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Gauss Interpolation Formulas and Totally Positive Kernels*

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Abstract. This paper simplifies and generalizes an earlier result of the author's on Gauss interpolation formulas for the one-dimensional heat equation. Such formulas approximate a function at a point (x^*, t^*) in terms of a linear combination of its values on an initial-boundary curve in the (x, t) plane. The formulas are characterized by the requirement that they be exact for as many basis functions as possible. The basis functions are generated from a Tchebycheff system on the line $t = 0$ by an integral kernel $K(x, y, t)$, in analogy with the way heat polynomials are generated from the monomials x^i by the fundamental solution to the heat equation. The total positivity properties of $K(x, y, t)$ together with the theory of topological degree are used to establish the existence of the formulas.

1. Introduction. In a recent paper [1] we discussed formulas of the form

$$u(x^*, t^*) \simeq \sum_{k=1}^{m} A_k u(x_k, t_k) \tag{1}$$

for approximating solutions to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

The function u is prescribed on an initial-boundary curve C in the (x, t) plane, and (x^*, t^*) is a fixed point where an approximate solution is desired. The formula (1), where the points (x_k, t_k) lie on C and the weights A_k are positive, is characterized by the requirement that it be exact for as many "basis functions" as possible. In [1] we proved the existence of m-point formulas which are exact for all heat polynomials of degree $n = 2m - 1$, and that this is best possible, in the sense that no m-point formula is exact for all heat polynomials of degree $n_1 > 2m - 1$. Such formulas were called *Gauss interpolation formulas*, because of their similarity to Gaussian quadrature formulas.

A heat polynomial of degree n is a linear combination of the functions

$$u_i(x, t) = i! \sum_{j=0}^{[i/2]} \frac{x^{i-2j} t^j}{(i - 2j)!\, j!}, \qquad i = 0, 1, \ldots, n,$$

where $[a]$ means the greatest integer less than or equal to a. Each $u_i(x, t)$ solves (2) and satisfies

$$u_i(x, 0) = x^i \equiv \phi_i(x).$$

If we now introduce the fundamental solution to the heat equation

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$$K(x, y, t) = \frac{1}{\sqrt{4\pi t}} \exp(-(x-y)^2/4t), \qquad t > 0,$$

we can express the heat polynomials as

$$(3) \qquad u_i(x, t) = \int_{-\infty}^{\infty} K(x, y, t)\phi_i(y)\, dy, \qquad t > 0.$$

Now, it is known that for each $t > 0$, the kernel K is *extended totally positive* (cf. Karlin [2], and below), and that the functions $\phi_i(x) = x^i$, $i = 0, 1, \ldots, n$, form a *Tchebycheff system* (cf. Karlin and Studden [3]). These facts can be used to simplify the proof of the result from [1] mentioned above. More significantly, the concept of total positivity allows considerable generalization of the results of [1]. For example, Karlin and McGregor have shown that fundamental solutions for a large class of one-dimensional parabolic problems are totally positive [5]. If such fundamental solutions can be shown to satisfy a little more, namely, the hypotheses of Theorem 1 below (as in fact some of the specific examples from [4] seem to do), then we will have shown the existence of Gauss interpolation formulas for solutions to these parabolic problems. In Section 3 we prove that one of the examples from [4] does indeed satisfy the hypotheses of Theorem 1; we also give a numerical example based on this case.

A further generalization incorporated into Theorem 1 is the consideration of formulas of the form

$$(4) \qquad u(x^*, t^*) \simeq \sum_{k=1}^{m} A_k \left(u + \alpha \frac{\partial u}{\partial x}\right) (x_k, t_k),$$

where $\alpha = \alpha(x, t)$ is a prescribed continuous function on the curve C. Such a formula could be used, for example, in the case where the data $au + bu_x = f$ is known on C, for a and b fixed continuous functions with $a > 0$. One first obtains the formula (4) corresponding to $\alpha = b/a$, and then applies it to the data f/a.

In the next section we state and prove the main results of this paper, Theorems 1 and 2. Theorem 2 is concerned with the linear independence of certain linear functionals on a space of functions of two variables, and can be thought of as providing a "zero-counting" procedure for such functions. The corollary to Theorem 2 is a key step in the proof of Theorem 1.

2. Main Results. We first introduce some notation for certain determinants associated with a function $K(x, y)$ (cf. Karlin [2, Chapter 2]). Here x and y take values in the totally ordered sets X and Y, respectively. Let $x_1 < x_2 < \cdots < x_r$ be selections from X, and $y_1 < y_2 < \cdots < y_r$ from Y. The determinant of the matrix whose (i, j)th entry is $K(x_i, y_j)$ will be denoted by

$$K\begin{pmatrix} x_1, x_2, \ldots, x_r \\ y_1, y_2, \ldots, y_r \end{pmatrix}.$$

If this determinant is nonnegative for all such choices of the x_i's and y_j's, then K is said to be *totally positive.*

If X is an open interval and $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_r$ are values from X and $y_1 < y_2 < \cdots < y_r$, then

$$K^*\begin{pmatrix} x_1, x_2, \ldots, x_r \\ y_1, y_2, \ldots, y_r \end{pmatrix}$$

will denote the determinant of the matrix described as follows: if $x_i = x_{i+1} = \cdots = x_{i+l}$ is a block of coincident x's, then the $(i + k)$th row of the matrix will have the entries

$$\frac{\partial^k}{\partial x^k} K(x_i, y_j), \qquad k = 0, 1, \ldots, l$$

(assuming K is sufficiently differentiable). If this determinant is always *positive* whenever at most r of the x's coincide, we say that K is *extended totally positive of degree r in the variable x.*

We now state the hypotheses for Theorem 1. Let $K(x, y, t)$ be a real valued function continuous on $X \times X \times (0, \infty)$, where X is an open interval. K is assumed to have the following properties:

K1. For each $t > 0$, K is extended totally positive of degree 3 in the variable x;

K2. There is a class of functions D_K which are integrable with respect to the measure $d\mu = K(x, y, t)\,dy$ over the interval X; furthermore, we assume that if $f \in D_K$, and if

$$(5) \qquad \begin{aligned} T_t f(x) &\equiv \int_X K(x, y, t) f(y)\,dy, \qquad t > 0, \\ &\equiv f(x), \qquad t = 0, \end{aligned}$$

then $T_t f(x)$ is continuous for $(x, t) \in X \times [0, \infty)$ if $f \in C(X)$; also, we assume that differentiation under the integral sign up to order 2 in x is legitimate:

$$\frac{\partial^\nu T_t f(x)}{\partial x^\nu} = \int_X \frac{\partial^\nu K(x, y, t)}{\partial x^\nu} f(y)\,dy, \qquad \nu = 0, 1, 2, t > 0;$$

K3. $\int_X K(x, y, t)\,dy = 1$ for $t > 0$.

K4. The family of linear operators T_t is a semigroup: $T_{s+t} = T_t \circ T_s$.

Now let n be a positive integer and let $\{\phi_i\}_{i=0}^n \subset C^2(X) \cap D_K$ be an *extended Tchebycheff system of order* 3 (cf. [3, p. 6]). This is equivalent modulo the sign of one of the functions ϕ_i, to the statement that any polynomial $p(x) = \Sigma_{i=0}^n \beta_i \phi_i(x)$ has at most n zeroes, counting multiplicities up to order 3. Also, we specify that $\phi_0(x) \equiv 1$.

We next define the family of functions

$$(6) \qquad u_i(x, t) \equiv T_t \phi_i(x), \qquad i = 0, 1, \ldots, n,$$

in analogy with (3). We can now state

THEOREM 1. *Let C: $\{(x(s), t(s))\colon 0 \leqslant s \leqslant 1\}$ be a Jordan arc in the (x, t) plane satisfying*:

(i) $x(s) \in X$, $0 \leqslant s \leqslant 1$;

(ii) $(x(0), t(0)) = (a, t^*)$, $(x(1), t(1)) = (b, t^*)$, *with* $a < b$ *and* $t^* > 0$;

(iii) $0 \leqslant t(s) < t^*$ *for* $0 < s < 1$.

Let $\alpha(s)$ be continuous for $0 \leqslant s \leqslant 1$, and consider α to be defined on C via the parametrization for C: $\alpha(x(s), t(s)) = \alpha(s)$. Assume $\alpha(a, t^*) \leqslant 0 \leqslant \alpha(b, t^*)$. Then for *any* $a < x^* < b$ *and* $n = 2m - 1$ $(m \geqslant 2)$, *there is a formula of the form*

$$u(x^*, t^*) \simeq \sum_{k=1}^{m} A_k \left(u + \alpha \frac{\partial u}{\partial x}\right) (x_k, t_k) \tag{7}$$

which is exact for all polynomials $p(x, t) = \Sigma_{i=0}^{n} \beta_i u_i(x, t)$. *The weights* A_k *are positive and the points* (x_k, t_k) *lie on C, with* $t_k < t^*$. *Furthermore, no such formula can hold for all polynomials of degree* $n_1 > n$.

The proof of Theorem 1 will be postponed until we establish some lemmas and Theorem 2.

LEMMA 1. *For* $f \in C^2(X)$, *let* $\widetilde{Z}_{(3)}(f)$ *be the number of zeroes of f, counting multiplicities up to order* 3, *and let* $S(f)$ *be the number of strict sign changes of f on X. Then for* $f \in C^2(X) \cap D_K$, $f \not\equiv 0$, *and* $t > 0$,

$$\widetilde{Z}_{(3)}(T_t f) \leqslant S(f). \tag{8}$$

Proof. The proof is identical to the proof of the (a) part of Theorem 3.2, p. 239 of [2]. The requirement there that f be bounded is obviated by our hypotheses on K and D_K.

COROLLARY. *If* $f \in D_K$, $f \not\equiv 0$ *and* $f \geqslant 0$, *then* $T_t f(x) > 0$ *for* $t > 0$ *and* $x \in X$.

Proof. $S(f) = 0$, so $T_t f$ has no zeroes, by the lemma. Hence, $T_t f$ is positive (it is nonnegative since $K(x, y, t)$ is, as follows from its total positivity).

For convenience, we introduce the vector notation

$$\mathbf{u}(x, t) = (u_0(x, t), u_1(x, t), \ldots, u_n(x, t))$$

and

$$\phi(x) = (\phi_0(x), \phi_1(x), \ldots, \phi_n(x)) = \mathbf{u}(x, 0).$$

THEOREM 2. *Let the functions* $\{u_i(x, t)\}$, $i = 0, 1, \ldots, n$, *be as in the hypotheses for Theorem* 1. *Let* (x_k, t_k), $k = 1, 2, \ldots, l + j$, $l \geqslant 0$, $j \geqslant 1$, *be distinct points in the half-plane* $t \geqslant 0$ *such the the first l of them have equal t-coordinates which are greater than or equal to those of the other points*; *i.e.*, $t_1 = t_2 = \cdots = t_l \geqslant t_k$, $k = l + 1, \ldots, l + j$. *Suppose also that* $n + 1 \geqslant l + 2j$. *Then the vectors*

$$\{\mathbf{u}(x_k, t_k)\}_{k=1}^{l+j} \cup \left\{\frac{\partial \mathbf{u}}{\partial x}(x_k, t_k)\right\}_{k=l+1}^{l+j}$$

are linearly independent.

Proof. We may assume that $n + 1 = l + 2j$, for otherwise we could adjoin points to the line $t = t_1$, increasing l, to achieve this. Suppose the theorem were false. Then there would exist a nontrivial polynomial $p(x, t) = \Sigma_{i=0}^{n} \beta_i u_i(x, t)$ satisfying

$$p(x_k, t_k) = 0, \quad k = 1, 2, \ldots, l + j,$$

and

$$P_x(x_k, t_k) = 0, \qquad k = l + 1, \ldots, l + j.$$

We will show that this is impossible.

For $t \geqslant 0$, let $\widetilde{Z}(t)$ denote the number of zeroes of $p(x, t)$ in x, counting multiplicities up to order three. Let $S(t)$ be the number of sign changes of $p(x, t)$ in x. Lastly, let $j = j_1 + j_2$, where j_1 is the number of the points (x_k, t_k), $k = l + 1, \ldots, l + j$, on the line $t = t_1$. Then clearly

$$\widetilde{Z}(t_1) \geqslant l + 2j_1 \tag{8}$$

and

$$\widetilde{Z}(0) \leqslant n, \tag{9}$$

the last inequality holding because the functions $u_i(x, 0)$ form an extended Tchebycheff system of order 3.

It will now be convenient to classify the types of zeroes that $p(x, t)$ may have. For $t \geqslant 0$, let $z_i(t)$, $i = 1, 2, 3$, denote the number of x's such that:

for $i = 1$, $p(x, t) = 0$, $p_x(x, t) \neq 0$;

for $i = 2$, $p(x, t) = p_x(x, t) = 0$, p does not change sign at x;

for $i = 3$, $p(x, t) = p_x(x, t) = 0$, p does change sign at x.

Thus, each (x_k, t_k) with $k = l + 1, \ldots, l + j$ is a zero of type z_2 or z_3. Furthermore, it is clear that

$$\widetilde{Z}(t) \geqslant z_1(t) + 2z_2(t) + 3z_3(t) \geqslant z_1(t) + z_3(t) = S(t). \tag{10}$$

By Lemma 1, we have for any $\delta > 0$,

$$\widetilde{Z}(t + \delta) = \widetilde{Z}_{(3)}(T_\delta p(\cdot, t)) \leqslant S(p(\cdot, t)) = S(t). \tag{11}$$

Combining (10) and (11), we obtain

$$\widetilde{Z}(t + \delta) \leqslant S(t) = z_1(t) + z_3(t) \leqslant \widetilde{Z}(t) \geqslant z_1(t) + 2z_2(t) + 3z_3(t). \tag{12}$$

Hence, $\widetilde{Z}(t)$ is nonincreasing in t and decreases by at least two due to each of the points (x_k, t_k), $k = l + 1, \ldots, l + j$, where $p = p_x = 0$. Thus we have

$$\widetilde{Z}(t_1) \leqslant \widetilde{Z}(0) - 2j_2 \leqslant n - 2j_2 = l + 2j_1 - 1, \tag{13}$$

which is a contradiction of (8).

COROLLARY. *Assuming all the hypotheses of Theorem* 1, *a necessary condition for the existence of a formula* (7) *with* $A_k \geqslant 0$ *and* $(x_k, t_k) \in C$ *which is exact for polynomials of degree* $n = 2m - 1$ *or less, is that all* m *points be distinct, and for* $k = 1, 2, \ldots, m$, $A_k > 0$ *and* $t_k < t^*$.

Proof. We first show that a formula with fewer than m points is impossible. Suppose that

$$\mathbf{u}(x^*, t^*) = \sum_{k=1}^{j} A_k(\mathbf{u} + \alpha \mathbf{u}_x)(x_k, t_k), \tag{14}$$

where $j < m$. If we now adjoin points to the line $t = t^*$, to give $l = 2(m - j)$ in all, and reindex, we see that (14) is impossible by Theorem 2.

To see that $t_k < t^*$ for all k, suppose first that $t_1 = t^*$ and $t_k < t^*$, $k = 2, \ldots, m$. If we take $l = 0$ and $j = m$, Theorem 2 implies the existence of a polynomial $p(x, t) = \Sigma_{i=0}^{n} \beta_i u_i(x, t)$ satisfying $p(x_k, t_k) = 0$, $k = 1, \ldots, m$; $p_x(x_k, t_k) = 0$, $k = 2, \ldots, m$; and $p_x(x_1, t_1) = 1$. But then the formula (7) gives

$$p(x^*, t^*) = \sum_{k=1}^{m} A_k(p + \alpha p_x)(x_k, t_k) = A_1\alpha(a, t^*) \leqslant 0.$$

This in turn implies that $p(x, t^*)$ has at least two distinct zeroes in x, which, by Theorem 2 with $l = 2$ and $j = m - 1$, implies $p \equiv 0$, a contradiction. The other two possibilities, $t_1 < t_m = t^*$ and $t_1 = t_m = t^*$ are dispensed with similarly. This completes the proof of the corollary.

LEMMA 2. *The vector* $q \equiv \mathbf{u}(x^*, t^*)$ *has a unique representation of the form*

$$q = \sum_{k=1}^{m} \lambda_k \phi(x_k), \tag{15}$$

where $x_k \in X$ *and* $\lambda_k > 0$, $k = 1, \ldots, m$.

Proof. Since $u_i(x^*, t^*) = \int_X K(x^*, y, t^*)\phi_i(y)\, dy$, $i = 0, 1, \ldots, n$, the components of q form a "moment sequence" with respect to the functions $\{\phi_i(x)\}$ on X. It follows by Theorem 1 of [7] that there is a representation for q of the form

$$q = \sum_{k=1}^{p} \gamma_k \phi(y_k) \tag{16}$$

with $\gamma_k > 0$, $y_k \in X$, $k = 1, \ldots, p$. Let c, d be such that $\{y_k\} \subset (c, d) \subset X$. Since the functions $\{\phi_i\}$, $i = 0, \ldots, n$, form a Tchebycheff system on $[c, d]$, we may appeal to the results of [3, Chapter 2]. The equation (16) shows that q belongs to the "moment cone" generated by the $\{\phi_i\}$ on $[c, d]$ (i.e., those vectors $\mathbf{c}$ whose components $c_i = \int_c^d \phi_i(s)\, d\mu(s)$ for some bounded, right-continuous function $\mu(s)$). Moreover, the Corollary to Lemma 1 implies that q is actually in the interior of the moment cone. The conclusion of Lemma 2 now follows from Corollary 3.1, p. 47 of [3].

Proof of Theorem 1. As in [1], the proof uses the concept of topological degree to establish the existence of a solution to a system of N equations in N unknowns. We begin by reviewing the needed properties of degree theory (see Schwartz [8] or Ortega and Rheinboldt [6]).

Let D be an open bounded set in the Euclidean space R^N, with $\bar{D}$ and ∂D denoting its closure and boundary, respectively. Let F: $\bar{D} \longrightarrow R^N$ be continuous. Then if $q \in R^N$ and $q \notin F(\partial D)$, the degree of F with respect to D and q is defined, has an integer value, and will be denoted by $\deg(F, D, q)$. The following are some basic properties of the degree:

(i) Suppose that $F \in C'(D)$, $q \notin F(\partial D)$, and that for each $z \in D$ where $F(z) = q$, it is true that $\det(F'(z)) \neq 0$. Then there are a finite number of points $z_i \in D$ where $F(z_i) = q$, and $\deg(F, D, q) = \Sigma_i \operatorname{sgn}(\det(F'(z_i)))$.

(ii) If $\deg(F, D, q) \neq 0$, there is at least one point $z \in D$ such that $F(z) = q$.

(iii) Let $F(z, \lambda)$ be continuous on $\bar{D}$. $[0, 2]$, such that $F(z, \lambda) \neq q$ for any $z \in \partial D$, $0 \leqslant \lambda \leqslant 2$. Then $\deg(F(\cdot, \lambda), D, q)$ is constant, independent of λ.

We will apply these properties by constructing a function $F(z, \lambda)$, a set D, and a point q, such that $\deg(F(\cdot, 0), D, q) = \pm 1$, $F(z, \lambda) \neq q$ for $z \in \partial D$, $0 \leqslant \lambda \leqslant 2$, and hence deduce that $\deg(F(\cdot, 2), D, q) = \pm 1$. This will imply that the equation $F(z, 2) = q$ has a solution in D, which will be equivalent to the existence statement of Theorem 1.

Let $N = 2m = n + 1$ and let $D \subset R^N$ be the set

$$D = \{z = (A_1, A_2, \ldots, A_m, s_1, s_2, \ldots, s_m): \qquad (17)$$
$$0 < s_1 < s_2 < \cdots < s_m < 1, 0 < A_k < 1, k = 1, \ldots, m\}.$$

Let $\mathbf{u}(x, t)$ be as before, and let $q = \mathbf{u}(x^*, t^*)$. Let C_0: $\{(x_0(s), t_0(s)), 0 \leqslant s \leqslant 1\}$ be a Jordan arc to be described below, and let C_1 be the curve C of Theorem 1 (parametrized by $x_1(s) \equiv x(s)$, $t_1(s) \equiv t(s)$). For $(s, \lambda) \in [0, 1] \times [0, 1]$, let C_λ be the curve

$$(x(s, \lambda), t(s, \lambda)) = (\lambda x_1(s) + (1 - \lambda)x_0(s), \lambda t_1(s) + (1 - \lambda)t_0(s)).$$

Thus, as λ varies from 0 to 1, C_λ is a continuous deformation of C_0 into C_1.

We define $F(z, \lambda)$ as

$$F(z, \lambda) = \sum_{k=1}^{m} A_k \mathbf{u}(x(s_k, \lambda), t(s_k, \lambda)), \qquad 0 \leqslant \lambda \leqslant 1,$$

$$= \sum_{k=1}^{m} A_k(\mathbf{u} + (\lambda - 1)\alpha \mathbf{u}_x)(x_1(s_k), t_1(s_k)), \qquad 1 < \lambda \leqslant 2.$$

Let C_0: $(x_0(s), t_0(s))$ be any Jordan arc (i.e., continuous, non-self-intersecting) satisfying:

(a) $(x_0(0), t_0(0)) = (a, t^*)$, $(x_0(1), t_0(1)) = (b, t^*)$;

(b) $(x_0(s), t_0(s)) \subset X \times [0, t^*)$ for $0 < s < 1$;

(c) C_0 includes an open interval of the x-axis which contains the points $\{x_k\}$ of Lemma 2, with the parametrization chosen so that if $x_k = x_0(s_k)$, then $dx_0(s_k)/ds = 1$.

LEMMA 3. *If* $z \in \partial D$ *and* $0 \leqslant \lambda \leqslant 2$, *then* $F(z, \lambda) \neq q$.

Proof. This is an immediate application of the Corollary to Theorem 2. Note that if $z \in \partial D$, one or more of the following is true:

(i) $s_1 = 0$ or $s_m = 1$,

(ii) $s_k = s_{k+1}$ for some $k = 1, 2, \ldots, m - 1$,

(iii) some $A_k = 0$,

(iv) some $A_k = 1$.

Thus, if $z \in \partial D$ and $F(z, \lambda) = q$ with one of the first three cases occurring, there would exist a formula (7) of a kind ruled out by the corollary (applied with a possibly different curve C or function α). In case (iv), we use the fact that $u_0(x, t) \equiv 1$ so that $\Sigma_{i=1}^m A_i = 1$, and conclude that case (iii) must also hold.

We now claim that

$$\deg(F(\cdot, 0), Dq) = \pm 1. \qquad (18)$$

The fact that $F(z, 0) = q$ has a unique solution in D follows immediately from Lemma 2. At this solution,

$$\frac{\partial F(z, 0)}{\partial z} = \left[\phi(x_1), \ldots, \phi(x_m), A_1 \frac{d\phi}{dx}(x_1), \ldots, A_m \frac{d\phi}{dx}(x_m)\right],$$

where $x_1, x_2, \ldots, x_m$ are the points of Lemma 2. The determinant of this matrix is nonzero, since the positive A_k's may be factored out and the $\{\phi_i\}$, $i = 0, \ldots, n$, form an extended Tchebycheff system of order three. The equality (18) now follows from property (i) of the degree.

Combining (18) and Lemma 3 with properties (ii) and (iii) of the degree, we deduce that the equation $F(z, 2) = q$ has a solution $z \in D$. This proves the existence statement of Theorem 1. The fact that such a formula cannot hold for all polynomials of degree $n_1 > n$ follows from Theorem 2.

3. **An Example.** We conclude with an example illustrating an instance of Theorem 1. It should be pointed out that the example is not arbitrary, but was chosen because the fundamental solution $K(x, y, t)$ and the family of solutions $\{u_i(x, t)\}$ used in Theorem 1 are known explicitly for this case.

Consider the diffusion equation [4, pp. 170–171] on $X = (-\infty, \infty)$

$$\frac{\partial u}{\partial t} = e^{x^2}\frac{\partial}{\partial x}\left(e^{-x^2}\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial u}{\partial x}. \tag{19}$$

The Cauchy problem for this equation has the fundamental solution

$$K(x, y, t) = a(t)\exp(-b(t)x^2)\exp(-b(t)y^2)\exp(c(t)xy), \tag{20}$$

where $a(t) = (\pi(1 - e^{-4t}))^{-1/2}$, $b(t) = e^{-4t}/(1 - e^{-4t})$, and $c(t) = 2e^{-2t}/(1 - e^{-4t})$. We first show that, for each $t > 0$, this kernel is extended totally positive, of arbitrary degree, in both x and y. Indeed, by Theorem 2.6, p. 55 of [2], it is sufficient to show that for $m = 1, 2, \ldots$,

$$K^*\begin{pmatrix} \overbrace{x, \ldots, x}^{m} \\ y, \ldots, y \end{pmatrix} > 0 \tag{21}$$

(where we suppress the dependence of K on t); (21) is established by arguing as in [2, pp. 99–100], where (21) is proved for $K(x, y) = e^{xy}$. Properties K2–K4 likewise can be shown to hold.

We take for the functions $\{u_i(x, t)\}$ the class of solutions to (19)

$$u_i(x, t) = e^{-2it}H_i(x), \qquad i = 0, 1, \ldots, \tag{22}$$

where $H_i(x) = (-1)^i e^{x^2} d^i/dx^i e^{-x^2}$ are the Hermite polynomials. We let the coefficient α in (7) be zero and take for the image of the curve C the set $(0, t)$: $0 \leqslant t \leqslant .1$, $(x, 0)$: $0 \leqslant x \leqslant 1$, and $(1, t)$, $0 \leqslant t \leqslant .1$. Formulas (7) were calculated for $t^* = .1$ and $x^* = .25$, $.5$, and $.75$, with $m = 2, 3, \ldots, 6$, by numerically solving $2m$ nonlinear equations in each case.

The formulas for $(x^*, t^*) = (.5, .1)$ are given in Table 1. Table 2 presents the result of applying these formulas to the problem of interpolating the function $u(x, t)$ which satisfies (19) and

$$u(0, t) = e^{2t},$$
$$u(x, 0) = e^{x^2}, \qquad 0 \leqslant x \leqslant 1,$$
$$u(1, t) = e^{2t+1},$$

this function being $u(x, t) = e^{2t+x^2}$.

TABLE 1

Interpolation formulas for $(x^*, t^*) = (.5, .1)$

m	A_k	x_k	t_k
2	.5	.33604788 (-2)	.0
	.5	.81537027	.0
3	.35269431	.0	.38492690 (-1)
	.41343840	.45103057	.0
	.23386729	1.0	.24033195 (-1)
4	.26874349	.0	.52478250 (-1)
	.26561142	.19419862	.0
	.27676509	.66696976	.0
	.18888000	1.0	.43361146
5	.20936116	.0	.60423741 (-1)
	.15094470	.0	.28924570 (-2)
	.35320546	.43742786	.0
	.13407306	.87996367	.0
	.15241562	1.0	.53738453 (-1)
6	.15282969	.0	.67010371 (-1)
	.16747343	.0	.20535510 (-1)
	.22581758	.27120875	.0
	.24931337	.63648509	.0
	.87309498 (-1)	1.0	.88217190 (-2)
	.11725642	1.0	.61648245 (-1)

TABLE 2

Interpolation formulas applied to $u(x, t) = \exp(2t + x^2)$

(x*,t*) =		(.25,.1)	(.5,.1)	(.75,.1)
m =	2	1.2629568	1.4720844	1.9516098
	3	1.2938350	1.5546484	2.1200954
	4	1.2992568	1.5660694	2.1394610
	5	1.3000442	1.5679052	2.1427821
	6	1.3001549	1.5682518	2.1435101
Exact		1.3001765	1.5683122	2.1436286

All calculations described in this paper were performed on the AMDAHL 470 computer at the Data Processing Center at Texas A & M University, using double precision arithmetic in FORTRAN, which carries approximately 16 significant digits.

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