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Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes

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The problem of simulation of multivariate Lévy processes is investigated. A method based on generalized shot noise series representations of Lévy processes combined with Gaussian approximation of the remainder is established in full generality. This method is applied to multivariate stable and tempered stable processes and formulae for their approximate simulation are obtained.

Keywords: Gaussian approximation; Lévy processes; shot noise series expansions; simulation; tempered stable processes

1. Introduction

Simulation of stochastic processes is widely used in science, engineering and economics to model complex phenomena. There is a vast literature on this subject; see the classical monographs of Kloeden and Platen (1992) on numerical solutions of stochastic differential equations and of Janicki and Weron (1994) on simulation of one-dimensional stable processes, for example. Applications of Lévy processes to stochastic finance (Cont and Tankov 2004) and physics created the need for efficient simulation schemes. Contrary to the one-dimensional case, closed formulae for simulation of increments of multidimensional Lévy processes are rarely available. Thus one needs to use approximate methods. The present paper develops a general framework for this in the multidimensional case.

At the first level of approximation of a Lévy process one can use an appropriate compound Poisson process. However, if the Lévy process has paths of infinite variation, then the error (the remainder process) of such approximation can be significant. In the one-dimensional case it was shown in Asmussen and Rosiński (2001) that the remainder process can often be approximated by a Brownian motion with small variance. Adding such a small-variance Brownian motion to a compound Poisson process introduces variability between the epochs of the latter process, improving the approximation in general; see Asmussen and Rosiński (2001).

There are several issues related to the extension of this method to multidimensional Lévy processes. The first one is how to choose for a given Lévy process a family of successive

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compound Poisson approximations that are easy to simulate. This choice determines the form of the remainder process which, in turn, we want to approximate by a Brownian motion with a small covariance matrix. The level of difficulty for proving that the remainder is asymptotically Gaussian may depend on the form of the remainder and that form, in turn, is determined by the type of Poisson approximation. Let ν be the Lévy measure of the Lévy process under consideration. In the one-dimensional case one usually takes compound Poisson processes with jump measures $\nu_{|\{|x|\geq\epsilon\}}$ and $\epsilon \searrow 0$. We found that this method is too restrictive in the multidimensional case and may lead to substantial technical and theoretical difficulties in concrete situations. We propose to use generalized shot noise series representations of Lévy processes (see Rosiński 2001) to generate a family of successive compound Poisson approximations. Such expansions can also be related to Lévy copulas (see Cont and Tankov 2004: Section 6.6), but we do not consider them here. Finally, let us mention a small issue which is of practical importance. The covariance matrix of the remainder process at time 1 serves as the covariance matrix of the asymptotic Brownian motion and thus it should be explicitly known for the purpose of simulation. However, this is not often the case in the multidimensional situation. Therefore, an additional requirement is that the remainder should be defined in such a way that the covariance matrix (or its square root, to be precise) is asymptotically equivalent to some explicitly known matrix. All these issues are usually non-essential, or trivial, in the one-dimensional case. They are present in the multidimensional situation as we demonstrate it on an example of simulation of multivariate tempered stable processes. These issues are even more acute in the case of simulation of operator stable Lévy processes, which we will consider in a forthcoming paper together with some other practical issues.

The present paper is organized as follows. The general problem of Gaussian approximation of Lévy processes is considered in Section 2. We find the necessary and sufficient conditions for such approximation to hold (Theorem 2.2) as well as some useful sufficient conditions (Theorems 2.4 and 2.5). In Section 3.1 we present a general set-up for the approximate simulation of multivariate Lévy processes (Theorem 3.1). We illustrate this method in Sections 3.2 and 3.3 by applying it to multivariate stable and tempered stable Lévy processes, respectively.

For the reader's convenience we list here some basic facts from matrix theory that will used in this paper and which can be found, for example, in Graybill (1983). The square root of a positive definite matrix A is a unique positive definite matrix B such that $A = B^2$ (Graybill 1983: 449), and we write $A^{1/2} := B$. The inverse of a positive definite matrix is positive definite (Graybill 1983: Theorem 12.2.1(3b)). The inequality $A \geq B$ between two symmetric matrices A and B of the same size means that $A - B$ is non-negative definite. If A and B are non-singular symmetric matrices and $A \geq B$, then $B^{-1} \geq A^{-1}$ (Graybill 1983: Theorem 12.2.14(2)).

2. Gaussian approximation

In this section we give necessary and sufficient conditions for normal approximation of multidimensional Lévy processes. Suppose that for every $\epsilon \in (0, 1]$ we are given an $\mathbb{R}^d$-valued Lévy processes $\mathbf{X}_\epsilon := \{X_\epsilon(t) : t \geq 0\}$ with characteristic function of the form

$$\mathrm{E}e^{\mathrm{i}\langle y, X_\epsilon(t)\rangle} = \exp\left\{ t\int_{\mathbb{R}^d} [e^{\mathrm{i}\langle y,x\rangle} - 1 - \mathrm{i}\langle y, x\rangle]\nu_\epsilon(\mathrm{d}x)\right\}, \tag{2.1}$$

where

$$\int_{\mathbb{R}^d} \|x\|^2\, \nu_\epsilon(\mathrm{d}x) < \infty. \tag{2.2}$$

Then $X_\epsilon(t)$ has zero mean and covariance matrix $\mathrm{E}[X_\epsilon(t)X_\epsilon(t)^\mathsf{T}] = t\Sigma_\epsilon$, where

$$\Sigma_\epsilon = \int_{\mathbb{R}^d} xx^\mathsf{T}\, \nu_\epsilon(\mathrm{d}x). \tag{2.3}$$

(A^T denotes the transpose of a matrix A.) We will assume that Σ_ϵ is non-singular. Because of the importance of this assumption we will first state some conditions equivalent to it.

Lemma 2.1. *The following conditions are equivalent for each $\epsilon > 0$:*

(i) *Σ_ϵ is non singular;*
(ii) *ν_ϵ is not concentrated on any proper linear subspace of $\mathbb{R}^d$;*
(iii) *for each (equivalently, some) $t > 0$, the distribution of $X_\epsilon(t)$ is not concentrated on any proper hyperplane of $\mathbb{R}^d$.*

Proof. A proof of the equivalence of (i) and (ii) is elementary and thus omitted. The equivalence of (ii) and (iii) follows from Proposition 24.17(ii) in Sato (1999). □

Throughout this paper $\mathbf{W} = \{W(t) : t \geqslant 0\}$ will denote a standard Brownian motion in $\mathbb{R}^d$. $\overset{(d)}{\to}$ will stand for the weak convergence of processes in the space $D([0, \infty), \mathbb{R}^d)$ of cadlag functions from $[0, \infty)$ into $\mathbb{R}^d$ equipped with the Skorokhod topology.

Theorem 2.2. *With the above notation, suppose that Σ_ϵ is non-singular for every $\epsilon \in (0, 1]$. Then, as $\epsilon \to 0$,*

$$\Sigma_\epsilon^{-1/2}\mathbf{X}_\epsilon \overset{(d)}{\to} \mathbf{W} \tag{2.4}$$

if and only if for every $\kappa > 0$,

$$\int_{\langle\Sigma_\epsilon^{-1}x,x\rangle>\kappa} \langle\Sigma_\epsilon^{-1}x, x\rangle\, \nu_\epsilon(\mathrm{d}x) \to 0. \tag{2.5}$$

Proof. Notice that $\Sigma_\epsilon^{-1/2}\mathbf{X}_\epsilon$ is a Lévy process with characteristic function

$$\begin{aligned}\mathrm{E}e^{\mathrm{i}\langle y,\Sigma_\epsilon^{-1/2}X_\epsilon(t)\rangle} &= \exp\left\{ t\int_{\mathbb{R}^d} [e^{\mathrm{i}\langle y,\Sigma_\epsilon^{-1/2}x\rangle} - 1 - \mathrm{i}\langle y, \Sigma_\epsilon^{-1/2}x\rangle]\, \nu_\epsilon(\mathrm{d}x)\right\}\\ &= \exp\left\{ \mathrm{i}t\langle y, b_\epsilon\rangle + t\int_{\mathbb{R}^d} [e^{\mathrm{i}\langle y,x\rangle} - 1 - \mathrm{i}\langle y, x\rangle\mathbf{1}(\|x\| \leqslant 1)]\, \tilde{\nu}_\epsilon(\mathrm{d}x)\right\},\end{aligned}$$

where $\tilde{\nu}_\epsilon = \nu_\epsilon \circ \Sigma_\epsilon^{1/2}$ is the push forward of ν_ϵ by the map $x \to \Sigma_\epsilon^{-1/2}x$ and

$$b_\epsilon = -\int_{\|x\|>1} x\tilde{\nu}_\epsilon(\mathrm{d}x) = -\int_{\|\Sigma_\epsilon^{-1/2}x\|>1} \Sigma_\epsilon^{-1/2}x\,\nu_\epsilon(\mathrm{d}x).$$

First we will prove that (2.5) implies (2.4). By a theorem due to Skorokhod (cf. Kallenberg 2002: Theorem 15.17), it is enough to show the convergence in distribution of $\Sigma_\epsilon^{-1/2}X_\epsilon(1)$ to $W(1)$. To this end it is enough to check three conditions of Theorem 15.14 in Kallenberg (2002): as $\epsilon \to 0$,

$$b_\epsilon \to 0; \tag{2.6}$$

$$\int_{\|x\|\leqslant 1} xx^\mathsf{T}\,\tilde{\nu}_\epsilon(\mathrm{d}x) \to I_d; \tag{2.7}$$

$$\tilde{\nu}_\epsilon(\|x\| \geqslant \kappa) \to 0, \qquad \forall\kappa > 0. \tag{2.8}$$

Indeed, (2.6) holds because

$$\begin{aligned}\|b_\epsilon\| &\leqslant \int_{\|\Sigma_\epsilon^{-1/2}x\|>1} \|\Sigma_\epsilon^{-1/2}x\|\,\nu_\epsilon(\mathrm{d}x) \leqslant \int_{\|\Sigma_\epsilon^{-1/2}x\|>1} \|\Sigma_\epsilon^{-1/2}x\|^2\,\nu_\epsilon(\mathrm{d}x)\\ &= \int_{\langle\Sigma_\epsilon^{-1}x,x\rangle>1} \langle\Sigma_\epsilon^{-1}x,\, x\rangle\,\nu_\epsilon(\mathrm{d}x) \to 0\end{aligned}$$

by assumption (2.5). Observe that

$$\begin{aligned}\int_{\mathbb{R}^d} xx^\mathsf{T}\,\tilde{\nu}_\epsilon(\mathrm{d}x) &= \int_{\mathbb{R}^d} (\Sigma_\epsilon^{-1/2}x)(\Sigma_\epsilon^{-1/2}x)^\mathsf{T}\,\nu_\epsilon(\mathrm{d}x)\\ &= \Sigma_\epsilon^{-1/2}\int_{\mathbb{R}^d} xx^\mathsf{T}\,\nu_\epsilon(\mathrm{d}x)\Sigma_\epsilon^{-1/2} = I_d.\end{aligned} \tag{2.9}$$

Denoting by $\|\cdot\|$ the operator norm, we obtain

$$\begin{aligned}\left\| I_d - \int_{\|x\|\leqslant 1} xx^\mathsf{T}\,\tilde{\nu}_\epsilon(\mathrm{d}x)\right\| &= \left\|\int_{\|\Sigma_\epsilon^{-1/2}x\|>1} (\Sigma_\epsilon^{-1/2}x)(\Sigma_\epsilon^{-1/2}x)^\mathsf{T}\,\nu_\epsilon(\mathrm{d}x)\right\|\\ &\leqslant \int_{\|\Sigma_\epsilon^{-1/2}x\|>1} \|\Sigma_\epsilon^{-1/2}x\|^2\,\nu_\epsilon(\mathrm{d}x) \to 0,\end{aligned}$$

as above, which proves (2.7). To obtain (2.8) we observe that

$$\begin{aligned}\tilde{\nu}_\epsilon(\|x\| > \kappa) &\leqslant \kappa^{-2}\int_{\|x\|>\kappa} \|x\|^2\,\tilde{\nu}_\epsilon(\mathrm{d}x)\\ &= \kappa^{-2}\int_{\|\Sigma_\epsilon^{-1/2}x\|>\kappa} \|\Sigma_\epsilon^{-1/2}x\|^2\,\nu_\epsilon(\mathrm{d}x) \to 0 \qquad \text{as } \epsilon \to 0.\end{aligned}$$

Therefore, we have proved that (2.5) implies (2.4).

To obtain the converse, we first notice that from Theorem 15.14 in Kallenberg (2002) we have, for all $\kappa > 0$,

$$\int_{\|x\|\leqslant\kappa} xx^{\mathsf{T}}\,\tilde{\nu}_\epsilon(\mathrm{d}x) \to I_d$$

as $\epsilon \to 0$. By (2.9) this is equivalent, for all $\kappa > 0$, to

$$\int_{\|x\|>\kappa} xx^{\mathsf{T}}\,\tilde{\nu}_\epsilon(\mathrm{d}x) \to 0.$$

The latter condition implies (2.5) because

$$\begin{aligned}\int_{\langle\Sigma_\epsilon^{-1}x,x\rangle>\kappa} \langle\Sigma_\epsilon^{-1}x,\, x\rangle\,\nu_\epsilon(\mathrm{d}x) &= \int_{\|\Sigma_\epsilon^{-1/2}x\|>\kappa^{1/2}} \|\Sigma_\epsilon^{-1/2}x\|^2\,\nu_\epsilon(\mathrm{d}x)\\ &= \int_{\|x\|>\kappa^{1/2}} \|x\|^2\,\tilde{\nu}_\epsilon(\mathrm{d}x) = \sum_{i=1}^d \int_{\|x\|>\kappa^{1/2}} \langle e_i,\, x\rangle^2\,\tilde{\nu}_\epsilon(\mathrm{d}x)\\ &= \sum_{i=1}^d \left\langle e_i,\, \int_{\|x\|>\kappa^{1/2}} xx^{\mathsf{T}}\,\tilde{\nu}_\epsilon(\mathrm{d}x)e_i\right\rangle \to 0,\end{aligned}$$

where $\{\mathrm{e}_i\}_{i=1}^d$ is an orthonormal basis in $\mathbb{R}^d$. The proof of Theorem 2.2 is complete. □

As the matrix Σ_ϵ can be quite complicated, a direct verification of (2.5) may be difficult or impossible. We propose a simple method which alleviates this difficulty in many cases.

Lemma 2.3. *Suppose that, for every* $\kappa > 0$*, there exist* $\epsilon(\kappa) > 0$ *and a family of positive definite matrices* $\{\tilde{\Sigma}_\epsilon : \epsilon \in (0, \epsilon(\kappa))\}$ *such that, for all* $\epsilon \in (0, \epsilon(\kappa))$*,*

$$\Sigma_\epsilon \geqslant \tilde{\Sigma}_\epsilon \tag{2.10}$$

and

$$\lim_{\epsilon\to 0}\int_{\langle\tilde{\Sigma}_\epsilon^{-1}x,x\rangle>\kappa} \langle\tilde{\Sigma}_\epsilon^{-1}x,\, x\rangle\,\nu_\epsilon(\mathrm{d}x) = 0. \tag{2.11}$$

Then (2.5) *holds.*

Proof. Recall that (2.10) means that $\langle\Sigma_\epsilon x,\, x\rangle \geqslant \langle\tilde{\Sigma}_\epsilon x,\, x\rangle$, for all $x \in \mathbb{R}^d$. Hence $\tilde{\Sigma}_\epsilon^{-1} \geqslant \Sigma_\epsilon^{-1}$ (see the last paragraph of the Introduction), which yields

$$\int_{\langle\Sigma_\epsilon^{-1}x,x\rangle>\kappa} \langle\Sigma_\epsilon^{-1}x,\, x\rangle\,\nu_\epsilon(\mathrm{d}x) \leqslant \int_{\langle\tilde{\Sigma}_\epsilon^{-1}x,x\rangle>\kappa} \langle\tilde{\Sigma}_\epsilon^{-1}x,\, x\rangle\,\nu_\epsilon(\mathrm{d}x) \to 0$$

as $\epsilon \to 0$. □

In typical applications of Theorem 2.2, Lévy measures ν_ϵ can often be written in the form

$$\nu_\epsilon(\mathrm{d}r,\, \mathrm{d}u) = \mu_\epsilon(\mathrm{d}r|u)\lambda(\mathrm{d}u), \qquad r > 0,\, u \in S^{d-1}, \tag{2.12}$$

in polar coordinates, where λ is a finite measure on the unit sphere S^{d-1} and, for each $\epsilon > 0$, $\{\mu_\epsilon(\cdot\,|u) : u \in S^{d-1}\}$ is a measurable family of Lévy measures on $(0, \infty)$. Define

$$\sigma_\epsilon^2(u) = \int_0^\infty r^2\,\mu_\epsilon(\mathrm{d}r\,|u). \tag{2.13}$$

Theorem 2.4. *Let ν_ϵ be Lévy measures on $\mathbb{R}^d$ given by* (2.12) *such that the support of λ is not contained in any proper linear subspace of $\mathbb{R}^d$. Suppose there exists a function $b : (0, 1] \mapsto (0, \infty)$ such that*

$$\liminf_{\epsilon\to 0} \frac{\sigma_\epsilon(u)}{b(\epsilon)} > 0, \qquad \lambda\text{-almost everywhere,} \tag{2.14}$$

and, for every $\kappa > 0$,

$$\lim_{\epsilon\to 0} b(\epsilon)^{-2} \int_{\|x\|>\kappa b(\epsilon)} \|x\|^2\,\nu_\epsilon(\mathrm{d}x) = 0. \tag{2.15}$$

Then Σ_ϵ is non-singular for sufficiently small ϵ and condition (2.5) *of Theorem 2.2 holds.*

Proof. Let

$$\Lambda = \int_{S^{d-1}} uu^{\mathsf{T}}\,\lambda(\mathrm{d}u).$$

Λ is non-singular by Lemma 2.1. Hence $\inf_{v\in S^{d-1}} \langle \Lambda v, v\rangle =: 2a > 0$. For any Borel set $B \subset S^{d-1}$, consider a positive definite matrix

$$\Lambda_B = \int_B uu^{\mathsf{T}}\,\lambda(\mathrm{d}u). \tag{2.16}$$

There exists a $\delta > 0$ such that $\|\Lambda - \Lambda_B\| < a$ whenever $\lambda(S^{d-1}\backslash B) < \delta$. From (2.14) we can find $\epsilon_0, \epsilon_1 \in (0, 1]$ such that the set B of the form

$$B := \{u \in S^{d-1} : \inf_{0<\epsilon<\epsilon_0} b(\epsilon)^{-2}\sigma_\epsilon^2(u) > \epsilon_1\}$$

satisfies $\lambda(S^{d-1}\backslash B) < \delta$. Hence, for any $v \in S^{d-1}$,

$$\langle \Lambda_B v, v\rangle \geqslant \langle \Lambda v, v\rangle - \|\Lambda - \Lambda_B\| > a,$$

which yields

$$\Lambda_B \geqslant aI_d. \tag{2.17}$$

Using (2.13) and (2.17), we obtain for any $\epsilon \in (0, \epsilon_0)$,

$$\Sigma_\epsilon = \int_{S^{d-1}} \int_0^\infty r^2\,\mu_\epsilon(\mathrm{d}r|u)uu^{\mathsf{T}}\,\lambda(\mathrm{d}u) \geqslant \int_B \sigma_\epsilon^2(u)uu^{\mathsf{T}}\,\lambda(\mathrm{d}u)$$
$$\geqslant \epsilon_1 b(\epsilon)^2\Lambda_B \geqslant a\epsilon_1 b(\epsilon)^2 I_d =: \tilde{\Sigma}_\epsilon.$$

Thus Σ_ϵ is non-singular and, for any $\kappa > 0$, we have

$$\int_{\langle \tilde{\Sigma}_\epsilon^{-1} x, x\rangle > \kappa} \langle \tilde{\Sigma}_\epsilon^{-1} x, x\rangle \, \nu_\epsilon(\mathrm{d}x) \leq a^{-1}\epsilon_1^{-1} b(\epsilon)^{-2} \int_{\|x\| > (a\epsilon_1\kappa)^{1/2} b(\epsilon)} \|x\|^2 \, \nu_\epsilon(\mathrm{d}x) \to 0$$

by (2.15), as $\epsilon \to 0$. Lemma 2.3 concludes the proof. □

A special but important case of (2.12) is when

$$\nu_\epsilon = \nu_{|\{\|x\|<\epsilon\}}, \tag{2.18}$$

where ν is a Lévy measure given in polar coordinates by

$$\nu(\mathrm{d}r, \mathrm{d}u) = \mu(\mathrm{d}r \,|u)\lambda(\mathrm{d}u), \qquad r > 0, \, u \in S^{d-1}. \tag{2.19}$$

Here $\{\mu(\cdot\,|u) : u \in S^{d-1}\}$ is a measurable family of Lévy measures on $(0, \infty)$ and λ is a finite measure on the unit sphere S^{d-1}. In this case we have μ_ϵ given by

$$\mu_\epsilon(\mathrm{d}r \,|u) = \mathbf{1}(r < \epsilon)\mu(\mathrm{d}r \,|u)$$

and

$$\Sigma_\epsilon = \int_{\|x\|<\epsilon} xx^{\mathsf{T}} \, \nu(\mathrm{d}x). \tag{2.20}$$

This is a direct multidimensional extension of the case studied in Asmussen and Rosiński (2001) for $d = 1$.

Theorem 2.5. *Let ν be a Lévy measure on $\mathbb{R}^d$ given by* (2.19) *such that the support of λ is not contained in any proper linear subspace of $\mathbb{R}^d$. Let ν_ϵ be given by* (2.18) *and Σ_ϵ by* (2.20). *If*

$$\lim_{\epsilon \to 0} \epsilon^{-2} \int_{(0,\epsilon)} r^2 \, \mu(\mathrm{d}r \,|u) = \infty, \; \lambda\text{-a.e.}, \tag{2.21}$$

then Σ_ϵ is non-singular and condition (2.5) *of Theorem 2.2 holds.*

Proof. We will show that the conditions of Theorem 2.4 are satisfied. Recall (2.13) and notice that

$$\sigma_\epsilon^2(u) = \int_{(0,\epsilon)} r^2 \, \mu(\mathrm{d}r \,|u).$$

Therefore, condition (2.21) says that

$$\lim_{\epsilon \to 0} \frac{\sigma_\epsilon(u)}{\epsilon} = \infty, \qquad \lambda\text{-a.e.} \tag{2.22}$$

We can choose a sequence $\epsilon_k \searrow 0$ such that the sets given by

$$B_k := \left\{ u \in S^{d-1} : \inf\left\{ \frac{\sigma_\epsilon(u)}{\epsilon} : 0 < \epsilon \leq \epsilon_k \right\} > k^2 \right\}$$

satisfy $\lambda(B_k) > \lambda(S^{d-1})(1 - 2^{-k})$ for every $k \geq 1$. Then $S_0 = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \mathrm{B}_k$ has full λ-measure. Put $b(\epsilon) = \epsilon k$ for $\epsilon_{k+1} < \epsilon \leq \epsilon_k$. It is clear that

$$\lim_{\epsilon\to 0}\frac{b(\epsilon)}{\epsilon}=\infty, \tag{2.23}$$

and for each $u \in S_0$ there is an n such that $u \in B_k$ for $k \geqslant n$. Hence, for $\epsilon_{k+1} < \epsilon \leqslant \epsilon_k \leqslant \epsilon_n$,

$$\frac{\sigma_\epsilon(u)}{b(\epsilon)}=\frac{\sigma_\epsilon(u)}{k\epsilon}>k,$$

which proves that

$$\lim_{\epsilon\to 0}\frac{\sigma_\epsilon(u)}{b(\epsilon)}=\infty, \qquad \lambda\text{-a.e.}$$

Thus (2.14) of Theorem 2.4 holds. Then, for each $\kappa > 0$ and sufficiently small $\epsilon > 0$,

$$\int_{\|x\|\geqslant\kappa b(\epsilon)}\|x\|^2\,\nu_\epsilon(\mathrm{d}x)=\int_{\{\|x\|\geqslant\kappa b(\epsilon),\,\|x\|<\epsilon\}}\|x\|^2\,\nu(\mathrm{d}x)=0$$

because the region of the integration is empty by (2.23). This verifies the remaining condition (2.15) of Theorem 2.4. Finally, we notice that if Σ_ϵ in (2.20) is non-singular for small ϵ then it is non-singular for all ϵ. The proof is complete. □

Remark 2.1. If $\mu(\mathrm{d}r\,|u)=f(r,u)\,\mathrm{d}r\lambda(\mathrm{d}u)$, where $f(\cdot,u)$ is a regularly varying function at zero with index $-\alpha(u)-1$, with $\alpha(u)\in(0,2)$ for almost every u, then (2.21) is fulfilled and hence condition (2.5) of Theorem 2.2 holds.

3. Approximate simulation of multivariate Lévy processes

In this section we present a general set-up for an approximate simulation of multivariate Lévy processes and its implementation to stable and tempered stable cases.

3.1. Approximation of Lévy processes

Consider a Lévy process $\mathbf{X}=\{X(t): t\geqslant 0\}$ in $\mathbb{R}^d$ determined by its characteristic function in the Lévy–Khinchine form,

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\langle y,X(t)\rangle}=\exp\left\{t\left[\mathrm{i}\langle a,y\rangle+\int_{\mathbb{R}^d}(\mathrm{e}^{\mathrm{i}\langle y,x\rangle}-1-\mathrm{i}\langle y,x\rangle\mathbf{1}(\|x\|\leqslant 1))\nu(\mathrm{d}x)\right]\right\}. \tag{3.1}$$

Suppose that, for every $\epsilon\in(0,1]$, we are given a decomposition

$$\nu=\nu_\epsilon+\nu^\epsilon, \tag{3.2}$$

where

$$\int_{\mathbb{R}^d}\|x\|^2\,\nu_\epsilon(\mathrm{d}x)<\infty \quad\text{and}\quad \nu^\epsilon(\mathbb{R}^d)<\infty.$$

Consider the corresponding decomposition of the Lévy process $\mathbf{X}$ into a sum of *independent* terms

$$\mathbf{X} \overset{(d)}{=} \mathbf{X}_\epsilon + \mathbf{N}^\epsilon + \mathbf{a}_\epsilon, \tag{3.3}$$

where $\mathbf{X}_\epsilon$ is a Lévy process as in (2.1) determined by

$$\mathrm{E}e^{i\langle y, X_\epsilon(t)\rangle} = \exp\left\{ t\left[\int_{\mathbb{R}^d} (e^{i\langle y,x\rangle} - 1 - i\langle y, x\rangle)\nu_\epsilon(\mathrm{d}x)\right]\right\},$$

$\mathbf{N}^\epsilon = \{N^\epsilon(t) : t \geqslant 0\}$ is a compound Poisson process with the jumps measure ν^ϵ, and $\mathbf{a}_\epsilon = \{ta_\epsilon : t \geqslant 0\}$ is a drift. To be precise,

$$a_\epsilon = a + \int_{\|x\|>1} x\,\nu_\epsilon(\mathrm{d}x) - \int_{\|x\|\leqslant 1} x\,\nu^\epsilon(\mathrm{d}x). \tag{3.4}$$

We will refer to $\mathbf{X}_\epsilon$ as the 'small-jump' part of $\mathbf{X}$. Let

$$\Sigma_\epsilon = \int_{\mathbb{R}^d} xx^\top\,\nu_\epsilon(\mathrm{d}x)$$

be a non-singular matrix, as in Theorem 2.2. If that theorem applies, then the small-jump part $\mathbf{X}_\epsilon$ can be approximated by $\Sigma_\epsilon^{1/2}\mathbf{W}$, where $\mathbf{W}$ is a standard Brownian motion in $\mathbb{R}^d$ independent of $\mathbf{N}^\epsilon$. Consequently, $\mathbf{X} \overset{(d)}{\approx} \Sigma_\epsilon^{1/2}\mathbf{W} + \mathbf{N}^\epsilon + \mathbf{a}_\epsilon$ when ϵ is small. Observe that we do not need to know the exact form of $\Sigma_\epsilon^{1/2}$ but only its asymptotics. The following theorem provides a theoretical basis for the approximate simulation method discussed and gives some information on the behaviour of the error.

Theorem 3.1. *Let* $\mathbf{X} = \{X(t) : t \geqslant 0\}$ *be a Lévy process in* $\mathbb{R}^d$ *determined by* (3.1) *and let decomposition* (3.2) *be given. Suppose that condition* (2.5) *holds and there exists a family of non-singular matrices* $\{A_\epsilon\}_{\epsilon\in(0,1]}$ *of rank d such that*

$$A_\epsilon^{-1}\Sigma_\epsilon(A_\epsilon^{-1})^\top \to I_d, \qquad \textit{as } \epsilon \to 0. \tag{3.5}$$

Let $\mathbf{W}$, $\mathbf{N}^\epsilon$, *and* $\mathbf{a}_\epsilon$ *be as above. Then for every* $\epsilon \in (0, 1]$ *there exists a cadlag process* $\mathbf{Y}_\epsilon = \{Y_\epsilon(t) : t \geqslant 0\}$ *such that*

$$\mathbf{X} \overset{(d)}{=} A_\epsilon\mathbf{W} + \mathbf{N}^\epsilon + \mathbf{a}_\epsilon + \mathbf{Y}_\epsilon \tag{3.6}$$

in the sense of equality of finite-dimensional distributions and such that, for each $T > 0$,

$$\sup_{t\in[0,T]} \|A_\epsilon^{-1}Y_\epsilon(t)\| \overset{(\mathbb{P})}{\to} 0, \qquad \textit{as } \epsilon \to 0. \tag{3.7}$$

Proof. Consider the polar decomposition

$$A_\epsilon^{-1}\Sigma_\epsilon^{1/2} = C_\epsilon U_\epsilon$$

where C_ϵ is positive definite and U_ϵ is orthogonal matrix; see, for example, Graybill (1983:

Theorem 12.2.22). Let $\epsilon_n \to 0$. There exists a subsequence $\{n_k\}$ such that $C_{\epsilon_{n_k}} \to C$ and $U_{\epsilon_{n_k}} \to U$, for some non-negative definite matrix C and an orthogonal matrix U. Since

$$C_\epsilon^2 = C_\epsilon C_\epsilon^\mathsf{T} = A_\epsilon^{-1}\Sigma_\epsilon(A_\epsilon^{-1})^\mathsf{T} \to I_d,$$

we obtain that $C^2 = I_d$ or $C = I_d$ by the uniqueness of the square root of a matrix. Hence

$$A_{\epsilon_{n_k}}^{-1}\Sigma_{\epsilon_{n_k}}^{1/2} \to U.$$

By Theorem 2.2, $\Sigma_\epsilon^{-1/2}X_\epsilon(1) \overset{(d)}{\to} \mathcal{N}(0, I_d)$. Hence

$$A_{\epsilon_{n_k}}^{-1}X_{\epsilon_{n_k}}(1) = \left(A_{\epsilon_{n_k}}^{-1}\Sigma_{\epsilon_{n_k}}^{1/2}\right)\Sigma_{\epsilon_{n_k}}^{-1/2}X_{\epsilon_{n_k}}(1) \overset{(d)}{\to} \mathcal{N}(0, UU^\mathsf{T}) = N(0, I_d).$$

We have proved that $A_\epsilon^{-1}X_\epsilon(1) \overset{(d)}{\to} \mathcal{N}(0, I_d)$ as $\epsilon \to 0$. By a theorem of Skorokhod (cf. Kallenberg, 2002: Theorem 15.17), there exist Lévy processes $\mathbf{Z}_\epsilon = \{Z_\epsilon(t) : t \geqslant 0\}$ such that $\mathbf{Z}_\epsilon \overset{(d)}{=} A_\epsilon^{-1}\mathbf{X}_\epsilon$ and

$$\sup_{t\in[0,T]} \|Z_\epsilon(t) - W(t)\| \overset{(\mathbb{P})}{\to} 0, \qquad \text{as } \epsilon \to 0, \tag{3.8}$$

for each $T > 0$. Making $\mathbf{W}$ and $\mathbf{N}^\epsilon$ depend on different coordinates of a large enough probability space, we may also assume that $\mathbf{Z}_\epsilon$ and $\mathbf{N}^\epsilon$ are independent. Put

$$\mathbf{Y}_\epsilon = A_\epsilon(\mathbf{Z}_\epsilon - \mathbf{W}).$$

Then

$$\mathbf{X} \overset{(d)}{=} \mathbf{X}_\epsilon + \mathbf{N}^\epsilon + \mathbf{a}_\epsilon \overset{(d)}{=} A_\epsilon\mathbf{Z}_\epsilon + \mathbf{N}^\epsilon + \mathbf{a}_\epsilon = A_\epsilon\mathbf{W}_\epsilon + \mathbf{N}^\epsilon + \mathbf{a}_\epsilon + \mathbf{Y}_\epsilon.$$

This proves (3.6). Expression (3.7) follows from (3.8). □

Under the conditions of Theorem 3.1 we have

$$\mathbf{X} \overset{(d)}{\approx} A_\epsilon\mathbf{W} + \mathbf{N}^\epsilon + \mathbf{a}_\epsilon. \tag{3.9}$$

In order to use (3.9) in practice, for an approximate simulation of $\mathbf{X}$, one needs to have decomposition (3.2) such that condition (2.5) holds, simulation of $\mathbf{N}^\epsilon$ is unproblematic and anasymptotics A_ϵ of $\Sigma_\epsilon^{1/2}$ is explicitely known.

We propose the following strategy. The compound Poisson process $\mathbf{N}^\epsilon$ can be generated from a generalized shot noise series representation of a Lévy process. Namely, let ν be the Lévy measure of $\mathbf{X}$ as in (3.1). By a generalized shot noise series representation of the Lévy process $\mathbf{X}$ on $[0, T]$ we mean the equation

$$X(t) = \sum_{j=1}^{\infty}\left[\mathbf{1}(U_j \leqslant t)H(\gamma_j, \xi_j) - tc_j\right], \qquad \text{almost surely,} \tag{3.10}$$

where the series converges almost surely uniformly in $t \in [0, T]$. Here $\{\gamma_j\}$ is the sequence of arrival times in a Poisson process of rate 1, $\{\xi_j\}$ is a sequence of independent and identically distributed (i.i.d.) random elements taking values in a Borel space S and having the common distribution F, $\{\tau_j\}$ is an i.i.d. sequence of uniform on $[0, T]$ random variables, and the sequences $\{\gamma_j\}$, $\{\xi_j\}$ and $\{\tau_j\}$ are independent of each other. Furthermore, $\{c_j\}$ is a sequence of vectors in $\mathbb{R}^d$ and $H : \mathbb{R}_+ \times S \mapsto \mathbb{R}^d$ is a measurable map such that the function $r \mapsto \|H(r, s)\|$ is non-increasing for each $s \in S$ and

$$\nu(B) = \int_0^\infty F(\{s \in S : H(r, s) \in B\backslash\{0\}\})\mathrm{d}r, \qquad B \in \mathcal{B}(\mathbb{R}^d). \tag{3.11}$$

There are many ways of representing a Lévy process $\mathbf{X}$ in the form (3.10). In fact, condition (3.11) is crucial; if it is satisfied then there are constants c_j such that (3.10) holds (see Rosiński 2001: Theorems 4.1 and 5.1).

To define processes $\mathbf{N}^\epsilon$ we choose a family $\{D_\epsilon\}_{\epsilon\in(0,1]}$ of Borel subsets of $\mathbb{R}_+ \times S$ such that $(Leb \otimes F)(D_\epsilon) < \infty$ and $D_\epsilon \nearrow \mathbb{R}_+ \times S$ as $\epsilon \searrow 0$. Define

$$N^\epsilon(t) := \sum_{\{j\,:\,(\gamma_j,\xi_j)\in D_\epsilon\}} \mathbf{1}(U_j \leqslant t)H(\gamma_j, \xi_j), \qquad t \in [0, T]. \tag{3.12}$$

Since $(Leb \otimes F)(D_\epsilon) < \infty$, this sum has only finitely many terms, thus is well defined. Using Theorem 4.1 in Rosiński (2001), it is easy to check that $\mathbf{N}^\epsilon$ is a compound Poisson process with

$$\mathrm{E}\exp \mathrm{i}\langle y, N^\epsilon(t)\rangle = \exp\left\{t\int_{\mathbb{R}^d}(\mathrm{e}^{\mathrm{i}\langle y,x\rangle} - 1)\nu^\epsilon(\mathrm{d}x)\right\},$$

where

$$\nu^\epsilon(B) = \int_0^\infty F(\{s \in S : H(r, s)\mathbf{1}_{D_\epsilon}(r, s) \in B\backslash\{0\}\})\mathrm{d}r, \qquad B \in \mathcal{B}(\mathbb{R}^d). \tag{3.13}$$

Thus $\nu^\epsilon(\mathbb{R}^d) < \infty$ and $\nu^\epsilon \nearrow \nu$ because D_ϵ are ascending to $\mathbb{R}_+ \times S$ as $\epsilon \searrow 0$. Now we define $\nu_\epsilon := \nu - \nu^\epsilon$. Condition (2.2) requires an extra assumption on D_ϵ following from (3.11) and (3.13), and then one needs to show that Σ_ϵ satisfies (2.5). The preceding section shows how to achieve this in several general cases.

In summary, a choice of the compound Poisson approximation determines ν_ϵ and Σ_ϵ. The above method based on (3.10) and (3.11) provides great flexibility in selecting the most convenient approximation.

3.2. Application to stable processes

The Lévy measure ν of an α-stable Lévy process $\mathbf{X}$ in $\mathbb{R}^d$ has the form

$$\nu(\mathrm{d}r, \mathrm{d}u) = \alpha r^{-\alpha-1}\,\mathrm{d}r\,\lambda(\mathrm{d}u) \tag{3.14}$$

in polar coordinates, where $\alpha \in (0, 2)$ and λ is a finite measure on S^{d-1}. Put $\|\lambda\| = \lambda(S^{d-1})$. Assume that $X(1)$ is not concentrated on a proper hyperplane of $\mathbb{R}^d$, which by Lemma 2.1 means that λ is not concentrated on a proper linear subspace of $\mathbb{R}^d$.

As explained above, the compound Poisson process $\mathbf{N}^\epsilon$ of (3.3) can be generated from generalized shot noise series representations of a stable process. For this purpose we take the well-known LePage representation

$$X(t) = (T\|\lambda\|)^{1/\alpha} \sum_{j=1}^{\infty} \Big(\mathbf{1}(\tau_j \leqslant t)\gamma_j^{-1/\alpha} v_j - tc_j \Big), \qquad t \in [0, T],$$

where $\{\gamma_j\}$, $\{\tau_j\}$ are as in (3.10) and $\xi_j := v_j$ are i.i.d. random vectors taking values in S^{d-1} with the common distribution $F := \lambda/\|\lambda\|$. Clearly this is a representation of the form (3.10) and it is easy to check (3.11). Define

$$N^{\epsilon}(t) = (T\|\lambda\|)^{1/\alpha} \sum_{\{j\,:\,\gamma_j \leqslant \epsilon^{-\alpha} T\|\lambda\|\}} \mathbf{1}(\tau_j \leqslant t)\gamma_j^{-1/\alpha} v_j, \qquad t \in [0, T]. \tag{3.15}$$

Comparing this with (3.12), we see that $D_\epsilon = (0, \epsilon^{-\alpha} T\|\lambda\|] \times S^{d-1}$ and (3.13) gives $\nu^{\epsilon} = \nu_{|\{\|x\| \geqslant \epsilon\}}$. Therefore, $\nu_{\epsilon} = \nu_{|\{\|x\| < \epsilon\}}$, as in (2.18). Theorem 2.5 applies with $\mu(\mathrm{d}r\,|u) = \alpha r^{-\alpha-1}\,\mathrm{d}r$ and

$$\Sigma_\epsilon = \frac{\alpha}{2-\alpha} \epsilon^{2-\alpha} \Lambda, \tag{3.16}$$

where

$$\Lambda = \int_{S^{d-1}} uu^{\mathsf{T}}\, \lambda(\mathrm{d}u). \tag{3.17}$$

Theorem 3.1 yields the following.

Proposition 3.2. *Let* $\mathbf{X}$ *be an* α*-stable Lévy process with characteristic function given by* (3.1) *and Lévy measure as in* (3.14). *Suppose that the support of* λ *is not contained in any proper linear subspace of* $\mathbb{R}^d$ *and* Λ *is as in* (3.17). *Let* $T > 0$ *be fixed and let* $\mathbf{N}^{\epsilon}$ *be defined by* (3.15). *Assume that* $\mathbf{W}$ *is a standard Brownian motion in* $\mathbb{R}^d$ *independent of* $\mathbf{N}^{\epsilon}$, *and* $\mathbf{a}_\epsilon$ *is determined by* (3.4). *Then, for every* $\epsilon \in (0, 1]$, *there exists a cadlag process* $\mathbf{Y}_\epsilon$ *such that, on* $[0, T]$,

$$\mathbf{X} \stackrel{(d)}{=} \mathbf{a}_\epsilon + \epsilon^{1-\alpha/2} \left(\frac{\alpha}{2-\alpha} \right)^{1/2} \Lambda^{1/2} \mathbf{W} + \mathbf{N}^{\epsilon} + \mathbf{Y}_\epsilon \tag{3.18}$$

in the sense of equality of finite-dimensional distributions and such that

$$\epsilon^{\alpha/2-1} \sup_{t \in [0,T]} \|Y_\epsilon(t)\| \stackrel{(\mathbb{P})}{\to} 0, \qquad \text{as } \epsilon \to 0. \tag{3.19}$$

Proof. We apply Theorem 3.1 to $A_\epsilon = \Sigma_\epsilon^{1/2}$, where Σ_ϵ is given by (3.16). We obtain (3.18) and

$$\|Y_\epsilon(t)\| \leqslant \|A_\epsilon\| \|A_\epsilon^{-1} Y_\epsilon(t)\| \leqslant \epsilon^{1-\alpha/2} \left(\frac{\alpha}{2-\alpha} \right)^{1/2} \|\Lambda^{1/2}\| \|A_\epsilon^{-1} Y_\epsilon(t)\|.$$

Therefore (3.7) implies (3.19). □

3.3. Application to tempered stable processes

Recall that the Lévy measure of a tempered α-stable process $\mathbf{X}$ in $\mathbb{R}^d$ is of the form

$$\nu(\mathrm{d}r, \mathrm{d}u) = \alpha r^{-\alpha-1} q(r, u)\mathrm{d}r\,\lambda(\mathrm{d}u) \tag{3.20}$$

in polar coordinates, where $\alpha \in (0, 2)$, λ is a finite measure on S^{d-1}, and $q : (0, \infty) \times S^{d-1} \mapsto (0, \infty)$ is a Borel function such that, for each $u \in S^{d-1}$, $q(\cdot, u)$ is completely monotone with $q(0+, u) = 1$ and $q(\infty, u) = 0$. Such processes are considered in Rosiński (2006) and called *proper tempered α-stable*. As in Section 3.2, assume that λ is not concentrated on a proper linear subspace of $\mathbb{R}^d$. Define a finite measure Q on $\mathbb{R}^d$ by

$$Q(A) := \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(ru) Q(\mathrm{d}r|u)\lambda(\mathrm{d}u),$$

where $\{Q(\cdot|u)\}_{u \in S^{d-1}}$ is a measurable family of probability measures on $\mathbb{R}_+$ determined by $q(r, u) = \int_0^\infty \mathrm{e}^{-rs} Q(\mathrm{d}s|u)$. Notice that $\|\lambda\| := \lambda(S^{d-1}) = Q(\mathbb{R}^d)$ and $Q(\{0\}) = 0$.

The compound Poisson process $\mathbf{N}^\epsilon$ of (3.3) will be generated from a generalized shot noise series representation of a tempered stable process given in Rosiński (2006: Theorem 5.3), which is of the form

$$X(t) = \sum_{j=1}^\infty \left\{ 1(\tau_j \leqslant t) \left[\left(\frac{\gamma_j}{T\|\lambda\|} \right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} \|v_j\|^{-1} \right] \frac{v_j}{\|v_j\|} - t c_j \right\}, \qquad t \in [0, T],$$

where $\{\gamma_j\}$, $\{\tau_j\}$ are as in (3.10) and $\xi_j := (e_j, u_j, v_j)$ are i.i.d. random elements taking values in $\mathbb{R}_+ \times [0, 1] \times \mathbb{R}^d$ with the common distribution $F := \mathcal{E}(1) \otimes \mathcal{U}[0, 1] \otimes Q/\|\lambda\|$. Here $\mathcal{E}(1)$ and $\mathcal{U}[0, 1]$ denote the exponential distribution with parameter 1 and the uniform distribution on [0, 1], respectively. Clearly this is a representation of the form (3.10). Define

$$N^\epsilon(t) = \sum_{\{j\,:\,\gamma_j \leqslant \epsilon^{-\alpha} T\|\lambda\|\}} \mathbf{1}(\tau_j \leqslant t) \left(\left(\frac{\gamma_j}{T\|\lambda\|} \right)^{-1/\alpha} \wedge \mathrm{e}_j u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|}, \qquad t \in [0, T]. \tag{3.21}$$

Comparing this with (3.12), we see that $D_\epsilon = (0, \epsilon^{-\alpha} T\|\lambda\|] \times \mathbb{R}_+ \times [0, 1] \times \mathbb{R}^d$ and (3.13) gives

$$\nu^\epsilon(\mathrm{d}r, \mathrm{d}u) = k^\epsilon(r, u)\mathrm{d}r\lambda(\mathrm{d}u)$$

in polar coordinates, where

$$k^\epsilon(r, u) = \begin{cases} \epsilon^{-\alpha}\alpha \left[r^{-1} q(r, u) - r^{\alpha-1} \int_r^\infty \alpha s^{-\alpha-1} q(s, u)\mathrm{d}s \right], & 0 < r < \epsilon, \\ \alpha r^{-\alpha-1} q(r, u), & r \geqslant \epsilon. \end{cases}$$

Hence $\nu_\epsilon = \nu - \nu^\epsilon$ is of the form

$$\nu_\epsilon(\mathrm{d}r, \mathrm{d}u) = k_\epsilon(r, u)\mathrm{d}r\lambda(\mathrm{d}u),$$

where

$$k_\epsilon(r, u) = \begin{cases} \alpha(r^{-\alpha-1} - \epsilon^{-\alpha} r^{-1}) q(r, u) + \epsilon^{-\alpha} \alpha r^{\alpha-1} \int_r^\infty \alpha s^{-\alpha-1} q(s, u) \mathrm{d}s, & 0 < r < \epsilon, \\ 0, & r \geqslant \epsilon. \end{cases}$$

Notice that ν_ϵ is as in (2.12) but not as in (2.18). We will use Theorem 2.4 to show condition (2.5). We begin with an estimate for $\sigma_\epsilon^2(u)$. Since $q(\cdot, u)$ is decreasing, we obtain

$$\sigma_\epsilon^2(u) = \int_0^\infty r^2 k_\epsilon(r, u) \mathrm{d}r \geqslant \alpha q(\epsilon, u) \int_0^\epsilon (r^{-\alpha+1} - \epsilon^{-\alpha} r) \mathrm{d}r = \frac{\alpha^2}{2(2-\alpha)} \epsilon^{2-\alpha} q(\epsilon, u).$$

Then for $b(\epsilon) := \epsilon^{1-\alpha/2}$ we obtain

$$\liminf_{\epsilon \to 0} \frac{\sigma_\epsilon(u)}{b(\epsilon)} \geqslant \frac{\alpha}{\sqrt{2(2-\alpha)}} q(0+, u) > 0$$

because $q(0+, u) = 1$. Hence condition (2.14) of Theorem 2.4 is satisfied. Since ν_ϵ is concentrated on a ball of radius ϵ we obtain

$$\int_{\|x\| > \kappa b(\epsilon)} \|x\|^2 \nu_\epsilon(\mathrm{d}x) = \int_{\{\|x\| > \kappa \epsilon^{1-\alpha/2}, \|x\| < \epsilon\}} \|x\|^2 \nu(\mathrm{d}x) = 0$$

for sufficiently small ϵ, as the region of integration becomes empty. This proves (2.15), and Theorem 2.4 shows that (2.5) holds.

We have

$$\Sigma_\epsilon = \int_{S^{d-1}} \sigma_\epsilon^2(u) u u^\mathsf{T} \lambda(\mathrm{d}u).$$

It is clear that $\Sigma_\epsilon^{1/2}$ may have no closed form in general. However, we will find its asymptotics, proving that

$$\epsilon^{\alpha-2} \Sigma_\epsilon \to \frac{\alpha}{2-\alpha} \Lambda, \qquad \text{as } \epsilon \to 0, \tag{3.22}$$

where Λ is given by (3.17). To establish (3.22) we will need bounds for $\sigma_\epsilon^2(u)$. First we notice that

$$\sigma_\epsilon^2(u) \leqslant \alpha \int_0^\epsilon r^{-\alpha+1} q(r, u) \mathrm{d}r \leqslant \frac{\alpha}{2-\alpha} \epsilon^{2-\alpha},$$

which yields

$$\epsilon^{\alpha-2} \Sigma_\epsilon = \epsilon^{\alpha-2} \int_{S^{d-1}} \sigma_\epsilon^2(u) u u^\mathsf{T} \lambda(\mathrm{d}u) \leqslant \frac{\alpha}{2-\alpha} \Lambda. \tag{3.23}$$

To obtain a lower bound, we write

$$\sigma_\epsilon^2(u) = \alpha\int_0^\epsilon r^2(r^{-\alpha-1}q(r,u) - \epsilon^{-\alpha}\ell_\epsilon(r,u))\mathrm{d}r$$

$$\geqslant q(\epsilon,u)\frac{\alpha}{2-\alpha}\epsilon^{2-\alpha} - \alpha\epsilon^{-\alpha}\int_0^\epsilon r^2\ell_\epsilon(r,u)\mathrm{d}r \tag{3.24}$$

where

$$\ell_\epsilon(r,u) = r^{-1}q(r,u) - r^{\alpha-1}\int_r^\infty \alpha s^{-\alpha-1}q(s,u)\mathrm{d}s.$$

Then we estimate the last term in (3.24):

$$\alpha\epsilon^{-\alpha}\int_0^\epsilon r^2\ell_\epsilon(r,u)\mathrm{d}r \leqslant \alpha\epsilon^{2-\alpha}\int_0^\epsilon \ell_\epsilon(r,u)\mathrm{d}r$$

$$= \epsilon^{2-\alpha}\int_0^\epsilon \frac{\partial}{\partial r}\left(-r^\alpha\int_r^\infty \alpha s^{-\alpha-1}q(s,u)\mathrm{d}s\right)\mathrm{d}r$$

$$= \epsilon^{2-\alpha}\left[1 - \epsilon^\alpha\int_\epsilon^\infty \alpha s^{-\alpha-1}q(s,u)\mathrm{d}s\right] =: \epsilon^{2-\alpha}t(\epsilon,u).$$

Notice that $0 \leqslant t(\epsilon,u) \leqslant 1$ and $\lim_{\epsilon\to 0}t(\epsilon,u) = 0$. Combining the above estimate with (3.24) gives

$$\sigma_\epsilon^2(u) \geqslant \epsilon^{2-\alpha}\left[q(\epsilon,u)\frac{\alpha}{2-\alpha} - t(\epsilon,u)\right],$$

which together with (3.23) yields

$$\epsilon^{\alpha-2}\Sigma_\epsilon = \epsilon^{\alpha-2}\int_{S^{d-1}}\sigma_\epsilon^2(u)uu^\mathsf{T}\,\lambda(\mathrm{d}u) \geqslant \int_{S^{d-1}}\left[q(\epsilon,u)\frac{\alpha}{2-\alpha} - t(\epsilon,u)\right]uu^\mathsf{T}\,\lambda(\mathrm{d}u).$$

Since $q(0+,u) = 1$, the dominated convergence theorem and (3.23) yield (3.22). Hence

$$A_\epsilon := \epsilon^{1-\alpha/2}\left(\frac{\alpha}{2-\alpha}\right)^{1/2}\Lambda^{1/2}$$

satisfies condition (3.5) of Theorem 3.1. Consequently, A_ϵ can be used in simulation of $\mathbf{X}$ instead of $\Sigma_\epsilon^{1/2}$. Applying Theorem 3.1, we obtain the following.

Theorem 3.3. *Let* $\mathbf{X}$ *be a tempered* α*-stable Lévy process with characteristic function given by* (3.1) *and Lévy measure as in* (3.20). *Suppose that the support of* λ *is not contained in any proper linear subspace of* $\mathbb{R}^d$ *and* Λ *is as in* (3.17). *Let* $T > 0$ *be fixed and let* $\mathbf{N}^\epsilon$ *be defined by* (3.21). *Let* $\mathbf{W}$ *be a standard Brownian motion in* $\mathbb{R}^d$ *independent of* $\mathbf{N}^\epsilon$ *and* $\mathbf{a}_\epsilon$ *be a shift determined by* (3.4). *Then, for every* $\epsilon \in (0, 1]$ *there exists a cadlag process* $\mathbf{Y}_\epsilon$ *such that, on* $[0, T]$,

$$\mathbf{X} \overset{(d)}{=} \mathbf{a}_\epsilon + \epsilon^{1-\alpha/2}\left(\frac{\alpha}{2-\alpha}\right)^{1/2}\Lambda^{1/2}\mathbf{W} + \mathbf{N}^\epsilon + \mathbf{Y}_\epsilon \tag{3.25}$$

in the sense of equality of finite-dimensional distributions and such that

$$\epsilon^{\alpha/2-1} \sup_{t\in[0,T]} \|Y_\epsilon(t)\| \stackrel{(\mathbb{P})}{\to} 0, \qquad \text{as } \epsilon \to 0. \tag{3.26}$$

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