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Short Communication

Gauss Legendre quadrature over a triangle

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Abstract

This paper presents a Gauss Legendre quadrature method for numerical integration over the standard triangular surface: $\{(x, y) \mid 0 \le x, y \le 1, x + y \le 1\}$ in the Cartesian two-dimensional (x, y) space. Mathematical transformation from (x, y) space to (x, h) space map the standard triangle in (x, y) space to a standard 2-square in (x, h) space: $\{(x, h) | -1 \le x, h \le 1\}$. This overcomes the difficulties associated with the derivation of new weight coefficients and sampling points and yields results which are accurate and reliable. Results obtained with new formulae are compared with the existing formulae.

Keywords: Finite-element method, numerical integration, Gauss Legendre quadrature, triangular elements, standard 2-square, extended numerical integration.

1. Introduction

In recent years, the finite-element method (FEM) has become a very powerful tool for the approximate solution of boundary-value problems governing diverse physical phenomena. Its use in industry and research is extensive and without it many practical problems in science and engineering would be incapable of solution. The triangular elements with either straight or curved sides are very widely used in finite-element analysis. The versatility of these triangular elements can be enhanced further by improved numerical integration schemes. In FEM, various integrals are to be determined numerically in the evaluation of the stiffness matrix, mass matrix, body force vector, etc.

The basic problem of integrating an arbitrary function of two variables over the surface of triangle was first given by Hammer *et al.* [1] and Hammer and Stroud [2, 3]. With the advent of the finite-element method, the triangular elements proved to be versatile. There has been considerable interest in the area of numerical integration schemes over triangles. Cowper [4] provided a table of Gaussian quadrature formulae for symmetrically placed integration points.

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Lyness and Jespersen [5] made an elaborate study of symmetric quadrature rules by formulating the problem in polar coordinates. Lannoy [6] discussed the symmetric 4-point integration rule [4]. Laurie [7] derived a 7-point integration rule and discussed the numerical error in integrating some functions. Laursen and Gellert [8] gave a detailed table of symmetric integration formulae and suggested some new higher-order formulae of precision up to degree 10. Lether [9], Hillion [10] and Lague and Baldur [11] considered the product formulae derivable from one-dimensional Gaussian quadrature rules. Reddy [12], and Reddy and Shippy [13] derived 3-, 4-, 6- and 7-point formulae which give improved accuracy. However, Lague and Baldur have not listed the explicit weighting coefficients and sampling points for numerical applications. The present work aims to provide this information in a systematic manner for future reference. There is a great need for higher-order quadrature rules over the triangular surface [14]. A similar suggestion was also made by Lague and Baldur [11].

2. Formulation of integrals over a triangular area

The finite-element method for two-dimensional problems with triangular elements requires the numerical integration of shape functions on a triangle. Since an affine transformation makes it possible to transform any triangle into the two-dimensional standard triangle T with coordinates (0,0), (0,1), (1,0) in Cartesian frame, we have to consider just the numerical integration on T. The integral of an arbitrary function, f, over the surface of a triangle T is given by

$$I = \iint_T f(x,y)\,dxdy = \int_0^1 dx \int_0^{1-x} f(x,y)\,dy = \int_0^1 dy \int_0^{1-y} f(x,y)\,dx. \tag{1}$$

It is now required to find the value of the integral by a quadrature formula:

$$I = \sum_{m=1}^{N} c_m f(x_{m,} y_m) \tag{2}$$

where c_m are the weights associated with specific points (x_m, y_m) and N is the number of pivotal points related to the required precision. One such accurate method known to the present authors is based on 13 integration points [4]. It is not likely that this technique will be extended further to give greater accuracy which may be demanded in future. The other method is the approximation of I by product formulae [10] which is of type (2) based on the roots and weights of Gauss Legendre and Gauss Jacobi polynomials. The precision of these formulae is limited to polynomials of degree seven; this is because the weights and roots of Jacobi polynomials are not tabulated in standard texts for sufficiently higher degree polynomials. The product formulae proposed in this paper and in the work of Lague and Baldur [11] are based only on the roots and weights of Gauss Legendre polynomials.

The integral I of eqn (1) can be transformed into an integral over the surface of the square: $\{(u, v)\,|0 \le u, v \le 1\}$ by the substitution:

$$x = u,\ y = (1 - u)v. \tag{3}$$

Then the determinant of the Jacobian and the differential area are:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = (1)\,(1-u) - 0\,(-v) = 1 - u$$

and

$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}dudv = (1-u)dudv. \tag{4}$$

Then, on using eqns (3) and (4) in eqn (1), we have

$$\int_0^1\int_0^{1-x} f(x,y)\,dydx = \int_0^1\int_0^1 f(u,(1-u)v)(1-u)\,dudv. \tag{5}$$

The integral I of eqn (5) can be transformed further into an integral over the standard 2-square: $\{(x, h)| -1 \le x, h \le 1\}$ by the substitution

$$u = (1 + x)/2,\ v = (1 + h)/2, \tag{6}$$

then clearly the determinant of the Jacobian and the differential area are:

$$\frac{\partial(u,v)}{\partial(x,h)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial h} - \frac{\partial u}{\partial h}\frac{\partial v}{\partial x} = (1/2)(1/2) - (0)\,(0) = 1/4$$

$$dudv = \frac{\partial(u,v)}{\partial(x,h)}dx\,dh = \frac{1}{4}dx\,dh\,. \tag{7}$$

Now, on using eqns (6) and (7) in eqn (5), we have:

$$I = \int_0^1\int_0^{1-x} f(x,y)\,dydx = \int_0^1\int_0^1 f(u,(1-u)v)(1-u)\,dudv$$

$$= \int_{-1}^1\int_{-1}^1 f\left(\frac{1+x}{2}, \frac{(1-x)(1+h)}{4}\right)\left(\frac{1-x}{8}\right)dx\,dh. \tag{8}$$

Equation (8) represents an integral over the surface of a standard 2-square: $\{(x, h)|-1 \le x, h \le 1\}$. Efficient quadrature coefficients are readily available in the literature so that any desired accuracy can be readily obtained [15].

From eqn (8), we can write:

$$I = \int_{-1}^1\int_{-1}^1 f(x(x,h), y(x,h))\left(\frac{1-x}{8}\right)dx\,dh,$$

$$I = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1-x_i}{8}\right)w_i w_j\, f(x(x_i,h_j), y(x_i,h_j)), \tag{9}$$

where x_i, h_i are Gaussian points in the x, h directions, respectively, and w_i and w_j are the corresponding weights.

We can rewrite eqn (9) as:

$$I = \sum_{k=1}^{N=n\times n} c_k\, f(x_k, y_k) \tag{10}$$

Table I
Gauss points and weighting coefficients over a triangle

x_i	y_i	c_i
$n = 2$		
0.211324865	0.166666667	0.197168783
0.211324865	0.622008467	0.197168783
0.788675134	0.044658198	0.052831216
0.788675134	0.166666667	0.052831216
$n = 3$		
0.112701665	0.100000000	0.068464377
0.112701665	0.443649167	0.109543004
0.112701665	0.787298334	0.068464377
0.500000000	0.056350832	0.061728395
0.500000000	0.250000000	0.098765432
0.500000000	0.443649167	0.061728395
0.887298334	0.012701665	0.008696116
0.887298334	0.056350832	0.013913785
0.887298334	0.100000000	0.008696116

where c_k, x_k and y_k can be obtained from the relations:

$$c_k = \frac{(1-x_i)}{8} w_i w_j, \qquad x_k = \frac{(1+x_i)}{2}, \qquad y_k = \frac{(1-x_i)(1+h_j)}{4}$$

$$(k = 1, 2, \ldots\ldots, n), (i, j = 1, 2, 3, \ldots.., n). \tag{11}$$

The weighting coefficients c_k and sampling points (x_k, y_k) of various orders can now be easily computed by formula of eqns (10) and (11). We have tabulated a sample of these weight coefficients and sampling points for $n = 2, 3$ (Table I).

3. Some numerical results

We consider some typical integrals with known exact values [13].

$$I_1 = \int_0^1 \int_0^{1-y} (x+y)^{\frac{1}{2}} dxdy = 0.400000000;$$

$$I_2 = \int_0^1 \int_0^{1-y} (x+y)^{\frac{-1}{2}} dxdy = 0.666666667;$$

$$I_3 = \int_0^1 \int_0^{y} (x^2+y^2)^{\frac{-1}{2}} dxdy = 0.881373587;$$

$$I_4 = \int_0^{\frac{p}{2}} \int_0^{y} \sin(x+y) dxdy = 1.000000000;$$

Table II
Numerical results of double integration

n	I_1	I_2	I_3	I_4	I_5
2	0.401077171	0.648611850	0.773333314	0.997971215	0.713057545
3	0.400179978	0.660068693	0.826947872	1.000010067	0.716488207
4	0.400049569	0.663549499	0.848619059	0.999999973	0.719259751
5	0.400017920	0.664954585	0.859506833	1.000000004	0.718518356
6	0.400007718	0.665627534	0.867623963	1.000000007	0.717469550
7	0.400003754	0.665989386	0.869644421	1.000000001	0.718432382
8	0.400002008	0.666201003	0.872247990	1.000000000	0.718568842
9	0.400001147	0.666332910	0.874071505	1.000000006	0.718126535
10	0.400000697	0.666354438	0.875398197	0.999999996	0.718253208
15	0.400000094	0.666589692	0.878533306	0.999999999	0.718352298

$$I_5 = \int_0^1 \int_0^y e^{|x+y-1|} dxdy \ = 0.71828183.$$

These integrals were evaluated using the integration schemes derived in the present paper and it is found that excellent convergence occurs to the exact values. The calculations give very accurate results and are reliable as proved by Lague and Baldur [11]. The results are summarized in Table II.

4. Conclusions

We have derived various orders of extended numerical integration rules based on classical Gauss Legendre quadrature over a triangle (Table I). This is made possible by transforming the triangular surface: $0 \le x, y \le 1,\ x + y \le 1$ to a standard 2-square; $-1 \le x, h \le 1$. Over the 2-square, the Gauss Legendre quadrature rule of all orders is applicable. It may be noted that a lot of mathematical effort is needed to derive the numerical integration rules over the triangular surface and the integration formulae available at this moment in the literature are confined to a precision of degree up to ten. With the proposed method, this restriction is removed and one can now obtain numerical integration rules of very high degree as the derivations proposed here rely on the standard Gauss Legendre quadrature rules. This is essential as the demand for higher-order integration rules in the finite-element method is increasing.

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