GAUSSIAN ELIMINATION TECHNIQUE FOR SOLVING THE DIFFUSION EQUATION FOR MOISTURE MOVEMENT IN UNSATURATED SOIL

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INTRODUCTION

Importance of Problem

Water movement in soil is one of the most important phenomenon that occurs in nature. It is the movement of water to plant roots and hence through the plant that enables the later to synthesize plant tissue which is the basic material for sustaining animal life. As a result of increased use of fertilizers, insecticides, herbicides, hormones, etc., soil moisture is rapidly becoming the limiting factor in crop production. In arid regions, agriculture has long depended upon the application of supplemental water through irrigation. The increased financial pressures placed upon agriculture has caused a large increase in the use of supplemental irrigation in humid regions in order to realize optimum yields.

In addition to irrigation of desert land, drainage of swamps has also added greatly to the areas available for food production and as sites for human habitation. In semi-arid regions where irrigation water is not available, a system of dryland agriculture has been developed which relies on moisture stored in the soil to sustain plant growth during the year. In this type of agriculture, it is extremely important to reduce evaporation losses to a minimum. Understanding and controlling the movement of water through the soil to the evaporating surface is of extreme importance.

Each day many acres of good agriculture land are being used as building sites, for industrial development and in making new and larger highways. This increased competition for land is making it necessary to utilize areas which were formerly thought unprofitable for agricultural use. Developing these new areas for agricultural purposes will involve many water problems that must be solved.

Increased competition for water among industrial, agricultural, recreational and culinary users is making it necessary to find ways of more efficiently utilizing water in the growing of crops. A knowledge of water movement is essential to solving this problem.

In addition to its importance in agriculture, water movement in porous media is important in many other areas. Drainage is extremely important in highway and airport construction. Water movement in and out of clay layers underneath building foundations can cause their settlement. Water movement through porous building materials can cause damp walls in the case of basements; or it may allow excess moisture created within a building to move out through the porous walls making a more comfortable atmosphere.

The living cells are surrounded by a cell wall made of porous material. The movement of water and water solutions into and out of these cells is a vital process. Fluid flow through porous media is also of great importance in removal of petroleum from underground deposits. These are but a few of the areas where there is interest in flow through porous material.

Most of the studies of water movement in porous media has been concerned with flow in a saturated condition. These problems are much easier to solve but are not of as much agricultural importance as problems dealing with flow in unsaturated media. Some soil problems, such as leaching soil for reclamation purposes, involve saturated flow. But the great bulk of the significant agricultural problems, vis. drainage with a falling water table, application of irrigation water, evaporation from soil and movement to plant roots, are essentially unsaturated flow phenomena.

Scope of Research

One of the major reasons why flow in unsaturated soil has not been successfully studied is that there are many complications involved in applying the equations which deal with this type flow. Much work has been done using the diffusion theory, which seems to be a fairly good mathematical model of moisture movement in unsaturated soil; however, the solutions which have been developed to date have involved certain assumptions in order to solve the equations. These assumptions have limited the application of the solution to problems with extremely limited boundary conditions which, in general, are not characteristic of actual problems that might be encountered in the field.

However, the success achieved through using these solutions to predict moisture movement under these limited conditions has increased the interest in extending the diffusion equation to cover problems with more varied boundary conditions. It is the purpose of this study to develop a numerical technique for solving the diffusion equation and to apply this technique to moisture movement conditions that more nearly represent those which are encountered in the field. Experimental results will also be compared with the computed results to determine whether the diffusion equation is an acceptable mathematical model for isothermal moisture movement in unsaturated soil under the more complicated boundary conditions.

REVIEW OF LITERATURE

It is not necessary to treat fluid flow in saturated porous media since there are several good, recent reviews of this subject available (3, 9, 46). Although much work has been done on saturated flow, only limited attention has been given to fluid flow in unsaturated porous media. In the Petroleum industry there is considerable interest in immiscible, multiple phase flow systems; e.g. oil-brine, gas-oil, and oil-gas-brine. Short treatments of simultaneous flow of immiscible fluids in porous material are given by Collins (9, p. 139-169) and Scheidegger (46, p. 215-255). These treatments are not broad enough to completely cover the problems of water flow in unsaturated soil.

A short literature review of water movement in unsaturated porous solids has been written by Philip (33, p. 152-155). Philip's review is general in nature and is rather condensed. Other short literature reviews are contained in articles written by Childs (4), Gardner (17), Klute (22), Klute et al. (23), Philip (34), and Wiegand and Taylor (50, p. 18-20). Although all of these reviews are good they are all rather abbreviated. It is, therefore, necessary to fill in some additional background material on unsaturated moisture movement.

Moisture Flow in Unsaturated Soil

Early developments

Since most of the physical, chemical, and biological processes in soil are affected either directly or indirectly by soil moisture, numerous schemes have been developed for characterizing soil water. Of these schemes, the most significant is the potential concept which was first proposed by Buckingham (2) in 1907. One of the greatest contributions of the potential concept is in dealing with problems of moisture movement. Since all moisture flow is a consequence of potential gradients, the potential concept is the foundation of moisture flow in unsaturated soil.

The capillary potential ψ as defined by Buckingham is the specific work (work per unit mass) that must be done to pull an infinitesimal increment of water $\delta\Theta$ from the soil. He also determined the relationship between ψ and Θ over the range $0 < \psi < 10^5$ ergs/g for several soils. The ψ vs. Θ relationship was determined using vertical soil columns in equilibrium with a water table at the lower end. Evaporation was prevented at the upper end of the column.

Buckingham also defined capillary conductivity $K(\Theta)$ by the equation

$$v = K(\Theta)\frac{\partial\psi}{\partial x} \tag{1}$$

or

$$v = K(\Theta)\frac{\partial\psi}{\partial\Theta}\,\frac{\partial\Theta}{\partial x} \tag{2}$$

where v is the volume flux, Θ is moisture content, and $\partial\psi/\partial x$ is the capillary potential gradient in the x direction. Equation (1), which only applies when gravity can be neglected as a driving force, has the same form as Darcy's law for flow in saturated soil. Buckingham recognized, however, that capillary conductivity is not a constant but is a function of moisture content.

Unfortunately, Buckingham was not able to measure capillary conductivity. He did, however, draw a curve relating $K(\Theta)$ and Θ. This curve was based on theoretical reasoning, and he stated that the proportions of the curve were qualitative and might be far from right. Later studies (8, 27, 42, 43, 44) showed that although the proportions were not accurate, the general shape of his curve was correct.

The work of Buckingham was advanced for the times, and few people in the field of soils recognized the contribution he had made. Gardner apparently recognized the significance of the potential concept and published several highly mathematical papers on moisture movement in unsaturated soil (13, 14, 15). Unfortunately, his investigations did not incorporate the full significance of Buckingham's work. Gardner assumed that capillary conductivity was constant rather than a function of moisture content. This assumption greatly decreased the value of his otherwise good work.

In 1931, Richards (42) developed a general partial differential equation to describe moisture flow in unsaturated porous media under isothermal conditions. In this general equation Richards assumed capillary conductivity to be a function of moisture content. His equation,

$$\frac{\partial \Theta}{\partial t} = \nabla \cdot K(\Theta) \nabla \Phi \tag{3}$$

where Φ is a potential function and t is time, forms the basis for the mathematical study of moisture flow in unsaturated soil. Although Richards developed a method for measuring capillary conductivity as a function of moisture content, the utility of his work was limited by his choice of variables. Since

$$\Phi = \psi + \varphi \tag{4}$$

where ψ is capillary potential and φ is gravitational potential, equation (3) contains two functionally related variables ψ and Θ. Richards selected ψ as his independent variable, although he recognized the possibility of an alternate formulation in terms of Θ. It was later shown that the equation in Θ is much nicer to handle mathematically (5, 6, 28).

In 1936 Childs suggested that moisture movement in porous material was a diffusion phenomenon, but he assumed moisture diffusivity to be a constant (5, 6, 28). Kirkham and Feng (20) later showed that the diffusion equation is not an acceptable mathematical model for the movement of water in unsaturated soil where diffusivity is considered constant for the calculations.

In 1950 Childs and Collis-George (8) applied Darcy's law to isothermal, steady state flow in vertical or semivertical columns of unsaturated porous media. This equation, based on the diffusion model with the diffusion coefficient considered a function of moisture content, was developed from equation (1) written in a more general form to include gravity,

$$v = K(\theta)\frac{\partial \Phi}{\partial z} \ . \qquad (5)$$

Combining equation (4) with equation (5) gives

$$v = K(\theta)\left[\frac{\partial \psi}{\partial z} + \frac{\partial \varphi}{\partial z}\right]. \qquad (6)$$

Since = gz where g is acceleration of gravity and z is vertical distance, equation (6) becomes

$$v = K(\theta)\left[\frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial z} + g\right]. \qquad (7)$$

Equation (7) is the same as Buckingham's equation (2) except for the addition of a gravitational term. Childs and Collis-George combined $K(\theta)$ and $\partial\psi/\partial\theta$ to form a new function which they called a coefficient of diffusion $D(\theta)$. Written with a coefficient of diffusion, equation (7) becomes

$$v = D(\theta)\frac{\partial \theta}{\partial x} + K(\theta)g \qquad (8)$$

They also suggested the existence of a general flow equation for unsaturated porous media.

Emergence of a general diffusion equation

In 1952, Klute (22) developed a general flow equation for moisture movement in unsaturated soil. His development was similar to that of Richards (42) in that both equations were developed by combining Darcy's law with the equation of continuity; however, Klute chose θ as the independent variable. He then combined capillary conductivity with the slope of the ψ vs. θ curve, as was done by Childs and Collis-George (8), to obtain a diffusion coefficient $D(\theta)$ which he called diffusivity. For one-dimensional, horizontal flow the non-linear diffusion equation developed by Klute is of the form

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x}\right] \tag{9}$$

Crank and Henry (11, 12) were confronted with equation (9) in their work on the drying of extruded synthetic fibers and developed a means of integrating it numerically. This method was utilized by Klute (21) in solving equation (9) for water movement into unsaturated soil. Philip (31) introduced a numerical method of solving equation (9) which is more rapidly convergent and quicker to calculate than the method of Crank and Henry. Later, he extended his numerical technique to include solutions when the gravitational term is present (32).

Philip, Klute, and Crank and Henry all made use of the Boltzmann transformation that introduces a new variable μ which is a combination of two variables x and t combined in a definite way

$$\mu = xt^{-1/2}. \tag{10}$$

This transformation converts the nonlinear partial differential equation (9) into a nonlinear ordinary differential equation

$$-\frac{\mu}{2}\frac{d\Theta}{d\mu} = \frac{d}{d\mu}\left[D(\Theta)\frac{d\Theta}{d\mu}\right]. \tag{11}$$

Multiplying both sides of equation (11) by $d\mu/d\Theta$ gives

$$-\frac{\mu}{2} = \frac{d}{d\Theta}\left[D(\Theta)\frac{d\Theta}{d\mu}\right] \tag{12}$$

which is the desired form of the transformed equation.

The Boltzmann transformation also gives transformed boundary conditions which are as simple as the non-transformed boundary conditions. However, it can only be applied to semi-infinite uniform media which have uniform initial moisture content. The studies of Kirkham and Feng (20) and Swartzendruber et al. (49) seem to justify use of the Boltzmann transform in soil moisture flow. They found that the rate of advance of a wetting front was proportional to $t^{1/2}$.

Applications of the diffusion equation

Various problems of moisture movement in soil have been studied; but most of these studies have involved the Boltzmann transformation which puts serious limitations on boundary and initial conditions. Methods of solving the diffusion equation for concentration dependent diffusivity are discussed in detail by Crank (10); many of these are applicable to problems of moisture movement in homogeneous soils.

By making a series of assumptions, Philip has simplified the diffusion equation for the case of water infiltration into a homogeneous soil. His analysis of the infiltration problem has appeared in a series of papers (35, 36, 37, 38, 39, 40, 41). Youngs (52) has also applied the diffusion equation to similar infiltration problems.

A complete solution of the flow equation for drying of soils has not been obtained; although Gardner (18) obtained solutions for some limited boundary conditions.

A mathematical solution has not been obtained for the redistribution of moisture in soil after infiltration; Youngs (53), however, has given a good qualitative treatment of this subject from the standpoint of the diffusion equation. His experimental results (53, 54) agree qualitatively with his analysis.

The present status of moisture flow in unsaturated soil can be summarized as follows: The diffusion equation appears to be the best mathematical model for describing isothermal moisture movement in unsaturated soil; however, this equation is a nonlinear partial differential equation which cannot be solved analytically at the present time. At least three different approaches have been utilized in order to obtain a solution of the diffusion equation. Each of these approaches has required certain assumptions which have seriously limited the application of the solutions.

The first approach limits the solution to steady state flow--a condition which is almost never met in soil. The second approach is to assume a constant diffusivity--a model which does not even approximate moisture flow in soils. The third approach is to use the

Boltzmann transform--an approach which limits the solution to a uniform medium which is semi-infinite in extent and is at a uniform initial moisture content.

There still remain problems with finite or layered media and with an arbitrary initial moisture content. Numerical approaches appear to provide the best methods of attack at the present time. The purpose of the investigation reported here was to develop and test a numerical method for solving the diffusion equation for moisture movement in unsaturated layered soils.

Methods for Determining Moisture Diffusivity

Before a mathematical solution can be checked against laboratory experiment, a method of determining moisture diffusivity as a function of moisture content must be chosen. The methods available can be divided into three groups; steady state, calculation, and transient techniques.

Steady state methods

In the steady state procedures, the capillary conductivity is determined at a given moisture content. This is then multiplied by the slope of the soil moisture-potential curve taken at the same moisture content. Since

$$D(\theta) = K(\theta)\frac{\partial\psi}{\partial\theta} , \tag{13}$$

the diffusivity is specified for the given moisture condition. By repeating this procedure at various moisture contents, a curve of moisture diffusivity versus moisture content can be developed.

Richards (42) used a steady state system to determine $K(\Theta)$ over a limited range of moisture potential. His apparatus consisted of a chamber containing soil which was in contact with a porous ceramic plate on two opposite sides. Each ceramic plate was sealed to a water reservoir on which a suction could be applied. A different suction was applied on the two water reservoirs creating a moisture potential gradient across the soil sample. Tensiometers were installed in the soil near the ceramic plates to measure the actual soil moisture potential gradient. The volume of water entering the soil was also recorded as a function of time. Knowing the volume flux v and the potential gradient $\nabla\Phi$, the capillary conductivity could be determined by the equation

$$K(\Theta) = -\frac{v}{\nabla\Phi} \quad . \tag{14}$$

There are several disadvantages to this method. First, since there is a potential gradient across the soil, there must also be a moisture content gradient; this follows from the fact that $\psi = \psi(\Theta)$. Therefore, the $K(\Theta)$ which is determined does not correspond to a given moisture content but to an average moisture content $\overline{\Theta}$. The capillary conductivity thus determined is, therefore, an average value $\overline{K(\Theta)}$. Second, the relationships between the variables ψ, Θ and $K(\Theta)$ are not linear. In fact, for many soils an exponential relationship is a better approximation than a linear. Therefore, the gradient across the soil must be small since a linear relationship is assumed in calculating $\overline{\theta}$. Third, the values of ψ over which this apparatus will operate is limited to the range of tensiometer operation.

Richards and Moore (44) modified the Richards apparatus to use pressure rather than vacuum control. The sample chamber along with the connected water reservoirs is placed inside a pressure cooker and air pressure is applied to desaturate the soil. This offers little improvement over the old apparatus except in ease of control.

Nielsen and Biggar (29) described an apparatus similar in principle to that of Richards but somewhat easier to construct. The porous ceramic plates were replaced with fritted glass-bead plates, which greatly reduces the plate impedence. However, this apparatus is still limited to operation in the tensiometer range.

A second steady state method was developed by Moore (27) in which a water table was kept at one end of a soil column and evaporation was allowed to occur at the other. The flux was measured at the inflow (water table) end of the column, and moisture potential was determined with tensiometers at 10 cm intervals. Using equation (14), an average conductivity could then be determined for each interval. With this apparatus it is difficult to keep a constant evaporation rate at the soil surface. Also the range of operation is again limited since potential is measured with tensiometers.

In a third method, Childs (7) showed that if a water table is placed in contact with the lower end of a soil column and water is added at the upper end at a rate which is constant but is less than the saturated flux, the moisture content is sensibly uniform over an appreciable length of the tube. The zone of variable moisture content is limited at the lower end to the neighborhood of the water table in a way that depends on the pore size distribution and at the upper end to a zone in which is localized any intermittence of water application. Since $\psi = \psi(\Theta)$ there is no capillary potential gradient $\nabla\psi$ over the zone of uniform moisture content. The gradient form of equation (4)

$$\nabla\Phi = \nabla\psi + \nabla\varphi \qquad (15)$$

then becomes

$$\nabla\Phi = \nabla\varphi \qquad (16)$$

Childs and Collis-George (8) utilized this principle to determine capillary conductivity. The moisture content of the constant moisture zone was determined by measuring the capacitance between two previously calibrated aluminum foil electrodes placed on opposite sides of the column; the volume flux was simply determined from the rate of water addition at the upper end of the column; and the gradient was determined from equation (16). Equation (14) was then used to calculate $K(\Theta)$ which in this case was an actual rather than average value. The moisture content of the column was changed by adjusting the rate at which water entered the flow column. This method is limited to rather moist soils since at low rates of application it is difficult to control evaporation. This causes the calculated values of $K(\theta)$ to become inaccurate at low moisture contents.

After $K(\theta)$ has been determined by one of the above techniques, it is also necessary to determine $\partial\psi/\partial\theta$ and apply equation (13) to obtain $D(\theta)$. A relationship between ψ and θ is fairly easily obtained for a drying soil by use of the pressure plate and the pressure membrane apparatus. However, ψ is not a unique function of θ; instead, there is hysteresis depending on whether the soil is being wetted or dryed. In drainage problems the slope obtained from the drying curve is adequate, but for irrigation studies, the slope should be obtained from a wetting curve. A curve of moisture content versus moisture potential is difficult to obtain for a wetting soil, however, which further limits the utility of steady state approaches.

Calculation methods

A method of calculating $K(\theta)$ from the soil moisture-potential curve was developed by Childs and Collis-George (8). The values of $K(\theta)$ are obtained using the equation

$$K(\theta) = M \sum_{\beta = 0}^{\beta = R} \sum_{\gamma = 0}^{\gamma = R} \gamma^2 f(\beta)\,\delta r\, f(\gamma)\,\delta r \qquad (17)$$

where $f(\beta)\,\delta r$ is the area devoted to pores of radius β to $\beta + \delta r$ and $f(\gamma)\ \delta\gamma$ is the area devoted to pores in the range γ to $\gamma + \ \delta r$.

The summation is stopped at the pore size R appropriate to the largest pore which remains full of water at moisture content Θ. The constant M is calculated by matching a single calculated value of $K(\Theta)$ with an experimental value. The calculations in this method are very laborious, and it also requires an experimental value of $K(\Theta)$.

A method was developed by Marshall (25) for calculating $K(\Theta)$ which does not require a matching constant. Values are obtained from the equation

$$K(\Theta) = 2.8 \times 10^{-3} \lambda^2 n^{-2} \left[h_1^{-2} + 3h_2^{-2} + 5h_3^{-2} + \cdots 2(n-1)\, h_n^{-2} \right] \quad (18)$$

where λ is fraction of total area occupied by water and h_1, h_2, $\cdots$ h_n represent the potentials in equal classes; h_1, corresponding to mean radius r_1, belongs to the class with the largest pores and h_n, corresponding to r_n, belongs to the class with the smallest pores.

Nielsen _et al_. (30) calculated capillary conductivity using the above methods and compared the calculated values with experimental values for four profiles on each of four different soils. They found that the values of $K(\Theta)$ obtained using the Marshall method were too high for all four soils. For one soil the calculated values were about 50,000 times higher than the measured ones. Values calculated by the Childs and Collis-George technique were near the experimental values for two soils but deviated considerably on the other two soils, except at the point of matching. A number of assumptions are made in developing equations (17) and (18). Evidently these assumptions are not met or, in some cases, met in only a limited way in soil.

Transient methods

Capillary conductivity can be determined from successive measurements of the moisture profile using a method developed by Staple and Lehans (47). The net amount of water that moves from one layer into the next is estimated from moisture profiles plotted at different times after water is added. The area between the profiles at two successive times represents the water that moves. The conductivity at any depth is determined by dividing the amount of water that moves past each level by the water potential gradient at that level. The gradient can be determined either by measuring the potential using tensiometers or gypsum blocks or estimating the potential from the moisture content. This profile method can only give approximate results because of generalization and approximations inherent in the method. It may be satisfactory for field work, however, since natural field variations are quite large.

Gardner (16) introduced a method for calculating $K(\theta)$ from pressure plate outflow data. In this method soil samples were placed in an ordinary pressure plate apparatus and allowed to come to equilibrium at a small pressure. The pressure was then increased by an increment and the outflow liquid was measured as a function of time. At equilibrium, the pressure was increased by another increment and outflow data were again recorded. From these data, K(θ) was calculated using an equation developed by Gardner.

In developing the equation it was assumed: a) that the capillary conductivity is constant over a pressure increment, b) that the relationship between moisture content and pressure is linear over the pressure increment and c) that the plate offers negligible resistance to flow. Except at low pressures (high moisture contents) where K(θ) is changing extremely rapidly with moisture content, assumptions a and b can be approximated if small increments of pressure are used. However, at lower moisture contents somewhat larger pressure increments must be used to get measurable outflow. The study was extended to pressure membrane apparatus to obtain values of K(θ) for smaller values of θ. Under these conditions it was necessary to use large pressure increments to get a measurable outflow. Also, some experimental difficulties are encountered with this method. The main difficulty is in measuring the total water coming from the soil because of the indefinite time required to reach equalibrium.

In the outflow method of determining K(θ), Gardner (16) assumed that the flow impedence of the membrane is negligable, an assumption which is very difficult to meet. Miller and Elrick (26) added a correction term to the pressure plate outflow equation which extends the application of the equation to cases in which the impedence of the supporting plate or membrane must be taken into account. From their theoretical results they developed a practical method for employing all of the experimental data (not just the exponential tail as is done in the Gardner method) for determining $K(\Theta)$. However, assumptions a and b made by Gardner are also made in the Miller and Elrick equation. These assumptions limit the validity of data obtained by this technique.

Bruce and Klute (1) developed a method of determining moisture diffusivity directly. They wrote the diffusion equation for one dimensional horizontal flow and applied the Boltzmann transform. The transformed equation was then solved for $D(\Theta)$ to give

$$D(\Theta_a) = -\frac{1}{2t}\left(\frac{dx}{d\Theta}\right)_{\Theta_a}\int_{\Theta_i}^{\Theta_a} x\,d\Theta \qquad (19)$$

where the subscript a indicates the distance at which the upper limit of integration and the derivative are evaluated. Equation (19) was integrated numerically from data obtained using a horizontal column of soil at initial moisture content Θ_i. Water was allowed to enter the soil from a water source. At time t the source was removed and the moisture distribution of the soil column was determined gravimetrically.

The authors point out that using this technique moisture diffusivity is calculated from the transformed diffusion equation; the calculated diffusivity is then put back into the transformed equation and used to calculate the moisture distribution. Comparing results obtained in this manner with experimental results appears to serve only as a check on the accuracy of the calculations. However, if the diffusivity function is computed using an experimentally determined moisture content distribution for time t_1 and this calculated diffusivity is in turn used to calculate a moisture distribution for a new time t_2, then a comparison of the calculated and experimental values at t_2 serves as a check on this method of calculating moisture diffusivity.

The Bruce and Klute method is easy to run and it does not involve the drastic assumption, inherent in the other transient methods, that $D(\Theta)$ is constant over a small pressure increment. It also gives values over the entire moisture range which is not true of the steady state methods. Therefore, the Bruce and Klute method was chosen for determining the moisture diffusivity function in this study.

THEORY

The Diffusion Equation for Moisture Flow

Darcy's law has been used for many years to characterize one-dimensional water flow in saturated soil. This law can be stated mathematically as

$$v = - K\frac{dH}{dz} \tag{20}$$

where v is the volume flux of moisture, K is a proportionality constant called the hydraulic conductivity, and dH/dz is the head gradient or driving force.

In developing the diffusion equation for moisture flow, it is necessary to assume that a law similar to the Darcy equation holds for steady state flow in unsaturated soil. For unsaturated flow, the proportionality constant, hydraulic conductivity, of Darcy's law must be replaced by capillary conductivity which is a function of moisture content. Although several workers (8, 19, 52) have presented evidence indicating that this assumption is correct, it appears to the author that this is probably the weakest assumption in the derivation of the diffusion equation for moisture movement in unsaturated soil.

If it is assumed that a "Darcy type" equation holds for unsaturated, one-dimensional, steady state flow, it can be written as

$$v = - K(\theta)\frac{\partial\Phi}{\partial z} \tag{21}$$

where v is volume flux, Θ is moisture content, $K(\Theta)$ is capillary conductivity and $\partial\Phi/\partial z$ is potential gradient.

The potential Φ can be broken into two components

$$\Phi = \psi + \varphi \tag{22}$$

where ψ is water potential and φ is gravitational potential. Substituting equation (22) into equation (21) results in

$$v = - K(\Theta)\left[\frac{\partial\psi}{\partial z} + \frac{\partial\varphi}{\partial z}\right] \tag{23}$$

since $\psi = \psi(\Theta)$, $\partial\psi/\partial z = (\partial\psi/\partial\Theta)(\partial\Theta/\partial z)$ can be substituted in equation (23) to give

$$v = - K(\Theta)\frac{\partial\psi}{\partial\Theta}\frac{\partial\Theta}{\partial z} - K(\Theta)\frac{\partial\varphi}{\partial z} \quad . \tag{24}$$

It is now possible to define a term called moisture diffusivity $D(\Theta)$ such that

$$D(\Theta) = K(\Theta)\frac{\partial\psi}{\partial\Theta} \; . \tag{25}$$

Also, the gravitational potential can be defined as

$$\varphi = gz \tag{26}$$

where g is the acceleration of gravity. Combining equations (24), (25) and (26) results in

$$v = - D(\Theta)\frac{\partial\Theta}{\partial z} - K(\Theta)g \; . \tag{27}$$

Since there is no gravitational potential gradient in horizontal flow, equation (27) can be written as

$$v = - D(\theta)\frac{\partial\theta}{\partial x} \qquad (28)$$

for flow in the x direction.

Inspection of equation (28) reveals that moisture diffusivity is a physical property relating to the readiness with which unsaturated soil transmits water. Since this definition could just as well apply to capillary conductivity, a distinction should be made between these two functions. Moisture diffusivity is the function relating volume flux (rate of flow) to the driving force when the driving force is expressed as a moisture concentration gradient; whereas, capillary conductivity is the function relating volume flux to the driving force when the driving force is expressed as a potential gradient.

Equations (20), (21), (23), (27) and (28) apply only to one-dimensional flow under steady state conditions, i.e. conditions where neither moisture content nor moisture potential are changing with time.

For multi-dimensional flow, Laplace's equation is frequently used in problems involving unsaturated soil. This equation states that the divergence of the mass flux V is equal to zero

$$\text{div } V = 0. \qquad (29)$$

The physical interpretation of the above equation is that the quantity of moisture flowing into a unit volume of soil in unit time is equal to the quantity flowing out of the same volume in unit time. Thus the moisture density is constant. Moisture flow of this type is referred to as steady state flow.

In multi-dimensional flow in unsaturated soil, the equation of continuity is used to characterize the flow. The equation of continuity states that the divergence of the mass flux is equal to the negative of the time rate of change of moisture density M

$$\operatorname{div} V = -\frac{\partial M}{\partial t} \quad . \tag{30}$$

In a physical sense, this equation expresses that the difference between the quantity of moisture flowing into a unit volume of soil in unit time and the quantity flowing out of the same volume in a unit time is equal to the quantity of moisture stored within the volume in the same unit time interval. Thus, the moisture density is changing with time. This type of moisture flow is referred to as transient flow.

The moisture density is

$$M = \rho_b \, \Theta' \tag{31}$$

where ρ_b is the bulk density of the soil and Θ' is moisture content expressed as dry-weight mass fraction. The mass flux V is related to the volume flux v by the equation

$$V = \rho v \tag{32}$$

where ρ is the density of water.

A combination of equations (31) and (32) with equation (30) results in

$$\frac{\partial}{\partial t} (\rho_b \theta') = - \text{div } \rho v. \tag{33}$$

If the density of water is assumed to be constant, ρ can be taken outside the divergence operator and equation (33) can be written

$$\frac{\partial}{\partial t}\left(\frac{\rho_b}{\rho} \theta'\right) = - \text{div } v. \tag{34}$$

But since

$$\frac{\rho_b}{\rho} \theta' = \theta \tag{35}$$

where θ is moisture content expressed as volume fraction, equation (34) becomes

$$\frac{\partial \theta}{\partial t} = - \text{div } v. \tag{36}$$

The divergence of a vector is defined as

$$\text{div } v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{37}$$

where v_x, v_y, and v_z are the components of v in the x, y, and z directions respectively. The result of combining equations (36) and (37) is

$$\frac{\partial \theta}{\partial t} = - \frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} - \frac{\partial v_z}{\partial z} . \tag{38}$$

The term v_x is defined by equation (28); v_y is defined by a similar equation in which x is replaced by y; and v_z is defined by equation (27). Putting equations (27) and (28) into equation (38) gives

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[D(\theta)\frac{\partial\theta}{\partial y}\right] + \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial\theta}{\partial z} + K(\theta)g\right] \tag{39}$$

or

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[D(\theta)\frac{\partial\theta}{\partial y}\right] + \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial\theta}{\partial z}\right] + g\frac{\partial K(\theta)}{\partial z} \; . \tag{40}$$

The vector form of equation (40) is

$$\frac{\partial\theta}{\partial t} = \nabla\cdot D(\theta)\nabla\theta + g\,\frac{\partial K(\theta)}{\partial z} \; . \tag{41}$$

Equation (40), or alternately equation (41), is called the diffusion equation for soil moisture. This name was given because of the similarity between equation (40) and the equation for diffusion of gases. The name is not meant to imply that the mechanism of water movement in unsaturated soil is due to random molecular motion as is the case in the diffusion of gases. Actually there are probably several mechanisms involved in moisture flow.

The importance of the various mechanisms is not known but their relative importance probably changes with each change in moisture content.

It is possible to write equation (40) in another alternate form. Since

$$\frac{\partial K(\theta)}{\partial z} = \frac{\partial K(\theta)}{\partial\theta}\,\frac{\partial\theta}{\partial z} \tag{42}$$

a new function $K'(\theta)$ can be defined as

$$K'(\theta) = \frac{\partial K(\theta)}{\partial\theta} \quad . \tag{43}$$

This allows equation (42) to be written as

$$\frac{\partial K(\theta)}{\partial z} = K'(\theta)\frac{\partial \theta}{\partial z} \quad . \tag{44}$$

If equation (44) is combined with equation (40), the result is

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[D(\theta)\frac{\partial \theta}{\partial y}\right] + \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] + gK'(\theta)\frac{\partial \theta}{\partial z} \tag{45}$$

In equation (45) all space derivatives have θ as the independent variable which is probably a more desirable form for machine computations than equation (40).

This study is being limited to one-dimensional horizontal flow. For these conditions equation (40) becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x}\right] \tag{46}$$

Difference Equations for One-Dimensional Flow in Homogeneous Soil

A numerical technique similar to that described by Richtmyer can (45, p. 89-120) be used to solve the differential equation for moisture flow. The implicit difference analogues for equation (46) are

$$\frac{\theta_j^{n+1} - \theta_j^n}{\Delta t} = \frac{D(\theta_{j+1/2}^{n+1/2})\,\dfrac{\theta_{j+1}^{n+1} - \theta_j^{n+1}}{\Delta x} - D(\theta_{j-1/2}^{n+1/2})\,\dfrac{\theta_j^{n+1} - \theta_{j-1}^{n+1}}{\Delta x}}{\Delta x} \tag{47}$$

where $j = 1, 2, 3, 4, \cdots k$; and $n = 0, 1, 2, 3, \cdots N$. The superscripts and subscripts denote time and space iterations respectively.

The notation is shown graphically in figure 1. Letting $\Delta t/(\Delta x)^2 = r$ and rearranging (47) gives

$$- rD(\theta_{j-1/2}^{n+1/2})\theta_{j-1}^{n+1} + \left[1 + rD(\theta_{j-1/2}^{n+1/2}) + rD(\theta_{j+1/2}^{n+1/2})\right]\theta_j^{n+1}$$
$$- rD(\theta_{j+1/2}^{n+1/2})\theta_{j+1}^{n+1} = \theta_j^n . \tag{48}$$

Equations (48) together with the boundary and the initial conditions,

$\theta_0^n = \theta_s$, $n \geq 0$ where θ_s is the moisture content at saturation

$\theta_j^0 = \theta_o$, $j > 0$ where θ_o is the initial moisture content,

form a non-linear system of equations.

Equations (48) are non-linear since the values of $D(\theta_{j\pm1/2}^{n+1/2})$ are dependent on the values of θ_j^{n+1} and $\theta_{j\pm1}^{n+1}$ for which solutions are being sought. However, if $\theta_{j\pm1/2}^{n+1/2}$ is approximated by $\theta_{j\pm1/2}^{*}$, then $D(\theta_{j\pm1/2}^{*})$ can be computed and a system of linear equations which can be solved is obtained by replacing $D(\theta_{j\pm1/2}^{n+1/2})$ in equations (48) by $D(\theta_{j\pm1/2}^{*})$.

Making the indicated substitutions in equations (48) results in

$$-rD(\theta_{j-1/2}^{*})\theta_{j-1}^{n+1} + \left[1 + rD(\theta_{j-1/2}^{*}) + rD(\theta_{j+1/2}^{*})\right]\theta_j^{n+1}$$
$$- rD(\theta_{j+1/2}^{*})\theta_{j+1}^{n+1} = \theta_j^n . \tag{49}$$

The function $\theta_{j\pm1/2}^{*}$ may be chosen with some freedom. In the computations presented in this paper the function used is $\theta_{j\pm1/2}^{*} = \theta_{j\pm1/2}^{n} + \alpha$, where α is an arbitrarily chosen small constant and

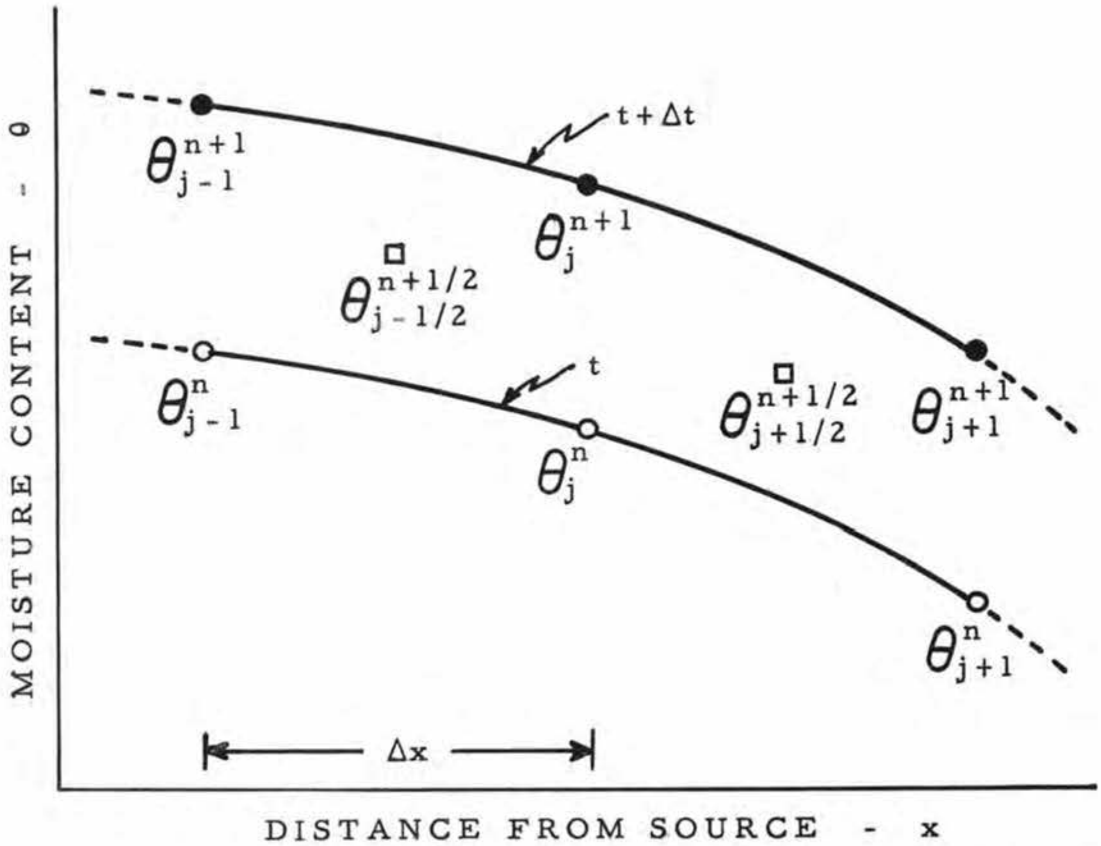

Figure 1. Small sections of two moisture distribution curves for a soil column at times t and t + Δt. The symbols used to show moisture content are; ○, values of θ which were calculated at the n time step; ●, values of θ to be calculated at the n+1 time step; and □, values of θ which are estimated by θ^*'s and are used to calculate values for moisture diffusivity.

$\theta^n_{j\pm 1/2}$ is an average of two successive moisture contents at x and $x\pm\Delta x$. In current calculations a value of 0.001 has been used for α. If Δt and Δx are small, good results can probably be obtained using $\theta^*_{j\pm 1/2} = \theta^n_{j\pm 1/2}$, however, the function $\theta^*_{j\pm 1/2} = \theta^n_{j\pm 1/2} + \alpha$ is a somewhat better approximation and gives very acceptable results. If more accuracy were desired, an iterative procedure might be used.

The functional relationship between $D(\theta)$ and θ was determined using the method described by Bruce and Klute (1). This method will be discussed in a following section.

The system of linear equations (49) is very convenient as the coefficient matrix is tri-diagonal.

$$\begin{vmatrix} b_1 & c_1 & & & & & \\ a_2 & b_2 & c_2 & & & & \\ & a_3 & b_3 & c_3 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & a_{m-1} & b_{m-1} & c_{m-1} \\ & & & & & a_m & b_m \end{vmatrix} \tag{50}$$

$a_i = -\,rD(\theta^*_{i-1/2})$, $b_i = 1 + rD(\theta^*_{i-1/2}) + rD(\theta^*_{i+1/2})$, $c_i = -\,rD(\theta^*_{i+1/2})$

The terms a_1 and c_m are missing from matrix (50). The reason these terms do not appear explicitly in the matrix is that θ_0^{n+1} and θ_{m+1}^{n+1} are known, since $\theta_0^{n+1} = \theta_s$ and $\theta_{m+1}^{n+1} = \theta_o$. Therefore, θ_0^{n+1} and θ_{m+1}^{n+1} can be combined with the known values on the right sides of the respective equations.

A tri-diagonal system of equations can be solved very conveniently by Gaussian elimination (51, p. 28-32). In the Gaussian elimination scheme, the first pair of equations is used to eliminate the first variable from the second equation. The elimination continues successively to the remaining variables until one equation in one unknown remains which can be solved. Working backwards, the solution of the last equation is substituted into the preceding equation to obtain the value of a second variable. This procedure is continued to obtain successively the values of the remaining variables.

An example, using numbers chosen for convenience in illustrating the method, follows:

Consider the set of equations

$$\begin{aligned} 3\theta_1 + 2\theta_2 \qquad\qquad\qquad &= 4.5 \\ 2\theta_1 + 3\theta_2 + \theta_3 \qquad\quad &= 4.75 \\ \theta_2 + 2\theta_3 + \theta_4 &= 2.0 \\ \theta_3 + 2\theta_4 &= 1.0 \end{aligned}$$

The coefficient matrix of the set of equations is obviously tri-diagonal. The steps in the solution by Gaussian elimination are:

1. Divide the first equation by its leading coefficient. Save the result.

2. Multiply the result of the first step by the leading coefficient of the second equation and subtract the result from

the second equation. This eliminates the first variable from the second equation.

3. Repeat steps one and two using the equation from which the variable has been eliminated as the first equation and its successor as the second.

At the end of this process the original set of equations is in the form

$$
\begin{aligned}
\Theta_1 + \tfrac{2}{3}\Theta_2 \qquad\qquad\qquad &= 1.5 \\
\Theta_2 + \tfrac{3}{5}\Theta_3 \qquad\quad &= 1.05 \\
\Theta_3 + \tfrac{5}{7}\Theta_4 &= \frac{4.75}{7} \\
\tfrac{9}{7}\Theta_4 &= \frac{2.25}{7}
\end{aligned}
$$

The solution for Θ_4 is now easily computed to be 0.25 and by substituting successively into the preceding equations, Θ_3, Θ_2, and Θ_1 are found to be 0.5, 0.75, and 1.0 respectively.

This method of solving the given system (49) has several very nice features for machine computation. All of the numbers involved in the computation stay nicely in scale; that is, they are all of nearly the same magnitude. The elimination process need only proceed until the value of Θ_j^n is at that of the initial moisture content Θ_o. Then the values of Θ_k^{n+1}; $k = j, j-1, \ldots 2, 1$; can be calculated. Thus, only meaningful points are calculated at any one time iteration, which appreciably reduces the computation time.

The implicit difference formulation is unconditionally stable (45, p. 169-170) for all values of Δt and Δx; however, it is desirable to keep these increments small in order to insure a reasonably small truncation error. In current calculations values used are: $\Delta t < 2.5$ sec. and $\Delta x < 0.5$ cm.

Calculated moisture content curves which meet the Boltzmann boundary conditions can be expanded or contracted by simply changing Δt and Δx in such a manner that the ratio $r = \Delta t/(\Delta x)^2$ is held constant. The values Θ which are obtained at the n+1 time step using the system of equations (48) depend only on: the values of Θ at the n time step, the method of estimating $D(\Theta^{n+1})$, and the value of r. Using a given set of values for Θ^n and a given method of estimating $D(\Theta^{n+1})$, the values of Θ^{n+1} will depend only on r. Now if r is held constant, the same moisture content will be obtained at each calculated point for the n+1 time step regardless of the size of the time step Δt; that is, if Δt and Δx are varied in such a way that the ratio r is held constant, the moisture content calculated at each point for the n+1 time step from a given set of data at the n time step will be the same for all values of Δt; but if Δt is varied, the spacing of the points will also vary. It should be realized, of course, that when a curve is expanded, any errors in the curve will be magnified in proportion to the expansion. Therefore, extreme expansion of curves computed for short times should be avoided.

Difference Equations for One-Dimensional Flow in Multi-Region Soil

The partial differential equation (46)

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\Theta)\frac{\partial \Theta}{\partial x}\right] , \tag{46}$$

with proper boundary conditions may be applied to media in which there are regions with dissimilar flow properties. The boundary conditions specify continuity of flow and continuity of potential. These conditions between two regions I and II are:

$$\left(D_I(\Theta)\frac{\partial \Theta}{\partial x}\right)_{x=b^-} = \left(D_{II}(\Theta)\frac{\partial \Theta}{\partial x}\right)_{x=b^+} \tag{53}$$

$$\psi_{b^-} = \psi_{b^+} \tag{54}$$

where $D_I(\Theta)$ and $D_{II}(\Theta)$ are the diffusivity functions for regions I and II respectively; b is the boundary between regions so that b^- is arbitrarily near the boundary in region I and b^+ is arbitrarily near the boundary in region II; ψ_{b^-} and ψ_{b^+} are the moisture potential functions relative to regions I and II.

When finite difference equations are used to approximate the differential equation, boundary condition (53) must be modified since the boundary lies in an increment of finite thickness Δx. The notation used in developing the finite difference equations is shown graphically in figure 2.

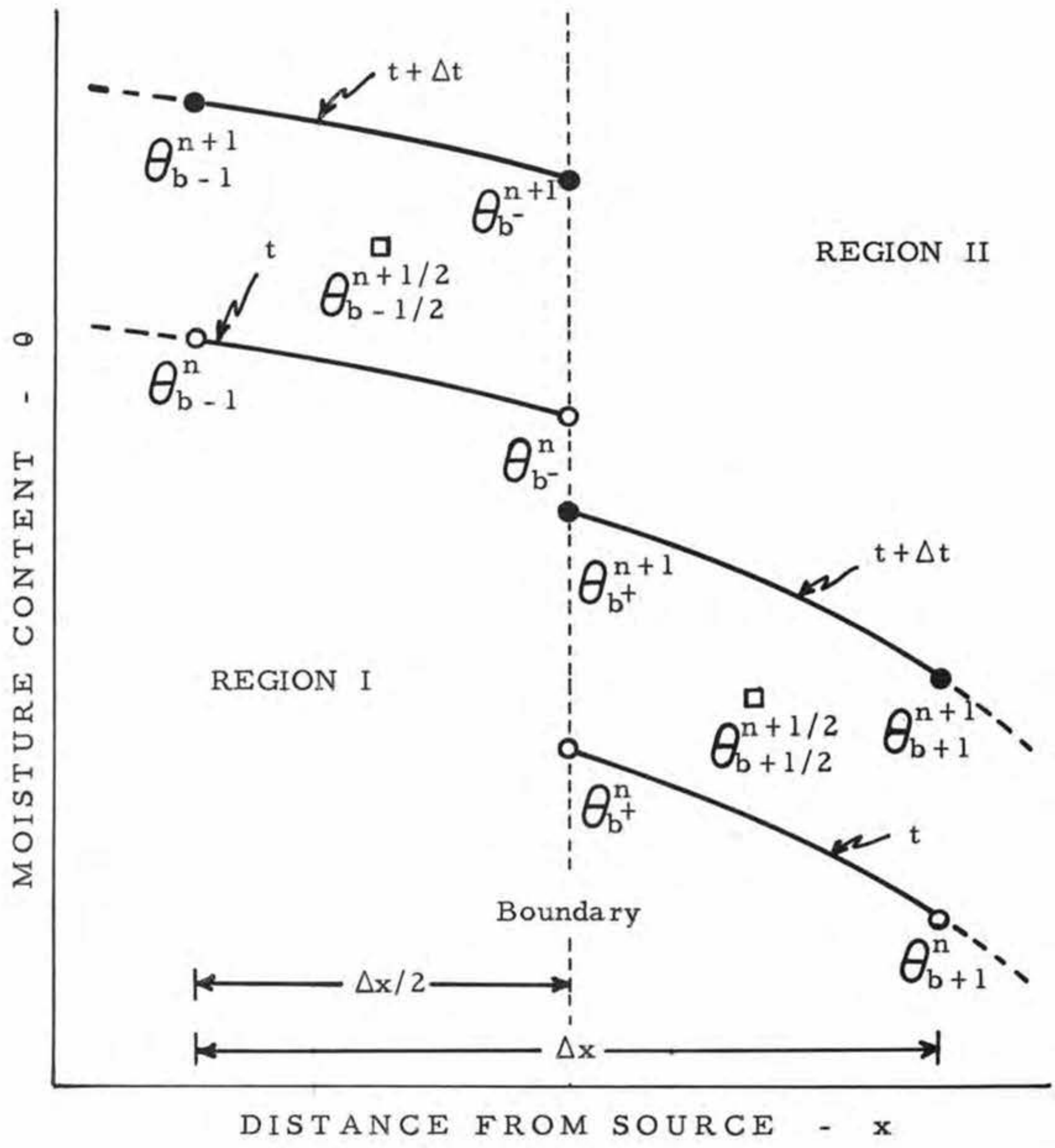

Figure 2. Two moisture content distribution curves at times t and t + Δt for the boundary section of a two region soil column. The broken line indicates the boundary between region I and region II. The symbols used to show moisture content are; O, values of θ which were calculated at the n time step; ●, values of θ to be calculated at the n+1 time step; and □, values of θ which are estimated by θ^*'s and are used to calculate values for moisture diffusivity.

The modified boundary condition corresponding to equation (53) is

$$Q_s = \frac{-D_I(\theta^*_{b-1/2})\left[\theta^{n+1}_{b^-} - \theta^{n+1}_{b-1}\right]}{\Delta x} + \frac{D_{II}(\theta^*_{b+1/2})\left[\theta^{n+1}_{b+1} - \theta^{n+1}_{b^+}\right]}{\Delta x} \tag{55}$$

where Q_s is the difference between the flux into and the flux out of the increment Δx and therefore may be thought of in terms of the quantity of water stored at any time increment, $\theta^*_{b-1/2}$ is an approximation of $(\theta^{n+1}_{b-1} + \theta^{n+1}_{b^-})/2$ and $\theta^*_{b+1/2}$ is an approximation of $(\theta^{n+1}_{b^+} + \theta^{n+1}_{b+1})/2$. The potential ψ_{b^-} and ψ_{b^+} can be related experimentally to the moisture contents θ_{b^-} and θ_{b^+} respectively.

The quantity of water stored in the increment about the boundary Q_s can be obtained by applying a difference equation to an increment $\frac{\Delta x}{2}$ on each side of the boundary and adding the results

$$Q_s = \frac{\Delta x}{2\Delta t}\left[(\theta^{n+1}_{b^-} - \theta^{n}_{b^-}) + (\theta^{n+1}_{b^+} - \theta^{n}_{b^+})\right] . \tag{56}$$

Equating equations (55) and (56) and letting $\Delta t/(\Delta x)^2 = r$ gives

$$r\left[-D_I(\theta^*_{b=1/2})\left(\theta^{n+1}_{b^-} - \theta^{n+1}_{b-1}\right) + D_{II}(\theta^*_{b+1/2})\left(\theta^{n+1}_{b+1} - \theta^{n+1}_{b^+}\right)\right]$$
$$= \frac{1}{2}\left[(\theta^{n+1}_{b^-} - \theta^{n}_{b^-}) + (\theta^{n+1}_{b^+} - \theta^{n}_{b+})\right] \tag{57}$$

On rearrangement, equation (57) becomes

$$-rD_I(\theta^*_{b-1/2})\theta^{n+1}_{b-1} + \left[rD_I(\theta^*_{b-1/2}) + \frac{1}{2}\right]\theta^{n+1}_{b^-} + \left[rD_{II}(\theta^*_{b+1/2}) + \frac{1}{2}\right]\theta^{n+1}_{b^+}$$
$$-rD_{II}(\theta^*_{b+1/2})\theta^{n+1}_{b+1} = \frac{1}{2}(\theta^{n}_{b^-} + \theta^{n}_{b^+}) \tag{58}$$

Equation (58) is one condition which must be satisfied on the boundary between regions. The second condition which must be met can be specified by equation (54). However, since all of the equations have been written using θ as the independent variable, it is also necessary to write this boundary condition in terms of moisture contents θ_{b^-} and θ_{b^+} which correspond to potentials ψ_{b^-} and ψ_{b^+} respectively. Since the relationship between θ and ψ is known for the soil in each region, a relationship between θ_{b^-} and θ_{b^+} can be obtained for the condition that $\psi_{b^-} = \psi_{b^+}$.

In order to use the Gaussian elimination technique for the solution of a multi-region problem, it is necessary to approximate the relationship between θ_{b^-} and θ_{b^+} with a piece-wise linear function; i.e., if $\theta_{b^+} = f(\theta_{b^-})$, then $f(\theta_{b^-})$ must be approximated by line segments such that

$$\theta_{b^+} = a_i \theta_{b^-} + c_i \qquad i = 1, 2, 3, \cdots n \tag{59}$$

where there are n line segments in the approximating function. Equation (59) thus becomes the second condition that must be satisfied between the two regions.

If equation (59) is substituted into equation (58) for $\theta_{b^+}^{n+1}$ and $\theta_{b^+}^{n}$, the result is

$$- rD_I(\theta^*_{b-1/2})\theta^{n+1}_{b-1} + \left[rD_I(\theta^*_{b-1/2}) + \frac{1}{2} + a_i\left(rD_I(\theta^*_{b+1/2}) + \frac{1}{2}\right)\right]\theta^{n+1}_{b^-}$$

$$- rD_{II}(\theta^*_{b+1/2})\theta^{n+1}_{b+1} = \frac{1}{2}(1 + a_i)\theta^{n+1}_{b^-} - c_i rD_{II}(\theta^*_{b+1/2}) \tag{60}$$

Both boundary conditions are now incorporated into this one equation. Since equation (60) involves only three unknowns, it fits into the tridiagonal system of equations and the system can be solved by Gaussian elimination as described in the previous section.

Thus, the system of equations which must be solved in a two region problem with boundary at $j = b$, and diffusivity functions D_I and D_{II} in regions I and II respectively is

$$- rD_I(\theta^*_{j-1/2})\theta^{n+1}_{j-1} + \left(1 + rD_I(\theta^*_{j-1/2}) + rD_I(\theta^*_{j+1/2})\right)\theta^{n+1}_j$$

$$- rD_I(\theta^*_{j+1/2})\theta^{n+1}_{j+1} = \theta^n_j \qquad \text{for } j < b \tag{61a}$$

$$- rD_I(\theta^*_{b-1/2})\theta^{n+1}_{j-1} + \left[rD_I(\theta^*_{b-1/2}) + \frac{1}{2} + a_i\left(rD_{II}(\theta^*_{b+1/2}) + \frac{1}{2}\right)\right]\theta^{n+1}_{j-}$$

$$- rD_{II}(\theta^*_{b+1/2})\theta^{n+1}_{j+1} = \frac{1}{2}(1 + a_i)\theta^n_{j-} - c_i rD_{II}(\theta^*_{b+1/2}) \qquad \text{for } j = b \tag{61b}$$

$$- a_i rD_{II}(\theta^*_{b+1/2})\theta^{n+1}_{j-1} + \left[1 + rD_{II}(\theta^*_{b+1/2}) + rD_{II}(\theta^*_{j+1/2})\right]\theta^{n+1}_j$$

$$- rD_{II}(\theta^*_{j+1/2}) = \theta^n_j + c_i rD_{II}(\theta^*_{b+1/2}) \qquad \text{for } j = b + 1 \tag{61c}$$

$$- rD_{II}(\theta^{*}_{j-1/2})\theta^{n+1}_{j-1} + \left[1 + rD_{II}(\theta^{*}_{j-1/2}) + rD_{II}(\theta^{*}_{j+1/2})\right]\theta^{n+1}_{j}$$

$$- rD_{II}(\theta^{*}_{j+1/2})\theta^{n+1}_{j+1} = \theta^{n}_{j} \qquad \text{for } j > b + 1 \tag{61d}$$

The value of $\theta^{n+1}_{b^+}$ is not calculated in a normal pass through the program at any one time iteration but must be calculated separately from

$$\theta^{n+1}_{b^+} = a_i \theta^{n+1}_{b^-} + c_i . \tag{62}$$

Theoretically, the values of $\theta^{n+1}_{b^+}$ calculated from equation (62) should all lie in the interval $\theta_{oII} \leq \theta^{n+1}_{b^+} \leq \theta_{sII}$ where θ_{oII} and θ_{sII} are initial and saturated moisture content respectively. Since the evaluation of a_i and c_i involves some experimental error, a value of $\theta^{n+1}_{b^+}$ occasionally falls outside this interval and the following adjustments must be made to prevent successive values of θ from diverging from the true solution: if $\theta^{n+1}_{b^+} < \theta_{oII}$, set $\theta^{n+1}_{b^+} = \theta_{oII}$; if $\theta^{n+1}_{b^+} > \theta_{sII}$, set $\theta^{n+1}_{b^+} = \theta_{sII}$.

When a fine textured soil is followed by a coarse textured soil, water does not enter the coarse textured soil until ψ has been reduced to a level ψ_p which depends on the pore size of the coarse textured soil. The potential ψ_p corresponds to a moisture content θ_p. In the computer program, flow across the boundary must be restricted until a barrier is reached, _i.e._, until $\theta_{b^-} = \theta_p$.

MATERIALS AND METHODS

Laboratory Techniques

The soils used in preparing flow columns were: Quincy loamy sand, Quincy fine sandy loam, Salkum clay loam, Olympic silty clay loam, and Chehalis loam. The mechanical analyses of these soils are given in table 1.

Table 1. Mechanical composition of five soils as determined by pipette analysis.

Soil	Clay $<2\mu$	Silt 2 to 50μ	Sand $>50\mu$
	%	%	%
Quincy loamy sand	9.58	6.01	84.41
Quincy fine sandy loam	11.58	31.16	57.26
Salkum clay loam	37.35	41.07	21.58
Olympic silty clay loam	32.53	55.23	12.24
Chehalis loam	20.58	41.06	38.36

The Quincy loamy sand was prepared by passing through a 1 mm sieve; the other four soils were crushed sufficient to pass through a 2 mm sieve.

The apparatus used to determine the moisture content distribution curves was a modification of that used by Bruce and Klute (1). A diagram of the apparatus is shown in figure 3. The flow columns consisted of a number of 1.06 cm sections of plastic tubing which

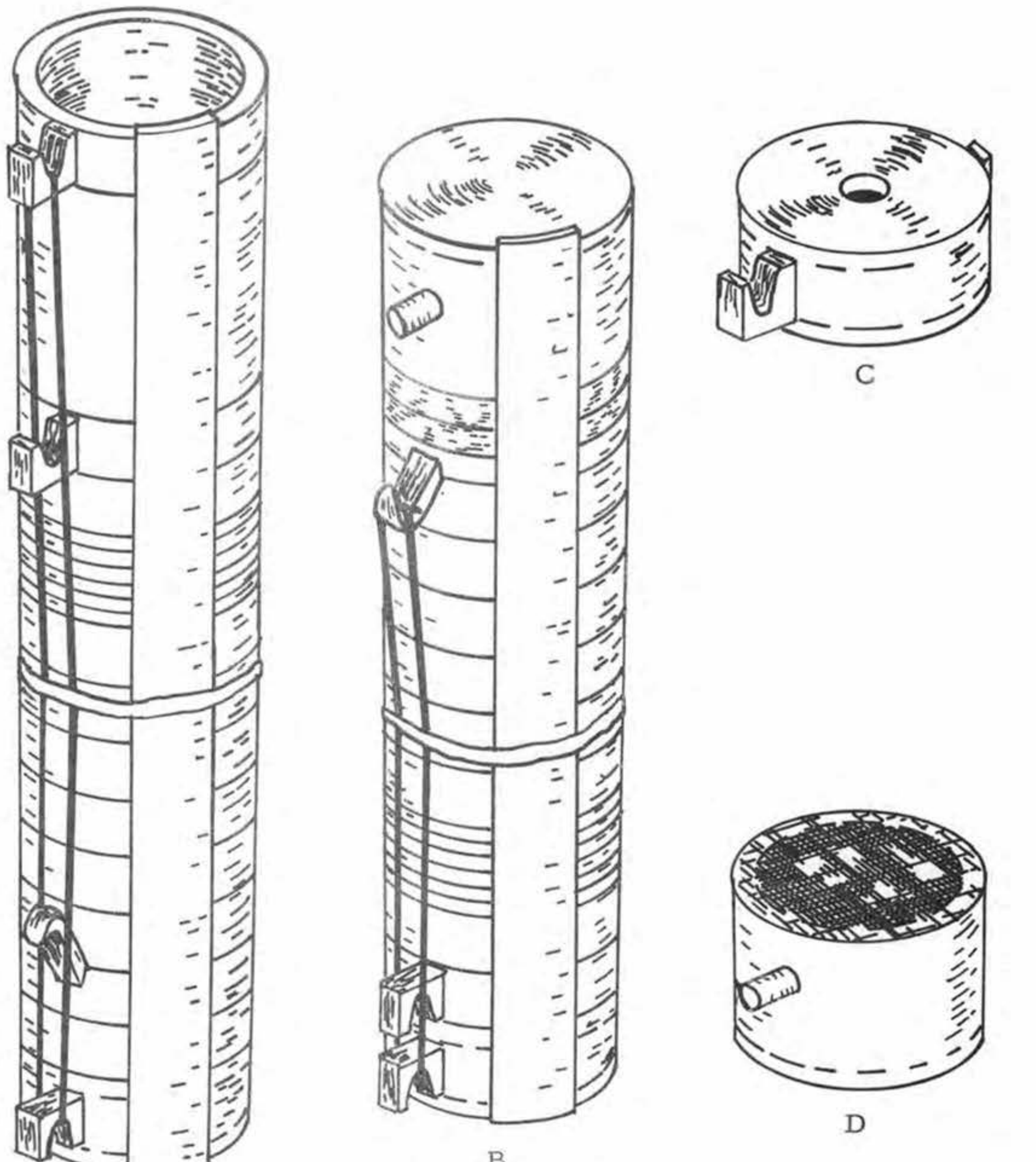

Figure 3. Sketches showing assembly of flow columns: A, assembled flow column ready for filling with soil; B, filled flow column with excess soil removed and water source attached with cellophane tape; C, end piece with hole for inserting vibrator; and D, water source showing the brass screen end.

had an inside diameter of 3.14 cm. To obtain more precision at the wetting front, six thin (0.24 cm) sections were placed in the flow column in the region where the wetting front would be at the time of soil sampling. The columns were held together with rubber bands placed around small plastic hooks which were cemented to the end sections. The flow column was made more rigid by placing two strips, cut from aluminum tubing, on opposite sides of the column. The aluminum strips were held in place with a rubber band. The assembled column was fitted with an end piece which had provisions for inserting a vibrator (figure 3C).

The flow column was filled using a tremie which consisted of a plastic funnel attached to a 40 cm section of 1.3 cm inside diameter, plastic tubing. The tremie was rested on the end piece and filled by taking small samples from bulk soil which had been thoroughly mixed. The filled tremie was slowly raised while being continuously moved in a random pattern inside the flow column. The continuous motion in the horizontal plane, as the tremie was raised, resulted in a very uniform layering of soil with very little size segregation of particles.

The filled column was set on the tip of a vibrator[1/], where it was held in place by a loose fitting clamp. The columns were vibrated five minutes for Chehalis, Olympic, Salkum, and Quincy fine sandy loam and eleven minutes for Quincy loamy sand. Also a slightly

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1/ A Burgess Vibro-graver was used in this study.

shorter vibrator stroke was used on the Quincy than on the other soils. It was found that vibration beyond these periods produced negligible additional packing.

The bulk density was less in the upper 2 cm of the column; therefore, after packing, the upper 5 cm section of soil was discarded and a solid cap was placed over the end of the column. The column was then inverted, the 1.06 cm section of soil which had been at the bottom of the column was removed, and a water source was attached with cellophane tape (figure 3B).

The water source consisted of a 2.5 cm section of tubing with a disk of plastic cemented over one end and a piece of brass screen attached to the other end (figure 3D). The brass screen was fused to the plastic with a hot soldering iron. Two short lengths of plastic tubing were cemented into holes on opposite sides of the water source to act as filling tubes.

The completely assembled flow column was placed in a constant temperature room and allowed to reach equilibrium with the room which was kept at $20 \pm 0.6^{\circ}$ C.

The flow column was placed in a swivel clamp which allowed rotating the flow column from a vertical to a horizontal position. A rubber cap was placed over one filling tube of the water source and the other was connected to a Mariotte bottle which was adjusted to maintain zero head at the center of the horizontal flow column.

The water source was filled by rotating the flow column into a vertical position and opening a pinch clamp on the tygon tubing connecting the water source and the Mariotte bottle. Since the water source was about 5 cm lower when the column was rotated to the vertical than in the horizontal, it filled under approximately 5 cm of head. The water level was carefully watched as it rose in the water source. The instant the water touched the screen the column was rotated to a horizontal position and simultaneously a stop watch was started. Using this procedure the flow column did not experience a positive head when flow was initiated.

Flow was allowed to proceed until the wetting front was at the fifth thin section. The column was then quickly detached from the Mariotte bottle, the reservoir was emptied and the stop watch was stopped. The sections of the flow column were quickly cut apart with a spatula and the soil from each section was placed in separate moisture cans. Moisture content was determined gravimetrically.

The sampling procedure required approximately 60 to 80 sec. It is possible that during this time some redistribution of moisture took place. However, the samples near the wetting front were removed first and, as indicated by the moisture content distribution curves, the moisture gradient was small behind the wetting front. It is therefore believed that the moisture redistribution was negligible except possibly in the section immediately next to the water source.

Flow columns for two region problems consisted of two short columns filled and packed as above. A strip of paper was placed on top of one column and was held in place with a spatula. This column was inverted and placed on top of the second column, and the two columns were secured together with rubber bands. The spatula was carefully removed; followed by careful removal of the paper. The completely assembled column was vibrated for 3 seconds to insure good contact between the two soils. The remaining steps in handling the layered-soil flow columns were the same as those described for the single-soil columns.

Computer Techniques

The moisture content distribution obtained from the flow columns was used to compute moisture diffusivity from equation (19)

$$D(\theta_a) = -\frac{1}{2t}\left(\frac{dx}{d\theta}\right)_{\theta_a} \int_{\theta_i}^{\theta_a} x\,d\theta \qquad (19)$$

where $\left(\frac{dx}{d\theta}\right)_{\theta_a}$ indicates the value of the derivative at $\theta = \theta_a$.

The first problem attacked was to write a computer program that takes the raw data from moisture distribution measurements and calculates the diffusivity from these data using equation (19). A functional relationship can then be determined between moisture diffusivity and moisture content for use in solving the diffusion equation.

Because of the derivative which appears in equation (19), it was considered necessary to include data smoothing techniques in the program. It is well established that the differentiation process tends to magnify experimental error just as integration tends to smooth out this type of error. The technique used was to combine the point x_k, which is to be smoothed, with the two neighboring points to the right and the two neighboring points to the left; and through these five points a parabola of second order was fitted by the method of least squares. The new, corrected value of x_k is taken as the value of the central point on the parabola. This process is then repeated on each successive point, x_{k+1}, x_{k+2}, ... x_{k-2}. There is a special formulation to handle the smoothing of the two points at each end of the data. At the conclusion of this process one has a new set of data which has been smoothed once.

The smoothed data are now subjected to a similar process with the derivative being calculated at the center point of the second order parabola at each step. In this way the data is smoothed twice in the calculation of each derivative. In cases where the experimental datum contains a value $x_k < x_{k+1}$, the derivative at x_{k+1} will be positive. This gives a negative diffusivity which is meaningless. It was, therefore, necessary to use a French curve to smooth any points x_{k+1} where $x_{k+1} > x_k$ before applying the machine smoothing techniques.

It should be noted that since the techniques used require equally spaced data and the given data are equally spaced in x, the derivative $\left(\frac{d\theta}{dx}\right)_{\theta_a}$ was computed and the reciprocal, $\frac{1}{\left(\frac{d\theta}{dx}\right)_{\theta_a}} = \left(\frac{dx}{d\theta}\right)_{\theta_a}$ was taken. The integral was calculated using Simpson's Formula. In this calculation use was made of the relation

$$\int_{\theta_i}^{\theta_a} x d\theta = (\theta_a - \theta_i)a - \int_{x_i}^{a} \theta dx \qquad (63)$$

where $x = a$ at $\theta = \theta_a$, and $x = x_i$ at $\theta = \theta_i$. This allows the integration to be taken along the x-axis where the data are equally spaced. A graphical representation of equation (63) is shown in figure 4.

In the system of equations (49) or (61), the function $D(\theta)$ enters as a factor in the various coefficients of the θ's. In order to solve either system, the function $D(\theta)$ or a reasonable approximation thereof must be available. Since there is no known analytic expression for $D(\theta)$ an approximation must be used. Using equation (19) values of $D(\theta)$ were calculated from experimental data as described above. The approximating function was then fitted to this set of data.

Several approximating functions were tried before one was found that fitted the data reasonably well. The first function tried was an exponential function of the form

$$D(\theta) = e^{P(\theta)} \qquad (64)$$

where $P(\theta)$ is a polynomial in θ, $P(\theta) = a_1 + a_2x + a_3x^2 + \cdots a_{n+1}x^n$. The best fitting polynomials up to degree 3 were obtained by the method of least squares. However, none of these exponentials gave reasonable fits.

Another function which was tried was of the form

$$\begin{aligned} D(\theta) &= c(\theta + d)^n && \text{for } \theta_o \le \theta \le 1 - d \\ D(\theta) &= c(2 - d - \theta)^n && \text{for } 1 - d < \theta \le \theta_s \end{aligned} \qquad (65)$$

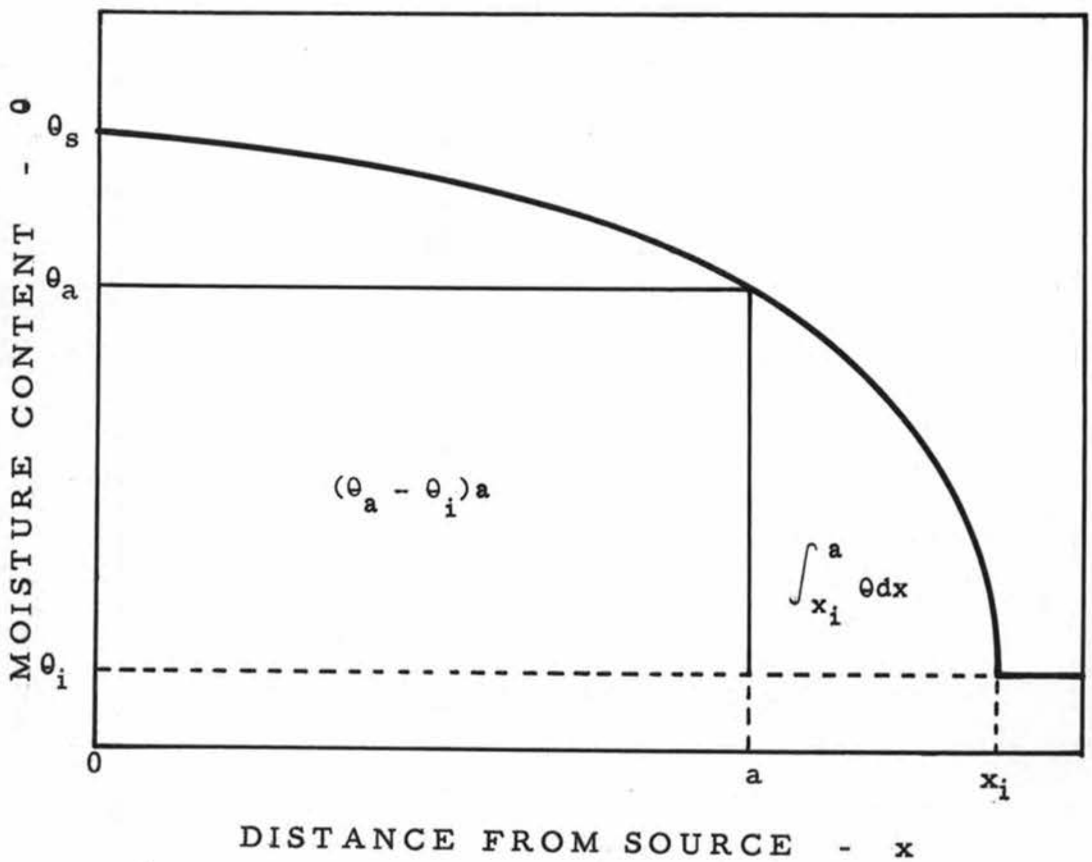

Figure 4. A graphical representation of equation (63) showing the area obtained by integrating along the x axis from $x = x_i$ to $x = a$ and the rectangular area which is added to the integral. The sum of these two areas is equal to that obtained by integrating along the θ axis from $\theta = \theta_i$ to $\theta = \theta_a$.

where $1 - d = \theta_{peak}$ and $c = D(\theta_{peak})$, the term θ_{peak} being the value of θ at which $D(\theta)$ reaches a maximum. The value of n was varied to obtain the best approximation. This function is like x^n for $\theta \leq x \leq 1$, which, for large n, is small for all values of x except those near 1. The fit with this function was relatively good near the peak; but it did not fit well for values of θ near θ_o, the fitted values of $D(\theta)$ being too low at low moisture contents.

Since the calculated values of $D(\theta)$ showed a narrow, sharp, rising peak, almost a discontinuity, it was decided that a better fit might be obtained by multiplying the $D(\theta)$ values by a factor $(\theta - a)^2$, where a is the value of θ at the peak, and fitting a function to the resulting data. This operation gave a set of data which was easily fitted by a piece-wise linear function. The resulting function can then be divided by $(\theta - a)^2$ to obtain the approximating function for $D(\theta)$, *i.e.*, if $D(\theta)$ is the symbol used for the approximating function (as well as the symbol of the theoretical function) then

$$D(\theta) = \frac{f(\theta)}{(\theta - a)^2} \qquad (66)$$

where $f(\theta)$ is now the function that must be fitted. For the type of data obtained for moisture diffusivity, the function $f(\theta)$ can be fitted piece-wise with linear exponentials,

$$f(\theta) = e^{a_i\theta + b_i} . \qquad (67)$$

The fit was obtained by the method of least squares. Combining equations (66) and (67) gives the final form of the equation used to obtain the approximating function D(θ),

$$D(\theta) = \frac{e^{a_i\theta + b_i}}{(\theta - a)^2} . \qquad (68)$$

It should be noted that with this technique a value slightly different from a must be used in the factor $(\theta - a)^2$ for the segment in which $\theta = a$. In solving for $f(\theta)$, two different values of a, $a \pm \epsilon$, were used in obtaining values for $(\theta - a)^2 D(\theta)$. Thus, $(\theta - (a + \epsilon))^2 D(\theta)$ was used for values of $\theta \leq a$ and $(\theta - (a - \epsilon))^2 D(\theta)$ for all values of $\theta > a$.

With the diffusivity function approximated by equation (68), the system of equations (49) and (61) can be used to calculate the moisture content distribution in a horizontal flow column with uniform and layered soil respectively. A computer program was written to handle the calculations involved in the solution of these systems of equations.[2/]

2/ The calculations were performed on an IBM 709 computer at the Western Data Processing Center at the University of California, Los Angeles.

RESULTS AND DISCUSSION

Comparison of Gaussian Elimination and Boltzmann Transform Techniques

In order to evaluate the Gaussian elimination technique for solving the difference analogues (49) of the diffusion equation (46), it was thought desirable to determine whether solutions obtained by Gaussian elimination predict experimental values as well or better than solutions obtained by the Boltzmann transform technique.

Diffusivity data reported by Gardner and Mayhugh (19) were used to calculate the moisture content distribution for several soils using the Gaussian elimination scheme. Figures 5 and 6 show the moisture content distributions of four soils calculated using Gaussian elimination; in addition, the moisture content distributions calculated by Gardner and Mayhugh using the Boltzmann transform are shown. Experimentally determined moisture contents are also included in these figures. The two methods of calculation gave very similar moisture distribution; in fact, the curves for Chino are almost superimposed. Yolo, Traver and Pachappa show slight differences between the curves obtained by the two methods of calculation. For these three soils, the curves calculated by the Boltzmann transform fit the experimentally determined points somewhat better than the curves calculated by Gaussian elimination. However, this does not necessarily mean that the Boltzmann technique

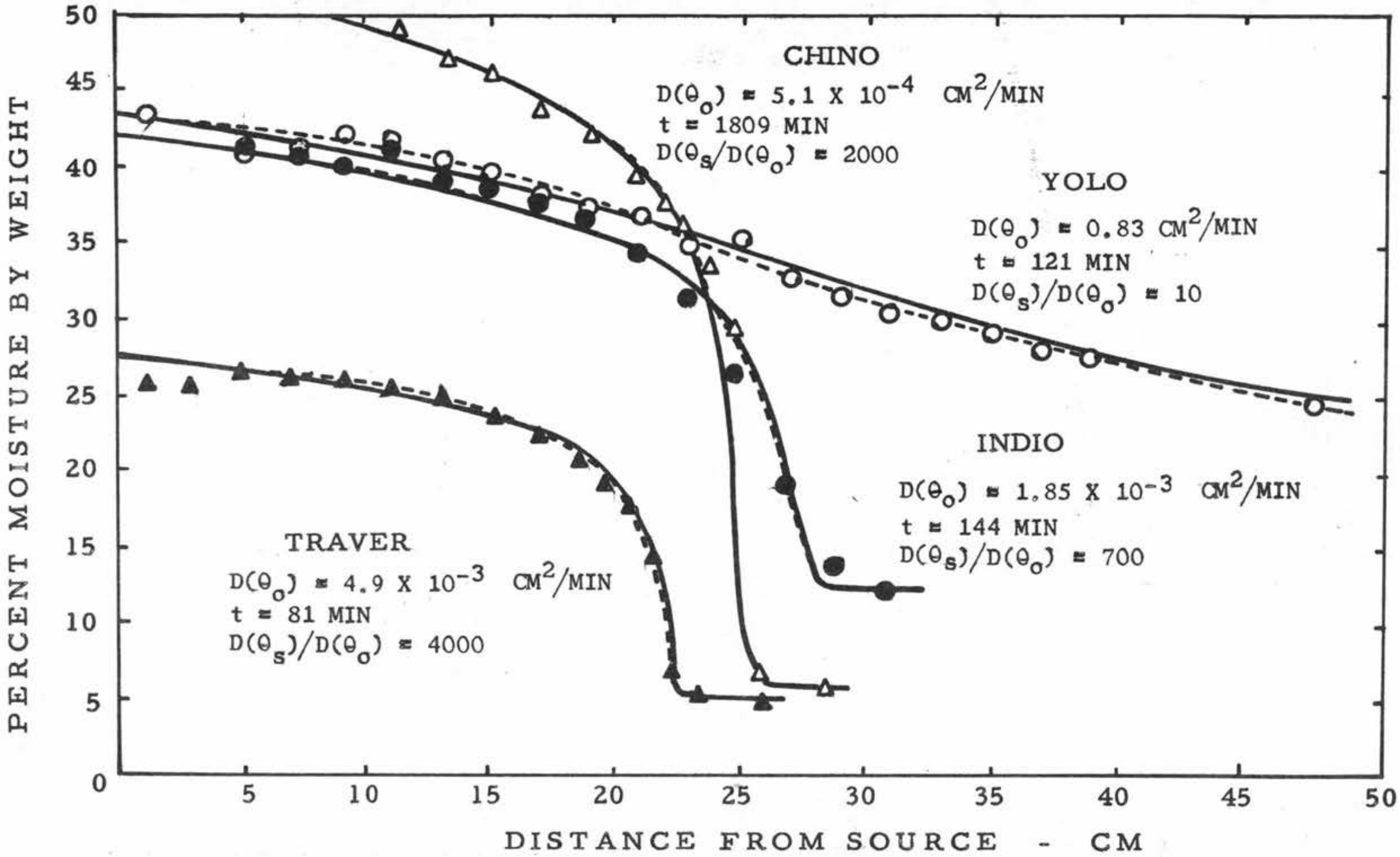

Figure 5. Moisture content distribution for the movement of water into several soils. The solid curves are solutions of the diffusion equation by a Gaussian elimination technique and the broken curves are solutions obtained by the Boltzmann transform technique using the data indicated. The broken curves and the data are those reported by Gardner and Mayhugh (19).

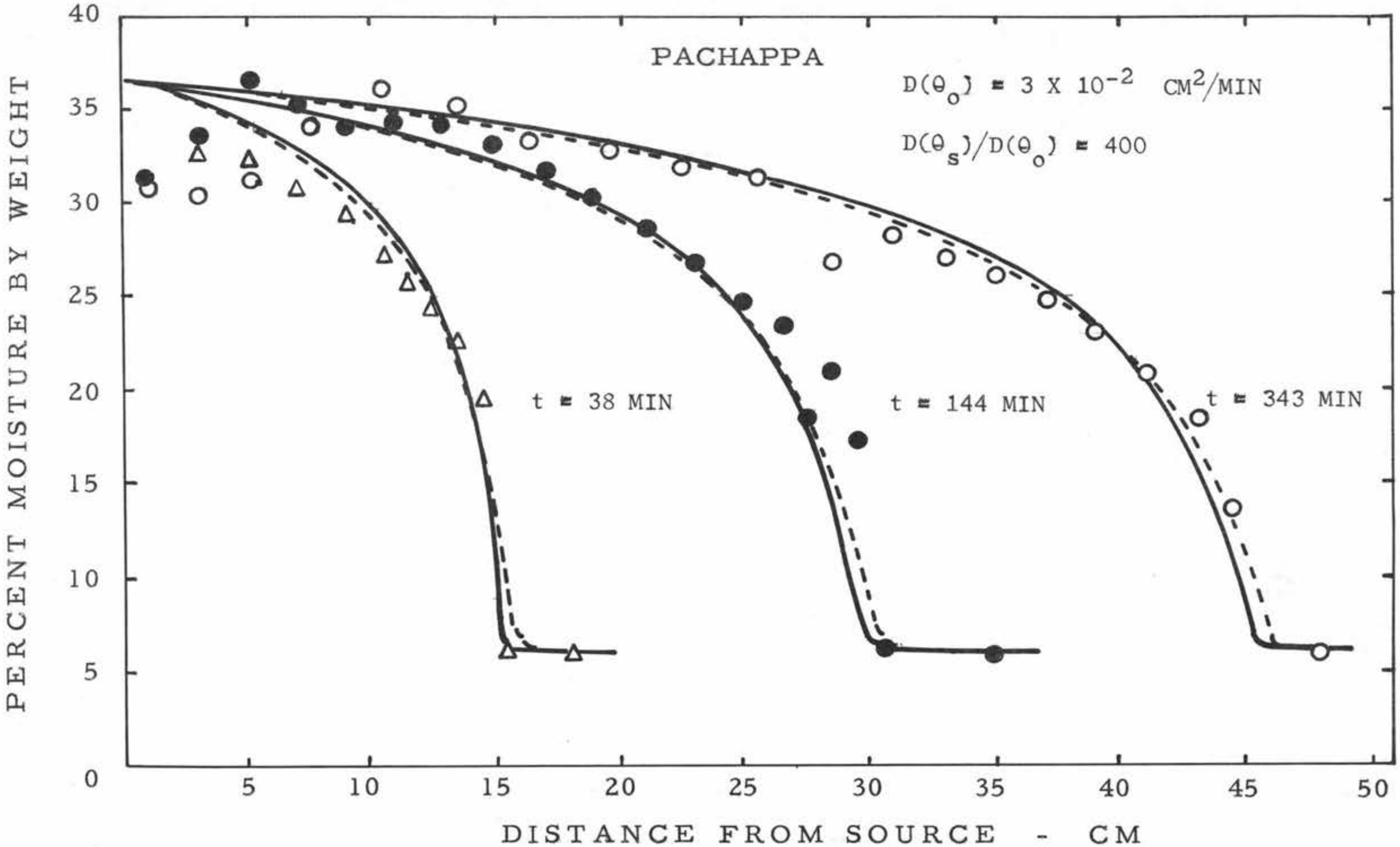

Figure 6. Moisture content distribution at three times for Pachappa soil. The solid curves represent solutions of the diffusion equation by a Gaussian elimination technique. The solid curve for 38 minutes also represents the contraction of the 144-minute curve, and the solid curve for 343 minutes represents the expansion of the 144-minute curve. The broken curves, which represent solutions obtained by the Boltzmann transform, and the data points were taken from Gardner and Mayhugh (19).

is preferable to Gaussian elimination.

Figure 6 shows moisture content distribution curves at three different times for Pachappa soil. In order to evaluate expansion and contractions of moisture content distribution curves, the curve for 144 minutes was contracted to 38 minutes and was also expanded to 343 minutes. Both the expanded and the contracted curves were essentially identical with the curves calculated for the same times by Gaussian elimination.

Calculation of Moisture Diffusivity Functions

From each experimentally determined moisture content distribution, moisture diffusivity was determined at various moisture contents using previously described methods. Examples of moisture content versus moisture diffusivity are shown in figures 7 and 8.

Since moisture diffusivity determined by the method of Bruce and Klute (1) should be independent of the length of the run, moisture diffusivity curves for several times of sampling were determined. Figure 7 shows moisture content-diffusivity curves of Chehalis loam for runs of 85.1, 24.5 and 14.4 minutes. Variation among runs is not extreme; nevertheless, there is some variation.

Figure 8 shows the moisture content-diffusivity curve for Quincy loamy sand. The length of the runs shown are 5.5, 2.9 and 1.1 minutes. There is considerably more variation among the curves for Quincy than among the curves for Chehalis. In fact, the Quincy loamy sand showed greater variation than any of the other soils.

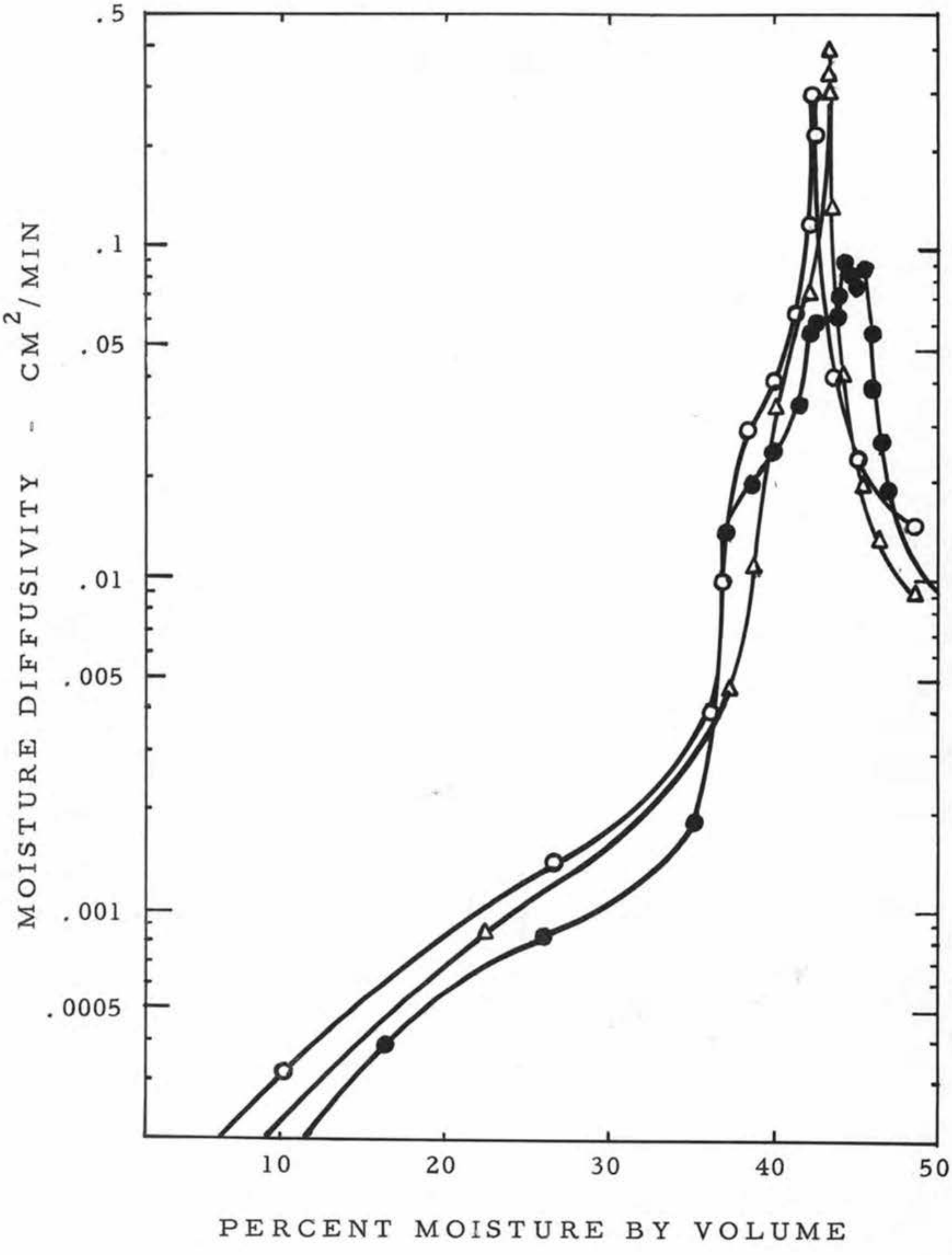

Figure 7. Moisture diffusivity as a function of moisture content for Chehalis loam. Moisture diffusivity was calculated from data collected in three separate runs, each run being sampled at a different time. The times for the runs were: ●, 85.1 minutes; O, 24.5 minutes; and Δ, 14.4 minutes.

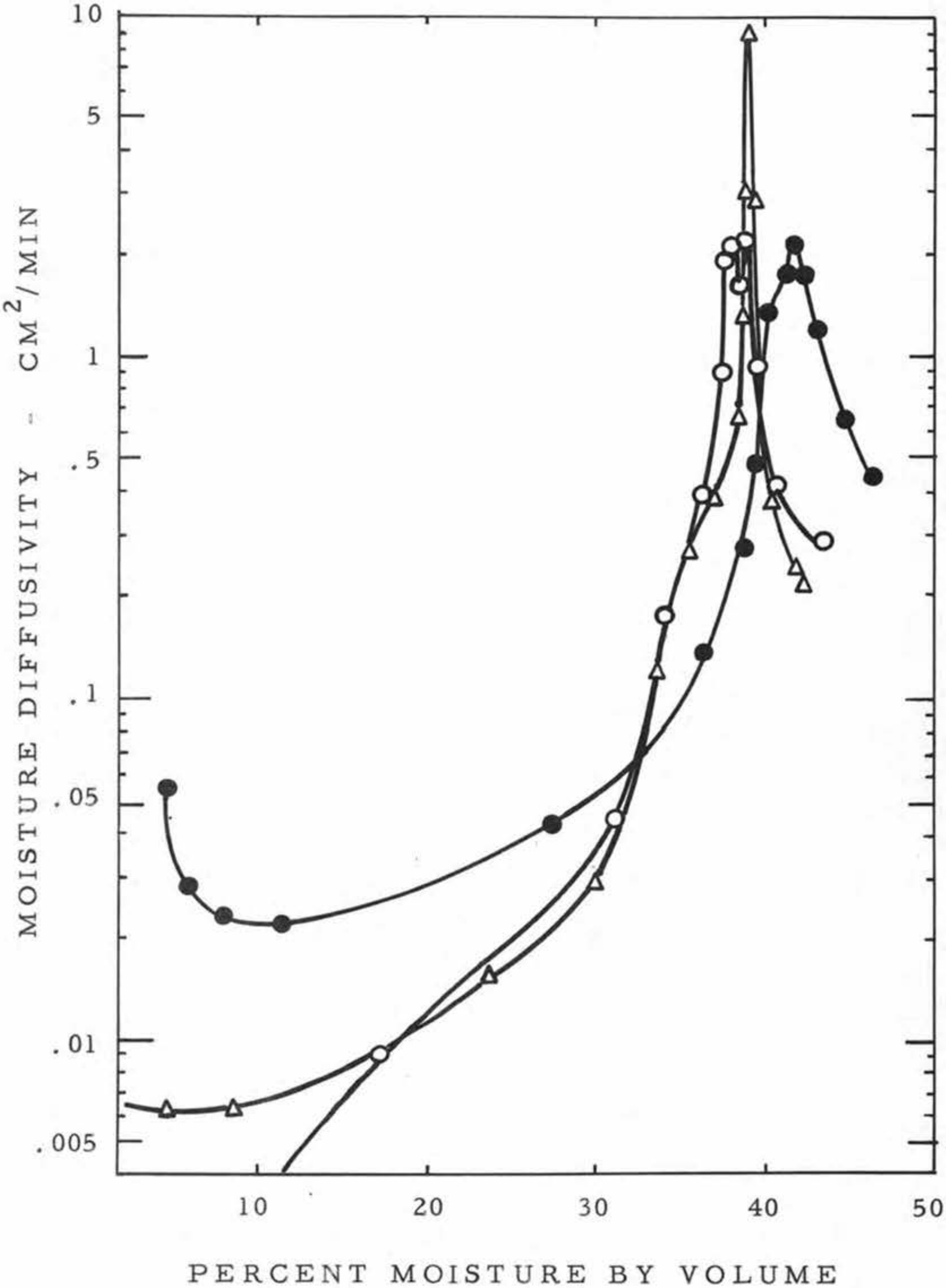

Figure 8. Moisture diffusivity as a function of moisture content for Quincy loamy sand. Moisture diffusivity was calculated from data collected in three separate runs; each run being sampled at a different time. The times for the runs were: ●, 1.1 minutes; O, 2.9 minutes; and Δ, 5.5 minutes.

The variation may have resulted from the high rate of water transmission in this soil. The wetting front moves into the Quincy loamy sand so rapidly that the percentage error in measuring time is much greater than for soils in which the wetting front does not move so rapidly. The high water transmission also causes a greater redistribution of moisture during sampling.

Since there was some variation among runs for various times, it was decided that the diffusivity approximating function should be fitted to a diffusivity curve which was an average of three runs with different sampling times. Figure 9 shows average diffusivity curves for the five soils. Each curve is an average of three runs. As all curves are plotted on the same figure, it is possible to compare diffusivity functions for the various soils. At low moisture contents the Quincy loamy sand has a diffusivity value approximately ten times as large as the Chehalis loam, and the maximum value is also approximately ten times larger. The curves are also of approximately the same shape, but the peak on Quincy occurs at a lower moisture content than the peak for Chehalis.

An approximating diffusivity function, equation (68), was fitted to the values used in plotting figure 9. Figure 10 shows values of $(\theta - a)^2 D(\theta)$ versus moisture ratio for Quincy fine sandy loam. The piece-wise exponential fit is shown as three straight line segments on the semilog plot used in this figure. Values of moisture diffusivity computed from equation (68) for Quincy loamy sand are shown in figure 11. Average diffusivity values calculated from experimental data are also shown. These average values are the

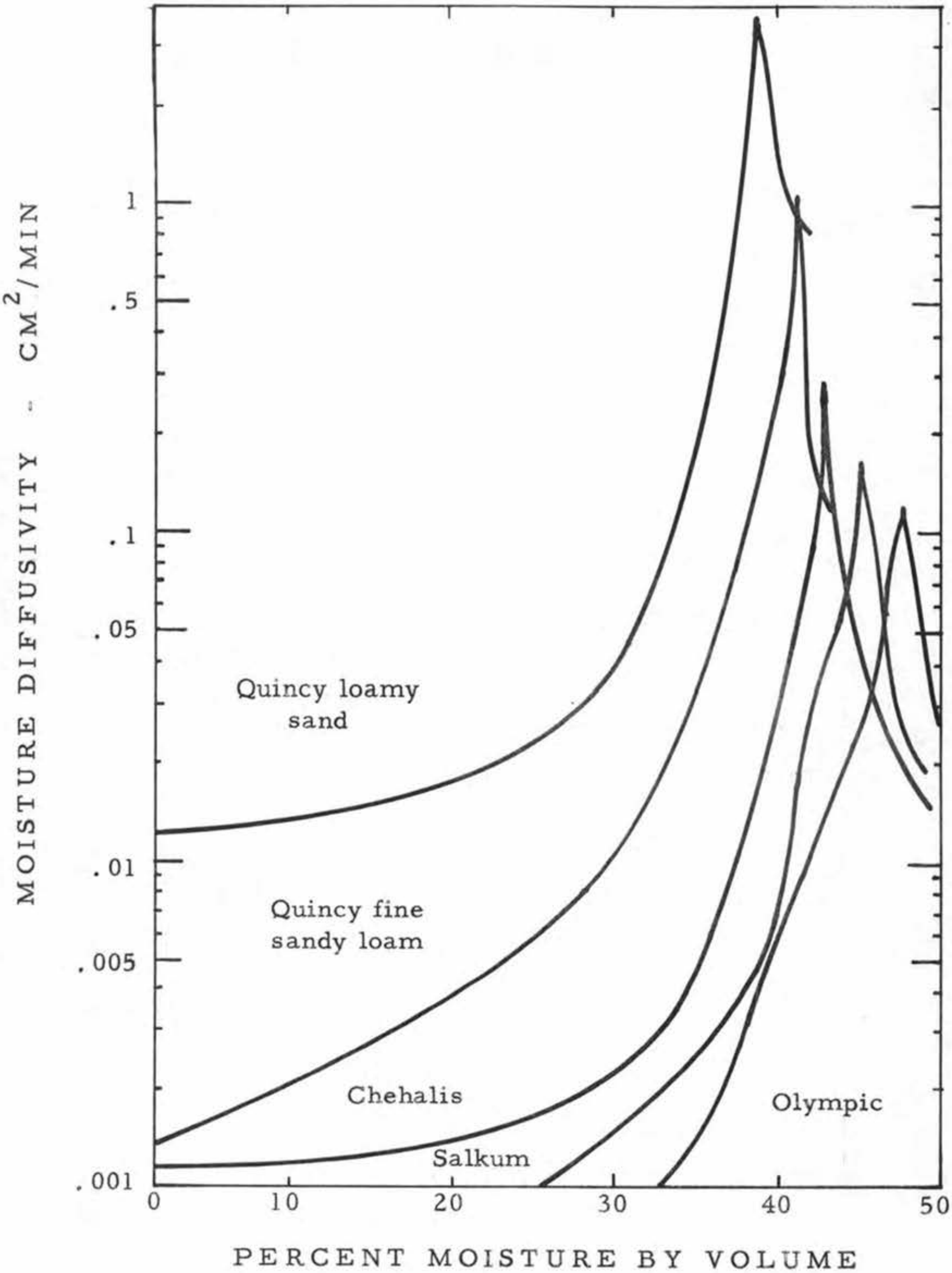

Figure 9. Average curves of moisture diffusivity versus moisture content for five soils. Each curve is an average of three runs for three different sampling times.

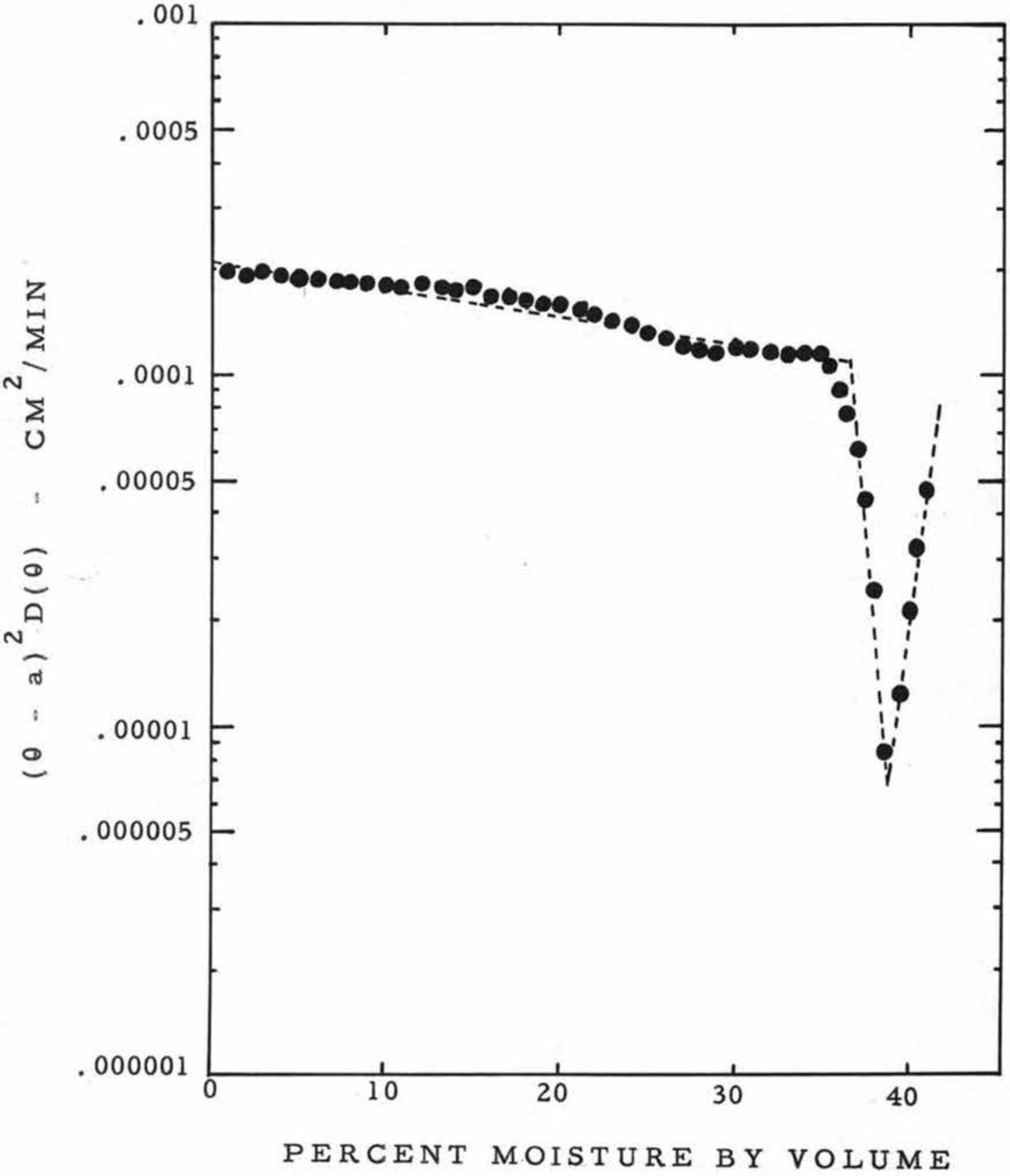

Figure 10. Values of $(\theta - a)^2 D(\theta)$ versus percent moisture for Quincy fine sandy loam. The points represent values computed from a diffusivity function which was an average of three separate runs. The line segments were fitted piecewise to the data by the method of least squares.

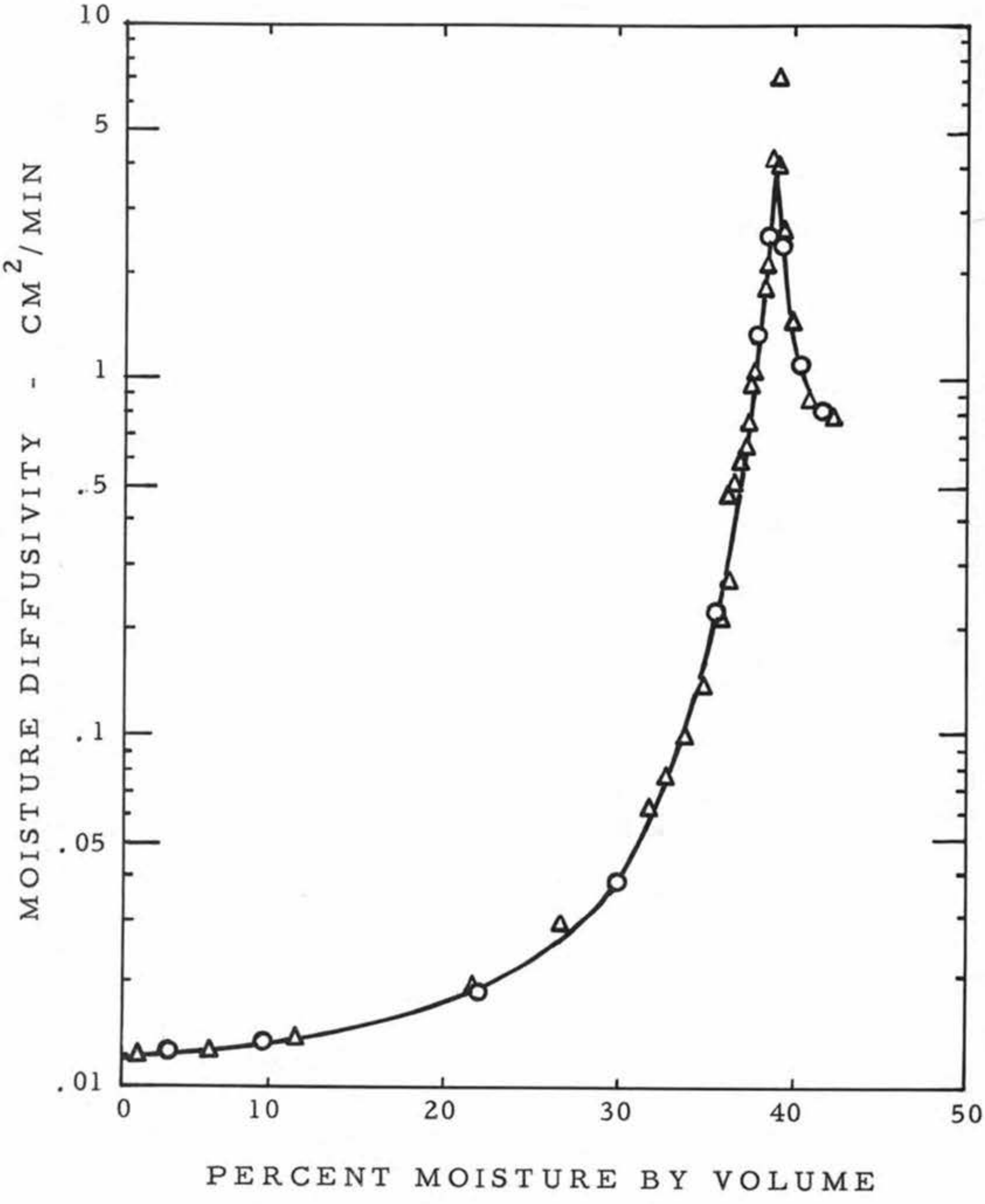

Figure 11. Moisture diffusivity versus moisture content for Quincy loamy sand. The triangles represent values calculated from equation (68) and the circles represent average values calculated from three experimental runs.

same as those used to plot the curve for Quincy loamy sand in figure 9; and they are also the values which were used to determine the parameters for equation (68). As can be seen from figure 11, the approximating diffusivity function gives a very good fit to the actual data. In fact, there is very much less scatter between experimental values and approximating values than among the experimental values for the various runs shown in figure 8. The fits on the other soils were equally as good.

Moisture content distributions were computed using the system of equations (49). Figure 12 shows the moisture content distribution for Chehalis loam at two different times. The calculated curves fit the experimental data rather well except at the wetting front. It was felt that the poor fit at the wetting front may have resulted from averaging the diffusivity functions calculated from three separate runs rather than from curve fitting errors. To check this, a moisture content distribution curve was calculated for an 85.0 minute run using diffusivity data which was also calculated from an 85.0 minute run. The comparison is shown in figure 13. This procedure gives a good fit to the experimental data.

If the diffusion equation is to be of value as a mathematical model for moisture movement in soil, the diffusivity function must be a property of the soil which is independent of the time at which the diffusivity function was determined. Therefore, curves for the other soils, which are shown in figure 14, were calculated using an approximating diffusivity function fitted to data which were averaged

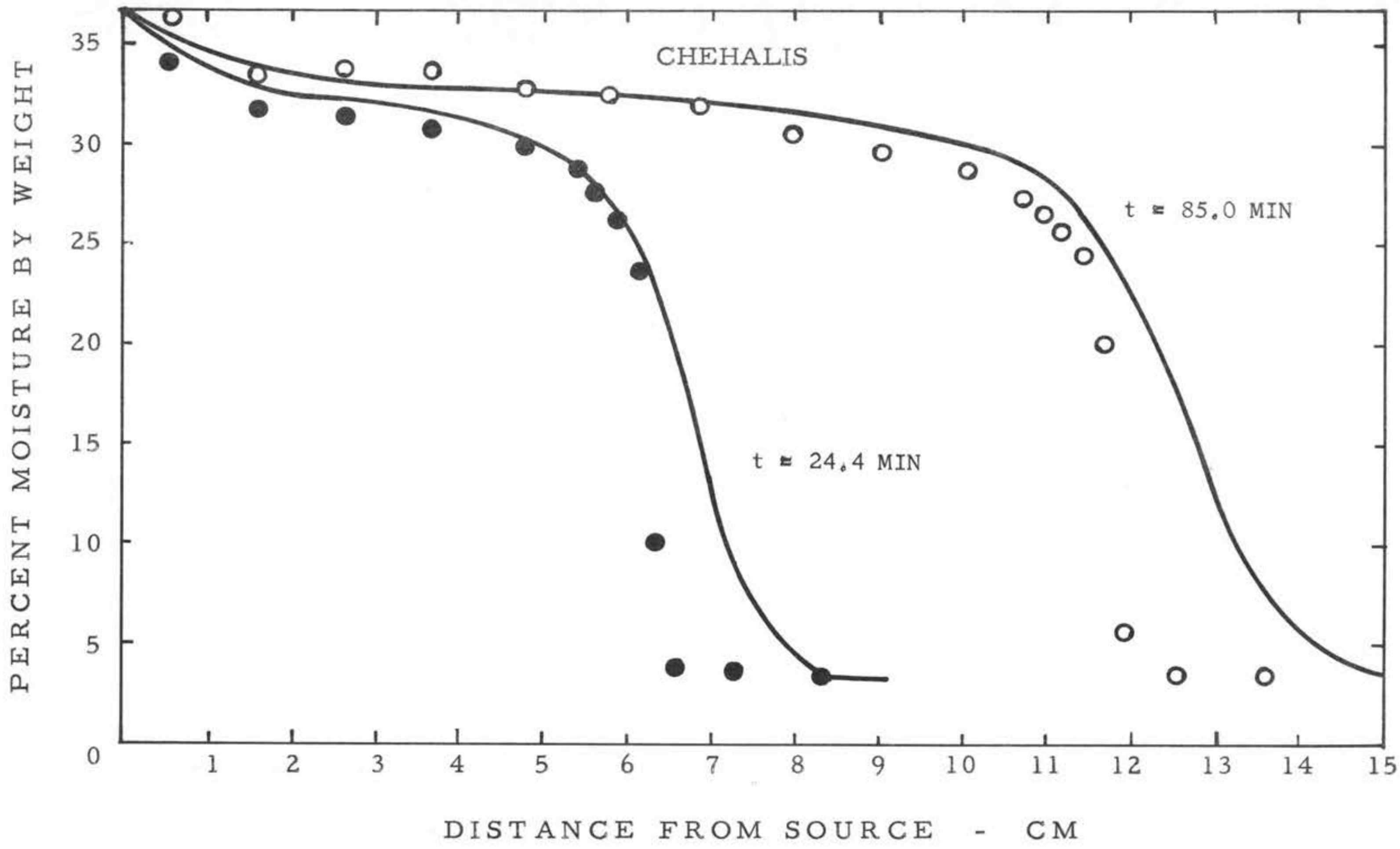

Figure 12. Moisture content distribution at two times for Chehalis loam. The solid curves represent solutions of the diffusion equations using a diffusivity function which was an average of three runs. The points indicate experimentally determined values.

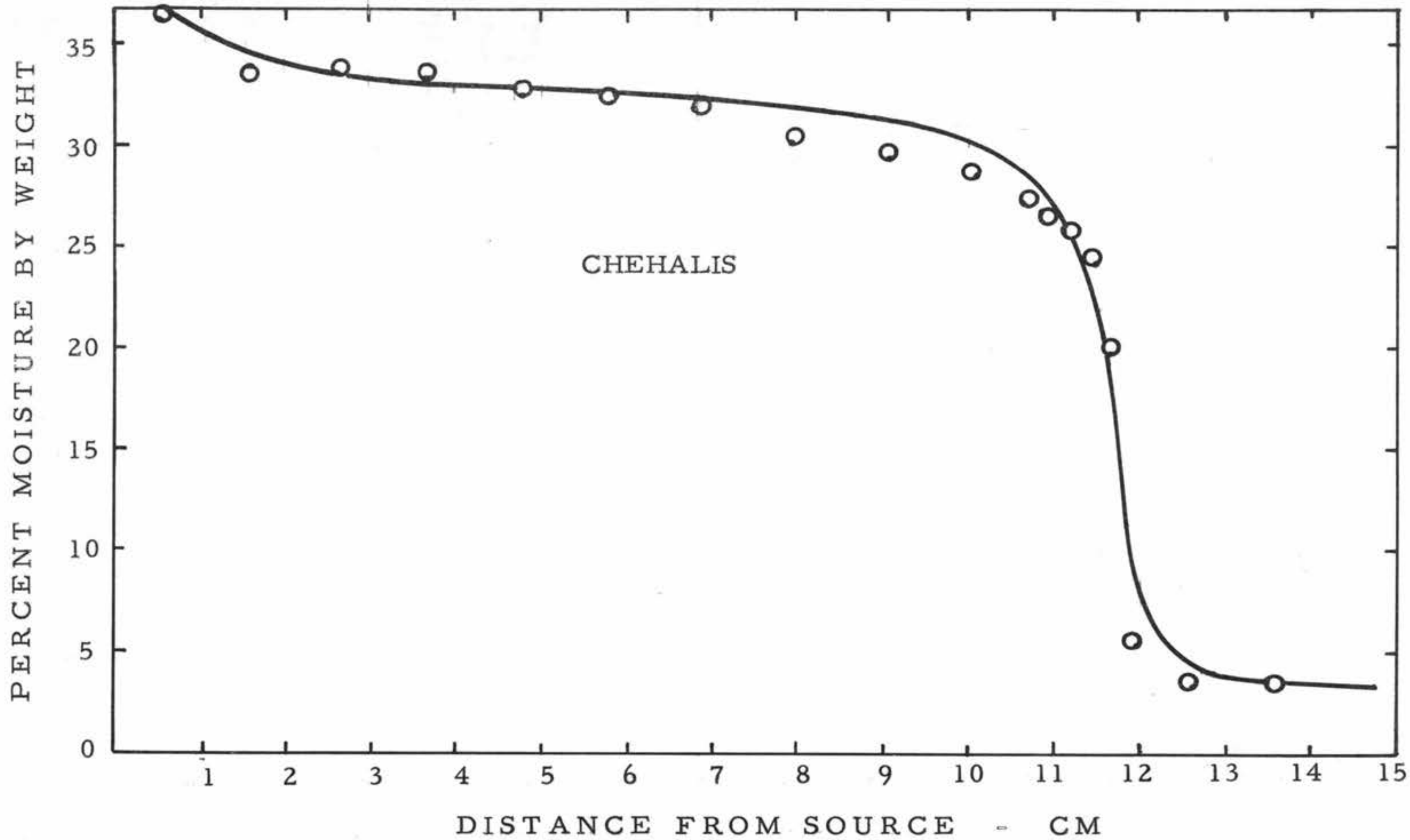

Figure 13. Moisture content distribution for Chehalis loam. The solid curve represents a solution of the diffusion equation using a diffusivity function which was determined from an 85.0 minute run. The points indicate experimently determined values for an 85.0 minute run.

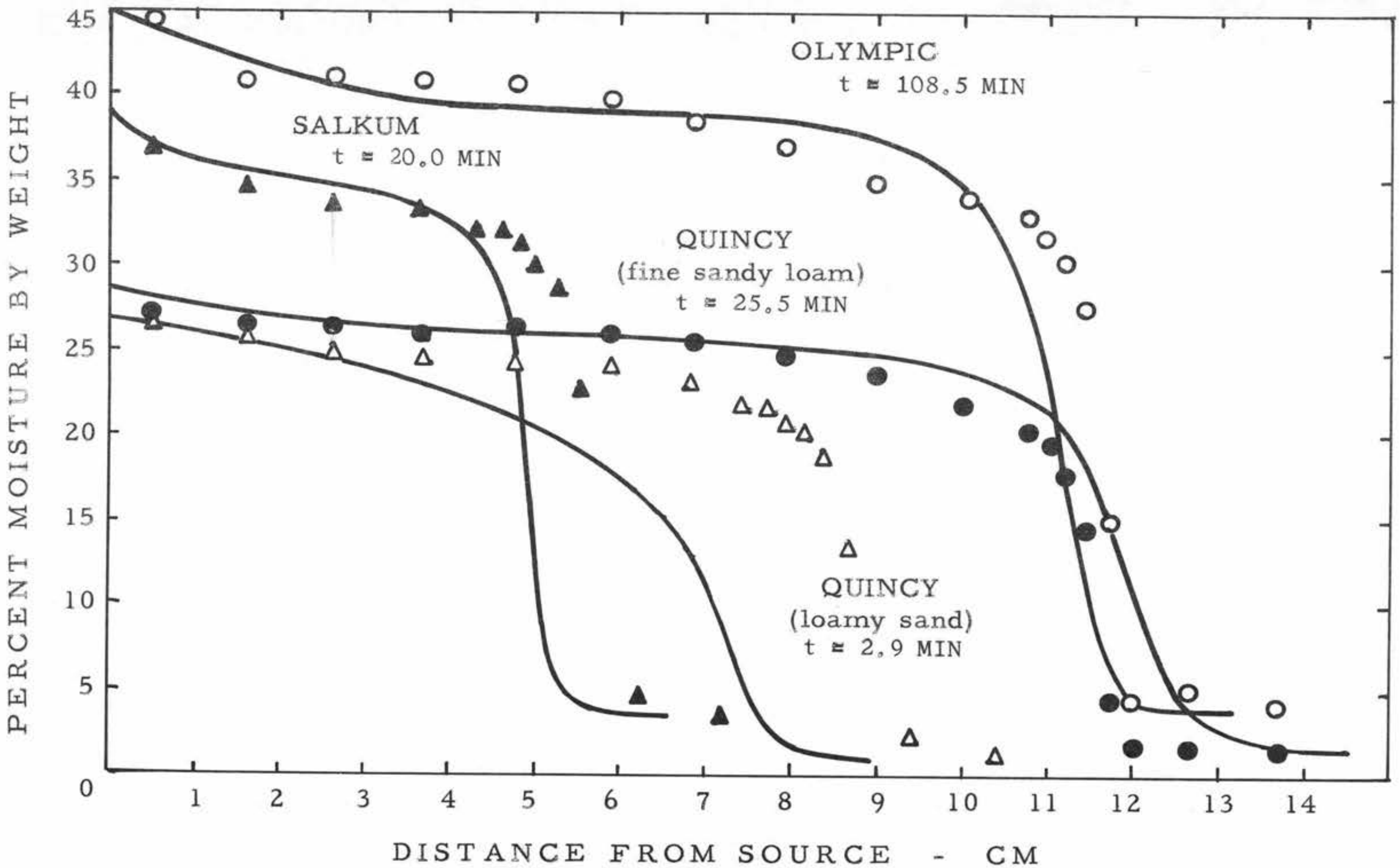

Figure 14. Moisture content distributions for four soils. The solid curves represent solutions of the diffusion equation using a diffusivity function which was an average of three runs. The points indicate experimentally determined values.

for three runs. The fit on all of these soils is poorest at the wetting front.

Quincy loamy sand showed a particularly poor fit. For this soil, even the shape of the curve was not correct. The experimental points indicate a very abrupt change in moisture content at the wetting front. The calculated curve shows a much more gradual change in moisture content near the wetting front. The variation in moisture diffusivity curves for different times was greatest for Quincy loamy sand. Since curves for three different times were averaged to determine the diffusivity function, it is not surprising that the calculated moisture content distribution for this soil should give the poorest fit to the experimental data.

Calculation of Moisture Distribution in a Layered Soil

For calculation of moisture distribution in layered soil, Chehalis loam and Quincy loamy sand were chosen. These two soils are quite different in their flow properties and should give a good test of the diffusion equation as a model of moisture movement in layered soils.

In order to satisfy the second boundary condition which must be met at the junction between the two soils, it is necessary to have a piece-wise linear function relating moisture contents of the two soils at equal water potentials. A curve relating moisture content for Quincy loamy sand and Chehalis loam is shown in figure 15.

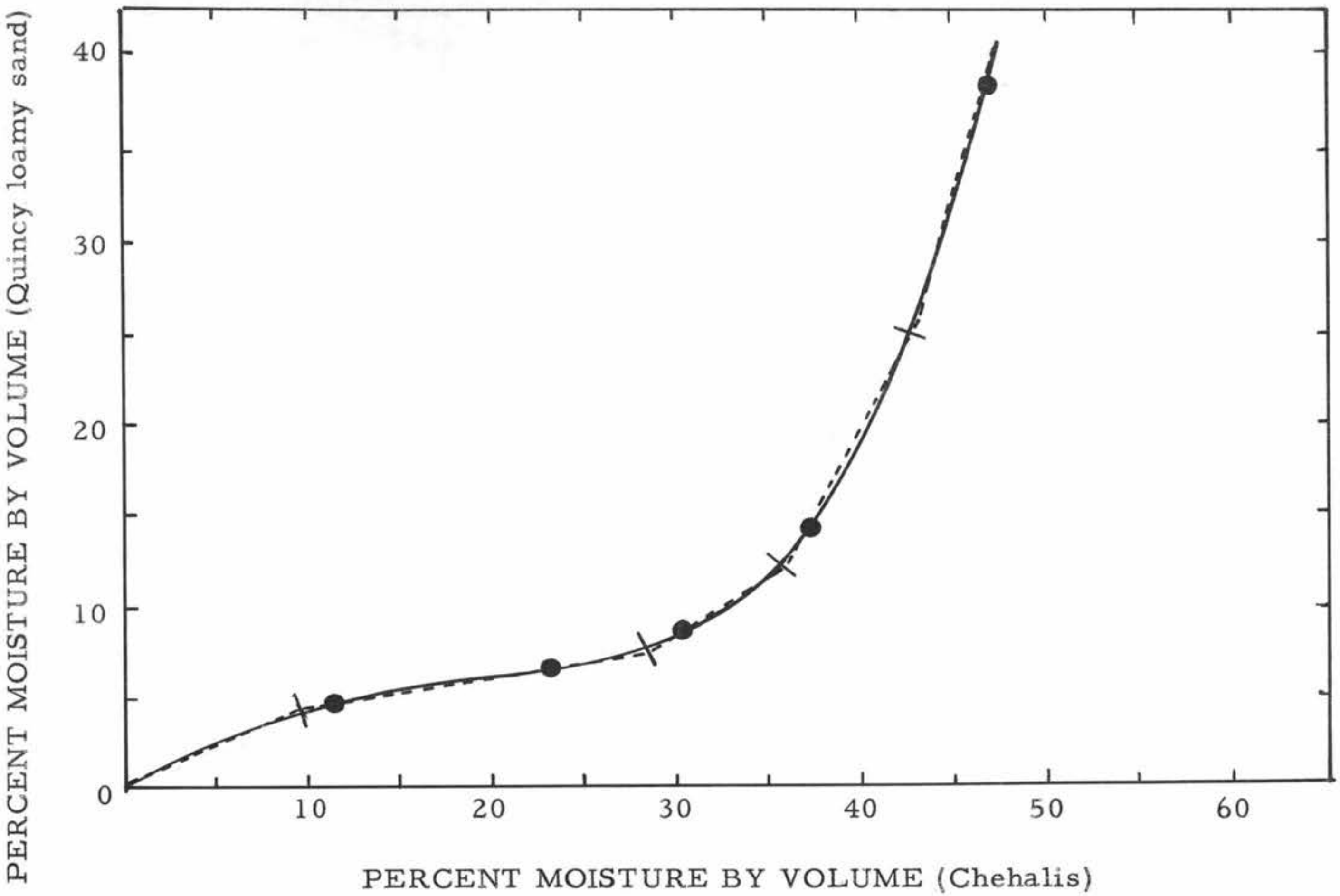

Figure 15. The moisture content of Chehalis loam plotted against the moisture content of Quincy loamy sand at corresponding water potentials. The broken curve is the piece-wise linear approximation used in the computer program.

A piece-wise linear fit is also shown.

Using the system of equations (61) and the diffusivity functions as previously calculated, the moisture content distribution was determined for combinations of Quincy loamy sand and Chehalis loam. Figure 16 shows the results of placing Quincy first followed by Chehalis. In general the curves are of about the same shape. However, the calculated moisture content was higher in the Quincy than the experimental values, and the calculated wetting front had not progressed as far in the Chehalis as the experimental wetting front. The moisture content distribution for Chehalis loam followed by Quincy loamy sand is shown in figure 17. The curve for the Chehalis portion of the soil column fits the experimental data fairly well, but for the Quincy portion the shape of the curve is quite different from the experimental values. Here, as in one-region flow, the calculated moisture distribution for Quincy loamy sand does not change as abruptly at the wetting front as the experimentally determined distribution.

The results obtained on layered soils indicate that the system of equations (61) solved by Gaussian elimination can be used to predict moisture content distribution for layered soil. Although the calculated distribution is not a good fit for the experimental data over the entire moisture range, there are several encouraging features about the fit. First, there is a sharp moisture discontinuity across the boundary between the two soils. Second, in agreement with experimental results, the calculated moisture jump across

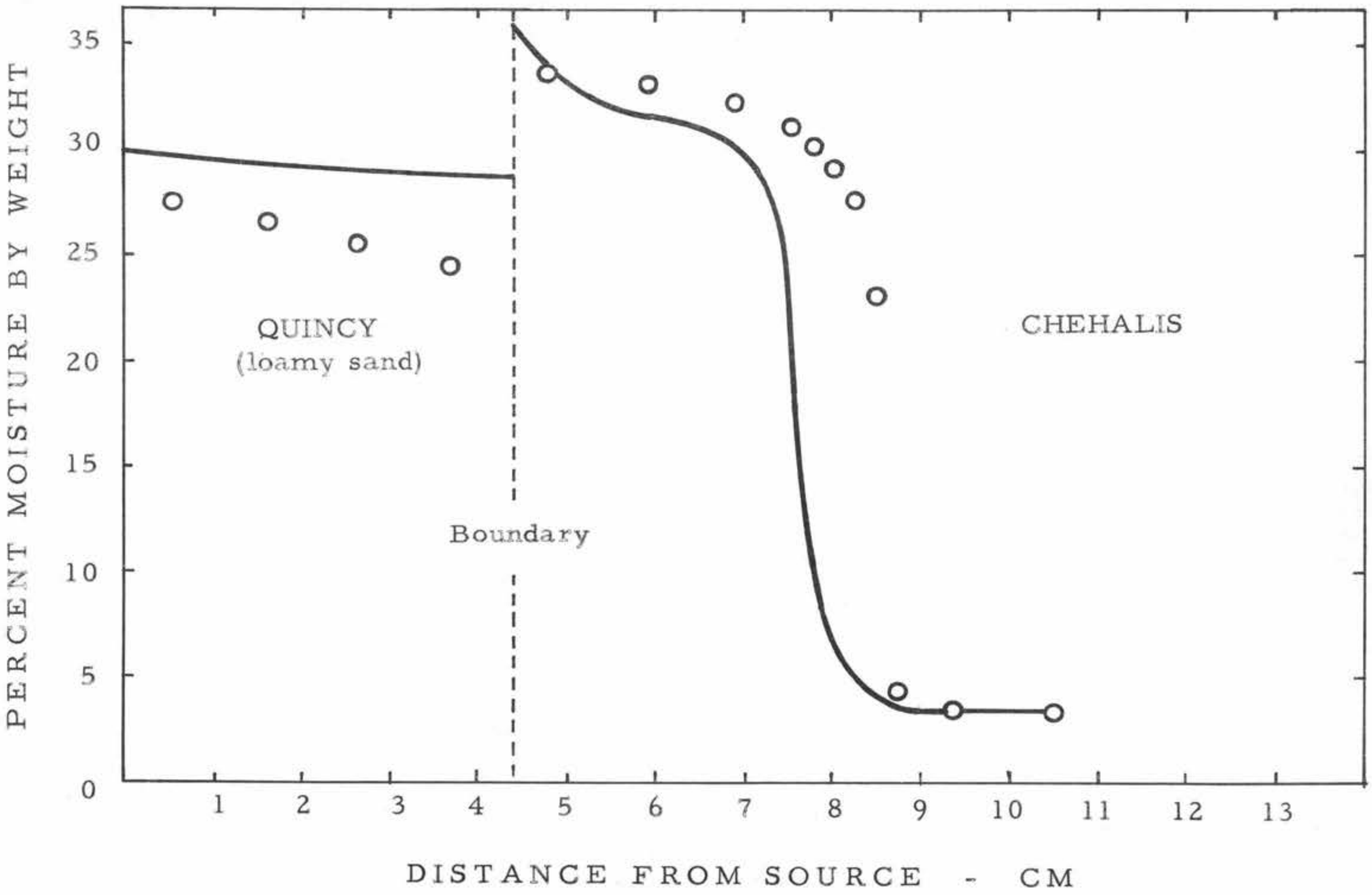

Figure 16. Moisture content distribution for Quincy loamy sand-Chehalis loam layered soil. The curves represent a solution of the diffusion equation and the points indicate experimental values. Both the solution and the experimental values were for a 9.6 minute run.

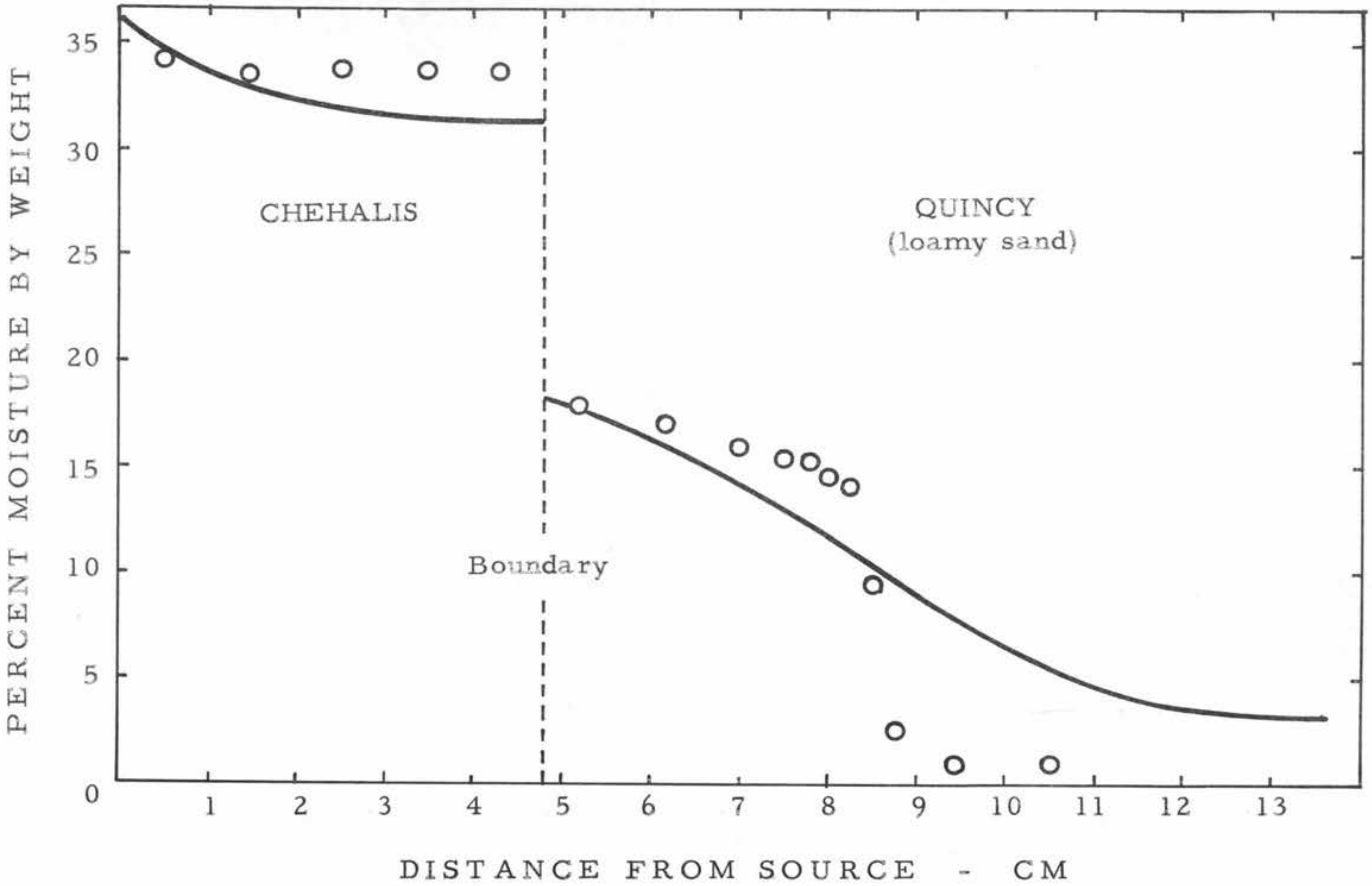

Figure 17. Moisture content distribution for Chehalis loam-Quincy loamy sand layered soil. The curves represent a solution of the diffusion equation and the points indicate experimental values. Both the solution and the experimental values were for a 34.0 minute run.

the boundary is much greater for the Chehalis-Quincy than for the Quincy-Chehalis combination. Third, the fit for the Chehalis portion of the column is much better than for the Quincy portion. This latter is to be expected since for one-region flow, the fit of Chehalis is much better than the fit of Quincy.

SUMMARY AND CONCLUSIONS

The diffusion equation has been successfully used by several investigators to describe moisture movement in soil. Their success depends largely upon the Boltzmann transformation which can only be used for semi-infinite, uniform regions.

A numerical technique is developed for solving the implicit difference analogues of the diffusion equation. This method involves 1) devising suitable difference analogues for the differential equation, 2) developing a method for choosing diffusivity to convert the non-linear difference analogues into linear equations, and 3) solving the system of linear equations by Gaussian elimination.

Boundary conditions for the junction between two media with dissimilar flow properties were also incorporated into the finite difference equations to give a special equation that can be used for horizontal movement in layered soil.

Solutions obtained by the Gaussian elimination technique were compared with solutions obtained by the Boltzmann transform technique. The results indicate that both methods of obtaining solutions give similar results for horizontal flow in semi-infinite, uniform media.

A data smoothing technique is described which greatly aids in calculating moisture diffusivity. An equation is also proposed for representing the diffusivity function and a method is given for determining the parameters in the equation.

A comparison of experimental results with solutions obtained by Gaussian elimination indicates that a solution obtained using a diffusivity function evaluated for a particular length of run is very similar to the experimental data used in evaluating the diffusivity function. However, a solution obtained using a diffusivity function that is an average for several lengths of run does not fit the experimental data as well.

Solutions were also obtained for layered soils. These solutions fit the experimental data almost as well as the solutions for uniform soil.

From this study it can be concluded that:

1) The diffusion equation seems to be a satisfactory model for isothermal moisture movement in unsaturated soil.

2) For moisture movement in soil, the accuracy of the solution is either limited by experimental difficulties in determining moisture diffusivity or assumptions made in the derivation of the diffusion equation are not met for unsaturated soil moisture flow.

3) Numerical techniques can be used to extend solutions of the diffusion equation with concentration dependent diffusivity to multi-region problems.

4) Studies should be initiated to further check the diffusion equation as a mathematical model of moisture movement in unsaturated soil for an arbitrary initial moisture content distribution, for flow in the vertical direction, and for two- and three-dimensional flow.

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