



Gauging scale symmetry and inflation: Weyl versus Palatini gravity

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Abstract We present a comparative study of inflation in two theories of quadratic gravity with *gauged* scale symmetry: (1) the original Weyl quadratic gravity and (2) the theory defined by a similar action but in the Palatini approach obtained by replacing the Weyl connection by its Palatini counterpart. These theories have different vectorial non-metricity induced by the gauge field (w_μ) of this symmetry. Both theories have a novel spontaneous breaking of gauged scale symmetry, in the absence of matter, where the necessary scalar field is not added ad-hoc to this purpose but is of geometric origin and part of the quadratic action. The Einstein-Proca action (of w_μ), Planck scale and metricity emerge in the broken phase after w_μ acquires mass (Stueckelberg mechanism), then decouples. In the presence of matter (ϕ_1), non-minimally coupled, the scalar potential is similar in both theories up to couplings and field rescaling. For small field values the potential is Higgs-like while for large fields inflation is possible. Due to their R^2 term, both theories have a small tensor-to-scalar ratio ($r \sim 10^{-3}$), larger in Palatini case. For a fixed spectral index n_s , reducing the non-minimal coupling (ξ_1) increases r which in Weyl theory is bounded from above by that of Starobinsky inflation. For a small enough $\xi_1 \leq 10^{-3}$, unlike the Palatini version, Weyl theory gives a dependence $r(n_s)$ similar to that in Starobinsky inflation, while also protecting r against higher dimensional operators corrections.

1 Motivation

In this work we present a comparative study of inflation in two theories of quadratic gravity that have a *gauged* scale symmetry also known as Weyl gauge symmetry. This symmetry was first present in the original Weyl quadratic gravity [1–3] (for a review [4]) that follows from an underlying Weyl conformal geometry. This is relevant in early cosmology when

effective theories at short distances may become conformal. Due to their symmetry, these theories have no mass scales or dimensionful couplings – these must be generated by the vacuum expectations values (vev) of the fields and this is the view we adopt here.

The first theory is the original Weyl quadratic gravity revisited recently in [5–7] with new results. This was in fact the first gauge theory (of scale invariance).¹ The second theory [8] has a similar action but in the Palatini formalism [9–13], which means replacing the Weyl connection by the Palatini connection. In the absence of matter the Lagrangian has the form

$$L_0 = \sqrt{g} \left\{ \frac{\xi_0}{4!} R(\tilde{\Gamma}, g)^2 - \frac{1}{4\alpha^2} R_{[\mu\nu]}(\tilde{\Gamma})^2 \right\} \quad (1)$$

where $\tilde{\Gamma}$ is the Weyl or Palatini connection, respectively and ξ_0 and α are constants. These terms involve the (scalar and tensor) curvatures R and $R_{\mu\nu}$ which are functions of $\tilde{\Gamma}$; note that $\tilde{\Gamma}$ is not determined by the metric $g_{\mu\nu}$. This is the minimal action with such gauge symmetry. More quadratic terms may be present in both cases, see later.

In both theories the connection ($\tilde{\Gamma}$) is Weyl gauge invariant. Hence this is not only a symmetry of the action, but also of the underlying geometry. Both theories have vectorial non-metricity which is due to the dynamics of the gauge field w_μ of scale symmetry;² w_μ is dynamical since for $\tilde{\Gamma}$ symmetric (which we assume to be the case) the term $R_{[\mu\nu]}^2 \sim F_{\mu\nu}^2$ is just a gauge kinetic term of w_μ . And if w_μ is not dynamical it can easily be integrated out and both theories are Weyl integrable and metric ($\tilde{\nabla}_\mu g_{\mu\nu} = 0$), see e.g. [5, 6, 8]. In both theories the Weyl gauge field is related to the *trace* of non-metricity: $w_\mu \propto g^{\alpha\beta} \tilde{\nabla}_\mu g_{\alpha\beta}$ where $\tilde{\nabla}$ is computed with the

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¹ The literature sometimes calls Weyl gravity the action of a Weyl-tensor-squared term (in Riemannian geometry). The Weyl action we discuss is the original one defined by Weyl geometry [1–4] but without Weyl's unfortunate interpretation of its gauge boson as the real photon, and it includes the aforementioned term.

² If matter fields are present, they can also induce non-metricity.

Weyl or Palatini connection. The two theories have however a different non-metricity *tensor*; this leads to different inflation predictions that we discuss. We thus have a link between non-metricity and inflation predictions.

Our study of these two theories with *gauged* scale symmetry is motivated by:

- (a) In the *absence of matter* both theories of quadratic gravity have spontaneous breaking of this symmetry as it was shown for the first time in [5,6] for Weyl quadratic theory and in [8] for the Palatini theories. In both cases the Einstein-Proca action of w_μ and the Planck scale emerge in the broken phase, after w_μ becomes *massive* by “eating” the Stueckelberg field (would-be Goldstone/dilaton); this is the field that “linearises” $R(\tilde{\Gamma}, g)^2$ in the action, as we shall detail. After w_μ decouples near the Planck scale $M \sim \langle \text{dilaton} \rangle$, the Einstein action is naturally obtained (together with metricity, see below).³ Thus, these theories provide a natural mass generation (Planck and w_μ masses) via a symmetry breaking mechanism. The above result is important since it shows a new mechanism of spontaneous breaking of scale symmetry (in the absence of matter) in which the necessary scalar field is not added ad-hoc to this purpose (as usually done); instead, the Stueckelberg field is here of *geometric* origin, being “extracted” from the $R(\tilde{\Gamma}, g)^2$ term. This situation is very different from previous studies that used instead e.g. modified versions of Weyl action that were linear-only in R and/or used additional matter field(s) to generate the Planck scale [19–29].
- (b) The breaking of Weyl gauge symmetry mentioned at a) is accompanied by a change of the underlying geometry (connection). For example in the Weyl theory after w_μ becomes massive it decouples, the Weyl connection becomes Levi-Civita, thus the underlying Weyl geometry becomes Riemannian and the theory becomes metric. A similar change of the underlying geometry happens in the Palatini case. Hence, the breaking of the Weyl gauge symmetry shown in [5,6,8] is not the result of a mere choice of a gauge (as it happens in Weyl or conformal theories with no Weyl gauge field), but is more profound: it is accompanied by both a Stueckelberg mechanism (as mentioned) and by transformations at a geometric level.
- (c) In both Weyl and Palatini theories w_μ has a large mass ($\sim M$) [5,6,8] so the associated non-metricity scale is very high; hence, non-metricity effects are suppressed by M . One thus avoids long-held criticisms [1] that had assumed a *massless* w_μ (implying metricity violation

at low scales or path dependence of clock’s rates/rod’s length, in contrast to experience [30,31]).

- (d) If matter is present e.g. a Higgs-like scalar is non-minimally coupled to $R(\tilde{\Gamma}, g)$, Weyl and Palatini theories have successful inflation, in addition to mass generation. The main goal of this work is to investigate comparatively their inflation predictions. We give new results in Sect. 3, such as the dependence $r(n_s)$ of the tensor-to-scalar ratio r on the spectral index n_s in Weyl and Palatini cases and their relation to Starobinsky inflation [32–34].
- (e) The standard model (SM) with a vanishing Higgs mass has a Weyl gauge symmetry. It is well-known that the fermions and gauge bosons do not couple to the gauge field w_μ [35] but scalars (Higgs) have couplings to w_μ . Having seen that w_μ is massive [5,6,8] it is worth studying the SM in Weyl quadratic gravity or its Palatini version.⁴ If the gauged scale symmetry is relevant for the mass hierarchy problem, it is intriguing that only the Higgs field couples directly to the gauge boson w_μ of scale symmetry.
- (f) w_μ is a dark matter candidate [36] and, being part of $\tilde{\Gamma}$, it could give a *geometric* solution to the dark matter problem. This brings together physics beyond SM and gravity.
- (g) The models with *gauged* scale symmetry do not have the unitarity issue (negative kinetic term) present in *local* scale invariant Lagrangians (without w_μ), when generating the Einstein action from such Lagrangians: $L = (-1/12)\sqrt{g}[\phi^2 R + 6(\partial_\mu \phi)^2]$. See [26] for a discussion on this issue in local scale invariant models⁵ [37–43]. In a *gauged* scale invariant model this negative kinetic term is cancelled and ϕ is “eaten” by w_μ which acquires mass [5,6,8] à la Stueckelberg [15,44] and decouples, to recover the Einstein action and gauge.
- (h) In the gauged [5,6,8] and global [45–47] cases there is an associated non-zero conserved current, unlike in some local scale invariant models where this current is trivial [48,49].
- (i) A *gauged* scale symmetry seems stable under black-hole physics unlike a global one [50], so it is preferable when building models that include gravity. Global models are easily made gauged scale invariant by replacing their Levi-Civita connection by e.g. Weyl connection. The theories discussed can give a gauged scale invariant version of Agravity global model [51,52].

³ This mechanism may be more general and could apply to metric affine theories [14–16] (see also [17]).

⁴ For the SM Lagrangian in Weyl quadratic gravity, see [[6], section 1.7] and [24,29,35].

⁵ Avoiding unitarity violation in local scale invariant cases may require ϕ have an imaginary vev [38–40] but then the associated conformal transformation involving $\Omega^2 \propto \phi^2$ seems to change the overall metric signature.

(j) Another motivation to study theories with Weyl gauge symmetry is their geodesic completeness, as emphasized in [26]. In conformal invariant theories geodesic completeness can be achieved without the Weyl gauge vector presence, in the (metric) Riemannian universe; there, geodesic completeness or incompleteness is related to a specific gauge choice (with singularities due to an unphysical conformal frame) [53–56]. But Weyl gauge symmetry seems more profound: it is more than a symmetry of the action since, (unlike in conformal/Weyl invariant theory with no w_μ), it is also a symmetry of the underlying geometry (of connection $\tilde{\Gamma}$). The geodesics in such case are determined by the affine structure. Differential geometry actually demands the existence of the Weyl gauge field [57] for the construction of the affine connection, because this ensures that geodesics are invariant (as required on physical arguments). After the breaking, w_μ decouples, see b) above, and we return to Riemannian geometry and geodesics are given by extremal proper time condition.⁶

The above arguments, a) to j), motivated our interest in theories beyond Standard Model (SM) with Weyl gauge symmetry. Section 2 reviews the two theories, showing their similarities and differences, see [5, 6, 8] for technical details. Section 3 studies comparatively their inflation predictions. The Appendix has technical details and an application to inflation.

2 Weyl versus Palatini quadratic gravity

2.1 The symmetry

Consider a Weyl local scale transformation $\Omega(x)$ of the metric $g_{\mu\nu}$ and of a scalar field ϕ_1 ⁷

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \sqrt{\hat{g}} = \Omega^4 \sqrt{g}, \quad \hat{\phi}_1 = \frac{1}{\Omega} \phi_1. \quad (2)$$

To this geometric transformation one associates a Weyl gauge field w_μ that transforms as

$$\hat{w}_\mu = w_\mu - \partial_\mu \ln \Omega^2. \quad (3)$$

Equations (2), (3) define a gauged scale transformation. The symmetry is a gauged dilation group isomorphic to R^+ (non-compact). It differs from internal gauge symmetries, since Ω is real.

What is the relation of the Weyl field to the underlying geometry which is defined by $g_{\mu\nu}$ and $\tilde{\Gamma}$? One can define

w_μ via the non-metricity, but it is more intuitive to define w_μ as a measure of the deviation of (the trace of) $\tilde{\Gamma}$ from the Levi-Civita connection:

$$w_\mu = (1/2) (\tilde{\Gamma}_\mu - \Gamma_\mu(g)), \quad (4)$$

with a notation $\tilde{\Gamma}_\mu = \tilde{\Gamma}_{\mu\nu}^\nu$ and $\Gamma_\mu = \Gamma_{\mu\nu}^\nu(g)$. $\Gamma_{\mu\nu}^\alpha(g)$ is the Levi-Civita connection for $g_{\mu\nu}$ while $\tilde{\Gamma}_{\mu\nu}^\alpha$ is the connection in either Weyl or Palatini gravity. We assume a symmetric connection $\tilde{\Gamma}_{\mu\nu}^\alpha = \tilde{\Gamma}_{\nu\mu}^\alpha$ (no torsion). Note that w_μ is a vector under coordinate transformation ($\tilde{\Gamma}_\mu$ and Γ_μ are not). Finally, $\tilde{\Gamma}_{\mu\nu}^\alpha$ and in particular $\tilde{\Gamma}_\mu$ is invariant under (2), (3), in both Weyl and Palatini gravity (see also the Appendix). To check this invariance use (3) in (4) and that $\Gamma_\mu(g) = \partial_\mu \ln \sqrt{g}$; then $\Gamma_\mu(\hat{g}) = \partial_\mu \ln(\sqrt{g} \Omega^4)$. The change of the metric is compensated by that of w_μ , leaving $\tilde{\Gamma}_\mu$ invariant.

2.2 The Lagrangian: Weyl versus Palatini

Consider next a Lagrangian with gauged scale invariance for a scalar field with non-minimal coupling, in Weyl and Palatini quadratic gravity. The analysis being similar, we present simultaneously both Weyl and Palatini theories. The main difference between them is in the coefficients $\tilde{\Gamma}_{\mu\nu}^\alpha$ which we do not need to specify right now. Consider then a (Higgs-like) scalar ϕ_1 with non-minimal coupling $\xi_1 > 0$:

$$L = \sqrt{g} \left[\frac{\xi_0}{4!} R(\tilde{\Gamma}, g)^2 - \frac{1}{4\alpha^2} F_{\mu\nu}(\tilde{\Gamma})^2 - \frac{1}{12} \xi_1 \phi_1^2 R(\tilde{\Gamma}, g) + \frac{1}{2} (\tilde{D}_\mu \phi_1)^2 - \frac{\lambda_1}{4!} \phi_1^4 \right], \quad (5)$$

with a scalar curvature $R(\tilde{\Gamma}, g)$ which depends on the Weyl or Palatini connection $\tilde{\Gamma}$:

$$R(\tilde{\Gamma}, g) = g^{\mu\nu} R_{\mu\nu}(\tilde{\Gamma}), \quad R_{\mu\nu}(\tilde{\Gamma}) = \partial_\lambda \tilde{\Gamma}_{\mu\nu}^\lambda - \partial_\mu \tilde{\Gamma}_{\lambda\nu}^\lambda + \tilde{\Gamma}_{\rho\lambda}^\lambda \tilde{\Gamma}_{\mu\nu}^\rho - \tilde{\Gamma}_{\rho\mu}^\lambda \tilde{\Gamma}_{\nu\lambda}^\rho. \quad (6)$$

$\tilde{\Gamma}$ is invariant under (2), (3) so $R_{\mu\nu}(\tilde{\Gamma})$ is invariant; $R(\tilde{\Gamma}, g)$ transforms covariantly, (A-10)

$$\hat{R}(\tilde{\Gamma}, \hat{g}) = (1/\Omega^2) R(\tilde{\Gamma}, g). \quad (7)$$

With (7), the first and third term in L are invariant under (2), (3).

Further, the second term in L is a gauge kinetic term of w_μ and involves

$$F_{\mu\nu}(\tilde{\Gamma}) = \tilde{\nabla}_\mu w_\nu - \tilde{\nabla}_\nu w_\mu = \partial_\mu w_\nu - \partial_\nu w_\mu = (\partial_\mu \tilde{\Gamma}_\nu - \partial_\nu \tilde{\Gamma}_\mu)/2. \quad (8)$$

$\tilde{\nabla}$ is defined by $\tilde{\Gamma}$ and in the second step we used that $\tilde{\Gamma}$ is symmetric. From (8) $F_{\mu\nu}$ is invariant under (2), (3), and one verifies that the second term in L is also invariant under these transformations. Since $F_{\mu\nu}(\tilde{\Gamma})^2 = R_{[\mu\nu]}(\tilde{\Gamma})^2$ where

⁶ Since the Weyl gauge field brings in non-metricity, geodesic completeness seems related to non-metricity.

⁷ Our conventions are those in the Appendix of [58] with metric (+, -, -, -), and $g \equiv |\det g_{\mu\nu}|$.

$R_{[\mu\nu]} \equiv (R_{\mu\nu} - R_{\nu\mu})/2$, a gauged scale symmetry is naturally present in the Palatini version of $R^2 + R^2_{[\mu\nu]}$ gravity.

The Weyl-covariant derivative of ϕ_1 in L and its transformation under (2), (3) are

$$\tilde{D}_\mu \phi_1 = (\partial_\mu - 1/2 w_\mu) \phi_1, \quad \hat{\tilde{D}}_\mu \hat{\phi}_1 = (1/\Omega) \tilde{D}_\mu \phi_1. \quad (9)$$

Therefore ϕ_1 is charged under the Weyl gauge symmetry. With (9) one checks that the kinetic term of ϕ_1 is invariant under (2), (3). Finally, $\lambda_1 \phi_1^4$ is the only potential term allowed by symmetry, so the entire L is invariant.

In the *absence of matter* (ϕ_1), L contains the first two terms only, giving the minimal action of the original Weyl quadratic gravity or its Palatini version; both actions have gauged scale symmetry and, after spontaneous breaking of this symmetry, one obtains the Einstein-Proca action for w_μ , see [5,6,8]. If only the first term is present in L , both theories are Weyl integrable (metric) and Einstein action is obtained with a positive cosmological constant.

Returning to L , we replace the first term in (5) by $\xi_0 R(\tilde{\Gamma}, g)^2 \rightarrow -\xi_0 (2\phi_0^2 R(\tilde{\Gamma}, g) + \phi_0^4)$ where ϕ_0 is an auxiliary scalar; using the equation of motion of ϕ_0 (of solution $\phi_0^2 = -R$) recovers onshell the term $\xi_0 R^2$ in (5). This gives a classically equivalent L , linear in R

$$L = \sqrt{g} \left[-\frac{1}{2} \rho^2 R(\tilde{\Gamma}, g) - \frac{1}{4\alpha^2} F_{\mu\nu}^2 + \frac{1}{2} (\tilde{D}_\mu \phi_1)^2 - \mathcal{V}(\phi_1, \rho) \right], \quad (10)$$

where

$$\mathcal{V}(\phi_1, \rho) = \frac{1}{4!} \left[\frac{1}{\xi_0} (6\rho^2 - \xi_1 \phi_1^2)^2 + \lambda_1 \phi_1^4 \right], \quad \text{and} \quad \rho^2 = \frac{1}{6} (\xi_1 \phi_1^2 + \xi_0 \phi_0^2). \quad (11)$$

We further replaced ϕ_0 by radial direction ρ in field space, so our new fields are now $\{\rho, \phi_1\}$.

L has similarities to a global scale invariant Higgs-dilaton model, Equations (2.9), (2.10) of [47] also [59,60]; ϕ_0 has a large coupling ($\xi_0 > 1$) to R since the R^2 term has a perturbative coupling $1/\sqrt{\xi_0} < 1$ and this corresponds to a Higgs of non-minimal coupling $\xi_h > 1$ in [47].

The action in (10) depends on $\tilde{\Gamma}$ through its first three terms. We have two cases:

- In Weyl quadratic gravity, $\tilde{\Gamma}$ is determined by $g_{\mu\nu}$ and the gauge field w_μ , see its expression in Eq. (A-5) in the Appendix. Using this one replaces the scalar curvature in (10) in terms of the Ricci scalar of Riemannian geometry, Eq. (A-11). The result is Eq. (12) below.
- In Palatini gravity, $\tilde{\Gamma}$ is simply determined by its equation of motion from the action in (10). After solving this Eq. [8], we obtain the connection shown in Eq. (B-2) in

the Appendix; $\tilde{\Gamma}$ differs from that in Weyl case, due to different non-metricity (accounted for by γ in Eq. (12)). With this $\tilde{\Gamma}$, one computes the scalar curvature, as usually done Eq. (B-5). Replacing this curvature back in action (10) one finds again L below (for $\tilde{\Gamma}$ onshell):

$$L = \sqrt{g} \left\{ \frac{-1}{2} \left[\rho^2 R(g) + 6(\partial_\mu \rho)^2 \right] + 3\gamma \rho^2 (w_\mu - \partial_\mu \ln \rho^2)^2 - \frac{1}{4\alpha^2} F_{\mu\nu}^2 + \frac{1}{2} (\tilde{D}_\mu \phi_1)^2 - \mathcal{V}(\phi_1, \rho) \right\} \quad (12)$$

where $\gamma = 1/4$ for Weyl case; $\gamma = 1$ for Palatini case. (13)

$R(g)$ is the Ricci scalar for the metric $g_{\mu\nu}$. This is a metric formulation equivalent to the initial Lagrangian Eq. (5), invariant under transformations (2), (3); under these $\ln \rho$ transforms with a shift, $\ln \rho \rightarrow \ln \rho - \ln \Omega$, so $\ln \rho$ acts like a would-be Goldstone (“dilaton”), see later.

2.3 Einstein-Proca action as a broken phase of Weyl or Palatini gravity

Since L has a gauged scale symmetry, we should “fix the gauge”. We choose the Einstein gauge corresponding to constant ρ ; this is obtained by using a transformation (2),(3) of a particular $\Omega = \rho/\langle \rho \rangle$ which is ρ -dependent and sets $\hat{\rho}$ to a constant $\hat{\rho} = \langle \rho \rangle$, and so introduces a mass scale. In terms of new variables (with a hat) Eq. (12) becomes

$$L = \sqrt{\hat{g}} \left\{ -\frac{1}{2} M^2 R(\hat{g}) + 3\gamma M^2 \hat{w}_\mu \hat{w}^\mu - \frac{1}{4\alpha^2} \hat{F}_{\mu\nu}^2 + \frac{1}{2} (\hat{\tilde{D}}_\mu \hat{\phi}_1)^2 - \mathcal{V}(\hat{\phi}_1, M) \right\}, \quad (14)$$

with $R(\hat{g})$ the Ricci scalar for metric $\hat{g}_{\mu\nu}$, $\hat{\tilde{D}}_\mu \hat{\phi}_1 = (\partial_\mu - 1/2 \hat{w}_\mu) \hat{\phi}_1$ and with $\nabla_\mu \hat{w}^\mu = 0$; we denoted $M = \langle \rho \rangle$ which we identify with the Planck scale. The potential now depends on $\hat{\phi}_1$ only, see (11). This is the Einstein-Proca action for \hat{w}_μ : this field has become massive of mass $m_w^2 = 6\alpha^2 \gamma M^2$ by absorbing the derivative of the Stueckelberg (would-be “dilaton”) field $\partial_\mu \ln \rho$; then the *radial* direction in field space (ρ) is not present anymore in the action. This is a spontaneous breaking of Weyl gauge symmetry; the number n of degrees of freedom other than the graviton ($n = 3$) is conserved during this breaking: the initial massless scalar ρ and massless vector w_μ are replaced by a massive gauge field w_μ .

Note that in the absence of matter (ϕ_1), the Stueckelberg field needed for breaking becomes $\ln \rho \propto \ln \phi_0$ and has a pure geometric origin, being simply “extracted” from the quadratic curvature term $R^2(\tilde{\Gamma}, g)$ in the initial, symmetric action. Therefore, one does not need to add this scalar field ad-hoc as usually done to this purpose, and the breaking and mass generation (m_w , Planck scale) takes place even in the

absence of matter [5, 6, 8]. Finally, unless one is tuning the coupling α to small values, the mass of \hat{w}_μ is near the Planck scale.⁸

2.4 Scalar potential

To obtain a standard kinetic term for $\hat{\phi}_1$, similar to the “unitarity gauge” in the electroweak case, we remove the coupling $\hat{w}^\mu \partial_\mu \hat{\phi}_1$ from the term $(\hat{D}_\mu \hat{\phi}_1)^2$ in (14) by a field redefinition

$$\begin{aligned}\hat{w}_\mu &\rightarrow \hat{w}_\mu + \partial_\mu \ln \cosh^2 \left[\frac{\varphi}{2M\sqrt{6\gamma}} \right], \\ \hat{\phi}_1 &\rightarrow 2M\sqrt{6\gamma} \sinh \left[\frac{\varphi}{2M\sqrt{6\gamma}} \right]\end{aligned}\quad (15)$$

In terms of the new fields Eq. (14) becomes

$$\begin{aligned}L = \sqrt{\hat{g}} \left\{ -\frac{1}{2} M^2 R(\hat{g}) + 3\gamma M^2 \cosh^2 \left[\frac{\varphi}{2M\sqrt{6\gamma}} \right] \hat{w}_\mu \hat{w}^\mu \right. \\ \left. - \frac{1}{4\alpha^2} \hat{F}_{\mu\nu}^2 + \frac{\hat{g}^{\mu\nu}}{2} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right\},\end{aligned}\quad (16)$$

which is ghost-free and

$$\begin{aligned}V(\varphi) = V_0 \left\{ \left[1 - (4\gamma) \xi_1 \sinh^2 \frac{\varphi}{2M\sqrt{6\gamma}} \right]^2 \right. \\ \left. + (4\gamma)^2 \lambda_1 \xi_0 \sinh^4 \frac{\varphi}{2M\sqrt{6\gamma}} \right\}, \quad V_0 \equiv \frac{3}{2} \frac{M^4}{\xi_0}.\end{aligned}\quad (17)$$

Lagrangian (16) describes Einstein gravity, a scalar field φ with canonical kinetic term and potential (17) that is γ -dependent, and a massive Proca field (\hat{w}_μ) that decouples near the Planck scale M . To make obvious the mass term of w_μ in (16) use that $\cosh^2 x = 1 + \sinh^2 x$. Equations (16), (17) can be extended to more scalar fields, see [[6], Eq. 24].

For *small field* values $\varphi \ll M$, the potential in (17) becomes (recall that $M = \langle \rho \rangle$):

$$\begin{aligned}V(\varphi) = \frac{3\langle \rho \rangle^4}{2\xi_0} - \frac{1}{2} \frac{\xi_1}{\xi_0} \langle \rho \rangle^2 \varphi^2 \\ + \frac{1}{4!} \left[\lambda_1 + \frac{\xi_1}{\xi_0} \left(\xi_1 - \frac{1}{6\gamma} \right) \right] \varphi^4 + \mathcal{O}(\varphi^6 / \langle \rho \rangle^2).\end{aligned}\quad (18)$$

In this case the potential is similar in Weyl and Palatini cases, up to a small γ -dependence of the quartic coupling, negligible for (ultra)weak couplings $\xi_1/\xi_0 \ll 1$; in this case also the quadratic coupling is suppressed (recall the perturbative couplings are $1/\sqrt{\xi_0} < 1$ and $\xi_1 < 1$).

If we identify φ with the Higgs field, we have electroweak symmetry breaking, since $\xi_1 > 0$. For a classical hierarchy $\xi_1/\xi_0 \ll 1$ one may be able to tune the mass of φ near the electroweak scale $m^2 = (\xi_1/\xi_0)\langle \rho \rangle^2$. Gravitational corrections to λ_1 may be negative but there is no instability: the exact form of $V(\varphi)$ is positive, even if the self-coupling $\lambda_1 = 0$!

For *large* φ the potential is different in Weyl and Palatini cases due to a different γ . This potential changed from initial (5) to (17) following two steps: the “linearisation” of the R^2 term by ϕ_0 that induced the ϕ_0^4 term, then transformation (15) which decoupled the (trace of) the connection from $\partial_\mu \phi_1$ and brought the presence of γ i.e. non-metricity dependence.

3 Inflation

3.1 Weyl versus Palatini

We can now use Lagrangian (16) and potential $V(\varphi)$ of (17) to study inflation with φ as the inflaton and compare its predictions for the Weyl ($\gamma = 1/4$) and Palatini ($\gamma = 1$) cases. For a previous study of inflation in the Weyl case, see⁹ [7, 76]. Lagrangian (16) describes a single scalar field in Einstein gravity and the usual formalism for a single-field inflation can be used. However, notice there exists a coupling of φ to the Weyl field \hat{w}_μ , the second term in (16). Hence, we must first show that this coupling and \hat{w}_μ do not affect inflation by φ .

Firstly, we do not consider here the possibility of the Weyl vector field itself as the inflaton¹⁰ since it could induce a substantial large-scale anisotropy [81] which would be in conflict with CMB isotropy. The anisotropy is obvious in the stress-energy tensor contribution of \hat{w}_μ which is not diagonal. This issue can be avoided if one considers a large number of randomly oriented vector fields or a triplet of mutually orthogonal vector fields [81], however this is not possible in the current fixed setup.

Secondly, one may ask if the Weyl field could play the role of a curvaton with φ as the inflaton. The scenario of a vector field as a curvaton was discussed in detail in [82, 83]; in such scenario the vector field does not drive inflation (to avoid large scale anisotropy) but becomes important after inflation when it may dominate the Universe and imprint its perturbation spectrum. A scale invariant spectrum can be generated by \hat{w}_μ provided that during inflation the mass-squared of \hat{w}_μ is *negative* and large in absolute value ($\sim H^2$) while after inflation is positive and the vector field engages in oscillations and behaves as pressureless matter; this means it does not lead to large-scale anisotropy when it dominates

⁸ This is preferable, since then one avoids metricity violation below the Planck scale (due to a lighter w_μ). Current non-metricity lower bounds could be as low as TeV [30, 31] but are model dependent.

⁹ For inflation in related Palatini models but without Weyl gauge symmetry, see [61–75].

¹⁰ Inflation by vector fields was suggested in [77–80].

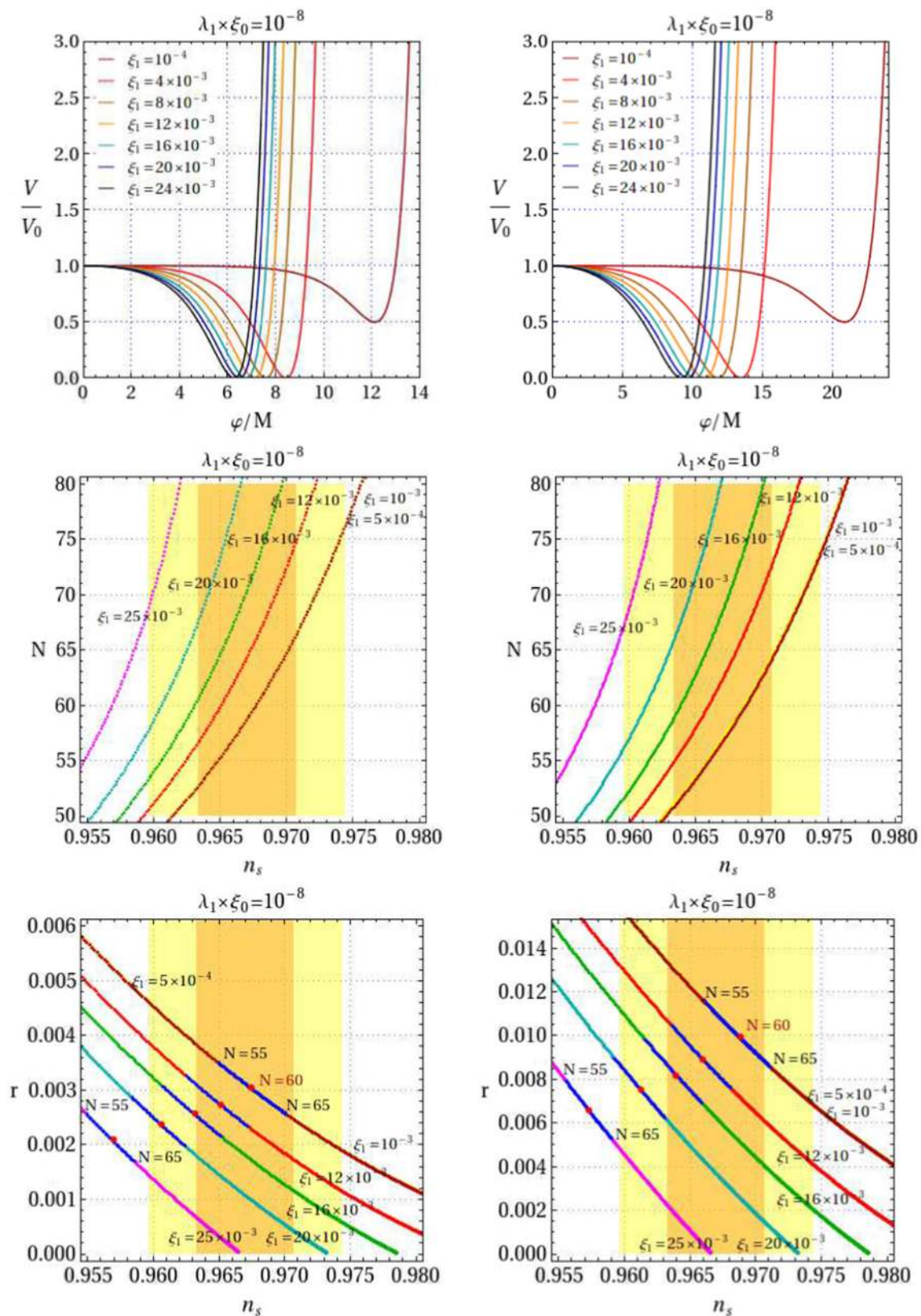


Fig. 1 Left column: Weyl inflation plots; Right column: Palatini inflation plots. All figures have $\lambda_1 \xi_0 = 10^{-8} \leq \xi_1^2$; Top plots: the potential V/V_0 in terms of φ/M for different ξ_1 ; larger ξ_1 moves the curves to the left; larger $\lambda_1 \xi_0$ lifts the minimum of the rightmost curves, see Eq. (17); middle plots: the plots (n_s, N) for different ξ_1 ; bottom plots:

the plots (n_s, r) for different ξ_1 ; curves of $\xi_1 = 5 \times 10^{-4}$ and 10^{-3} are degenerate; along these curves the value of N changes: the blue line segment has points of $55 \leq N \leq 65$; red dots have $N = 60$. The yellow (orange) area corresponds to the measured value of n_s at 95% CL (68%), respectively

[82, 83]. This scenario cannot apply here since m_w^2 is always positive. Indeed, the second term in (16)

$$\Delta L = \frac{1}{2} \sqrt{\hat{g}} f(\varphi) \hat{w}_\mu \hat{w}^\mu, \quad f(\varphi) = 6\gamma M^2 \cosh^2 \frac{\varphi}{2M\sqrt{6\gamma}}. \quad (19)$$

has $f(\varphi) > 0$, for any value of φ and the effective mass-squared of \hat{w}_μ is always positive.

Finally, in Friedmann–Robertson–Walker (FRW) universe $\hat{g}_{\mu\nu} = (1, -a(t)^2, -a(t)^2, -a(t)^2)$, the vector field background compatible with the metric is $\hat{w}_\mu(t) = (\hat{w}_0(t), 0, 0, 0)$. However, from the equation of motion of \hat{w}_μ one immediately sees that $\hat{w}_\mu(t) = 0$, (see also Eq. (C-13) for details). In this case ΔL is vanishing. Therefore, we are left with potential (17) and the usual formalism of single-field inflation in Einstein gravity applies, with φ as inflaton.

One may ask what happens at the perturbations level? One easily sees that perturbations $\delta\varphi$ of φ do not mix with perturbations $\delta\hat{w}_\mu$ (of longitudinal mode/Stueckelberg field ρ) of massive \hat{w}_μ . Such mixing is in principle possible, with potential impact on inflation predictions, but it vanishes since it is proportional to $\hat{w}_\mu(t) (= 0)$, as seen from expanding ΔL to quadratic level in perturbations: $\Delta L \propto \hat{w}_\mu(t) \delta\varphi \delta\hat{w}_\mu + \dots$.¹¹ As a result, the coupling ΔL does not affect $\delta\varphi$ and the predictions of inflation by φ . For further discussion on perturbations $\delta\varphi$ and $\delta\hat{w}_\mu$ see Appendix 1 which supports these results.

The above arguments justify our use below of single-field slow-roll formulae¹²

$$\epsilon = \frac{M^2}{2} \left\{ \frac{V'(\varphi)}{V(\varphi)} \right\}^2 = \frac{4}{3} \gamma \xi_1^2 \sinh^2 \frac{\varphi}{M\sqrt{6\gamma}} + \mathcal{O}(\xi_1^3), \quad (20)$$

$$\eta = M^2 \frac{V''(\varphi)}{V(\varphi)} = -\frac{2}{3} \xi_1 \cosh \frac{\varphi}{M\sqrt{6\gamma}} + \frac{8}{3} \gamma \xi_1^2 \sinh^2 \frac{\varphi}{2M\sqrt{6\gamma}} + \mathcal{O}(\xi_1^3), \quad (21)$$

The number of e-folds is

$$N = \frac{1}{M^2} \int_{\varphi_e}^{\varphi_*} d\varphi \frac{V(\varphi)}{V'(\varphi)} = \left\{ -\frac{3}{4\xi_1} \ln \tanh^2 \frac{\varphi}{2M\sqrt{6\gamma}} + 3\gamma \ln \cosh^2 \frac{\varphi}{2M\sqrt{6\gamma}} \right\} \Big|_{\varphi=\varphi_e}^{\varphi=\varphi_*}. \quad (22)$$

with the last step in (20), (21), (22) valid in the leading approximation $\lambda_1 \xi_0 \ll \xi_1^2$ needed for a deep enough minimum for inflation; φ_e is determined by $\epsilon(\varphi_e) = 1$ and φ_* is

the initial value of the scalar field. Further, the scalar spectral index

$$n_s = 1 + 2\eta_* - 6\epsilon_* = 1 - \frac{4}{3} \xi_1 \cosh \frac{\varphi_*}{M\sqrt{6\gamma}} + \frac{8}{3} \xi_1^2 \gamma \times \left[\cosh^2 \frac{\varphi_*}{M\sqrt{6\gamma}} - 1 \right] + \mathcal{O}(\xi_1^3). \quad (23)$$

With the tensor-to-scalar ratio $r = 16\epsilon_*$, then from (20), (21), (23)

$$r = 12\gamma(1 - n_s)^2 - \frac{64\gamma}{3} \xi_1^2 + \mathcal{O}(\xi_1^3). \quad (24)$$

The non-minimal coupling is reducing r , for fixed n_s . If we ignore the term $\propto \xi_1^2$ and higher orders, then the Palatini case ($\gamma = 1$) has a larger r than Weyl theory ($\gamma = 1/4$), for the same n_s . This is confirmed by exact numerical results, see later. From (22), we also find

$$r \approx \frac{48\gamma}{\bar{N}^2} + \frac{64\gamma}{\bar{N}} \times \mathcal{O}(\xi_1); \quad n_s \approx 1 - \frac{2}{\bar{N}} + \mathcal{O}(\xi_1) \quad (25)$$

with $\bar{N} \approx N + 9$ and $\gamma = 1/4$ in the Weyl case and $\bar{N} \approx N + 28$ and $\gamma = 1$ for the Palatini case. Equations (25) are only an approximation and ignore some ξ_1 dependence in \bar{N} , but give an idea of the exact behaviour (see later, Fig. 2).

There is an additional constraint on the parameters space of Weyl/Palatini models, from the normalization of the CMB anisotropy $V_0/(24\pi^2 M^4 \epsilon_*) = \kappa_0$, $\kappa_0 = 2.1 \times 10^{-9}$ and with $r < 0.07$ [86] then $\xi_0 = 1/(\pi^2 r \kappa) \geq 6.89 \times 10^8$. With this bound, condition $\lambda_1 \xi_0 \ll \xi_1^2$ is respected for any perturbative ξ_1 , $1/\xi_0$, by choosing an ultraweak $\lambda_1 \ll \xi_1^2/\xi_0$.

Let us compare Eq. (24) to that in the Starobinsky model of

$$\mathcal{L} = (-1/2)M^2 R + (\xi_0/4!) R^2, \quad (26)$$

giving $V = V_0(1 - e^{-\varphi\sqrt{2/3}/M})^2$ with $V_0 = 3M^4/(2\xi_0)$; then $r \approx 12/N^2$, $n_s \approx 1 - 2/N$ and

$$r = 3(1 - n_s)^2. \quad (27)$$

Interestingly, in Eq. (24) with $\xi_1 \sim 10^{-3}$ or smaller, the term $\propto \xi_1^2$ and higher powers have a negligible correction to r and $(1 - n_s)^2$ and can be ignored; therefore Weyl inflation ($\gamma = 1/4$) recovers relation (27) of Starobinsky model [32–34, 86]. For larger values of ξ_1 and fixed n_s , ξ_1 reduces r of Weyl inflation below that of Starobinsky model. In the Palatini case relation (27) is not possible (unless ξ_1 is tuned for every n_s) – the slope of $r(n_s)$ is different.

3.2 Numerical results

Our exact numerical results (with no expansion in powers of ξ_1) are given by the plots of potential, (n_s, N) , (n_s, r) , presented in Figs. 1 and 2 for Weyl and Palatini cases. Their differences are due to different γ . The results show a value of

¹¹ The absence of such mixing is also due to the FRW metric and to the fact that ρ (radial direction) and $\varphi \sim \hat{\phi}_1$ were orthogonal directions in field space (that do not mix) and similar for their perturbations.

¹² With $M \sim \langle \rho \rangle$ a simple phase transition scale, values of the field $\varphi \geq M$ are natural.

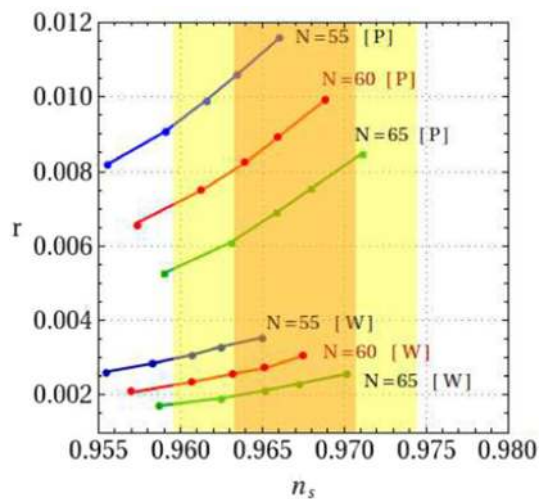


Fig. 2 The dependence $r(n_s)$ for various curves of constant N (as shown), for the Palatini theory ([P]) and Weyl theory ([W]). The dots stand for ξ_1 values corresponding to the curves in the last two plots in Fig. 1. The yellow (orange) area corresponds to the range measured for n_s at 95%CL (68%), respectively. The curves show the largest range of values for r with $\lambda_1 \xi_0 \ll \xi_1^2$; this range shrinks towards smaller r if $\lambda_1 \xi_0$ increases to $\lambda_1 \xi_0 \sim \xi_1^2$

r smaller in the Weyl case than in Palatini case, for relevant n_s . For $n_s = 0.9670 \pm 0.0037$ (68%CL) (TT, TE, EE + low E + lensing + BK14 +BAO) [86] one finds

$$\text{Palatini : } N = 60, \quad 0.00794 \leq r \leq 0.01002, \quad (28)$$

$$\text{Weyl : } N = 60, \quad 0.00257 \leq r \leq 0.00303. \quad (29)$$

and for n_s at 95%CL one has

$$\text{Palatini : } N = 60, \quad 0.00700 \leq r \leq 0.01002, \quad (30)$$

$$\text{Weyl : } N = 60, \quad 0.00227 \leq r \leq 0.00303. \quad (31)$$

The case of Starobinsky model for $N = 60$ corresponds to the upper limit of r (0.003) of the Weyl model (top curve in Fig. 1 and highest r in Fig. 2 for $N = 60$), while in the Palatini case a larger r is allowed for the same n_s, N .

While the plots in Fig. 1 have $\lambda_1 \xi_0 = 10^{-8}$, they are actually more general. In the extreme case of $\lambda_1 \sim 0$, corresponding to a simplified potential (without the last term in (17)), the same range of values for (n_s, r) shown in this figure remains valid. However, if we increase $\lambda_1 \xi_0$ to $\lambda_1 \xi_0 \approx \xi_1^2$, the last term in (17) becomes relatively large, the rightmost curves of V (of smallest ξ_1) have their minimum lifted and the range for (n_s, r) in (28) to (31) is reduced: the smaller values $\xi_1 \sim 10^{-3}$ in Fig. 1, cannot then have successful inflation.

The main results of this work are summarised in Fig. 2; in this figure the dependence $r(n_s)$ is shown for different curves of constant N , that respect the required parametric constraint $\lambda_1 \xi_0 \leq \xi_1^2$. The curves $r(n_s)$ give a numerically exact representation of the dependence in Eq. (24); they are extended even outside the 95% CL range for n_s . In all cases,

the Palatini case has r larger than in the Weyl case. This aspect and the different slope of the curves $r(n_s)$ can be used to distinguish these models from each other and from other models in future experiments.

The small r predicted by both Weyl and Palatini gravity models may be reached by the next generation of CMB experiments: CMB-S4, LiteBIRD, PICO, PIXIE [87–92], that will be sensitive to values as low as $r \sim 5 \times 10^{-4}$. Therefore they will be able to test these two inflation models.

3.3 Corrections and other models

Compared to another model with Weyl gauge symmetry [29] (Fig. 2) which is linear in $R(\tilde{\Gamma}, g)$ and had $r \sim 0.04$ – 0.06 , we see that the presence of the $R^2(\tilde{\Gamma}, g)$ term in the Weyl theory reduced r significantly (for a fixed n_s). Reducing r by an R^2 term also exists in the Palatini models without Weyl gauge symmetry [70]. Therefore, a small measured $r \sim 10^{-3}$ may indicate a preference for quadratic gravity models of inflation.

The Weyl inflation case of $\xi_1 = 10^{-3}$ or smaller is similar to the Starobinsky model, with the difference that it needs an additional scalar field (the Higgs field¹³) to play the role of the inflaton. This is because the other scalar in the Weyl theory (radial direction ρ in the ϕ_0, ϕ_1 space) is used to generate the Planck scale and the mass of w_μ . Briefly, Weyl gravity gives a relation $r(n_s)$ similar to that in the Starobinsky model, with similar, large ξ_0 , while also providing protection against corrections to r from higher dimensional operators; these are forbidden since their effective scale violates the symmetry; the Stueckelberg field cannot play the role of this scale since it was eaten by the Weyl field to all orders. Another benefit for Weyl inflation is the minimal approach: one only needs to consider the SM Higgs field in the Weyl conformal geometry; the underlying geometry provides the spontaneous breaking of the Weyl quadratic gravity action to the Einstein action and the Planck scale generation.

Despite this similarity of the Weyl and the Starobinsky models, it is possible to distinguish between them; it may happen that a curve $r(n_s)$ corresponding to $\xi_1 > 10^{-3}$ is preferred by data (see $r(n_s)$ curves in Fig. 1), in which case it is shifted below that of the Starobinsky model for the same n_s – the two models are distinguishable. Also the Weyl model has an additional coupling, see ΔL in (19). While ΔL does not mix linear perturbations of δw_μ and of $\delta\varphi$, it can lead however to cubic interactions of the form $f'(\varphi)\delta\varphi\delta w_\mu\delta w^\mu$. These can result in different predictions for the inflationary bispectrum compared to the pure single-field case. This can be used to further distinguish the Weyl case from the Starobinsky R^2 inflation (in case of $\xi_1 \sim 10^{-3}$). The analysis of non-Gaussianities is thus interesting for further research.

¹³ See discussion in the text after Eq. (18).

The above results are subject to corrections from other operators of $d = 4$ that may exist and are Weyl gauge invariant, as we discuss below.

In the Weyl case the Weyl-tensor-squared operator of Weyl geometry may be present $(1/\zeta)\tilde{C}_{\mu\nu\rho\sigma}^2$. This can be re-written in a metric description as the Weyl-tensor-squared term of Riemannian geometry $(1/\zeta)C_{\mu\nu\rho\sigma}^2$ plus a gauge kinetic term of w_μ which gives a threshold correction to our coupling α . The Weyl tensor term is invariant under Weyl gauge transformations performed to reach the Einstein-Proca action, hence one simply adds it to the final action, Eq. (14). This operator has an impact on the value of r that we found numerically and in Eq. (25) with $\gamma = 1/4$ for Weyl case. The overall impact of the Weyl tensor term is essentially a rescaling of r into¹⁴ $r_c = r(1 + 8/(\zeta\xi_0))^{1/2}$ [7]. Since our ξ_0 is large, only a low $|\zeta| \sim 1/\xi_0$ can increase r and this comes with an instability since the mass of the associated spin-two ghost (or tachyonic) state that this operator brings is $m^2 \sim \zeta M^2$, where M is the Planck scale. Therefore, a stable Weyl gravity model up to the Planck scale will not modify the value of r . Other operators in Weyl gravity are topological and do not affect r (classically).

In the Palatini case one should consider the remaining quadratic operators of $d = 4$ [93] that are Weyl gauge invariant and have a symmetric connection. They modify the equation of motion of $\tilde{\Gamma}$ and the vectorial non-metricity (B-6); unfortunately, it does not seem possible to find in this case an analytical solution to this equation due to its modified, complex structure and new states present (ghosts, etc). Additional simplifying assumptions would be needed, making the analysis model dependent. We only mention here the interesting possibility that for a symmetric $\tilde{\Gamma}$, the solution $\tilde{\Gamma}$ may become equal to that in Weyl-geometry (A-5); if so, the Palatini approach would provide an “offshell” version of Weyl quadratic gravity that is recovered for $\tilde{\Gamma}$ onshell.

4 Conclusions

We made a comparative study of the action and inflation in two theories of quadratic gravity with Weyl gauge symmetry: the original Weyl gravity action and the Palatini version of the same action, obtained by replacing the Weyl connection

by Palatini connection. The actions of these theories are non-minimally coupled to a (Higgs-like) field ϕ_1 , necessary for inflation.

Given the symmetry, there is no scale in these theories. Mass scales are generated by an elegant spontaneous breaking of gauged scale symmetry that happens even *in the absence of matter*: the necessary scalar field (Stueckelberg field ϕ_0) is not added *ad-hoc* as usually done to this purpose, but is of geometric origin and is “extracted” from the $R(\tilde{\Gamma}, g)^2$ term in the action. If matter (ϕ_1) is present, the Stueckelberg field is actually the radial direction (ρ) in the field space of ϕ_0 and ϕ_1 ; the field ρ is then eaten by the Weyl gauge field w_μ which acquires mass $m_w \sim \langle \rho \rangle$ near the Planck scale. The breaking conserves the number of degrees of freedom and generates in the broken phase the Einstein-Proca action for w_μ . In both theories, below the mass of w_μ the connection becomes Levi-Civita and Einstein gravity is recovered, with an “emergent” Planck scale $M \sim \langle \rho \rangle$ and a scalar potential (of the remaining angular-variable field).

The potential $V(\varphi)$ is controlled by the symmetry of the theory together with effects from the non-trivial connection $\tilde{\Gamma}$, different in the two theories. For *small* field values, V is similar in both theories; the scalar field can act as the Higgs field, in which case the potential displays electroweak symmetry breaking. For *large* field values, the potential has the same form in Weyl and Palatini theories up to couplings and field rescaling (due to different non-metricity) and gives successful inflation.

Our main results, comparing inflation predictions in the two theories and summarised in Fig. 2, showed how different non-metricity impacts on inflation predictions. In Weyl gravity the scalar-to-tensor ratio $0.00257 \leq r \leq 0.00303$, which is smaller than in Palatini case, $0.00794 \leq r \leq 0.01002$, for measured n_s at 68% CL and $N = 60$ e-folds. Similar results exist for n_s at 95%CL or mildly different N , etc. Such values of r will be measured by new CMB experiments that can then test and distinguish Weyl and Palatini quadratic gravity.

There are similarities of inflation in Weyl and Palatini cases to Starobinsky inflation ($R + \xi_0 R^2$). In Weyl and Palatini theories one also has an R^2 term with a large ξ_0 that reduces r , but one needs in addition a non-minimally coupled scalar field (ϕ_1) for inflation; this is because the other (radial) combination enabled the breaking of gauged scale symmetry and the generation of mass scales (Planck, w_μ mass). In both Weyl and Palatini theory, for a fixed n_s , reducing the non-minimal coupling (ξ_1) increases r which in Weyl theory is bounded from above by that of Starobinsky inflation. Unlike in the Palatini theory, Weyl gravity for $\xi_1 \leq 10^{-3}$ gives a dependence $r(n_s)$ essentially similar to that in Starobinsky inflation, while also protecting r against higher dimensional operators corrections.

¹⁴ For an extended analysis of the role of the Weyl tensor term on inflation (in particular in R^2 inflation) see [94–97]. The above mentioned rescaling effect of a Weyl tensor squared term on the value of r found in its absence (e.g. Starobinsky result) is more general; for example, for a non-local Weyl-tensor-squared term (of Riemannian geometry), the effect is again a rescaling of r value found in its absence, albeit by an overall factor different from that above [97]; the different factor is due to the more general structure of this term. Such Weyl tensor-dependent operator cannot appear here since it is forbidden by the Weyl gauge symmetry.

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Appendix

A Weyl gravity

We include here basic information on Weyl gravity used in the text. First, in the (pseudo)-Riemannian case (Einstein gravity) $\nabla_\mu g_{\alpha\beta} = 0$ with ∇_μ defined by the Levi-Civita connection

$$\Gamma_{\mu\nu}^\rho(g) = (1/2) g^{\rho\beta} (\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}). \quad (\text{A-1})$$

Setting $\nu = \rho$ and summing over gives $\Gamma_\mu \equiv \Gamma_{\mu\nu}^\nu = \partial_\mu \ln \sqrt{g}$ used in the text.

In Weyl gravity and conformal geometry the theory has vectorial non-metricity, i.e.

$$\tilde{\nabla}_\lambda g_{\mu\nu} = -w_\lambda g_{\mu\nu}, \quad (\text{A-2})$$

so $w_\lambda = (-1/4) g^{\mu\nu} \tilde{\nabla}_\lambda g_{\mu\nu}$; here $\tilde{\nabla}_\mu$ is defined by the Weyl connection $\tilde{\Gamma}_{\mu\nu}^\rho$:

$$\tilde{\nabla}_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \tilde{\Gamma}_{\mu\lambda}^\rho g_{\rho\nu} - \tilde{\Gamma}_{\nu\lambda}^\rho g_{\mu\rho}. \quad (\text{A-3})$$

Write this for cyclic permutations of the indices and combine the three equations to find

$$\tilde{\Gamma}_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho(g) + (1/2) g^{\rho\lambda} (\tilde{\nabla}_\lambda g_{\mu\nu} - \tilde{\nabla}_\mu g_{\nu\lambda} - \tilde{\nabla}_\nu g_{\lambda\mu}), \quad (\text{A-4})$$

which with (A-2) gives the Weyl connection

$$\tilde{\Gamma}_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho(g) + (1/2) [\delta_\mu^\rho w_\nu + \delta_\nu^\rho w_\mu - g_{\mu\nu} w^\rho]. \quad (\text{A-5})$$

$\tilde{\Gamma}_{\mu\nu}^\rho$ are symmetric ($\tilde{\Gamma}_{\mu\nu}^\rho = \tilde{\Gamma}_{\nu\mu}^\rho$) i.e. there is no torsion. $\tilde{\Gamma}$ is invariant under transformations (2), (3) since the variation of the metric is compensated by that of w_μ . Using that

$g^{\alpha\beta} \tilde{\nabla}_\lambda g_{\alpha\beta} = 2 \tilde{\nabla}_\lambda \ln \sqrt{g}$ one finds for the Weyl field

$$w_\lambda = (-1/2) \tilde{\nabla}_\lambda \ln \sqrt{g}. \quad (\text{A-6})$$

Setting $\nu = \rho$ in (A-5) and summing over, we recover our definition (3) in the text:

$$\tilde{\Gamma}_\mu = \Gamma_\mu(g) + 2 w_\mu. \quad (\text{A-7})$$

The Riemann and Ricci tensors in Weyl geometry are defined as in Riemannian geometry but with the replacement of the Levi-Civita connection $\Gamma_{\mu\nu}^\rho(g)$ by the new $\tilde{\Gamma}_{\mu\nu}^\rho$

$$R_{\mu\nu}^\lambda(\tilde{\Gamma}, g) = \partial_\nu \tilde{\Gamma}_{\mu\sigma}^\lambda - \partial_\sigma \tilde{\Gamma}_{\mu\nu}^\lambda + \tilde{\Gamma}_{\nu\rho}^\lambda \tilde{\Gamma}_{\mu\sigma}^\rho - \tilde{\Gamma}_{\sigma\rho}^\lambda \tilde{\Gamma}_{\mu\nu}^\rho, \quad (\text{A-8})$$

and

$$R_{\mu\sigma}(\tilde{\Gamma}, g) = R_{\mu\lambda\sigma}^\lambda(\tilde{\Gamma}, g), \quad R(\tilde{\Gamma}, g) = g^{\mu\sigma} \tilde{R}_{\mu\sigma}(\tilde{\Gamma}, g). \quad (\text{A-9})$$

Since $\tilde{\Gamma}$ is invariant under transformations (2), (3), then the Riemann and Ricci tensors of Weyl geometry are also invariant. Since the Weyl scalar curvature $R(\tilde{\Gamma}, g)$ contains $g^{\mu\nu}$, it transforms covariantly

$$\hat{R}(\tilde{\Gamma}, g) = (1/\Omega^2) R(\tilde{\Gamma}, g). \quad (\text{A-10})$$

This helps build Weyl gauge invariant operators. Using the expression of $\tilde{\Gamma}$, one shows

$$R(\tilde{\Gamma}, g) = R(g) - 3 \nabla_\mu w^\mu - \frac{3}{2} g^{\mu\nu} w_\mu w_\nu, \quad (\text{A-11})$$

where $R(g)$ is the Riemannian scalar curvature and $\nabla_\mu w^\mu$ is defined by Levi-Civita connection. Equation (A-11) was used in the text, in going from (10) to (12) for the Weyl case.

B Palatini gravity

We present here the connection and the scalar curvature for the Palatini approach to gravity, used in the text. In this case, similarly to Weyl gravity, $\tilde{\Gamma}$ is not determined by the metric (a priori is independent of it), hence it is invariant under rescaling $g_{\mu\nu}$. The connection is determined by its equation of motion from the Lagrangian of Eq. (10). Solving this equation of motion one finds [8, (Eqs. 23, 25, 39)]

$$\tilde{\nabla}_\lambda (\rho^2 g_{\mu\nu}) = (-2) \rho^2 (g_{\mu\nu} V_\lambda - g_{\mu\lambda} V_\nu - g_{\nu\lambda} V_\mu), \quad (\text{B-1})$$

where V_λ is some arbitrary vector, related to w_λ (see below). One writes (B-1) for cyclic permutations of the three indices, then combines the equations obtained and uses the symmetry $\tilde{\Gamma}_{\mu\nu}^\alpha = \tilde{\Gamma}_{\nu\mu}^\alpha$, to find

$$\tilde{\Gamma}_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha(\rho^2 g) - (3 g_{\mu\nu} V_\lambda - g_{\nu\lambda} V_\mu - g_{\lambda\mu} V_\nu) g^{\lambda\alpha}, \quad (\text{B-2})$$

where

$$\Gamma_{\mu\nu}^{\alpha}(\rho^2 g) = \Gamma_{\mu\nu}^{\alpha}(g) + 1/2 (\delta_{\nu}^{\alpha} \partial_{\mu} + \delta_{\mu}^{\alpha} \partial_{\nu} - g^{\alpha\lambda} g_{\mu\nu} \partial_{\lambda}) \ln \rho^2, \quad (\text{B-3})$$

with $\Gamma_{\mu\nu}^{\alpha}(g)$ the Levi-Civita connection for $g_{\mu\nu}$. Setting $\nu = \alpha$ in (B-2) one then finds $\tilde{\Gamma}_{\mu} = \Gamma_{\mu}(\phi^2 g) + 2 V_{\mu}$ and from (B-3): $\Gamma_{\mu}(\phi^2 g) = \Gamma_{\mu}(g) + 2 (\partial_{\mu} \ln \rho^2)$. From these two equations and with the definition $w_{\lambda} = 1/2 (\tilde{\Gamma}_{\mu} - \Gamma_{\mu}(g))$, then $V_{\lambda} = w_{\lambda} - \partial_{\lambda} \ln \rho^2$. Using this relation and that found by contracting (B-1) by $g^{\mu\nu}$, then

$$w_{\lambda} = (-1/2) \tilde{\nabla}_{\lambda} \ln \sqrt{g}. \quad (\text{B-4})$$

similar to (A-6), but with different $\tilde{\Gamma}$. Finally, Eqs. (B-2), (B-3) together with $V_{\lambda} = w_{\lambda} - \partial_{\lambda} \ln \rho^2$, give the expression of $\tilde{\Gamma}$ in terms of $g_{\mu\nu}$, ρ and w_{λ} and verifies that $\tilde{\Gamma}$ is indeed invariant under a gauged scale transformation (2), (3). This is obvious since $\rho^2 g_{\mu\nu}$ and V_{μ} are invariant in (B-2). With $\tilde{\Gamma}$ a function of w_{λ} , ϕ , $g_{\mu\nu}$, one computes the Ricci tensor $R_{\mu\nu}(\tilde{\Gamma})$ for Palatini gravity, then the scalar curvature $R(\tilde{\Gamma}, g) = g^{\mu\nu} R_{\mu\nu}(\tilde{\Gamma}, g)$. The result is [8]:

$$R(\tilde{\Gamma}, g) = R(g) - 6g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \ln \rho - 6(\nabla_{\mu} \ln \rho)^2 - 12(\nabla_{\lambda} V^{\lambda} + V^{\lambda} \partial_{\lambda} \ln \rho^2) - 6V_{\mu} V^{\mu}, \quad (\text{B-5})$$

with $R(g)$ the Ricci scalar (Riemannian case), and $V_{\lambda} \equiv w_{\lambda} - \partial_{\lambda} \ln \rho^2$. Replacing (B-5) in Eq. (10) for the Palatini case, one finds after some algebra Eq. (12) in the text with $\gamma = 1$. At the same time, the vectorial non-metricity becomes

$$\tilde{\nabla}_{\lambda} g_{\mu\nu} = (-2) (g_{\mu\nu} w_{\lambda} - g_{\mu\lambda} w_{\nu} - g_{\nu\lambda} w_{\mu}), \quad (\text{B-6})$$

which is different from (A-2) of Weyl geometry, but has the same trace $g^{\mu\nu} \tilde{\nabla}_{\lambda} g_{\mu\nu} = -w_{\lambda}/4$.

C Inflation: perturbations to the scalar and vector fields

We discuss in detail the scalar ($\delta\phi$) and vector (δw_{μ}) fields perturbations in a FRW universe $g_{\mu\nu} = (1, -a(t)^2, -a(t)^2, -a(t)^2)$ and show that ΔL of (19) does not affect inflation by ϕ . To simplify notation hereafter we remove the 'hat' (^) on w_{μ} , $g_{\mu\nu}$ when we refer to action (16).

• Let us first review the usual case of a single scalar field, see e.g. [84], needed later. Consider

$$\mathcal{L}_{\phi} = \int \sqrt{g} \left[\frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - \mathcal{V}(\phi) \right]. \quad (\text{C-1})$$

The equation of motion $\nabla_{\mu} \nabla^{\mu} \phi + \mathcal{V}'(\phi) = 0$ gives for a FRW metric:

$$\ddot{\phi}(\vec{x}, t) + 3H\dot{\phi}(\vec{x}, t) - \frac{1}{a(t)^2} \partial_j \partial_j \phi(\vec{x}, t) + \mathcal{V}'(\phi) = 0 \quad (\text{C-2})$$

Expanding about $\phi(t)$, with $\phi(\vec{x}, t) = \phi(t) + \delta\phi(\vec{x}, t)$, one has at linear level

$$\delta\ddot{\phi}(\vec{x}, t) + 3H\delta\dot{\phi}(\vec{x}, t) - \frac{1}{a(t)^2} \partial_j \partial_j \delta\phi(\vec{x}, t) + \mathcal{V}''(\phi(t)) \delta\phi(\vec{x}, t) = 0 \quad (\text{C-3})$$

Using mode expansion $\delta\phi(\vec{x}, t) = \int d^3k / (2\pi)^{3/2} \delta\phi_k(t) \exp(i\vec{x}\vec{k})$, then

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + [k^2/a^2 + \mathcal{V}''(\phi(t))] \delta\phi_k = 0 \quad (\text{C-4})$$

or, with a notation $\delta\phi_k = \delta\chi_k/a(t)$

$$\delta\ddot{\chi}_k + H\delta\dot{\chi}_k + [k^2/a^2 - \dot{H} - 2H^2 + \mathcal{V}''(\phi(t))] \delta\phi_k = 0. \quad (\text{C-5})$$

In conformal time (η) via $dt = a(t)d\eta$, this equation becomes

$$\delta\chi_k'' + [k^2 - (1/\eta^2)(v^2 - 1/4)] \delta\phi_k = 0, \quad v^2 = 9/4 - \mathcal{V}''/H^2. \quad (\text{C-6})$$

where we used that with $a \sim e^{Ht}$ then $a(\eta) = -1/(H\eta)$ and $2/\eta^2 = a''/a$ with $H \sim \text{constant}$. In the subhorizon limit $-\eta k \gg 1$ the solution should be $\delta\chi_k = e^{-ik\eta}/\sqrt{2k}$. With this boundary condition, the solution is

$$\delta\chi_k = \frac{\sqrt{\pi}}{2} e^{i(v+1/2)\pi/2} \sqrt{-\eta} H_v^{(1)}(-\eta k) \quad (\text{C-7})$$

where $H^{(1)}$ is the Hankel function of first kind. This leads to the usual power spectrum, with

$$|\delta\phi_k|^2 \approx \frac{H^2}{2k^3} \left(\frac{k}{aH} \right)^{2\eta_{\phi}}, \quad P_{\delta\phi_k} = \frac{k^3}{2\pi^2} |\delta\phi_k|^2 = \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{2\eta_{\phi}} \quad (\text{C-8})$$

with $\eta_{\phi} = 3/2 - v \approx \mathcal{V}''/(3H^2) = M_p^2 \mathcal{V}''/\mathcal{V} \ll 1$. This gives $n_{\phi} = 1 + 2\eta_{\phi}$ ($H \sim \text{constant}$).

For later use, we also consider solution (C-7) when $\mathcal{V}''(\phi) > (9/4)H^2$ i.e. v is imaginary, $v = i\tilde{v}$, \tilde{v} real. In the (superhorizon) limit $(-\eta k) \ll 1$ one finds:

$$\delta\chi_k = \frac{(1+i)2^{-3/2-i\tilde{v}}e^{-\tilde{v}\pi/2}}{\sqrt{\pi k}\Gamma(1+i\tilde{v})} \times \left[\pi(-\eta k)^{2i\tilde{v}}(1 + \coth \pi \tilde{v}) + 2^{2i\tilde{v}}\tilde{v}\Gamma(i\tilde{v})^2 \right]. \quad (\text{C-9})$$

Returning to $\delta\phi_k$ notation, one finds (see e.g. [84])

$$P_{\delta\phi_k} = \frac{\pi e^{-\pi\tilde{v}}}{2} \frac{H^2}{(2\pi)^2} \left(\frac{k}{aH} \right)^3 \{ \dots \} \quad (\text{C-10})$$

where the brackets $\{ \dots \}$ stand for terms that vanish when $\tilde{v} \rightarrow \infty$ ($\mathcal{V}''(\phi) \gg H^2$). Therefore, modes $\delta\phi_k$ of v imaginary are exponentially suppressed [85]; this is expected since they are too massive to be excited.

• Consider now our action (16); its φ -dependence is described by \mathcal{L}_φ by replacing in (C-1)

$$\mathcal{V}(\varphi) \rightarrow \mathcal{V}(\varphi, w) = V(\varphi) - \frac{1}{2} f(\varphi) w_\mu w^\mu \quad (\text{C-11})$$

with $V(\varphi)$ of Eq. (17) and $f(\varphi)$ of Eq. (19). In this case (C-2) and the equation for $\varphi(t)$ receive a correction from the last term in the rhs of (C-11). Then Eq. (C-4) for the perturbations $\delta\varphi_k$, also with $w_\mu(\vec{x}, t) = w_\mu(t) + \delta w_\mu(\vec{x}, t)$, is now modified into

$$\begin{aligned} \delta\ddot{\varphi}_k + 3H\delta\dot{\varphi}_k + [k^2/a^2 + \mathcal{V}''(\varphi(t), w(t))] \delta\varphi_k \\ = f'(\varphi(t)) w_\mu(t) \delta w^\mu(\vec{k}, t), \end{aligned} \quad (\text{C-12})$$

where the second derivative \mathcal{V}'' is with respect to φ and $\delta w_\mu(\vec{k}, t)$ are the Fourier modes of $\delta w_\mu(\vec{x}, t) = \int d^3k/(2\pi)^{3/2} \delta w_\mu(\vec{k}, t) \exp(i\vec{k}\vec{x})$. Next, the background $w_\mu(t)$ compatible with the FRW metric is $w_\mu(t) = (w_0(t), 0, 0, 0)$, while from (16), the equation of motion of w_μ gives

$$\frac{1}{\sqrt{g}} \partial_\rho [\sqrt{g} F^{\rho\mu}] + f(\varphi) w^\mu = 0. \quad (\text{C-13})$$

One has a trivial solution $w_\mu(t) = 0$ ($f(\varphi) \neq 0$). Therefore, in (C-12) we must replace $\mathcal{V}''(\varphi(t), w(t)) \rightarrow \mathcal{V}''(\varphi(t), 0) = V''(\varphi)$ while the rhs of (C-12) is vanishing. Therefore equation (C-12) of $\delta\varphi_k$ is actually independent of w_μ and δw_μ and there is no mixing of $\delta\varphi$ to δw_μ ¹⁵. Then the calculation of $\delta\varphi_k$ proceeds as earlier but for potential $V(\varphi)$, see (C-8) for $\mathcal{V}(\varphi) \rightarrow V(\varphi)$. Thus ΔL does not impact on φ -inflation and the usual formulae of single-field inflation in Einstein gravity apply, as used in Sect. 3.1.

• We saw above that the perturbations $\delta\varphi$ do not mix with those of w_μ and φ -inflation decouples from w_μ in a FRW universe. While somewhat beyond the purpose of this work, we also examine below the vector field perturbations, following [82, 83], in the approximation $H \sim \text{constant}$. Compatibility with the FRW metric demands computing the perturbations about a background $w_\mu(t) = 0$ as seen earlier. In fact we may take a more general background, if initially the vector field contribution to the stress-energy tensor is negligible relative to that of the scalar, in an isotropic universe; we shall then consider a quasi-homogeneous field $\partial_i w^\alpha = 0$. Our FRW case is always restored by setting anywhere below $w_\mu(t) = 0$. Then from (C-13) for $\mu = 0$ and $\mu = i$, respectively

$$w_0(t) = 0 \quad \text{and} \quad \ddot{w}_i(t) + H\dot{w}_i(t) + f(\varphi) w_i(t) = 0, \quad (\text{C-14})$$

In an expanding FRW universe the relevant physical quantity is not w_i ($i = 1, 2, 3$) but $q_i = w_i/a$, as also seen from the norm $w_\mu w^\mu = w_0^2 - (w_i w_i)/a^2$ (sum over i) and from the

¹⁵ A priori a mixing may exist of longitudinal mode and φ , and of their perturbations ($\delta\varphi, \delta w_\mu$).

stress energy tensor [81, 82]. Then the last equation becomes

$$\ddot{q}_i + 3H\dot{q}_i + (2H^2 + f(\varphi)) q_i = 0. \quad (\text{C-15})$$

Denote $m^2 = 2H^2 + f(\varphi(t))$ where $f(\varphi) > 0$ since $f(\varphi(t)) = 6\gamma M^2 \{1 + \sinh^2[\varphi(t)/(2M\sqrt{\gamma})]\}$, Eq. (19). Ignoring the time dependent part in $f(\varphi)$, the solution is

$$\begin{aligned} q_i(t) \propto 1/a(t)^{3/2} (c_1 e^{-\alpha t/2} + c_2 e^{\alpha t/2}), \\ \alpha = \sqrt{H^2 - 4m^2}. \end{aligned} \quad (\text{C-16})$$

with constants $c_{1,2}$. Since α is purely imaginary, during inflation the vector field is massive with damped oscillations (up to corrections due to $\varphi(t)$). Its contribution to the stress energy tensor (T_ν^μ) is anisotropic; the spatial part of this tensor contains off-diagonal entries of comparable size to the diagonal ones and can be made diagonal for a particular direction of the vector field. However, with $q_i(t) \sim 1/a(t)^{3/2}$, the contribution of the vector field to T_ν^μ during inflation is suppressed by the scale factor $1/a(t)^3$ relative to that of¹⁶ φ .

Consider now the equations for perturbations, with $w_\mu(\vec{x}, t) = w_\mu(t) + \delta w_\mu(\vec{x}, t)$. Then Eq. (C-13) for $\mu = i$ and $\mu = 0$ give

$$\begin{aligned} \ddot{w}_i + H\dot{w}_i - \frac{1}{a(t)^2} [\partial_j \partial_j w_i - \partial_i \partial_j w_j] + f(\varphi) w_i \\ = \partial_i \dot{w}_0 + H\partial_i w_0 \end{aligned} \quad (\text{C-18})$$

$$\partial_i \dot{w}_i - \partial_i \partial_j w_0 + a(t)^2 f(\varphi) w_0 = 0 \quad (\text{C-19})$$

with $w_\mu = w_\mu(\vec{x}, t)$ and $\varphi = \varphi(\vec{x}, t)$. By applying ∂_μ on (C-13) we find

$$i\partial_i \dot{w}_0 - \frac{1}{a(t)^2} \partial_i \partial_j w_j + 3H\partial_i w_0 + \partial_i D = 0 \quad (\text{C-20})$$

where $D = D(\vec{x}, t)$ and $D = w_0 \partial_0 \ln f(\varphi) - 1/a(t)^2 w_j \partial_j \ln f(\varphi)$. Adding (C-18), (C-20) then

$$\ddot{w}_i + H\dot{w}_i - \frac{1}{a(t)^2} \partial_j \partial_j w_i + f(\varphi) w_i = -2H\partial_i w_0 - \partial_i D. \quad (\text{C-21})$$

This gives for perturbations δw_μ a linear differential equation:

$$\begin{aligned} \delta\ddot{w}_i + H\delta\dot{w}_i - \frac{1}{a(t)^2} \partial_j \partial_j \delta w_i + f(\varphi(t)) \delta w_i \\ = -(2H + \partial_0 \ln f(\varphi(t))) \partial_i \delta w_0 \\ - f'(\varphi(t)) w_j(t) \sigma_{ij}(t) \end{aligned} \quad (\text{C-22})$$

¹⁶ A diagonal stress energy tensor can be obtained if we take e.g. $q_\mu = (0, 0, 0, q)$:

$$\begin{aligned} T_0^0 &= [\dot{\varphi}^2/2 + V(\varphi)] + [(\dot{q} + qH)^2 + 2q^2 f(\varphi)]/2; \\ -T_k^k &= [\dot{\varphi}^2/2 - V(\varphi)] \pm [(\dot{q} + qH)^2 - 2q^2 f(\varphi)]/2; \end{aligned} \quad (\text{C-17})$$

with the contribution of w_μ having opposite signs for $k = 1, 2$ (+) and for $k = 3$ (-) and $T_j^i = 0, i \neq j$. The contribution of $q(t) \sim 1/a(t)^{3/2}$ (C-16) is suppressed by $a(t)^3$ relative to that of φ .

Here $\sigma_{ij}(\vec{x}, t) = [\delta_{ij} + 1/(a(t)^2 f(\varphi(t))) \partial_i \partial_j] \delta\varphi(\vec{x}, t)$; notice that in general case of $w_i(t) \neq 0$ (C-22) is “mixing” $\delta\varphi$ and δw_μ . However, this mixing is absent in our FRW case of $w_\mu(t) = 0$.

Further, from remaining (C-19)

$$\partial_i \delta \dot{w}_i(\vec{x}, t) - \partial_i \partial_i \delta w_0(\vec{x}, t) + a(t)^2 f(\varphi(t)) \delta w_0(\vec{x}, t) = 0. \quad (\text{C-23})$$

or, in Fourier modes

$$\delta w_0(\vec{k}, t) = -i \frac{k_j \delta \dot{w}_j(\vec{k}, t)}{k^2 + a^2 f(\varphi(t))}. \quad (\text{C-24})$$

We separate the perturbations into parallel and orthogonal directions to \vec{k} (taken along OZ):

$$\delta \vec{w}^\parallel = \frac{\vec{k}(k_i \delta w_i)}{k^2}, \quad \delta \vec{w}^\perp = \delta \vec{w} - \delta \vec{w}^\parallel. \quad (\text{C-25})$$

We introduce the physical perturbations $\delta q_\mu(\vec{x}, t) = (1/a(t)) \delta w_\mu(\vec{x}, t)$ and express (C-22) in terms of the Fourier modes $\delta q_\mu(\vec{k}, t)$ defined by $\delta q_\mu(\vec{x}, t) = \int d^3k/(2\pi)^{3/2} \delta q_\mu(k, t) \exp(i\vec{k}\vec{x})$. We find for the Fourier modes of parallel $\delta q_z^\parallel(\vec{k}, t)$ and orthogonal $\delta q_j^\perp(\vec{k}, t)$ directions¹⁷

$$\delta \ddot{q}_z^\parallel + \delta \dot{q}_z^\parallel [3H + \theta_1] + \left[\frac{k^2}{a^2} + 2H^2 + H\theta_1 + f(\varphi(t)) \right] \delta q_z^\parallel = \theta_2, \quad (\text{C-26})$$

$$\delta \ddot{q}_j^\perp + 3H \delta \dot{q}_j^\perp + \left[\frac{k^2}{a^2} + 2H^2 + f(\varphi(t)) \right] \delta q_j^\perp = 0, \quad j = 1, 2. \quad (\text{C-27})$$

where

$$\theta_1 = \frac{k^2(2H + \partial_0 \ln f(\varphi(t)))}{k^2 + a^2 f(\varphi(t))}, \quad \theta_2 = -\delta\varphi_k q_z(t) f'(\varphi(t)) \left(1 - \frac{k^2}{a^2 f(\varphi(t))} \right). \quad (\text{C-28})$$

Equations (C-26), (C-27) are similar to those in [82, Eqs. 21, 22, 67] except an extra φ -dependent correction to the mass ($\sim M^2$) of w_μ that induces θ_2 and a time derivative in θ_1 .

Equation (C-27) is similar to that for the scalar field perturbations, Eq. (C-4). We expect perturbations δq_j^\perp be generated if their effective mass $m^2 = 2H^2 + f(\varphi(t)) < H^2$. This condition is not respected since $f(\varphi) > 0$. The power spectrum is exponentially suppressed, as for the scalar field, Eq. (C-4) with $\mathcal{V}'' \rightarrow f(\varphi(t)) + 2H^2$ with v imaginary and Eq. (C-10).

Similar considerations apply to the perturbations to the parallel (longitudinal) mode of w_μ . For our FRW-compatible

background $q_z^\parallel(t) = 0$ ($w_\mu(t) = 0$), hence $\theta_2 = 0$. Therefore, there is no mixing of δq_z^\parallel and $\delta\varphi_k$ perturbations in (C-26), in agreement with the earlier similar finding, see discussion around Eq. (C-12). Note also that if $k^2 \ll a^2 f(\varphi)$, δq_z^\parallel has an equation similar to the transverse modes, with $\theta_1 \sim 0$ (with $H \sim \text{constant}$, $\dot{\varphi}^2 \sim -2\dot{H}^2 M^2$). Similar to the transverse case, the effective mass $m^2 = 2H^2 + H\theta_1 + f(\varphi(t))$ of δq_z^\parallel is again larger than H and its generation is exponentially suppressed. We see again that in the FRW case one can ignore the effect of δw_μ and of coupling of $w_\mu - \varphi$ on $\delta\phi_k$.

In a general background case $q_z^\parallel(t) \neq 0$, then $\theta_2 \neq 0$; then a mixing of perturbations of φ and of longitudinal mode of w_μ exists in (C-26) due to coupling $f(\varphi) w_\mu w^\mu$, Eq. (19). However, even in this case, q_z^\parallel is suppressed by the scale factor, due to Eq. (C-16), and thus the same is true for the mixing.

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¹⁷ The equations for δw^\parallel (δw^\perp) are similar to those for δq^\parallel (δq^\perp) but with coefficient $3H$ replaced by H and without any H -dependence inside the brackets multiplying δq^\parallel (δq^\perp) respectively.

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