J Geophys (1983) 53:1–15

Journal of Geophysics

Gaussian Beams and Paraxial Ray Approximation in Three-Dimensional Elastic Inhomogeneous Media

V. Červený [1] and I. Pšenčík [2]

[1] Institute of Geophysics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czechoslovakia
[2] Geophysical Institute, Czechoslovakian Academy of Sciences, Boční II, 141 31 Praha 4, Czechoslovakia

Abstract. The elastodynamic Gaussian beams in 3D elastic inhomogeneous media are derived as asymptotic high-frequency one-way solutions of the elastodynamic equation concentrated close to rays of *P* and *S* waves. In this case, the elastodynamic equation is reduced to a parabolic (Schrödinger) equation which further leads to a matrix Riccati equation and the transport equation. Both these equations can be simply solved along the ray, the first numerically and the other analytically. The amplitude profile of the principal displacement component of the elastodynamic Gaussian beams is Gaussian in the plane perpendicular to the ray, with its maximum at the ray. The Gaussian beams are regular along the whole ray, even at caustics. As a limiting case of infinitely broad Gaussian beams, the paraxial ray approximation is obtained. The properties and possible applications of Gaussian beams and paraxial ray approximations in the numerical modelling of seismic wave fields in 3D inhomogeneous media are discussed.

Key words: Seismic waves – 3 D elastodynamic equation – Paraxial ray approximation – Riccati equation – Elastodynamic Gaussian beams

Introduction

To solve various problems of 2D and 3D structural seismology, the high-frequency asymptotic methods, such as the ray method and its modifications, have been found very useful. Numerical procedures and program packages to evaluate rays, travel-times, ray amplitudes and ray synthetic seismograms in rather general 2D media have been developed. These programs, however, have certain limitations regarding their accuracy in singular regions, sensitivity to the fine details of the medium, etc. Moreover, the generalization of the 2D methods to 3D structures is not straightforward, mainly due to difficulties with the time-consuming two point ray tracing.

A powerful generalization of the ray method, based on the procedure of Gaussian beams, was suggested recently. This procedure combines the broad possibilities of the ray method and the accuracy of wave methods. The Gaussian beams represent high-frequency asymptotic solutions of the wave (or elastodynamic) equation, which are concentrated close to rays of *P* or *S* waves. Their properties along the ray are controlled by the parabolic (Schrödinger) equation. The distribution of the amplitude of the principal component of the displacement of the beam in the profile perpendicular to the ray is bell-shaped (Gaussian). The Gaussian beams are regular along the whole ray, even at caustics.

The basic concepts of the solution of the wave equation concentrated close to rays were first suggested by Babich (1968). For other details and references see Babich and Buldyrev (1972), Babich and Kirpichnikova (1979), Červený et at. (1982b). These references consider only the scalar Gaussian beams, which are asymptotic solutions of the scalar wave equation. In seismology, however, we are interested in the solutions of the elastodynamic equation. The high-frequency vectorial solutions of the elastodynamic equation concentrated close to rays of *P* and *S* waves were first investigated from a mathematical point of view by Kirpichnikova (1971), see also Popov (1982). In this paper, we are interested mainly in the seismological aspects of the Gaussian beams and in their relationship to the ray method. Our presentation follows mainly the paper by Červený and Pšenčík (1983) in which the elastodynamic Gaussian beams in two-dimensional elastic inhomogeneous media were investigated. Here the results of that paper are generalized for three-dimensional media. Note that scalar acoustic beams in 3D media were investigated by Babich and Popov (1981).

Gaussian beams can find various applications both in structural seismology and in the investigation of seismic sources. In seismology, as in other branches of physics, the Gaussian beams themselves, as a physical reality, may play an important role. They can, however, also be used to simulate complete wave fields generated by various seismic sources. In the latter case, the complete wave field is evaluated as an integral superposition of Gaussian beams, the relevant integrals being valid asymptotically (for high frequencies). The procedure is as follows: the wave field generated by a source is expanded into Gaussian beams. Each beam is continued along a ray. The properties of Gaussian beams vary along the ray due to diffusion, spreading, and reflection/transmission at interfaces. The final wave field at any point of the medium is then obtained as a superposition of all beams passing in some neighbourhood of the receiver. The procedure based on the

Gaussian beams removes most of the difficulties of the ray method: a) It is uniformly valid everywhere. The singularities at caustics are removed automatically as the Gaussian beams are regular along the whole ray. The integral superposition of beams removes most of the remaining singularities of the wave field associated with interfaces (critical region, etc.). b) The procedure is not so sensitive to the details of the medium as the ray method, mainly due to some frequency-dependent smoothing effect contained in the expansion integrals. c) The procedure does not require the time-consuming two-point ray tracing. As soon as a sufficiently dense system of supporting Gaussian beams covering a region is evaluated, the wave field at any point of this region can be obtained very efficiently, practically without any additional effort.

The procedure based on Gaussian beams may be applied both in the frequency domain and in the time domain. Several approaches can be used to rewrite the final expressions for the wave field from the frequency domain to the time domain. The most efficient of these approaches to evaluating the synthetic seismograms is based on the procedure of wave packets, which propagate along rays. For a detailed description of the evaluation of synthetic seismograms in laterally varying layered structures using the Gaussian beam method see Červený (1983).

The Gaussian beam approach is closely connected with the so-called dynamic ray tracing and the paraxial ray approximation. The paraxial ray approximation offers a possibility to evaluate the travel-time field and ray amplitudes not only directly on the ray (as in the standard ray method) but also in its close neighbourhood. Thus, when we wish to determine the travel-times and ray amplitudes at any point A close to the ray Ω, it is not necessary to seek a new ray which passes through A; the wave field at A can be computed directly from the quantities determined along the ray Ω. The two-point ray tracing is not involved in the procedure. The paraxial ray approximation, however, does not remove the singularities of the ray field, as is done in the Gaussian beam approach. Contrary to the Gaussian beam approach, however, the complete wave field at A is obtained directly, not by expansion integrals. Various modifications of the paraxial ray approximations are possible, see details in Červený (1983). The application of the paraxial ray approximation substantially increases the efficiency of the ray methods. In a 2D medium, it considerably decreases the computing time necessary to evaluate synthetic seismograms. However, it will find applications mainly in 3D media, where the cumbersome and time consuming procedures of two-point ray tracing have prevented the development of effective programs for synthetic seismograms. Program packages based on the paraxial ray approximation are now available to evaluate the synthetic seismograms for 3D media, even if the source is situated outside the profile, see Klimeš (1982a), Červený et al. (1982a). Similarly as in the Gaussian beam approach, the ray synthetic seismograms at any point of a region D on the Earth's surface can be computed easily by the paraxial ray approximation as soon as a system of supporting rays is evaluated which covers the region D at the Earth's surface with a sufficient density.

The above applications of Gaussian beams and paraxial ray approximations show the importance of these approaches. Before they can be fully realized, a detailed study of the behaviour of individual Gaussian beams and paraxial ray approximations must be performed.

In this paper, we study the 3D elastodynamic Gaussian beams and paraxial ray approximations in smooth, infinite 3D inhomogeneous media without interfaces. The behaviour of Gaussian beams and of the paraxial ray approximation in 3D media with curved interfaces will be described elsewhere.

Three-Dimensional Elastodynamic Equation in Ray-Centred Coordinates

Let us consider an isotropic medium described by the Lamé elastic parameters λ and μ and by the density ρ. We assume that λ, μ and ρ are continuous functions of coordinates together with at least the first and the second spatial derivatives.

We shall investigate the solutions of the elastodynamic equation which are concentrated close to a ray of a P or an S wave. For such a purpose it is very useful to introduce a special system of coordinates, which is centred at the specified ray. In the first part of this section, we shall introduce the ray centred coordinate system (s, n, m) and in the second part, we shall express the elastodynamic equation in these coordinates.

Ray-Centred Coordinate System

We consider an arbitrary ray Ω corresponding to a compressional (P) or a shear (S) wave and introduce an orthogonal coordinate system (s, n, m) connected with this ray. The coordinate s measures the arclength along the ray from an arbitrary reference point, n and m form a 2D Cartesian coordinate system in the plane perpendicular to Ω at s, with origin at Ω. In the coordinate system (s, n, m), the equation of the ray Ω is $n=m=0$. The vector basis of the coordinate system is formed by a right-handed system of three unit vectors $\mathbf{t}, \mathbf{e}_n, \mathbf{e}_m$, where $\mathbf{t}$ is the unit tangent to the ray Ω and the vectors $\mathbf{e}_n$ and $\mathbf{e}_m$ are perpendicular to the ray Ω. They are introduced in such a way as to make the ray centred coordinate system orthogonal. It was shown by Popov and Pšenčík (1978a, 1978b) that this condition was satisfied when the vectors $\mathbf{e}_n$ and $\mathbf{e}_m$ were introduced as follows:

$$\mathbf{e}_n=\mathbf{n}\cos\theta-\mathbf{b}\sin\theta, \qquad \mathbf{e}_m=\mathbf{n}\sin\theta+\mathbf{b}\cos\theta, \tag{1}$$

where $\mathbf{n}$ is the unit normal and $\mathbf{b}$ the unit binormal to the ray Ω and $\theta(s)=\int_{s_0}^{s} T(\zeta)\,d\zeta+\theta(s_0)$, T being the torsion of the ray. The integral in the expression for $\theta(s)$ is taken along the ray Ω, s_0 denotes the s-coordinate of a reference point on the ray Ω. The orientation of the mutually perpendicular unit vectors $\mathbf{e}_n, \mathbf{e}_m$ in the plane perpendicular to the ray Ω is controlled by the angle $\theta(s_0)$, which may be chosen arbitrarily.

We shall denote by V_P and V_S the velocities of compressional (P) and shear (S) waves, respectively,

$$V_P(s,n,m)=\left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}, \qquad V_S(s,n,m)=\left(\frac{\mu}{\rho}\right)^{1/2}. \tag{2}$$

Similarly as λ, μ and ρ, the velocities of P and S waves, V_P and V_S, are smooth functions of coordinates s, n, m. In the following equations, we shall often use the velocities of P and S waves directly at the ray Ω, i.e. $V_P(s,0,0)$ and $V_S(s,0,0)$. To simplify the equations, we shall use special symbols for these quantities,

$$\alpha(s)=V_P(s,0,0), \qquad \beta(s)=V_S(s,0,0). \tag{3}$$

A similar notation will be used even for the partial derivatives of V_P and V_S in the direction perpendicular to the ray,

$$\alpha_{,n}(s)=\left[\frac{\partial V_P(s,n,m)}{\partial n}\right]_{n=m=0}, \qquad \beta_{,n}(s)=\left[\frac{\partial V_S(s,n,m)}{\partial n}\right]_{n=m=0}, \tag{4a}$$

$$\alpha_{,nm}(s)=\left[\frac{\partial^2 V_P(s,n,m)}{\partial n\,\partial m}\right]_{n=m=0}, \qquad \beta_{,nm}(s)=\left[\frac{\partial^2 V_S(s,n,m)}{\partial n\,\partial m}\right]_{n=m=0} \tag{4b}$$

(and similarly for $\alpha_{,m}$, $\alpha_{,nn}$, $\alpha_{,mm}$, $\beta_{,m}$, $\beta_{,nn}$, $\beta_{,mm}$).

Thus, all the quantities denoted by symbols with α or β are functions of s only, not of n and m.

In this section and the two following we shall not specify whether the considered ray corresponds to a P wave or an S wave, all the equations in these sections are valid for both waves. The specification will be done only in the fifth section. Therefore, we introduce a special symbol $v(s)$ for the velocity of propagation along the ray Ω. If the ray corresponds to a P wave, we put

$$v=\alpha, \qquad v_{,n}=\alpha_{,n}, \qquad v_{,nm}=\alpha_{,nm}, \quad \text{etc.} \tag{5}$$

Similarly, if the ray Ω corresponds to an S wave, we put

$$v=\beta, \qquad v_{,n}=\beta_{,n}, \qquad v_{,nm}=\beta_{,nm}, \quad \text{etc.} \tag{6}$$

We should also use a special notation for $\rho(s,0,0)$, the density measured directly at the ray Ω. However, we shall use only one symbol ρ for the density throughout the paper, both for $\rho(s,0,0)$ and for $\rho(s,n,m)$. Note that any ρ in this and the next two sections has the meaning of $\rho(s,n,m)$ and any ρ in other sections denotes $\rho(s,0,0)$.

In the ray-centred coordinate system (s,n,m), we can write for the infinitesimal length element dr the expression

$$dr^2=\mathbf{dr}\cdot\mathbf{dr}=h^2\,ds^2+dn^2+dm^2, \tag{7}$$

where h is given by the formula

$$h=1+v^{-1}v_{,n}n+v^{-1}v_{,m}m, \tag{8}$$

see Popov and Pšenčík (1978a, 1978b), Červený and Hron (1980). Equation (7) shows that the ray-centred coordinate system (s,n,m) is orthogonal, with the scale factors

$$h_s=h, \qquad h_n=1, \qquad h_m=1. \tag{9}$$

In the case when the ray Ω is curved, the perpendicular planes to Ω intersect at certain distances from Ω. In other words, the ray centred coordinate system (s,n,m) is not regular at large distances from Ω. In the following, we shall consider only a region along Ω, in which the ray-centred coordinate system (s,n,m) is regular, and we shall call it the regularity region.

Elastodynamic Equation in Ray-Centred Coordinates

Consider a right-handed orthogonal curvilinear coordinate system c_1, c_2, c_3 and denote the corresponding scale factors by h_1, h_2, h_3. (We shall write all indices as subscripts.) The displacement components in a coordinate system c_1, c_2, c_3 are denoted by $u_1(x_i,t)$, $u_2(x_i,t)$, $u_3(x_i,t)$. The elastodynamic equation can be then written in the following form (Aki and Richards, 1980),

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=\frac{1}{h_1h_2h_3}\sum_{p=1}^{3}\sum_{q=1}^{3}\frac{\partial}{\partial c_q}(\tau_{pq}\mathbf{n}_p h_1h_2h_3/h_q). \tag{10}$$

In (10), t is the time and the other symbols have the following meaning:

$$\tau_{pq}=\lambda\delta_{pq}\Theta+2\mu e_{pq}, \qquad \Theta=\sum_{r=1}^{3}e_{rr},$$

$$e_{pq}=\frac{1}{2}\left[\frac{h_p}{h_q}\frac{\partial}{\partial c_q}\left(\frac{u_p}{h_p}\right)+\frac{h_q}{h_p}\frac{\partial}{\partial c_p}\left(\frac{u_q}{h_q}\right)\right]+\frac{\delta_{pq}}{h_q}\sum_{r=1}^{3}\frac{u_r}{h_r}\frac{\partial h_p}{\partial c_r}, \tag{11}$$

$$\mathbf{n}_p=h_p\nabla c_p, \qquad \frac{\partial\mathbf{n}_p}{\partial c_q}=\frac{1}{h_p}\mathbf{n}_q\frac{\partial h_q}{\partial c_p}-\delta_{pq}\sum_{r=1}^{3}\frac{1}{h_r}\mathbf{n}_r\frac{\partial h_p}{\partial c_r}.$$

Here δ_{pq} denotes the Kronecker symbol, $\delta_{pq}=1$ for $p=q$, $\delta_{pq}=0$ for $p\neq q$; $\mathbf{n}_p$ is the unit normal to the surface $c_p=\text{const.}$, τ_{pq} and e_{pq} are the components of the stress and strain tensor, respectively. Body forces are not considered in (10).

To simplify the following equations, we shall use commas between subscripts to denote the derivatives of the vector and tensor components with respect to coordinates c_i and time, e.g.

$$u_{p,tt}=\partial^2u_p/\partial t^2, \qquad u_{p,pq}=\partial^2u_p/\partial c_p\partial c_q, \qquad u_{p,q}=\partial u_p/\partial c_q. \tag{12}$$

Similar notation for derivatives will be also used for the derivatives of scalar quantities, e.g.

$$\lambda_{,q}=\partial\lambda/\partial c_q, \qquad \mu_{,q}=\partial\mu/\partial c_q. \tag{12'}$$

Under the coordinates (c_1, c_2, c_3) we shall understand the ray-centred coordinates (s,n,m) introduced above i.e. $c_1=s$, $c_2=n$, $c_3=m$. It is, however, useful to continue to work with the notation (c_1, c_2, c_3) for a while because of the summation convenience. We shall only specify the scale factors according to (9). Then we can rewrite Eqs. (10) and (11) in a component form,

$$\rho u_{p,tt}=\frac{\lambda}{h_p}\Theta_{,p}+2\mu\sum_{q=1}^{3}\frac{1}{h_q}e_{pq,q}+\frac{1}{h_p}\lambda_{,p}\Theta$$
$$-\frac{2\mu}{hh_p}e_{11}h_{,p}+\frac{1}{h}\sum_{q=1}^{3}\frac{e_{pq}}{h_p}(2\mu hh_p/h_q)_{,q}, \tag{13}$$

with

$$e_{pq}=\frac{1}{2}\left(\frac{1}{h_q}u_{p,q}+\frac{1}{h_p}u_{q,p}\right)-\frac{1}{2h_ph_q}(u_ph_{p,q}+u_qh_{q,p})$$
$$+\frac{1}{h_q}\delta_{pq}\sum_{r=1}^{3}\frac{u_r}{h_r}h_{p,r}. \tag{14}$$

Inserting (14) into (13) and separating the terms that contain the second derivatives of the displacement components from other terms yields

$$\rho u_{p,tt}=\sum_{q=1}^{3}\left(\frac{\lambda+\mu}{h_ph_q}u_{q,pq}+\frac{\mu}{h_q^2}u_{p,qq}\right)+C_p, \tag{15}$$

where C_p is given by the relation

$$C_p=\sum_{q=1}^{3}\left(\sum_{r=1}^{3}D_{pqr}u_{q,r}+E_{pq}u_q\right). \tag{16}$$

In expression (16), we have 27 quantities D_{pqr} and nine quantities E_{pq}. In the following, however, we shall need only five of these quantities. All other quantities D_{pqr} and E_{pq} will vanish in the high-frequency ($\omega\to\infty$) approximation of the elastodynamic equation, as will be shown later. We shall therefore specify here only the five quantities D_{pqr} mentioned:

$$D_{111}=h^{-1}[h^{-1}(\lambda+2\mu)]_{,1}, \qquad D_{231}=D_{321}=0,$$
$$D_{221}=D_{331}=h^{-1}(\mu/h)_{,1}. \tag{17}$$

Now we shall rewrite the elastodynamic Eq. (15) in the ray-centred coordinate system (s, n, m). Remember that $u_1=u_s$ denotes the component of the displacement vector $\mathbf{u}$ in the direction of $\mathbf{t}$ (along the ray), $u_2=u_n$ the component in the direction of $\mathbf{e}_n$ and $u_3=u_m$ the component in the direction of $\mathbf{e}_m$. We obtain

$$\rho u_{s,tt}=h^{-2}(\lambda+2\mu)u_{s,ss}+h^{-1}(\lambda+\mu)(u_{n,sn}+u_{m,sm})$$
$$+\mu(u_{s,nn}+u_{s,mm})+C_s,$$
$$\rho u_{n,tt}=h^{-2}\mu u_{n,ss}+(\lambda+\mu)(h^{-1}u_{s,sn}+u_{n,nn}+u_{m,nm})$$
$$+\mu(u_{n,nn}+u_{n,mm})+C_n,$$
$$\rho u_{m,tt}=h^{-2}\mu u_{m,ss}+(\lambda+\mu)(h^{-1}u_{s,sm}+u_{n,nm}+u_{m,mm})$$
$$+\mu(u_{m,nn}+u_{m,mm})+C_m. \tag{18}$$

Here we have used the subscripts s, n and m to denote the partial derivatives with respect to s, n and m.

The quantities C_s, C_n and C_m correspond to C_1, C_2 and C_3, respectively, given by (16), where both q and r take values $1=s$, $2=n$, $3=m$.

Equations (18) represent a final form of the elastodynamic Eq. (10), written in the ray-centred coordinates (s, n, m). They are still fully equivalent to (10), as no approximation was used to derive them.

Equations (18) may be used for arbitrary time-dependent displacement vector $\mathbf{u}$. In this paper, however, we consider the time-harmonic solutions. We assume that the time-harmonic factor is $\exp(-i\omega t)$, where ω is the angular frequency.

One-way Elastodynamic Equation

We shall study the solutions of the elastodynamic Eqs. (18), corresponding to the wave propagating along the ray Ω *in the direction of increasing s*. For this purpose, it is suitable to make the following substitution

$$u_j(s,n,m,t)=\exp\left\{-i\omega\left(t-\int_0^s v^{-1}(\xi)\,d\xi\right)\right\}U_j(s,n,m). \tag{19}$$

Here $j=s$, n or m. The integral is taken along the ray Ω. The lower limit in the integral may be an arbitrary real-valued constant; without loss to generality we put the constant equal to zero.

As Eq. (19) describes only the solution of the elastodynamic equation corresponding to the wave which propagates along Ω in the direction of increasing s (not in the opposite direction), we shall call it the *one-way solution.*

Inserting (19) into (18) yields

$$-\rho\omega^2U_s=\frac{\lambda+2\mu}{h^2}\left[\left(-\frac{\omega^2}{v^2}+i\omega\left(\frac{1}{v}\right)_{,s}\right)U_s+\frac{2i\omega}{v}U_{s,s}+U_{s,ss}\right]$$
$$+\frac{\lambda+\mu}{h}\left[\frac{i\omega}{v}(U_{n,n}+U_{m,m})+U_{n,sn}+U_{m,sm}\right]$$
$$+\mu(U_{s,nn}+U_{s,mm})$$
$$+\frac{i\omega}{v}(U_sD_{sss}+U_nD_{sns}+U_mD_{sms})+\tilde{C}_s,$$

$$-\rho\omega^2U_n=\frac{\mu}{h^2}\left[\left(-\frac{\omega^2}{v^2}+i\omega\left(\frac{1}{v}\right)_{,s}\right)U_n+\frac{2i\omega}{v}U_{n,s}+U_{n,ss}\right]$$
$$+(\lambda+\mu)\left(\frac{i\omega}{hv}U_{s,n}+\frac{1}{h}U_{s,sn}+U_{n,nn}+U_{m,nm}\right)$$
$$+\mu(U_{n,nn}+U_{n,mm})$$
$$+\frac{i\omega}{v}(U_sD_{nss}+U_nD_{nns})+\tilde{C}_n. \tag{20}$$

The third equation (for U_m) is not written here as it is obtained from the second equation merely by interchanging the indices n and m. The quantities $\tilde{C}_s$, $\tilde{C}_n$ and $\tilde{C}_m$ are again given by (16), where $u_{q,r}$ are replaced by $U_{q,r}$ and u_q by U_q. We must, of course, again consider $1\to s$, $2\to n$, $3\to m$ in the relevant indices.

High-frequency Approximation of One-way Elastodynamic Equation

It is well known that the high-frequency elastic wave field propagates mostly along rays. In this and the next sections, we shall show that the one-way elastodynamic equation reduces to the parabolic equation in the high-frequency approximation. This will give us a possibility to study the high-frequency elastic wave field concentrated close to the ray by the parabolic equation method.

We shall assume that ω is high and keep only the terms containing the highest degrees of ω in Eqs. (20). We shall retain only the terms of the order ω^δ, with $\delta \geqq 1$, and neglect all the terms of the order ω^δ, $\delta<1$.

Before doing that, we shall assume that $n=O(\omega^{-\gamma})$, $m=O(\omega^{-\gamma})$ with $\gamma=\frac{1}{2}$. This assumption expresses the fact that for large ω the investigation of the wave field can be restricted to a "boundary layer" along Ω. It is then suitable to introduce new variables ν, η instead of n, m:

$$\nu=\omega^{1/2} n, \qquad \eta=\omega^{1/2} m. \tag{21}$$

Note that $\nu=O(1)$, $\eta=O(1)$ with respect to the frequency $\omega\to\infty$. Substituting (21) into (20), dividing the equations by ρ and neglecting the terms with lower powers of ω (i.e., with $\omega^{1/2}$, ω^0, $\omega^{-1/2}$, ...), we obtain

$$\begin{aligned}
-\omega^2 U_s=&\frac{\lambda+2\mu}{\rho h^2}\left[\left(-\frac{\omega^2}{v^2}+i\omega\left(\frac{1}{v}\right)_{,s}\right)U_s+\frac{2i\omega}{v}U_{s,s}\right]\\
&+\frac{i\omega^{3/2}}{\rho h v}(\lambda+\mu)(U_{n,\nu}+U_{m,\eta})+\frac{\mu\omega}{\rho}(U_{s,\nu\nu}+U_{s,\eta\eta})\\
&+\frac{i\omega}{\rho v}(U_s D_{sss}+U_n D_{sns}+U_m D_{sms}),\\
-\omega^2 U_n=&\frac{\mu}{\rho h^2}\left[\left(-\frac{\omega^2}{v^2}+i\omega\left(\frac{1}{v}\right)_{,s}\right)U_n+\frac{2i\omega}{v}U_{n,s}\right]\\
&+\frac{\lambda+\mu}{\rho}\frac{i\omega^{3/2}}{hv}U_{s,\nu}+\frac{\lambda+\mu}{\rho}\omega(U_{n,\nu\nu}+U_{m,\nu\eta})\\
&+\frac{\mu}{\rho}\omega(U_{n,\nu\nu}+U_{n,\eta\eta})+\frac{i\omega}{\rho v}(U_s D_{nss}+U_n D_{nns}).
\end{aligned} \tag{22}$$

The third equation for U_m is again obtained from the second equation by interchanging the indices n with m and ν with η. It would be simple to show that the expressions $\tilde{C}_s$, $\tilde{C}_n$ and $\tilde{C}_m$ are of the order $\omega^{1/2}$ and less so that they do not contribute to (22).

We shall now expand U_i into asymptotic series in inverse powers of $\omega^{1/2}$,

$$U_i=U_i^0+\omega^{-1/2}U_i^1+\omega^{-1}U_i^2+\dots, \tag{23}$$

for $i=s$, n or m. Inserting (23) into (22) and keeping only the terms which contain the factors ω^2, $\omega^{3/2}$ and ω^1, we get

$$\begin{aligned}
-\omega^2&\left(\frac{\lambda+2\mu}{\rho h^2 v^2}-1\right)(U_s^0+\omega^{-1/2}U_s^1+\omega^{-1}U_s^2)+\frac{i\omega^{3/2}(\lambda+\mu)}{hv\rho}\\
&\cdot[U_{n,\nu}^0+U_{m,\eta}^0+\omega^{-1/2}(U_{n,\nu}^1+U_{m,\eta}^1)]\\
&+\omega\left\{i\frac{\lambda+2\mu}{\rho h^2}\left[\left(\frac{1}{v}\right)_{,s}U_s^0+\frac{2}{v}U_{s,s}^0\right]\right.\\
&+\frac{\mu}{\rho}(U_{s,\nu\nu}^0+U_{s,\eta\eta}^0)\\
&\left.+\frac{i}{v\rho}(U_s^0 D_{sss}+U_n^0 D_{sns}+U_m^0 D_{sms})\right\}=0,\\
-\omega^2&\left(\frac{\mu}{\rho h^2 v^2}-1\right)(U_n^0+\omega^{-1/2}U_n^1+\omega^{-1}U_n^2)\\
&+\frac{i\omega^{3/2}(\lambda+\mu)}{hv\rho}(U_{s,\nu}^0+\omega^{-1/2}U_{s,\nu}^1)\\
&+\omega\left\{i\frac{\mu}{\rho h^2}\left[\left(\frac{1}{v}\right)_{,s}U_n^0+\frac{2}{v}U_{n,s}^0\right]\right.\\
&+\frac{\lambda+\mu}{\rho}(U_{n,\nu\nu}^0+U_{m,\nu\eta}^0)+\frac{\mu}{\rho}(U_{n,\nu\nu}^0+U_{n,\eta\eta}^0)\\
&\left.+\frac{i}{v\rho}(U_s^0 D_{nss}+U_n^0 D_{nns})\right\}=0.
\end{aligned} \tag{24}$$

The third equation can be again obtained from the second equation by interchanging the indices n with m and ν with η.

This is the final form of the high-frequency approximation of the one-way elastodynamic equation, valid for solutions concentrated close to the rays of both P wave and S wave. Let us note that all the functions λ, μ, ρ and h in (24) depend generally an all the three coordinates s, n, m. To simplify further the above equations, we must consider the cases of P and S waves separately.

Parabolic Equations

It is now easy to specify the asymptotic high-frequency elastodynamic equations separately for the rays of P waves and S waves. In this section, we shall derive these equations and show that they correspond to the well-known parabolic equations.

For simplicity, we shall use the term "component of the displacement vector" not only for u_s, u_n and u_m, but also for the quantities U_s, U_n, U_m, see (19).

Parabolic Equation for a P Wave

In the zero-order approximation of the ray method, the only non-vanishing component of the displacement vector of a P wave has a direction tangent to the ray. The components perpendicular to the ray vanish. From this we can expect that in our case also the component along the ray Ω will be more important. Thus we shall call the component U_s *the principal component* and the components U_n and U_m *the additional components.*

To find the components U_s, U_n and U_m, we shall specify Eqs. (24) for case that the ray Ω corresponds to a P wave, i.e. for $v=\alpha$. Since the coefficients in (24) depend generally on n and m, we shall expand them in terms of n and m. As $n=\omega^{-1/2}\nu$, $m=\omega^{-1/2}\eta$, these expansions automatically represent the expansions in $\omega^{-1/2}$. In all cases, we shall again keep in (24) only the terms with powers of ω higher or equal to ω^1.

First let us expand the most important coefficient in (24), $(\lambda+2\mu)/(\rho h^2\alpha^2)-1$. We can write

$$\frac{\lambda+2\mu}{\rho h^2\alpha^2}-1=\frac{V_P^2(s,n,m)}{h^2\alpha^2}-1,$$

where $V_P(s, n, m)$ is given by (2). Expanding V_P in powers of n and m, we get

$$V_P(s,n,m)\sim\alpha(s)+\alpha_{,n}n+\alpha_{,m}m+\tfrac{1}{2}(n^2\alpha_{,nn}+2nm\alpha_{,nm}+m^2\alpha_{,mm}).$$

This can be rewritten in a more convenient matrix form

$$V_P(s,n,m) \sim \alpha(s) + \alpha_{,n} n + \alpha_{,m} m + \tfrac{1}{2} q^T \mathbf{V}^\alpha q, \tag{25}$$

where

$$\mathbf{V}^\alpha = \begin{pmatrix} \alpha_{,nn} & \alpha_{,nm} \\ \alpha_{,nm} & \alpha_{,mm} \end{pmatrix}, \quad q = \begin{pmatrix} n \\ m \end{pmatrix}, \quad q^T = (n \;\; m). \tag{26}$$

Thus, $\mathbf{V}^\alpha$ is a 2×2 real-valued symmetric matrix of second derivatives of the velocity of compressional waves in the plane perpendicular to the ray Ω. From (25) we easily get

$$V_P(s,n,m) \sim \alpha h \left(1 + \frac{1}{2\alpha h} q^T \mathbf{V}^\alpha q\right),$$

where h is given by (8). This gives

$$\frac{\lambda + 2\mu}{\rho h^2 \alpha^2} - 1 \sim \frac{1}{\alpha h} q^T \mathbf{V}^\alpha q. \tag{27}$$

We shall now use the coordinates v and η instead of n and m in (27). Let us denote

$$g = \omega^{1/2} q = \begin{pmatrix} v \\ \eta \end{pmatrix}, \quad g^T = \omega^{1/2} q^T = (v \;\; \eta). \tag{28}$$

Then we get finally

$$\frac{\lambda + 2\mu}{\rho h^2 \alpha^2} - 1 \sim \frac{1}{\alpha h \omega} g^T \mathbf{V}^\alpha g. \tag{29}$$

Thus, we can see that the first term in the first equation of (24) is not of the order ω^2, but of the order ω. The situation is different for the first term in the second equation of (24). Similarly we get

$$\frac{\mu}{\rho h^2 \alpha^2} - 1 = \frac{\beta^2 - \alpha^2}{\alpha^2} - \omega^{-1/2} H_1^\alpha + \omega^{-1} H_2^\alpha + \dots \tag{30}$$

where

$$H_1^\alpha = \frac{2\beta^2}{\alpha^2} \left[\left(\frac{\alpha_{,n}}{\alpha} - \frac{\beta_{,n}}{\beta} \right) v + \left(\frac{\alpha_{,m}}{\alpha} - \frac{\beta_{,m}}{\beta} \right) \eta \right]. \tag{31}$$

The expressions $\alpha_{,n}$, $\beta_{,n}$, $\alpha_{,m}$ and $\beta_{,m}$ are given by (4a). The expression for H_2^α is not given here as we shall not need it.

Now we insert (29) and (30) into (24) and keep only the terms of the order ω^γ, $\gamma \geqq 1$. After some simple manipulations, we obtain

$$\begin{aligned} &\frac{i\omega^{3/2}}{\alpha} (U_{n,v}^0 + U_{m,\eta}^0) + \frac{\omega}{\alpha^2 - \beta^2} \Big\{ U_s^0 \left[i\rho^{-1} (\rho\alpha)_{,s} - \frac{1}{\alpha} g^T \mathbf{V}^\alpha g \right] \\ &\quad + 2i\alpha U_{s,s}^0 + \beta^2 (U_{s,vv}^0 + U_{s,\eta\eta}^0) \\ &\quad + \frac{i}{\alpha} (\alpha^2 - \beta^2)(U_{n,v}^1 + U_{m,\eta}^1) \\ &\quad + \frac{i}{\alpha\rho} (U_n^0 D_{sns} + U_m^0 D_{sms}) \Big\} = 0, \\ &\omega^2 \frac{\alpha^2 - \beta^2}{\alpha^2} U_n^0 + \omega^{3/2} \left\{ \frac{\alpha^2 - \beta^2}{\alpha} \left(\frac{1}{\alpha} U_n^1 + i U_{s,v}^0 \right) + H_1^\alpha U_n^0 \right\} = 0, \\ &\omega^2 \frac{\alpha^2 - \beta^2}{\alpha^2} U_m^0 \\ &\quad + \omega^{3/2} \left\{ \frac{\alpha^2 - \beta^2}{\alpha} \left(\frac{1}{\alpha} U_m^1 + i U_{s,\eta}^0 \right) + H_1^\alpha U_m^0 \right\} = 0. \end{aligned} \tag{32}$$

In the second and the third equation of (32), we did not write the last terms (with ω), as we shall not need them in the following. Note that the first equation was multiplied by $\rho h/(\lambda + \mu)$. In the second terms of all the three equations, only the following approximations were used:

$$\begin{aligned} &(\lambda + 2\mu)/\rho \sim \alpha^2, \quad \mu/\rho \sim \beta^2, \quad \rho(s,n,m) \sim \rho(s,0,0), \\ &h \sim 1, \quad h_{,s} \sim 0. \end{aligned} \tag{33}$$

They correspond to the first terms of Taylor expansions of the relevant quantities in powers of $1/\omega^{1/2}$. Thus, the density ρ in (32) does not depend on the coordinates n, m; $\rho = \rho(s,0,0)$, see the discussion above. The corresponding approximation for the quantity $D_{sss} (= D_{111})$ was $(\rho\alpha^2)_{,s}$, see (17). The quantities D_{sns} and D_{sms} are not specified in (32), as they will not contribute to the final results.

From the last two equations in (32) we simply obtain

$$\begin{aligned} &U_n^0 = 0, \quad U_n^1 = -i\alpha U_{s,v}^0, \\ &U_m^0 = 0, \quad U_m^1 = -i\alpha U_{s,\eta}^0. \end{aligned} \tag{34}$$

Thus, the first terms of the asymptotic series for both the additional components (perpendicular to Ω) vanish, and the leading terms are U_n^1 and U_m^1. The additional components are given by relations, see (23) and (34),

$$U_n = -i\alpha\omega^{-1/2} U_{s,v}^0, \quad U_m = -i\alpha\omega^{-1/2} U_{s,\eta}^0. \tag{35}$$

Now we return to the first equation of (32). From the first term we immediately obtain $U_{n,v}^0 + U_{m,\eta}^0 = 0$. This relation, however, does not give us anything new, it follows immediately from (34). More important is the second term, with ω. Multiplying the term by α^{-2}, inserting $U_n^0 = 0$, $U_m^0 = 0$ and $U_{n,v}^1 + U_{m,\eta}^1 = -i\alpha(U_{s,vv}^0 + U_{s,\eta\eta}^0)$, see (34), we obtain

$$\begin{aligned} &\frac{2i}{\alpha} U_{s,s}^0 + U_{s,vv}^0 + U_{s,\eta\eta}^0 \\ &\quad + U_s^0 \left\{ i\alpha^{-1} [\ln(\alpha\rho)]_{,s} - \frac{1}{\alpha^3} g^T \mathbf{V}^\alpha g \right\} = 0. \end{aligned} \tag{36}$$

This is the parabolic equation for U_s^0 we have sought. The Eq. (36) can be further simplified by the substitution

$$U_s^0(s,v,\eta) = [\alpha(s)\rho(s)]^{-1/2} W^\alpha(s,v,\eta). \tag{37}$$

Inserting (37) into (36) we obtain the final form of the *parabolic equation for P waves*,

$$\frac{2i}{\alpha} W_{,s}^\alpha + W_{,vv}^\alpha + W_{,\eta\eta}^\alpha - \frac{1}{\alpha^3} (g^T \mathbf{V}^\alpha g) W^\alpha = 0. \tag{38}$$

In the conclusion of this section we shall write the final expression for the vectorial time-harmonic solution of the three-dimensional elastodynamic equation, concentrated close to a ray of a P wave. It reads

$$\mathbf{u}(s,\nu,\eta,t)=u_s(s,\nu,\eta,t)\,\mathbf{t}+u_n(s,\nu,\eta,t)\,\mathbf{e}_n + u_m(s,\nu,\eta,t)\,\mathbf{e}_m, \tag{39}$$

where

$$u_s(s,\nu,\eta,t)=\frac{1}{\sqrt{\alpha(s)\rho(s)}}\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\alpha(\zeta)}\right]\right\}W^\alpha(s,\nu,\eta),$$

$$u_n(s,\nu,\eta,t) = -i\omega^{-1/2}\sqrt{\frac{\alpha(s)}{\rho(s)}}\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\alpha(\zeta)}\right]\right\}\frac{\partial W^\alpha(s,\nu,\eta)}{\partial \nu},$$

$$u_m(s,\nu,\eta,t) = -i\omega^{-1/2}\sqrt{\frac{\alpha(s)}{\rho(s)}}\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\alpha(\zeta)}\right]\right\}\frac{\partial W^\alpha(s,\nu,\eta)}{\partial \eta}, \tag{40}$$

and where $W^\alpha(s,\nu,\eta)$ is a solution of the parabolic Eq. (38). Note that the function W^α depends on frequency ω, as $\nu=\omega^{1/2}n$, $\eta=\omega^{1/2}m$.

Parabolic Equation for an S Wave

In case of vectorial solutions of the 3D elastodynamic equation concentrated close to a ray Ω of an S wave, the components of the displacement vector U_n and U_m will be referred to as the principal components, the component U_s as the additional component. Thus, the principal components are perpendicular to the ray, and the additional component is parallel to the ray.

As the derivation of all the equations is now very similar to that above, we shall be brief. We put $v=\beta$ and get approximately

$$\frac{\mu}{\rho h^2\beta^2}-1\sim\frac{1}{\beta h\omega}g^T\mathbf{V}^\beta g, \tag{41}$$

$$\frac{\lambda+2\mu}{\rho h^2\beta^2}-1\sim\frac{\alpha^2-\beta^2}{\beta^2}+\omega^{-1/2}H_1^\beta+\omega^{-1}H_2^\beta, \tag{42}$$

where $H_1^\beta=(\alpha/\beta)^4H_1^\alpha$. The expression for H_2^β is not presented here as it does not influence the final results. The symbol $\mathbf{V}^\beta$ denotes the 2×2 real-valued matrix of the second derivatives of the velocities of shear waves in the plane perpendicular to the ray Ω

$$\mathbf{V}^\beta=\begin{pmatrix}\beta_{,nn} & \beta_{,nm}\\ \beta_{,nm} & \beta_{,mm}\end{pmatrix}, \tag{43}$$

see (4b).

Inserting (41) and (42) into (24) and using similar approximations as above we obtain

$$-\omega^2\frac{\alpha^2-\beta^2}{\beta^2}U_s^0 +\omega^{3/2}\left\{\frac{\alpha^2-\beta^2}{\beta}\left[i(U_{n,\nu}^0+U_{m,\eta}^0)-\frac{1}{\beta}U_s^1\right]-H_1^\beta U_s^0\right\}=0,$$

$$\frac{i\omega^{3/2}}{\beta}U_{s,\nu}^0+\frac{\omega}{\alpha^2-\beta^2}\left\{U_n^0\left[\frac{i}{\rho}(\beta\rho)_{,s}-\frac{1}{\beta}g^T\mathbf{V}^\beta g\right]+2i\beta U_{n,s}^0 +(\alpha^2-\beta^2)(U_{n,\nu\nu}^0+U_{m,\eta\nu}^0)+\beta^2(U_{n,\nu\nu}^0+U_{n,\eta\eta}^0) +\frac{i}{\beta\rho}U_s^0D_{nss}+\frac{i}{\beta}U_{s,\nu}^1(\alpha^2-\beta^2)\right\}=0,$$

$$\frac{i\omega^{3/2}}{\beta}U_{s,\eta}^0+\frac{\omega}{\alpha^2-\beta^2}\left\{U_m^0\left[\frac{i}{\rho}(\beta\rho)_{,s}-\frac{1}{\beta}g^T\mathbf{V}^\beta g\right] +2i\beta U_{m,s}^0+(\alpha^2-\beta^2)(U_{n,\eta\nu}^0+U_{m,\eta\eta}^0) +\beta^2(U_{m,\nu\nu}^0+U_{m,\eta\eta}^0)+\frac{i}{\beta\rho}U_s^0D_{mss} +\frac{i}{\beta}U_{s,\eta}^1(\alpha^2-\beta^2)\right\}=0. \tag{44}$$

It immediately follows from the first equation in (44)

$$U_s^0=0,\qquad U_s^1=i\beta(U_{n,\nu}^0+U_{m,\eta}^0). \tag{45}$$

Likewise, from the second and third equation we obtain identical equations for U_n^0 and U_m^0,

$$U_n^0(s,\nu,\eta)=\frac{1}{\sqrt{\beta(s)\rho(s)}}W_n^\beta(s,\nu,\eta), \tag{46}$$

$$U_m^0(s,\nu,\eta)=\frac{1}{\sqrt{\beta(s)\rho(s)}}W_m^\beta(s,\nu,\eta),$$

where $W_n^\beta(s,\nu,\eta)$ and $W_m^\beta(s,\nu,\eta)$ are solutions of the *parabolic equation for S waves*,

$$\frac{2i}{\beta}W_{,s}^\beta+W_{,\nu\nu}^\beta+W_{,\eta\eta}^\beta-\frac{1}{\beta^3}(g^T\mathbf{V}^\beta g)\,W^\beta=0. \tag{47}$$

The final expressions for the vectorial time-harmonic solutions of the three-dimensional elastodynamic equation concentrated close to a ray Ω of an S wave are again given by (39), where u_s, u_n and u_m are given by equations

$$u_s(s,\nu,\eta,t)=i\omega^{-1/2}\sqrt{\frac{\beta(s)}{\rho(s)}}\exp\left\{-i\omega\left[t-\int_0^s\frac{d\xi}{\beta(\zeta)}\right]\right\} \times\left(\frac{\partial W_n^\beta(s,\nu,\eta)}{\partial\nu}+\frac{\partial W_m^\beta(s,\nu,\eta)}{\partial\eta}\right),$$

$$u_n(s,\nu,\eta,t) =\frac{1}{\sqrt{\beta(s)\rho(s)}}\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\beta(\zeta)}\right]\right\}W_n^\beta(s,\nu,\eta),$$

$$u_m(s,\nu,\eta,t) =\frac{1}{\sqrt{\beta(s)\rho(s)}}\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\beta(\zeta)}\right]\right\}W_m^\beta(s,\nu,\eta), \tag{48}$$

and W_n^β and W_m^β are solutions of (47).

Solutions of the Parabolic Equation

As we have seen, the parabolic equation for both P and S waves can be written in the same form

$$\frac{2i}{v} W_{,s} + W_{,vv} + W_{,\eta\eta} - \frac{1}{v^3}(g^T \mathbf{V} g) W = 0. \tag{49}$$

For P waves, we must insert $v = \alpha$, $\mathbf{V} = \mathbf{V}^\alpha$ and $W = W^\alpha$. Similarly, for S waves, we put $v = \beta$, $\mathbf{V} = \mathbf{V}^\beta$ and $W = W^\beta$. Thus, we investigate the solution of the parabolic equation for both P and S waves together.

We shall seek the solution of (49) in the following form

$$W(s, v, \eta) = A(s) \exp\left[\frac{i}{2}(g^T \mathbf{M}(s) g)\right], \tag{50}$$

where $\mathbf{M}(s)$ is a 2×2 generally complex-valued symmetric matrix, and $A(s)$ a complex-valued scalar. A similar form of the solution of the parabolic equation is known from quantum mechanics and was also used by Kirpichnikova (1971). We assume that both A and $\mathbf{M}$ depend on s only, not on v and η. For the derivatives of W we simply obtain

$$W_{,s} = \left[\frac{dA}{ds} + \frac{iA}{2}\left(g^T \frac{d\mathbf{M}}{ds} g\right)\right] \exp\left[\frac{i}{2}(g^T \mathbf{M} g)\right],$$

$$W_{,vv} + W_{,\eta\eta} = [iA\,\mathrm{tr}(\mathbf{M}) - A(g^T \mathbf{M}^2 g)] \exp\left[\frac{i}{2}(g^T \mathbf{M} g)\right].$$

Inserting these expressions into (49) yields

$$\left(\frac{2}{v}\frac{dA}{ds} + A\,\mathrm{tr}(\mathbf{M})\right) + iA g^T \left(\frac{1}{v}\frac{d\mathbf{M}}{ds} + \mathbf{M}^2 + \frac{1}{v^3}\mathbf{V}\right) g = 0. \tag{51}$$

The simplest possibility to satisfy Eq. (51) is to put

$$\frac{d}{ds}\mathbf{M} + v\mathbf{M}^2 + \frac{1}{v^2}\mathbf{V} = \mathbf{0}, \tag{52}$$

where $\mathbf{0}$ is a 2×2 null matrix, and

$$\frac{d}{ds} A + \tfrac{1}{2} A v\,\mathrm{tr}(\mathbf{M}) = 0. \tag{53}$$

When $\mathbf{M}$ satisfies Eq. (52) and A Eq. (53), then the function W given by (50) is a solution of the parabolic equation (49). Note that more general solutions of the parabolic equation (49) (higher modes) contain Hermite polynomials. Due to this fact, the corresponding beams are called the Hermite-Gaussian beams. The complete system of linearly independent solutions of (49), i.e. the complete system of Hermite-Gaussian beams is derived and discussed in Klimeš (in press 1982b). Here we shall pay attention only to the Gaussian beams (50), which correspond to the zeroth mode in the above described complete system of solutions.

Equation (52) for $\mathbf{M}$ is a matrix non-linear ordinary differential equation of the first order of the Riccati type. This equation is of basic importance in investigating the solutions concentrated close to rays. Equation (53) can be used to evaluate $A(s)$ as soon as $\mathbf{M}(s)$ is known. Similarly as in the ray method, we shall call it the transport equation. We shall now investigate both Eqs. (52) and (53) in greater detail.

Solution of the Matrix Riccati Equation

It is simple to rewrite the non-linear equation (52) as linear equations. We shall derive several linear forms of (52) in the following.

Let us put

$$\mathbf{M} = v^{-1} \frac{d\mathbf{Q}}{ds} \mathbf{Q}^{-1}, \tag{54}$$

where $\mathbf{Q}$ is a new 2×2 matrix. It will be shown later that this matrix is regular along the whole ray Ω, if the initial conditions for (52) are properly chosen. Taking into account that $\frac{d}{ds}\mathbf{Q}^{-1} = -\mathbf{Q}^{-1}\frac{d\mathbf{Q}}{ds}\mathbf{Q}^{-1}$, we get from (52),

$$v \frac{d^2}{ds^2}\mathbf{Q} - v_{,s}\frac{d}{ds}\mathbf{Q} + \mathbf{V}\mathbf{Q} = \mathbf{0}. \tag{55}$$

Thus, we have obtained a matrix linear ordinary differential equation of the second order. It can be rewritten as a system of two matrix linear ordinary differential equations of the first order. We put

$$\mathbf{P} = v^{-1} \frac{d}{ds}\mathbf{Q}. \tag{56}$$

Then we get from (55) and (56) the system

$$\frac{d}{ds}\mathbf{Q} = v\mathbf{P}, \quad \frac{d}{ds}\mathbf{P} = -v^{-2}\mathbf{V}\mathbf{Q}. \tag{57}$$

When we determine $\mathbf{P}$ and $\mathbf{Q}$ from (57), the matrix $\mathbf{M}$ is given by the relation, see (56) and (54),

$$\mathbf{M} = \mathbf{P}\mathbf{Q}^{-1}. \tag{58}$$

The system (57) seems to be most effective form of the equations which can be used to find $\mathbf{M}$. It has also some other advantages, as we shall see later.

Let us now denote by $\mathbf{X}$ a 2×4 matrix

$$\mathbf{X} = \begin{pmatrix} \mathbf{Q} \\ \mathbf{P} \end{pmatrix}. \tag{59}$$

Then the system (57) can be rewritten in the form

$$\frac{d}{ds}\mathbf{X} = \mathbf{H}\mathbf{X}, \tag{60}$$

where $\mathbf{H}$ is a 4×4 real-valued matrix, given by the relation,

$$\mathbf{H} = \begin{pmatrix} \mathbf{0} & v\mathbf{I} \\ -v^{-2}\mathbf{V} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & v & 0 \\ 0 & 0 & 0 & v \\ -v^{-2}V_{11} & -v^{-2}V_{12} & 0 & 0 \\ -v^{-2}V_{12} & -v^{-2}V_{22} & 0 & 0 \end{pmatrix}. \tag{61}$$

Here $\mathbf{0}$ is again a 2×2 null matrix and $\mathbf{I}$ the 2×2 identity matrix. The solutions of the above system of equations (60) will be discussed in greater detail later.

The ordinary differential equations derived above may be simplified if we introduce a new variable σ along the central ray Ω instead of s;

$$d\sigma = v\,ds, \quad \text{i.e.} \quad \sigma = \sigma_0 + \int_{s_0}^{s} v(\zeta)\,d\zeta. \tag{62}$$

Then Eq. (55) takes the form of a matrix one-dimensional Helmholtz equation

$$\frac{d^2}{d\sigma^2}\mathbf{Q} + v^{-3}\mathbf{V}\mathbf{Q} = 0. \tag{63}$$

Similarly, (57) yields

$$\frac{d}{d\sigma}\mathbf{Q} = \mathbf{P}, \quad \frac{d}{d\sigma}\mathbf{P} = -v^{-3}\mathbf{V}\mathbf{Q}. \tag{64}$$

Finally, (60) gives

$$\frac{d}{d\sigma}\mathbf{X} = \tilde{\mathbf{H}}\mathbf{X}, \quad \tilde{\mathbf{H}} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -v^{-3}\mathbf{V} & \mathbf{0} \end{pmatrix}. \tag{65}$$

The new coordinate σ has no simple physical meaning, but it may be useful to use it instead of s in numerical computations. In the following, however, we shall use only the differential equations with the variable s, as they have a straightforward physical meaning.

Equations (52), (55) and (57) have been well known for some time in the ray method, especially in the evaluation of geometrical spreading. For details see Popov and Pšenčík (1978a, 1978b), Červený and Hron (1980), Červený et al. (1977), Hubral (1979, 1980), etc. In the ray method we consider **M, Q, P** to be real-valued, but here we allow **M, Q, P** to be complex-valued.

Transport Equation

Using (54), we can easily prove the relation

$$\mathrm{tr}(\mathbf{M}) = v^{-1}\frac{d}{ds}[\ln(\det\mathbf{Q})]. \tag{66}$$

Then the transport equation (53) takes the form

$$\frac{d}{ds}A + \tfrac{1}{2}A\frac{d}{ds}[\ln(\det\mathbf{Q})] = 0. \tag{67}$$

The transport equation (67) can be solved to give

$$A(s) = \frac{\Psi}{(\det\mathbf{Q}(s))^{1/2}}, \tag{68}$$

where Ψ is generally a complex-valued constant, independent of s.

Gaussian Beams and Paraxial Approximation

Inserting (68) into (50), we obtain the solution of the parabolic equation in the following form

$$W(s, v, \eta) = \frac{\Psi}{(\det\mathbf{Q}(s))^{1/2}}\exp[\tfrac{1}{2}i(g^T\mathbf{M}(s)\,g)], \tag{69}$$

or, using (58),

$$W(s, v, \eta) = \frac{\Psi}{(\det\mathbf{Q}(s))^{1/2}}\exp\left[\frac{i}{2}g^T\mathbf{P}\mathbf{Q}^{-1}g\right]. \tag{70}$$

Here **M** is a solution of the Riccati equation (52), **Q** and **P** are solutions of Eqs. (57). All the matrices **M, Q, P** are generally complex-valued. Thus we can rewrite (69) in the form

$$W(s, v, \eta) = \frac{\Psi}{(\det\mathbf{Q}(s))^{1/2}} \cdot \exp\left[\frac{i}{2}g^T(\mathrm{Re}\,\mathbf{M}(s))\,g - \tfrac{1}{2}g^T(\mathrm{Im}\,\mathbf{M}(s))\,g\right]. \tag{71}$$

For *complex-valued* **Q**, **P** *and* **M**, the solutions may, under certain conditions, represent Gaussian beams. We shall call these conditions *the existence conditions of Gaussian beams*. There are three existence conditions:

1) The condition of the concentration of the Gaussian beam: $\mathrm{Im}\,\mathbf{M}(s)$ must be a positive definite matrix along the whole ray Ω. When the condition is satisfied, the solution (71) is concentrated close to Ω.

2) The condition of symmetry of the matrix **M**: The matrix $\mathbf{M}(s)$ must be symmetrical along the whole ray, even though both **Q** and **P** may be non-symmetrical.

3) The condition of regularity of the Gaussian beam: The matrix $\mathbf{Q}(s)$ must be regular along the whole ray, i.e.,

$$\det\mathbf{Q}(s) \neq 0. \tag{72}$$

Thus, the matrix $\mathbf{M}(s) = \mathbf{P}\mathbf{Q}^{-1}$ exists and the solution (71) is regular along the whole ray, even at caustics and a point source.

We shall show in the next section that the existence conditions of Gaussian beams are fulfilled along the whole ray Ω as soon as they are satisfied at any point $s = s_0$ of the ray Ω. This will help us to choose properly the initial conditions for the system of differential equations given above.

For *real-valued solutions* **Q**, **P** *and* **M**, we have $\mathrm{Im}\,\mathbf{M}(s) = 0$. In this case we shall call the solutions (69)–(71) the *paraxial ray solutions* or *paraxial ray approximations*. The real-valued matrix $\mathbf{M}(s)$ represents the matrix of the second derivatives of the travel-time field and $(\det\mathbf{Q})^{1/2}$ gives the geometrical spreading. Note that the geometrical spreading vanishes at caustics, so that the solution $W(s, v, \eta)$ is singular there. There is only one condition for the existence of the paraxial ray solution: the real-valued matrix **M** must be symmetrical.

Gaussian Beams

By the term Gaussian beams we shall understand the solutions concentrated close to rays which satisfy the existence conditions of Gaussian beams, specified above. The purpose of this section is to show how to select the initial conditions for (60) or (57) at $s = s_0$ to obtain the Gaussian beams.

Matrix of Initial Parameters of Gaussian Beams

Let us rewrite the 2×4 matrix **X** given by (59) in the following form

$$\mathbf{X}=(\mathbf{X}_1\,\mathbf{X}_2) \tag{73}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are the column 1×4 matrices. Both $\mathbf{X}_1$ and $\mathbf{X}_2$ satisfy the same system of differential equations

$$\frac{d\mathbf{X}_i}{ds}=\mathbf{H}\mathbf{X}_i, \tag{74}$$

where $i=1,2$, see (60). We shall consider, for a while, one of these systems. It has four linearly independent real-valued solutions. Let us denote these solutions by $\boldsymbol{\Pi}_i$ $(i=1,2,3,4)$, where $\boldsymbol{\Pi}_i$ is again a 1×4 column matrix. The fundamental 4×4 real-valued matrix of linearly-independent solutions of (74) can be then introduced as follows

$$\boldsymbol{\Pi}(s)=(\boldsymbol{\Pi}_1\,\boldsymbol{\Pi}_2\,\boldsymbol{\Pi}_3\,\boldsymbol{\Pi}_4). \tag{75}$$

We shall specify the fundamental matrix at a reference point $s=s_0$ of the ray Ω by the relation

$$\boldsymbol{\Pi}(s_0)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}. \tag{76}$$

According to Liouville's theorem (see Kamke, 1959), $\det\boldsymbol{\Pi}(s)=\det\boldsymbol{\Pi}(s_0)$ for arbitrary s, since $\operatorname{tr}\mathbf{H}(s)=0$. Moreover, as all the elements of the matrix $\mathbf{H}$ are continuous functions of s (including the second derivatives of velocity, $v_{,ij}$), the solutions $\mathbf{X}_1$ and $\mathbf{X}_2$ exist for arbitrary s.

Any real or complex solution of (74) can be then expressed as a linear combination of linearly independent solutions $\boldsymbol{\Pi}_1,\boldsymbol{\Pi}_2,\boldsymbol{\Pi}_3,\boldsymbol{\Pi}_4$, multiplied by arbitrary real or complex constants. Thus, we can write

$$\mathbf{X}(s)=\boldsymbol{\Pi}(s)\,\mathbf{C}, \tag{77}$$

where $\mathbf{C}$ is a 2×4 matrix of constants. As we shall consider here the Gaussian beams, we must consider complex-valued constants. We shall call these constants the initial parameters of Gaussian beams and call matrix $\mathbf{C}$ *the matrix of the initial parameters of Gaussian beams.*

We introduce the notation

$$\boldsymbol{\Pi}(s)=\begin{pmatrix}\boldsymbol{\Pi}_{11}&\boldsymbol{\Pi}_{12}\\\boldsymbol{\Pi}_{21}&\boldsymbol{\Pi}_{22}\end{pmatrix},\qquad \mathbf{C}=\begin{pmatrix}\mathbf{C}_1\\\mathbf{C}_2\end{pmatrix}, \tag{78}$$

where $\boldsymbol{\Pi}_{ij}$ and $\mathbf{C}_i$ $(i,j=1,2)$ are 2×2 regular matrices. Then Eq. (77) can be rewritten in the following form

$$\begin{aligned}\mathbf{Q}(s)&=\boldsymbol{\Pi}_{11}\mathbf{C}_1+\boldsymbol{\Pi}_{12}\mathbf{C}_2,\\\mathbf{P}(s)&=\boldsymbol{\Pi}_{21}\mathbf{C}_1+\boldsymbol{\Pi}_{22}\mathbf{C}_2.\end{aligned} \tag{79}$$

Note that both the matrices $\mathbf{P}(s)$ and $\mathbf{Q}(s)$ exist for arbitrary s. For the matrix $\mathbf{M}$ we then obtain

$$\mathbf{M}=\mathbf{P}\mathbf{Q}^{-1}=(\boldsymbol{\Pi}_{21}\mathbf{C}_1+\boldsymbol{\Pi}_{22}\mathbf{C}_2)(\boldsymbol{\Pi}_{11}\mathbf{C}_1+\boldsymbol{\Pi}_{12}\mathbf{C}_2)^{-1}. \tag{80}$$

It is easy to see that the matrix $\mathbf{M}(s)$ is not changed when both $\mathbf{P}$ and $\mathbf{Q}$ are multiplied from the RHS by a non-singular 2×2 constant matrix $\tilde{\mathbf{C}}$. Let us put, e.g., $\tilde{\mathbf{C}}=\mathbf{C}_1^{-1}$. For simplicity, denoting the matrix $\mathbf{C}_2\tilde{\mathbf{C}}$ again by $\mathbf{C}_2$, we get for $\mathbf{M}$

$$\mathbf{M}=(\boldsymbol{\Pi}_{21}\mathbf{I}+\boldsymbol{\Pi}_{22}\mathbf{C}_2)(\boldsymbol{\Pi}_{11}\mathbf{I}+\boldsymbol{\Pi}_{12}\mathbf{C}_2)^{-1}. \tag{81}$$

We shall now give the interpretation of $\mathbf{C}_2$. At $s=s_0$, we have $\boldsymbol{\Pi}_{11}=\boldsymbol{\Pi}_{22}=\mathbf{I}$, $\boldsymbol{\Pi}_{12}=\boldsymbol{\Pi}_{21}=\mathbf{0}$, see (76). Then Eq. (81) gives

$$\mathbf{C}_2=\mathbf{M}(s_0), \tag{82}$$

and the matrix of the initial parameters of Gaussian beams $\mathbf{C}$ is given as follows

$$\mathbf{C}=\begin{pmatrix}\mathbf{I}\\\mathbf{M}(s_0)\end{pmatrix}. \tag{83}$$

We assume that $\mathbf{M}(s_0)$ is a regular symmetrical matrix with a positive-definite imaginary part.

In the same way, when we put $\tilde{\mathbf{C}}=\mathbf{C}_2^{-1}$, we get $\mathbf{C}_1=\mathbf{M}^{-1}(s_0)$, $\mathbf{C}_2=\mathbf{I}$, and

$$\mathbf{C}=\begin{pmatrix}\mathbf{M}^{-1}(s_0)\\\mathbf{I}\end{pmatrix}. \tag{84}$$

It is our choice which matrix of initial parameters $\mathbf{C}$ will be used in the following. In this section, we shall use Eq. (83).

Note that the special choice of the matrix $\mathbf{C}$ does not influence the matrix $\mathbf{M}(s)$, but it changes separately the matrices $\mathbf{Q}(s)$ and $\mathbf{P}(s)$, see (79). We can see from (69) that the solution of the parabolic equation contains only the matrix $\mathbf{M}$ and the factor $\det\mathbf{Q}$. Since $\det(\mathbf{Q}\tilde{\mathbf{C}})=(\det\mathbf{Q})\,(\det\tilde{\mathbf{C}})$, the special choice of the matrix $\mathbf{C}$ introduces only the multiplicative constant factor $(\det\tilde{\mathbf{C}})^{-1/2}$ into the solution (69). Without any loss of generality, the factor $(\det\tilde{\mathbf{C}})^{-1/2}$ may be included into the constant Ψ, see (69). Thus, the special choice of $\mathbf{C}$ does not influence the expressions for Gaussian beams.

As $\mathbf{M}(s_0)$ is a regular symmetrical 2×2 matrix with a positive-definite imaginary part, it is fully specified by three complex-valued constants. In other words, the Gaussian beam is fully specified by three complex-valued initial parameters, i.e. by six real-valued initial parameters. The imaginary parts of the complex-valued parameters are not quite arbitrary, they must form a *positive-definite* matrix $\operatorname{Im}\mathbf{M}(s_0)$.

Invariant Expressions along the Central Ray

It is not difficult to show that certain expressions formed from the matrices $\mathbf{Q}$ and $\mathbf{P}$ do not change along the ray Ω. We shall consider two such expressions, viz., $\mathbf{Q}^*\mathbf{P}-\mathbf{Q}\mathbf{P}^*$ and $\mathbf{Q}^T\mathbf{P}-\mathbf{Q}\mathbf{P}^T$, where the asterisk denotes the Hermitian conjugate and T the transpose.

From (57) we easily get

$$\frac{d}{ds}(\mathbf{Q}^*\mathbf{P}-\mathbf{P}^*\mathbf{Q})=-v^{-2}(\mathbf{Q}^*\mathbf{V}\mathbf{Q}-\mathbf{Q}^*\mathbf{V}^*\mathbf{Q})=\mathbf{0},$$

as $\mathbf{V}$ is a real symmetric matrix, so that $\mathbf{V}=\mathbf{V}^*$. Similarly,

$$\frac{d}{ds}(\mathbf{Q}^T\mathbf{P}-\mathbf{P}^T\mathbf{Q})=\mathbf{0}.$$

We determine the values of both the invariants at $s = s_0$, considering (83). In this way we obtain

$$\mathbf{Q}^* \mathbf{P} - \mathbf{P}^* \mathbf{Q} = 2i \operatorname{Im} \mathbf{M}(s_0) \tag{85}$$

and

$$\mathbf{Q}^T \mathbf{P} - \mathbf{P}^T \mathbf{Q} = \mathbf{0}. \tag{86}$$

Note that the Eqs. (85) and (86) are valid even for the paraxial ray approximation.

Existence Conditions of Gaussian Beams

In this section, we shall prove that the three existence conditions of Gaussian beams presented above are satisfied along the whole Gaussian beam as soon as they are satisfied at one arbitrary point $s = s_0$ of the beam.

The Condition of Regularity

We wish to prove that the matrix $\mathbf{Q}$ is regular, i.e. that $\det \mathbf{Q}(s) \neq 0$ along the whole ray Ω. We assume that $\operatorname{Im} \mathbf{M}(s_0)$ is a positive definite matrix.

If the matrix $\mathbf{Q}$ is singular, then a non-zero vector $\mathbf{b}$ exists, for which

$$\mathbf{Q}\,\mathbf{b} = 0. \tag{87}$$

Multiplying (85) by $\mathbf{b}$ from the RHS and by $\mathbf{b}^*$ from the LHS, we get

$$(\mathbf{Q}\,\mathbf{b})^* \mathbf{P}\,\mathbf{b} - (\mathbf{P}\mathbf{b})^* \mathbf{Q}\,\mathbf{b} = 2i\,\mathbf{b}^* (\operatorname{Im} \mathbf{M}(s_0))\,\mathbf{b}. \tag{88}$$

The LHS of (88) vanishes because of (87). The RHS, however, is positive for any choice of $\mathbf{b}$, since $\operatorname{Im}\mathbf{M}(s_0)$ is positive definite. From this it immediately follows that the non-zero vector $\mathbf{b}$ does not exist and that the matrix $\mathbf{Q}$ is regular.

Note that this condition is not satisfied for the paraxial ray approximation.

The Condition of Symmetry of the Matrix $\mathbf{M}$

We multiply (86) by $(\mathbf{Q}^T)^{-1}$ from the LHS and by $\mathbf{Q}^{-1}$ from RHS. This gives

$$(\mathbf{Q}^T)^{-1} \mathbf{Q}^T \mathbf{P}\mathbf{Q}^{-1} - (\mathbf{Q}^T)^{-1} \mathbf{P}^T \mathbf{Q}\mathbf{Q}^{-1} = \mathbf{0},$$

which yields

$$\mathbf{P}\mathbf{Q}^{-1} - (\mathbf{P}\mathbf{Q}^{-1})^T = \mathbf{0}. \tag{89}$$

From Eq. (89) it immediately follows that the matrix $\mathbf{M} = \mathbf{P}\mathbf{Q}^{-1}$ is symmetrical.

Note that this condition is satisfied even for the paraxial ray approximation.

The Condition of the Concentration of the Gaussian Beam

We multiply (85) by $(\mathbf{Q}^*)^{-1}$ from the LHS and by $\mathbf{Q}^{-1}$ from the RHS. It gives

$$\mathbf{P}\mathbf{Q}^{-1} - (\mathbf{P}\mathbf{Q}^{-1})^* = 2i(\mathbf{Q}^*)^{-1} (\operatorname{Im} \mathbf{M}(s_0))\,\mathbf{Q}^{-1}. \tag{90}$$

As

$$\operatorname{Im} \mathbf{M}(s) = \frac{1}{2i} [\mathbf{P}\mathbf{Q}^{-1} - (\mathbf{P}\mathbf{Q}^{-1})^*],$$

Eq. (90) yields

$$\operatorname{Im} \mathbf{M}(s) = (\mathbf{Q}^*)^{-1} (\operatorname{Im} \mathbf{M}(s_0))\,\mathbf{Q}^{-1}. \tag{91}$$

We shall now show that $\operatorname{Im} \mathbf{M}(s)$ is positive definite for any s. We take into account that $\mathbf{Q}^{-1}$ is regular as the matrix $\mathbf{Q}$ exists for arbitrary s.

As $\operatorname{Im}\mathbf{M}(s_0)$ is positive definite, it can be factorized in the following form: $\operatorname{Im}\mathbf{M}(s_0) = \mathbf{N}_0^T \mathbf{N}_0$. Then we can write

$$\operatorname{Im} \mathbf{M}(s) = (\mathbf{Q}^*)^{-1} \mathbf{N}_0^T \mathbf{N}_0 \mathbf{Q}^{-1} = (\mathbf{N}_0 \mathbf{Q}^{-1})^* (\mathbf{N}_0 \mathbf{Q}^{-1}).$$

It follows from the above equation that $\operatorname{Im}\mathbf{M}(s)$ is positive definite, as the product $\mathbf{A}^*\mathbf{A}$ is positive definite for any regular matrix $\mathbf{A}$.

Note that this condition is not satisfied for the paraxial ray approximation.

Properties of 3D Elastodynamic Gaussian Beams and of the Paraxial Ray Approximation

We shall now discuss the properties of Gaussian beams. As a limiting case of Gaussian beams, we shall also shortly discuss the paraxial ray approximation.

We shall return from the variables (ν, η) to the length variables (n, m). For simplicity, we shall call the Gaussian beam concentrated close to a ray of a P wave the compressional Gaussian beam (or P-Gaussian beam), and the beam concentrated close to a ray of an S wave the shear Gaussian beam (or S-Gaussian beam). In both cases, we can write for the displacement vector $\mathbf{u}(s, n, m, t)$ the expression

$$\mathbf{u}(s, n, m, t) = u_s(s, n, m, t)\,\mathbf{t} + u_n(s, n, m, t)\,\mathbf{e}_n + u_m(s, n, m, t)\,\mathbf{e}_m, \tag{92}$$

see (39). We shall also introduce the notation

$$u_\perp(s, n, m, t) = \begin{pmatrix} u_n(s, n, m, t) \\ u_m(s, n, m, t) \end{pmatrix}. \tag{93}$$

Compressional Gaussian Beams

Inserting the solutions W of the parabolic equation (49) into (40) yields

$$\begin{aligned} u_s(s, n, m, t) &= \frac{\Psi_P}{[\alpha(s)\,\rho(s) \det \mathbf{Q}(s)]^{1/2}} \cdot \exp\left\{ -i\omega \left[t - \int_0^s \frac{d\zeta}{\alpha(\zeta)} \right] + \frac{i\omega}{2} q^T \mathbf{M}(s)\, q \right\}, \\ u_n(s, n, m, t) &= \Psi_P \left[\frac{\alpha(s)}{\rho(s) \det \mathbf{Q}(s)} \right]^{1/2} (M_{11} n + M_{12} m) \cdot \exp\left\{ -i\omega \left[t - \int_0^s \frac{d\zeta}{\alpha(\zeta)} \right] + \frac{i\omega}{2} q^T \mathbf{M}(s)\, q \right\}, \\ u_m(s, n, m, t) &= \Psi_P \left[\frac{\alpha(s)}{\rho(s) \det \mathbf{Q}(s)} \right]^{1/2} (M_{12} n + M_{22} m) \cdot \exp\left\{ -i\omega \left[t - \int_0^s \frac{d\zeta}{\alpha(\zeta)} \right] + \frac{i\omega}{2} q^T \mathbf{M}(s)\, q \right\}. \end{aligned} \tag{94}$$

Here Ψ_P denotes a complex-valued quantity which is constant along the ray Ω. The principal component $u_s\mathbf{t}$ has a direction parallel to the central ray Ω. The additional components $u_n\mathbf{n}$ and $u_m\mathbf{m}$ are perpendicular to the ray Ω; they vanish at the central ray. Their importance increases with the increasing distance from the ray Ω.

Using notation (93), the expressions for the additional components can be written in a more compact form,

$$u_\perp(s,n,m,t)=\Psi_P\left[\frac{\alpha(s)}{\rho(s)\det\mathbf{Q}(s)}\right]^{1/2}\mathbf{M}(s)q \cdot\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\alpha(\zeta)}\right]+\frac{i\omega}{2}q^T\mathbf{M}(s)q\right\}. \tag{95}$$

Shear Gaussian Beams

Inserting (69) into (48) yields

$$u_s(s,n,m,t)=-\left[\frac{\beta(s)}{\rho(s)\det\mathbf{Q}(s)}\right]^{1/2}\cdot[\Psi_n(M_{11}n+M_{12}m)+\Psi_m(M_{12}n+M_{22}m)]\cdot\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\beta(\zeta)}\right]+\frac{i\omega}{2}q^T\mathbf{M}(s)q\right\},$$

$$u_n(s,n,m,t)=\frac{\Psi_n}{[\beta(s)\rho(s)\det\mathbf{Q}(s)]^{1/2}}\cdot\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\beta(\zeta)}\right]+\frac{i\omega}{2}q^T\mathbf{M}(s)q\right\},$$

$$u_m(s,n,m,t)=\frac{\Psi_m}{[\beta(s)\rho(s)\det\mathbf{Q}(s)]^{1/2}}\cdot\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\beta(\zeta)}\right]+\frac{i\omega}{2}q^T\mathbf{M}(s)q\right\}. \tag{96}$$

Here Ψ_n and Ψ_m are some complex-valued quantities which are constant along the ray Ω. Let us denote

$$\Psi=\begin{pmatrix}\Psi_n\\ \Psi_m\end{pmatrix}. \tag{97}$$

Using (97) and (93), Eqs. (96) can be then written in a more compact form

$$u_s(s,n,m,t)=-\left[\frac{\beta(s)}{\rho(s)\det\mathbf{Q}(s)}\right]^{1/2}(\Psi^T\mathbf{M}(s)q)\cdot\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\beta(\zeta)}\right]+\frac{i\omega}{2}q^T\mathbf{M}(s)q\right\},$$

$$u_\perp(s,n,m,t)=\frac{\Psi}{[\beta(s)\rho(s)\det\mathbf{Q}(s)]^{1/2}}\cdot\exp\left\{-i\omega\left[t-\int_0^s\frac{d\zeta}{\beta(\zeta)}\right]+\frac{i\omega}{2}q^T\mathbf{M}(s)q\right\}. \tag{98}$$

The principal components $u_n\mathbf{n}$ and $u_m\mathbf{m}$ have a direction perpendicular to the central ray Ω. The additional component $u_s\mathbf{t}$ is parallel to the ray Ω; it vanishes directly at the central ray. Its importance increases with increasing distance from the ray Ω.

Paraxial Ray Approximation

As shown above, the solutions (92)–(98) include as a limiting case the paraxial ray solution for real-valued $\mathbf{M}$, $\mathbf{Q}$, $\mathbf{P}$. In other words, the paraxial ray solution is obtained if the matrix of the initial parameters $\mathbf{C}$ in (78) is chosen real-valued. In some special cases of practical interest, it is not necessary to evaluate all four linearly independent solutions $\boldsymbol{\Pi}_i$ ($i=1, 2, 3, 4$). For example, for the point source solutions, we can evaluate only $\boldsymbol{\Pi}_3$ and $\boldsymbol{\Pi}_4$, and for the plane source solutions we can evaluate only $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$. The paraxial ray approximation for arbitrary other initial conditions (arbitrarily curved wavefront at $s=s_0$) can be then obtained as a linear combination of these two solutions.

We can easily see from the above equations that they give standard ray solutions at the central ray Ω for real-valued $\mathbf{M}$, $\mathbf{Q}$ and $\mathbf{P}$. The above equations, however, describe the wave field even in some neighbourhood of the central ray Ω (i.e., for $n\neq 0$ and/or $m\neq 0$). To determine the wave field at a point A close to the central ray Ω, it is not necessary to evaluate a new ray which passes through the point A, but we can compute the wave field at A by the paraxial ray approximation corresponding to the ray Ω.

The quantity $[\det\mathbf{Q}(s)]^{1/2}$ is real-valued in the paraxial ray approximation and has the meaning of geometrical spreading, well-known in the ray method, see Červený et al. (1977), Popov and Pšenčík (1978a, b), Červený and Hron (1980). It vanishes at caustics and causes infinite ray amplitudes there. Thus, the paraxial ray approximation has the same singularity at caustics as the standard ray method.

In the paraxial ray approximation, the real-valued matrix $\mathbf{M}(s)$ represents *the matrix of the second derivatives of the travel-time field* and the real-valued matrix $\mathbf{K}(s)=v(s)\mathbf{M}(s)$ is called the *curvature matrix*, see Červený and Hron (1980). The matrix $\mathbf{M}(s)$ fully describes the travel-time field in the vicinity of Ω accurate up to the second order terms in n and m. The matrix $\mathbf{K}(s)$ describes the geometric properties of the wavefront in the same vicinity of the ray Ω. As the matrices $\mathbf{M}(s)$ and $\mathbf{K}(s)$ are symmetric, they have real eigenvalues. For simplicity, we shall consider only the matrix $\mathbf{K}(s)$ in the following. (Similar conclusions regarding the matrix $\mathbf{M}(s)$ are straightforward.) Let us denote the eigenvalues of $\mathbf{K}(s)$ by $K_1(s)$ and $K_2(s)$. They represent the principal curvatures of the wavefront on Ω at s. The principal directions of the curvature of the wavefront are determined by the corresponding eigenvectors, $\mathbf{e}_1^K$ and $\mathbf{e}_2^K$. The three unit vectors $\mathbf{t}$, $\mathbf{e}_1^K$, $\mathbf{e}_2^K$ are mutually orthogonal. Instead of the principal curvatures $K_1(s)$ and $K_2(s)$ we can also introduce the principal radii of the curvature of the wavefront on Ω at s, $R_1=1/K_1$ and $R_2=1/K_2$. The quantities $K_{1,2}$ and $R_{1,2}$ may attain arbitrary real values, including 0 and ∞. For $K_{1,2}\neq 0$ and $R_{1,2}\neq 0$, the wavefront is ellipsoidal (for $K_1K_2>0$) or hyperboloidal (for $K_1K_2<0$) in the vicinity of Ω at s. The quadratic curve $q^T \mathbf{K}(s) q = 1$ is an ellipse or a hyperbola. We shall call it the wavefront ellipse or the wavefront hyperbola. The main axes of the *wavefront ellipse* (or *hyperbola*) are determined by the eigenvectors $\mathbf{e}_1^K$ and $\mathbf{e}_2^K$. Along the wavefront ellipse (hyperbola) the travel time is constant, the travel-time difference with respect to the relevant point s at Ω equals $(2v(s))^{-1}$.

For $K_1(s) = K_2(s) = 0$, the wavefront is a plane, perpendicular to Ω at s. Similarly, for $R_1(s) = R_2(s) = 0$, we get a point source singularity on Ω at s. The 2D cylindrical wavefronts are described by $K_1(s) = 0$ or $K_2(s) = 0$. For $K_1(s) = 0$ and $R_2(s) = 0$ or $K_2(s) = 0$ and $R_1(s) = 0$ we get a line source singularity on Ω at s.

It is not difficult to see that the displacement vector $\mathbf{u}$ of the paraxial ray approximation for the P wave is *linearly polarized*, even for $n, m \neq 0$. It is approximately perpendicular to the wavefront at a given point (s, n, m) in the close vicinity of the central ray Ω. Thus, the additional components u_n and u_m in (94) represent corrections to the principal component u_s which keep the displacement vector $\mathbf{u}$ perpendicular to the wavefront for $n \neq 0$ and/or $m \neq 0$.

The polarization of $\mathbf{u}$ of the paraxial ray approximation for an S wave is, however, more complicated. Let us first consider the situation directly at the central ray Ω $(n = m = 0)$, where the additional component u_s vanishes. In this case the displacement vector $\mathbf{u}$ is generally *elliptically polarized*, the polarization ellipse being perpendicular to Ω, i.e., tangent to the wavefront. The elliptical polarization degenerates into the linear polarization only if the phases of complex-valued constants Ψ_n and Ψ_m differ by $k\pi$, where k is an arbitrary integer (e.g. if both Ψ_n and Ψ_m are real-valued). The parameters of the polarization ellipse can be simply determined by well-known methods, see Born and Wolf (1959), Kravcov and Orlov (1980). The orientation of the polarization ellipse is controlled mainly by the phase difference between Ψ_n and Ψ_m. As this phase difference does not change along a smooth ray (even if it touches a caustic), the polarization ellipse does not change its orientation with respect to $\mathbf{e}_n$ and $\mathbf{e}_m$ along the ray. (It may change its orientation only if the ray impignes supercritically at an interference, but the interfaces are not considered in this paper.)

It is obvious that the vector $\mathbf{u}$ is elliptically polarized even for $n \neq 0$ and/or $m \neq 0$. The polarization ellipse is approximately tangent to the wavefront at the relevant point. The polarization ellipse of the S wave degenerates into linear polarization even for $n \neq 0$ and/or $m \neq 0$, if the phases of Ψ_n and Ψ_m differ by $k\pi$, where k is an integer, e.g. if both Ψ_n and Ψ_m are real-valued, see Eqs. (96).

Thus, we have four significant pairs of unit vectors in the plane perpendicular to the ray Ω at any point of Ω in the paraxial ray approximation. Each of these pairs is formed by two mutually orthogonal unit vectors and determines two specific directions. They are as follows:

1) The unit normal $\mathbf{n}$ and binormal $\mathbf{b}$ to the ray Ω.

2) The unit vectors $\mathbf{e}_n$ and $\mathbf{e}_m$, which form (together with $\mathbf{t}$) the basis of the ray-centred coordinate system (s, n, m). The unit vectors $\mathbf{e}_n$ and $\mathbf{e}_m$ rotate with respect to $\mathbf{n}, \mathbf{b}$ along a 3D ray with a torsion as the wave progresses, see (1). For plane rays (without torsion), the mutual orientation of both systems remains preserved.

3) The eigenvectors of the curvature matrix $\mathbf{e}_1^K$ and $\mathbf{e}_2^K$, which determine the principal curvature directions of the wavefront. They generally rotate both with respect to $\mathbf{e}_n, \mathbf{e}_m$ and with respect to $\mathbf{n}, \mathbf{b}$ as the wave progresses.

4) The directions of the main axes of the polarization ellipse of S waves at Ω. The polarization ellipse does not change its orientation with respect to $\mathbf{e}_n$, $\mathbf{e}_m$ in a smooth medium. It rotates, however, with respect to $\mathbf{n}, \mathbf{b}$, and with respect to $\mathbf{e}_1^K$ and $\mathbf{e}_2^K$ at the same rate as $\mathbf{e}_n$ and $\mathbf{e}_m$ as the wave progresses.

Complex Curvature Matrix of Gaussian Beams

The behaviour of Gaussian beams is controlled by the matrix $\mathbf{M}(s)$. In the case of Gaussian beams, the matrices $\mathbf{M}$, $\mathbf{Q}$ and $\mathbf{P}$ are complex-valued. We shall call $\mathbf{M}$ the *complex matrix of the second derivatives of the travel-time field.* We again introduce a complex-valued 2×2 symmetric matrix $\mathbf{K}(s) = v(s)\,\mathbf{M}(s)$ and call it the *complex curvature matrix.* A similar notation was used by Arnaud and Kogelnik (1969) in the case of light Gaussian beams in optical systems.

Both the matrices $\mathbf{M}(s)$ and $\mathbf{K}(s)$ can be formally divided into the real and imaginary parts. For simplicity, we shall use the following notations

$$\mathbf{M}(s) = \mathbf{M}^R(s) + i\mathbf{M}^I(s), \qquad \mathbf{K}(s) = \mathbf{K}^R(s) + i\mathbf{K}^I(s). \tag{99}$$

The interpretation of matrices $\mathbf{M}^R(s)$ and $\mathbf{K}^R(s)$ is quite similar to the interpretation of matrices $\mathbf{M}(s)$ and $\mathbf{K}(s)$ in the paraxial ray approximation. For example, $\mathbf{K}^R(s)$ describes the geometric properties of the wavefront of the Gaussian beam. The eigenvalues K_1^R, K_2^R of the matrix $\mathbf{K}^R$ and the eigenvalues M_1^R, M_2^R of the matrix $\mathbf{M}^R$ are always real. K_1^R and K_2^R determine the principal curvatures of the wavefront of the Gaussian beam on Ω at s. For finite $K_{1,2}^R \neq 0$, the wavefront of the Gaussian beam is ellipsoidal (when $K_1^R K_2^R > 0$) or hyperboloidal (when $K_1^R K_2^R < 0$). The principal directions of the curvature of the wavefront are determined by the corresponding eigenvectors of $\mathbf{K}^R(s)$, $\mathbf{e}_1^R$ and $\mathbf{e}_2^R$. The quadratic curve $q^T \mathbf{K}^R(s) q = 1$ will be called the *phase ellipse* or *phase hyperbola.* All other conclusions regarding the phase front can be obtained in the same way as above.

Now we shall consider the symmetric 2×2 real-valued positive-definite matrices $\mathbf{M}^I(s)$ and $\mathbf{K}^I(s)$. They are of basic importance in the investigation of Gaussian beams. They control the amplitude profile of the Gaussian beam in a cross-section orthogonal to Ω at s. The amplitude profile of the principal component of the displacement vector is controlled by the factor

$$\exp\left\{-\frac{\omega}{2}\, q^T \mathbf{M}^I q\right\} = \exp\left\{-\frac{\omega}{2}\,(n^2 M_{11}^I(s) + 2nm M_{12}^I(s) + m^2 M_{22}^I(s))\right\}. \tag{100}$$

A similar factor can be written in terms of $\mathbf{K}^I(s)$, using $\mathbf{M}^I(s) = v^{-1}(s)\,\mathbf{K}^I(s)$. Thus, we can see that the amplitude profile in any plane section containing the tangent to Ω at s is bell-shaped (Gaussian). This is the reason why these solutions are called Gaussian beams.

The width of the beam depends on the frequency; it decreases with increasing frequency. Note that the bell-shaped Gaussian profile is generally deformed by the additional components of the displacement vector, mainly at larger distances from Ω. For simplicity, we shall speak about Gaussian bell-shaped profiles in the following.

As the matrices $\mathbf{K}^I(s)$ and $\mathbf{M}^I(s)$ are positive definite, they have real-valued positive eigenvalues, K_1^I, K_2^I and M_1^I, M_2^I, where $K_{1,2}^I = vM_{1,2}^I$. The principal directions are determined by the corresponding unit eigenvectors $\mathbf{e}_1^I$ and $\mathbf{e}_2^I$ (which are the same for $\mathbf{K}^I$ and $\mathbf{M}^I$). The three unit vectors $\mathbf{t}, \mathbf{e}_1^I, \mathbf{e}_2^I$ are mutually orthogonal.

We can again use any of the two matrices $\mathbf{K}^I(s)$ and $\mathbf{M}^I(s)$ to investigate the amplitude profiles. We shall call the matrix $\mathbf{M}^I(s)$ the matrix of the amplitude profile of the Gaussian beam, or shortly the amplitude profile matrix. The quadratic curve $q^T\mathbf{K}^I(s)\,q=1$ always represents an ellipse, with axes $2/K_1^I$ and $2/K_2^I$. Following Arnaud and Kogelnik (1969), we shall call it the *spot ellipse*. The amplitude of the principal component of the Gaussian beam is constant along this ellipse and equals $\exp[-\frac{1}{2}\omega v^{-1}(s)]\,A(s)$. We can also introduce the frequency-dependent spot ellipse by the relation $q^T\mathbf{K}^I(s)\,q=2\omega^{-1}v(s)$, or equivalently by $q^T\mathbf{M}^I(s)\,q=2\omega^{-1}$. The amplitude along this frequency-dependent spot ellipse is $e^{-1}A(s)$. The dimensions of the frequency dependent spot ellipse decrease with increasing frequency.

Generally speaking, the Gaussian beam is more concentrated to the ray Ω for larger eigenvalues K_1^I and K_2^I. To describe this property, it may be useful to introduce a new matrix $\mathbf{L}(s)$ by the relation

$$\mathbf{L}(s)=\left[\frac{\omega}{2}\,\mathbf{M}^I(s)\right]^{-1/2}=\left[\frac{\omega}{2v(s)}\,\mathbf{K}^I(s)\right]^{-1/2}. \tag{101}$$

We shall call $\mathbf{L}(s)$ the matrix of the half-width of the Gaussian beam, or shortly *the half-width matrix*. Its eigenvalues L_1 and L_2 (always real and positive) are given by the relation,

$$L_{1,2}(s)=\left[\frac{\omega}{2}\,M_{1,2}^I(s)\right]^{-1/2}=\left[\frac{\omega}{2v(s)}\,K_{1,2}^I(s)\right]^{-1/2}. \tag{102}$$

It is obvious that the quantities $2L_1$ and $2L_2$ measure the length of axes of the frequency dependent spot ellipse. Thus, they determine the minimum and maximum distances from Ω at which the amplitude drops from $A(s)$ on the central ray Ω to $e^{-1}A(s)$. Thus, the Gaussian beam is more concentrated close to Ω for smaller L_1 and L_2.

The matrix $\mathbf{L}$ introduced above and its eigenvalues L_1, L_2 depend on frequency. We can also introduce a *frequency-independent half-width matrix*, namely the half-width matrix for the frequency $f=1$ Hz. We denote it by $\mathbf{L}^0(s)$,

$$\mathbf{L}^0(s)=[\pi\mathbf{M}^I(s)]^{-1/2}=\left[\frac{\pi}{v(s)}\,\mathbf{K}^I(s)\right]^{-1/2}. \tag{103}$$

The matrix $\mathbf{L}^0(s)$ has the eigenvalues $L_1^0(s)$ and $L_2^0(s)$ given by the relation

$$L_{1,2}^0(s)=[\pi M_{1,2}^I(s)]^{-1/2}=\left[\frac{\pi}{v(s)}\,K_{1,2}^I(s)\right]^{-1/2}. \tag{104}$$

The half-widths for the frequency $f=1$ Hz, L_1^0 and L_2^0, complemented by an angle which specifies the orientation of the spot ellipse, can be suitably chosen as the initial parameters of the Gaussian beam to characterize the matrix $\operatorname{Im}\mathbf{M}(s_0)$.

Polarization of Gaussian Beams

The displacement vectors $\mathbf{u}$ of both P and S Gaussian beams are generally elliptically polarized. The only exception is the linear polarization of $\mathbf{u}$ of the P Gaussian beam directly on the central ray. Outside the central ray, however, even $\mathbf{u}$ for the P Gaussian beam is elliptically polarized (in contradiction to the paraxial approximation of P-wave). The parameters of the polarization ellipse depend generally on all the three coordinates (s, n, m). The polarization ellipse of the S Gaussian beam directly on the ray is again perpendicular to the central ray Ω, similarly as in the ray approximation.

Astigmatism of Gaussian Beams

It was shown earlier in this section that four significant pairs of unit vectors perpendicular to the ray Ω could be introduced in the paraxial ray approximation. In case of Gaussian beams, an additional pair of unit vectors perpendicular to Ω plays an important role, namely the eigenvectors $\mathbf{e}_1^I$ and $\mathbf{e}_2^I$. They specify the orientation of the spot ellipse.

Generally, the unit vectors $\mathbf{e}_1^I$ and $\mathbf{e}_2^I$ rotate with respect to $\mathbf{e}_n$, $\mathbf{e}_m$ and with respect to $\mathbf{n}$, $\mathbf{b}$ as the wave progresses. They do not coincide (except for certain limiting situations) with the unit vectors $\mathbf{e}_1^R$ and $\mathbf{e}_2^R$, which specify the orientation of the phase ellipse. Thus, the orientation of the spot ellipse is generally different from the orientation of the phase ellipse. Such Gaussian beams are called *astigmatic Gaussian beams*, see Arnaud and Kogelnik (1969). When the unit vectors $\mathbf{e}_1^R$, $\mathbf{e}_2^R$ and $\mathbf{e}_1^I$, $\mathbf{e}_2^I$ coincide along the whole ray Ω (i.e., the spot ellipse and the phase ellipse keep the same orientation along the whole ray), the beam is called *the Gaussian beam with a simple astigmatism.* When the spot ellipse and the phase ellipse are circular along the whole ray Ω, the Gaussian beam is called *stigmatic (or circular)*.

The stigmatic Gaussian beams and the beams with simple astigmatism are common in homogeneous media. In homogeneous media, we have

$$\mathbf{M}^{-1}(s)=\mathbf{M}^{-1}(s_0)+\mathbf{I}\int_{s_0}^{s} v(\zeta)\,d\zeta.$$

Hence it simply follows that the Gaussian beam remains stigmatic along the whole ray if it is stigmatic at least at one point of the ray. The same is valid for the Gaussian beam with simple astigmatism.

Such Gaussian beams, however, are only of limited significance in general 3D laterally inhomogeneous media. Even if we locally specify the Gaussian beam by the initial conditions as stigmatic, it becomes astigmatic due to inhomogeneities.

References

Aki, K., Richards, P.: Quantitative seismology. San Francisco: Freeman 1980

Arnaud, J.A., Kogelnik, H.: Gaussian light beams with general astigmatism. Applied Optics **8**, 1687-1693, 1969

Babich, V.M.: Eigenfunctions, concentrated in the vicinity of closed geodesics. In: Mathematical problems of theory of propagation of waves, Vol. 9, V.M. Babich, ed.: pp. 15-63. Leningrad: Nauka 1968 (in Russian)

Babich, V.M., Buldyrev, N.J.: Asymptotic methods in problems of diffraction of short waves. Moscow: Nauka 1972

Babich, V.M., Kirpichnikova, N.Y.: The boundary layer method in diffraction problems. Berlin: Springer 1979

Babich, V.M., Popov, M.M.: Propagation of concentrated acoustical beams in three-dimensional inhomogeneous media. Akust. Zh. **27**, 828-835, 1981 (in Russian)

Born, M., Wolf, E.: Principles of optics. New York: Pergamon Press 1959

Červený, V.: Synthetic body wave seismograms for laterally varying layered structures by the Gaussian beam method. Geophys. J.R. Astron. Soc., **73**, 389-426, 1983

Červený, V., Hron, F.: The ray series method and dynamic ray tracing systems for 3D inhomogeneous media. Bull. Seismol. Soc. Am. **70**, 47-77, 1980

Červený, V., Klimeš, L., Pšenčík, I.: Synthetic seismic wave fields in 2D and 3D inhomogeneous structures. In: Proc. of the 27th Int. Geophysical Symposium, Bratislava 198, Vol. A(I), G. Plíva, ed.: pp. 17-28. Brno: N.E. Geofyzika 1982a

Červený, V., Molotkov, I.A., Pšenčík, I.: Ray method in seismology. Praha: Universita Karlova 1977

Červený, V., Popov, M.M., Pšenčík, I.: Computation of wave fields in inhomogeneous media - Gaussian beam approach. Geophys. J.R. Astron. Soc., **70**, 109-128, 1982b

Červený, V., Pšenčík, I.: Gaussian beams in two-dimensional elastic inhomogeneous media. Geophys. J.R. Astron. Soc., **72**, 417-433, 1983

Hubral, P.: A wave front curvature approach to computing ray amplitudes in inhomogeneous media with curved interfaces. Stud. Geophys. Geod. **23**, 131-137, 1979

Hubral, P.: Wave front curvatures in 3D laterally inhomogeneous media with curved interfaces. Geophysics **45**, 905-913, 1980

Kamke, E.: Differentialgleichungen. Lösungsmethoden und Lösungen. Vol. 1: Gewöhnliche Differentialgleichungen. Leipzig: 1959

Kirpichnikova, N.J.: Construction of solutions concentrated close to rays for the equations of elasticity theory in an inhomogeneous isotropic space. In: Mathematical problems of theory of diffraction and propagation of waves, Vol. 1, V.M. Babich, ed.: pp. 103-113. Leningrad: Nauka 1971 (in Russian, English translation by American Mathematical Society, 1974)

Klimeš, L.: Mathematical modelling of seismic wave fields in 3D laterally inhomogeneous media. Research Report No. 63, Institute of Geophysics, Charles University, Prague 1982a (in Czech)

Klimeš, L.: Hermite-Gaussian beams in inhomogeneous elastic media. Studia Geophys. Geod., in press 1982b

Kravcov, Y.A., Orlov, Y.I.: Geometrical optics of inhomogeneous media. Moscow: Nauka 1980 (in Russian)

Popov, M.M.: Method of composition of Gaussian beams in isotropic theory of elasticity. Preprint LOMI AN SSSR, P-3-82, Leningrad, 1982

Popov, M.M., Pšenčík, I.: Ray amplitudes in inhomogeneous media with curved interfaces. In: Geofys. Sb., Vol. 24, A. Zátopek, ed.: pp. 111-129. Praha: Academia 1978a

Popov, M.M., Pšenčík, I.: Computation of ray amplitudes in inhomogeneous media with curved interfaces. Stud. Geophys. Geod. **22**, 248-258, 1978b

Received December 17, 1982; Accepted March 21, 1983