The quotient

$$\mathbb{Z}_4 \sim \frac{\mathbb{Z}_{20} \times \mathbb{Z}_8}{\mathbb{Z}_{40}} \tag{6.26}$$

also forms a subgroup, which can be taken as

$$\psi \to e^{2\pi i \frac{2k}{8}}\psi = e^{2\pi i \frac{k}{4}}\psi\,; \qquad \chi \to e^{-2\pi i \frac{5k}{20}}\chi = e^{-2\pi i \frac{k}{4}}\chi\,, \tag{6.27}$$

or as

$$\delta A_\psi^{(0)} = \frac{2\pi k}{4}\,, \qquad \delta A_\chi^{(0)} = -\frac{2\pi k}{4}\,, \qquad k = 1, 2, 3, 4\,, \tag{6.28}$$

in (6.21), (6.22).

The action of $\mathbb{Z}_{40}$ on the $4D$ partition function can be obtained by setting $k = \ell = 1, 2, \ldots, 40$, in eq. (6.22), or the chiral transformations, eq. (6.27). The anomaly is proportional to

$$-\left(8 \cdot \frac{2\pi k}{8} - 20 \cdot \frac{2\pi k}{20}\right) \frac{1}{8\pi^2} \left[\int \mathrm{tr} \tilde{F}^2 - 6\,(B_c^{(2)})^2\right] = 0 \ : \tag{6.29}$$

i.e., $\mathbb{Z}_{40}$ remains nonanomalous, even after 1-form gauging of $\mathbb{Z}_2^C$ is done.

On the other hand, $\mathbb{Z}_4$ is affected by the gauging of the center $\mathbb{Z}_2^C$ symmetry. From (6.28) and (6.21) one finds that the $4D$ anomaly is given by

$$-3 \cdot 2\pi k \, \frac{1}{8\pi^2} \left[\int \mathrm{tr} \tilde{F}^2 - 6\,(B_c^{(2)})^2\right] = 2\pi k \cdot \left(\mathbb{Z} + 3 \cdot 6 \cdot \frac{\mathbb{Z}}{4}\right) . \tag{6.30}$$

Clearly $\mathbb{Z}_4$ is reduced to $\mathbb{Z}_2$ $(k = 2, 4)$ by the 1-form gauging of $\mathbb{Z}_2^C$.

Having learned the fates of the discrete symmetries

$$\mathbb{Z}_{20} \times \mathbb{Z}_8 \sim \mathbb{Z}_{40} \times \mathbb{Z}_4 \tag{6.31}$$

under the gauged 1-form center symmetry $\mathbb{Z}_2^C$, let us discuss now what their implications on the possible condensate formation in the infrared are. Restricting ourselves to the three types of bifermion condensates,

$$\psi\chi\,, \qquad \psi\psi, \qquad \chi\chi\,, \tag{6.32}$$

the MAC criterion might suggest some condensates in the channels

$$\begin{aligned}
&A: \quad \psi\,(\ydiagram{2})\;\psi\,(\ydiagram{2}) \quad &&\text{forming} \quad \ydiagram{2,2}\,;\\
&B: \quad \chi\left(\bar{\ydiagram{1,1}}\right)\chi\left(\bar{\ydiagram{1,1}}\right) \quad &&\text{forming} \quad \bar{\ydiagram{1,1,1,1}}\,;\\
&C: \quad \psi\,(\ydiagram{2})\;\chi\left(\bar{\ydiagram{1,1}}\right) \quad &&\text{forming adjoint representation}\,;
\end{aligned} \tag{6.33}$$

with the one-gluon exchange strengths $(N = 6)$ proportional to,

$$\begin{aligned}
A: &\quad \frac{2(N^2-4)}{N} - \frac{(N+2)(N-1)}{N} - \frac{(N+2)(N-1)}{N} = -\frac{2(N+2)}{N}\,;\\
B: &\quad \frac{2(N+1)(N-4)}{N} - \frac{(N+1)(N-2)}{N} - \frac{(N+1)(N-2)}{N} = -\frac{4(N+1)}{N}\,;\\
C: &\quad N - \frac{(N+2)(N-1)}{N} - \frac{(N+1)(N-2)}{N} = -\frac{N^2-4}{N}\,,
\end{aligned} \tag{6.34}$$

i.e., $16/6, 28/6, 32/6$, respectively. Among these the last channel is most attractive, and it is tempting to assume that the only condensate in the infrared is

$$\langle(\psi\chi)_{\mathrm{adj}}\rangle \neq 0\,. \tag{6.35}$$

However, the mixed anomalies studied above require that at least two different types of condensates be formed in the infrared. The breaking of $\mathbb{Z}_8$ (acting on ψ), $\mathbb{Z}_{20}$ (acting on χ), and of $\mathbb{Z}_4$ (acting on both ψ and χ but not on the composite $\psi\chi$), precludes the possibility that only one type of condensate, for instance, (6.35), is formed. Assuming that any two of the condensates (6.33) or all of them, are formed, the discrete symmetries $\mathbb{Z}_8$, $\mathbb{Z}_{20}$, and $\mathbb{Z}_4$ are all broken to $\mathbb{Z}_2$ and consistency with the implication of the 1-form gauging of $\mathbb{Z}_2^C$ is attained.

Actually, we cannot logically exclude the possibility that only one condensate (one of (6.33)) is formed,[17] part of the color and flavor symmetries survive unbroken, and the associated massless composite fermions in the infrared might induce, through its own mixed anomalies, breaking of the remaining unbroken part of discrete symmetries which is "not accounted for" by the unique bifermion condensate. There are however too many unknown factors in such an argument (which symmetry breaking pattern, which set of massless baryons, etc.), to justify a more detailed discussion on this point.

In any case, we conclude that the color symmetry is broken at least partially (dynamical Higgs phase), together with (part of) flavor symmetry. Other than this, the information we possess at the moment is unfortunately not powerful enough to indicate in more detail the infrared physics of this system.[18]

6.2 SU(8) theory with $\underline{\mathbf{36}} \oplus 3 \times \underline{\mathbf{28}}^*$

The classical continuous favor symmetry of this model is

$$\mathrm{U}(1)_\psi \times \mathrm{U}(1)_\chi \times \mathrm{SU}(3)\,. \tag{6.36}$$

A nonanomalous $\mathrm{U}(1)_{\psi\chi} \subset \mathrm{U}(1)_\psi \times \mathrm{U}(1)_\chi$ symmetry has associated charges:

$$(Q_\psi, Q_\chi) = (9, -5)\,. \tag{6.37}$$

The system has nonanomalous discrete groups:

$$\mathrm{U}(1)_\psi \longrightarrow \mathbb{Z}_{10}^\psi\,, \qquad \mathrm{U}(1)_\chi \longrightarrow \mathbb{Z}_{18}^\chi\,. \tag{6.38}$$

To check the overlap between $\mathbb{Z}_{10} \times \mathbb{Z}_{18}$ and $\mathrm{U}(1)_{\psi\chi}$ set

$$e^{2\pi i 9\alpha}\psi = e^{\frac{2\pi i}{10}m}\psi\,, \qquad e^{-2\pi i 5\alpha}\chi = e^{\frac{2\pi i}{18}n}\chi\,. \tag{6.39}$$

If we write $\alpha = \frac{k}{90}$ we get $k = m + 10\ell$ and $-k = n + 18m$ $(k, \ell, m \in \mathbb{Z})$. This has solution for each $m + n$ even. This means that if $m + n$ is even k can be chosen such that the $U_{\psi\chi}$ transformation cancel the discrete transformation. If instead $m + n$ is odd k can be

[17] The possibility of symmetric vacuum with no condensate formation has been already excluded.

[18] Just for completeness, let us comment also on the conventional matching condition for the discrete $\mathbb{Z}_4$ symmetry, independently of the $\mathbb{Z}_4 - \mathbb{Z}_2^C$ mixed anomalies discussed above. If $\mathbb{Z}_4$ is to remain unbroken in the infrared, (3.20) requires for $N = 4$, that $A_{\mathrm{IR}} - A_{\mathrm{UV}} = 0 \mod 4$. But $A_{\mathrm{UV}}([\mathbb{Z}_4^3]) = 21 - 5 \cdot 15 = -54 \neq 0$, mod 4, so a unique confining vacuum with mass gap (no condensates) is not consistent with the conventional $[\mathbb{Z}_4]$ anomaly matching conditions either. On the other hand, such a vacuum have been already excluded on the basis of the standard anomaly matching conditions involving $U_{\psi\chi}(1)$ and SU(5).

chosen to erase only one of them, e.g., choose k to cancel m and take $n-k$ to be 9. So the anomaly free symmetry group is:

$$\mathrm{SU}(3)\times\mathrm{U}(1)_{\psi\chi}\times\mathbb{Z}_2 \tag{6.40}$$

where $\mathbb{Z}_2$ act as:

$$\begin{array}{ccc} \psi\to-\psi & & \psi\to\psi \\ & \text{or} & \\ \chi\to\chi & & \chi\to-\chi \end{array} \tag{6.41}$$

the two representation are equivalent because a $\mathrm{U}(1)_{\psi\chi}$ transformation takes the former to the latter.

The anomaly coefficients in the UV are:

$$\begin{aligned} A_{\mathrm{UV}}\big([\mathrm{SU}(3)]^3\big) &= 28\,,\\ A_{\mathrm{UV}}\big(\,\mathrm{U}(1)_{\psi\chi}\,[\mathrm{SU}(3)]^2\big) &= \frac{1}{2}\cdot(-5)\cdot 28=-140\,,\\ A_{\mathrm{UV}}\big([\mathrm{U}(1)_{\psi\chi}]^3\big) &= 36\cdot 9^3-28\cdot 3\cdot 5^3=15744\,. \end{aligned} \tag{6.42}$$

In our system no gauge invariant spin $\frac{1}{2}$ baryons can be formed by using three fundamental fermions. Barring the possibilities that some massless baryons made of more than three fermion components, perhaps with gauge fields, saturate such anomalies, one is forced to conclude that $\mathrm{SU}(3)\times\mathrm{U}(1)_{\psi\chi}$ symmetry is broken in the infrared.

Let us first check the 1-form symmetry: this system has an exact $\mathbb{Z}_2^C$ center symmetry. As usual we can introduce:

- A_ψ: $\mathbb{Z}_{10}^{\psi}$ 1-form gauge field,
- A_χ: $\mathbb{Z}_{18}^{\chi}$ 1-form gauge field,
- $B_c^{(2)}$: $\mathbb{Z}_2^C$ 2-form gauge field,

where again:

$$2B_c^{(2)}=dB_c^{(1)}\,,\qquad B_c^{(1)}\to B_c^{(1)}+2\lambda\,,\qquad B_c^{(2)}\to B_c^{(2)}+d\lambda\,. \tag{6.43}$$

The anomaly polynomial is:

$$\begin{aligned} &\frac{1}{24\pi^2}\mathrm{tr}_{\square\square}\big(\tilde F-B_c^{(2)}-dA_\psi\big)^3+\frac{1}{24\pi^2}\mathrm{tr}_{\tiny\begin{array}{c}\square\\[-4pt]\square\end{array}}\big(\tilde F-B_c^{(2)}-dA_\chi\big)^3\\ &=-\frac{10}{8\pi^2}\mathrm{tr}\big(\tilde F-B_c^{(2)}\big)^2\wedge dA_\psi-\frac{18}{8\pi^2}\mathrm{tr}\big(\tilde F-B_c^{(2)}\big)^2\wedge dA_\chi+\dots\,. \end{aligned} \tag{6.44}$$

The relevant part of the 5D WZW action is therefore

$$-\frac{10}{8\pi^2}\left[\int\mathrm{tr}\tilde F^2-8\,(B_c^{(2)})^2\right]A_\psi-\frac{18}{8\pi^2}\left[\int\mathrm{tr}\tilde F^2-8\,(B_c^{(2)})^2\right]A_\chi\,. \tag{6.45}$$

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The integration over closed 4 cycles give this time

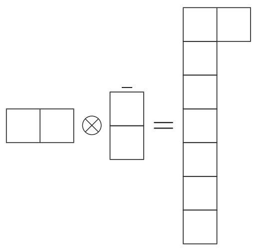
$$\frac{1}{8\pi^2}\int \mathrm{tr}\,\tilde{F}^2 \in \mathbb{Z}\,, \qquad \frac{1}{8\pi^2}\int (B_c^{(2)})^2 \in \frac{\mathbb{Z}}{4}\,. \tag{6.46}$$

Note that, in contrast to all other cases studied in this paper (except for the SU(4) model in section 3.2), the 1-form gauging of the exact $\mathbb{Z}_2^C$ center symmetry this time does not lead to the mixed anomalies, i.e., does not imply breaking of the discrete $\mathbb{Z}_{10}^{\psi}$ or $\mathbb{Z}_{18}^{\chi}$ symmetries. It does not give any new information on the infrared dynamics.

Because of the difficulties in satisfying the conventional 't Hooft anomaly constraints, one is led to believe that the symmetry of the model (6.40) is spontaneously broken in the infrared, by some condensate. It is possible that the vacuum (or vacua) is (are) characterized by four-fermion condensates, but the simplest scenario seems to be dynamical Abelianization, triggered by a bi-fermion condensate,

$$\langle \psi\chi \rangle \tag{6.47}$$

in color contraction



$$+ \cdots\,, \tag{6.48}$$

i.e., in the adjoint representation of color SU(8). The global symmetries could be broken as

$$\mathrm{SU}(3)\times \mathrm{U}(1)_{\psi\chi}\times \mathbb{Z}_2 \longrightarrow \mathrm{SU}(2)\times \mathrm{U}(1)'_{\psi\chi}\,, \tag{6.49}$$

where $\mathrm{U}(1)'_{\psi\chi}$ is an unbroken combination of SU(3) and $\mathrm{U}(1)_{\psi\chi}$. In this case the system may dynamically Abelianize completely [25, 26]. The conventional 't Hooft anomaly conditions are satisfied by the fermion components which do not condense and remain massless, in a simple manner, as in the model considered in the previous subsection.

7 Discussion

In this paper symmetries and dynamics of several gauge theories which possess an exact center symmetry have been studied. Let us consider a general setup which includes all the cases studied here. We consider an $\mathrm{SU}(N)$ gauge theory with matter content consisting of Weyl fermions ψ_i in representations R_i and multiplicities $N_{f,i}$ with $i = 1, \ldots, n_R$ where n_R is the number of different representations (in the cases discussed n_R has been at most 2). We consider only cases in which the gauge anomaly cancels and where b_0 is positive (asymptotically free theories). Each $\mathrm{U}(1)_{\psi_i}$ global symmetry is broken due to instantons as

$$\mathrm{U}(1)_{\psi_i} \longrightarrow \mathbb{Z}^{\psi}_{2T_{R_i}N_{f,i}}\,. \tag{7.1}$$

Every representation of $\mathrm{SU}(N)$ has a certain N-ality associated to it; that is the way ψ_i transforms under the center of the gauge group $\mathbb{Z}_N^C$ and corresponds to the number of

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boxes in the Young tableaux modulo N. Let $n(R_i)$ be the N-ality of the representation R_i. For example the fundamental representation has N-ality 1, the two-index representations have N-ality 2 and the adjoint has N-ality 0. We then consider the greatest common divisor between N and all the N-alities of the various representations

$$k = \gcd\left(N, n(R_1), n(R_2), \dots, n(R_{n_R})\right) . \tag{7.2}$$

If k is greater than 1, then we have a nontrivial 1-form center symmetry $\mathbb{Z}_k^C$. A toron can be constructed with $n_{12} = n_{34} = \frac{N}{k}$ units of 't Hooft fluxes and it has topological charge equal to

$$\frac{n_{12}\, n_{34}}{N} = \frac{1}{N}\frac{N^2}{k^2} = \frac{N}{k^2} \tag{7.3}$$

that of the instanton. This can be combined with some integer (instanton) number to yield the minimal possible topological charge

$$\frac{1}{\tilde{k}} = \frac{\gcd\left(\frac{N^2}{k^2}, N\right)}{N} . \tag{7.4}$$

To see this, set

$$N = k\, n\, , \qquad n \in \mathbb{Z}\, , \tag{7.5}$$

then

$$\tilde{k} = \frac{N}{\gcd\left(\frac{N^2}{k^2}, N\right)} = \frac{kn}{\gcd\left(n^2, kn\right)} = \frac{k}{\gcd\left(n, k\right)} \in \mathbb{Z}\, . \tag{7.6}$$

Now the toron charge is

$$\frac{N}{k^2} = \frac{n}{k} = \frac{n/\gcd(n,k)}{k/\gcd(n,k)} = \frac{m}{k/\gcd(n,k)} = \frac{m}{\tilde{k}}\, , \qquad m \equiv \frac{n}{\gcd(n,k)} \in \mathbb{Z}\, , \tag{7.7}$$

where m and $\tilde{k}$ are coprime. Combining this with some instanton number q, one has

$$\exists p, \exists q \in \mathbb{Z}\, , \qquad \frac{N}{k^2}\cdot p + q = \frac{m}{\tilde{k}}\cdot p + q = \frac{mp + \tilde{k}q}{\tilde{k}} = \frac{1}{\tilde{k}}\, , \tag{7.8}$$

due to Bézout's lemma.

If this is a fractional number, i.e. if $\tilde{k}$ is an integer larger than 1, we have generalized (mixed) anomalies of the type $\mathbb{Z}^{\psi}_{2T_{R_i}N_{f,i}}[\mathbb{Z}_k^C]^2$, and the discrete symmetry is further broken as

$$\mathbb{Z}^{\psi}_{2T_{R_i}N_{f,i}} \longrightarrow \mathbb{Z}_{2T_{R_i}N_{f,i}/\tilde{k}}\, . \tag{7.9}$$

That $\tilde{k}$ is a divisor of $2T_R$ can be shown, case by case, by using the formulas given in appendix A, as has been verified in all cases encountered.

Note that the existence of a nontrivial center symmetry does not necessarily imply the presence of generalized (mixed) anomalies, as k and $\tilde{k}$ may be different. We have seen three examples in this paper, the SU(4) model in subsection 3.2, SU(4ℓ) cases in section 5, and the SU(8) chiral model in subsection 6.2, where $\tilde{k} = 1$ and where no mixed anomalies arise.

In all cases studied in the paper the presence of some bi-fermion condensates would explain the anomaly matching by breaking the discrete chiral symmetry down to a sufficiently small subgroup. Most of the time this subgroup is smaller than the minimal one required, e.g., from matching of the mixed anomalies. In any event, the use of mixed and conventional 't Hooft anomaly matching constraints in general provides us with significant, if not decisive, indications about the infrared dynamics of the theories.

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A The Dynkin index of some SU(N) representations

The Dynkin index T_R is defined by

$$\mathrm{tr}\left(t_R^a t_R^b\right) = T_R\, \delta^{ab}\,, \tag{A.1}$$

where t_R^a are the generators of SU(N) in the representation R. Summing over $a = b$, one gets

$$d(R)C_2(R) = T_R\,(N^2-1)\,, \qquad \sum_a t_R^a t_R^a = C_2(R)\mathbb{1}_{d(R)}\,, \tag{A.2}$$

where $d(R)$ is the dimension of the representation and $C_2(R)$ is the quadratic Casimir. For the fundamental representation one has

$$C_2(F) = \frac{N^2-1}{2N}\,, \qquad d(F) = N\,, \qquad T_F = \frac{1}{2}\,, \tag{A.3}$$

and for the adjoint,

$$C_2(\mathrm{adj}) = N\,, \qquad d(\mathrm{adj}) = N^2-1\,, \qquad T_{\mathrm{adj}} = N\,; \tag{A.4}$$

these two are quite familiar. For a rectangular Young tableau the quadratic Casimir is[19]

$$C_2(\overbrace{f,\ldots,f}^{k},0,\ldots) = \frac{kf(N+f)(N-k)}{2N}\,, \tag{A.5}$$

where f is the number of the boxes in a row and k the number of rows.

For the order n-antisymmetric representation, $f=1$, $k=n$, it is

$$C_2(R) = \frac{n(N-n)(N+1)}{2N}\,. \tag{A.6}$$

Taking into account the multiplicity

$$d(R) = \frac{N(N-1)\cdots(N-n+1)}{n!} = \frac{N!}{n!(N-n)!}\,, \tag{A.7}$$

[19] See for example the book of Barut and Raczka [49], p. 259 (apart from a factor 1/2 which is included, so that $C_2(F) = (N^2-1)/2N$ for the fundamental). See also [50] for reference.

the Dynkin index of totally antisymmetric single column representation of height n is given by

$$T_R = \frac{(N-2)(N-3)\cdots(N-n)}{2(n-1)!}\,. \tag{A.8}$$

For the special cases (of relevance in section 3) with $N = 2n$ even, we have

$$C_2(R) = \frac{N(N+1)}{8}\,, \qquad d(R) = \binom{N}{N/2}\,, \tag{A.9}$$

and

$$2\,T_R = \binom{N-2}{N/2-1} = \binom{2n-2}{n-1}\,. \tag{A.10}$$

By using this expression it is easy to see that for $N = 4\ell$, $n = 2\ell$, $2\,T_R$ is a multiple of ℓ, whereas for $N = 4\ell+2$, $2\,T_R$ contains $2\ell+1$ as a divisor. To prove it, note that the general combinatoric number

$$\binom{m}{r} = \frac{m(m-1)\dots(m-r+1)}{r!} = \frac{m!}{r!(m-r)!}\,, \tag{A.11}$$

is always an integer. But

$$\binom{m}{r} = \binom{m}{r-1}\cdot\frac{m-r+1}{r} \tag{A.12}$$

and both $\binom{m}{r}$ and $\binom{m}{r-1}$ are integers. Therefore $m-r+1$ is a divisor of $\binom{m}{r}$. Applying this for $N = 4\ell$ one finds that

$$2\,T_R = \binom{4\ell-2}{2\ell-1} \tag{A.13}$$

has a divisor 2ℓ hence ℓ. For $N = 4\ell+2$,

$$2\,T_R = \binom{4\ell}{2\ell} \tag{A.14}$$

has a divisor, $2\ell+1$.

For the symmetric representation of rank 2, $f = 2$, $k = 1$, so

$$C_2(R) = \frac{(N+2)(N-1)}{N}\,. \tag{A.15}$$

By taking into account the multiplicity,

$$d(R) = \frac{N(N+1)}{2}\,, \tag{A.16}$$

one finds

$$2\,T_R = 2\,\frac{d(R)C_2(R)}{N^2-1} = N+2\,. \tag{A.17}$$

For the symmetric representation of rank m, $f = m$, $k = 1$, so

$$C_2(R) = \frac{m(N+m)(N-1)}{2N}\,. \tag{A.18}$$

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By taking into account the multiplicity,

$$d(R) = \frac{(N+m-1)!}{m!\,(N-1)!}\,, \tag{A.19}$$

one finds

$$2\,T_R = \frac{d(R)C_2(R)}{N^2-1} = \frac{(N+m)!}{(N+1)!\,(m-1)!}\,. \tag{A.20}$$

$2\,T_R$ is a multiple of m, as can be shown following a similar consideration as (A.11)–(A.14).

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