Gaussian limits for multidimensional random sequential packing at saturation

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November 2006

Abstract

Consider the random sequential packing model with infinite input and in any dimension. When the input consists of non-zero volume convex solids we show that the total number of solids accepted over cubes of volume λ is asymptotically normal as $\lambda \to \infty$. We provide a rate of approximation to the normal and show that the finite dimensional distributions of the packing measures converge to those of a mean zero generalized Gaussian field. The method of proof involves showing that the collection of accepted solids satisfies the weak spatial dependence condition known as stabilization.

1 Main results

Given $d \in \mathbb{N}$ and $\lambda \geq 1$, let $U_{1,\lambda}, U_{2,\lambda}, \ldots$ be a sequence of independent random d-vectors uniformly distributed on the cube $Q_\lambda := [0, \lambda^{1/d})^d$. Let S be a fixed bounded closed convex set in $\mathbb{R}^d$ with non-empty interior (i.e., a 'solid') with centroid at the origin $\mathbf{0}$ of $\mathbb{R}^d$ (for example, the unit ball), and for $i \in \mathbb{N}$, let $S_{i,\lambda}$ be the translate of S with centroid at $U_{i,\lambda}$. So $\mathcal{S}_\lambda := (S_{i,\lambda})_{i \geq 1}$ is an infinite sequence of solids arriving at uniform random positions in Q_λ (the centroids lie in Q_λ but the solids themselves need not lie wholly inside Q_λ).

Let the first solid $S_{1,\lambda}$ be *packed*, and recursively for $i = 2, 3, \ldots$, let the i-th solid $S_{i,\lambda}$ be packed if it does not overlap any solid in $\{S_{1,\lambda}, \ldots, S_{i-1,\lambda}\}$ which has already been packed. If not packed,

American Mathematical Society 2000 subject classifications. Primary 60F05, Secondary 60D05, 60K35

Key words and phrases. Random sequential packing, central limit theorem, infinite input, stabilizing measures

[1] Research partially supported by the Polish Minister of Scientific Research and Information Technology grant 1 P03A 018 28 (2005-2007)

[2] Research supported in part by NSF grant DMS-0203720

the i-th solid is discarded; we sometimes use *accepted* as a synonym for 'packed'. This process, known as *random sequential adsorption (RSA) with infinite input*, is irreversible and terminates when it is not possible to accept additional solids. At termination, we say that the sequence of solids $\mathcal{S}_\lambda$ *jams* Q_λ or *saturates* Q_λ. The *jamming* number $N_\lambda := N_\lambda(\mathcal{S}_\lambda)$ denotes the number of solids accepted in Q_λ at termination. We use the words 'jamming' and 'saturation' interchangeably in this paper.

Jamming numbers N_λ arise naturally in the physical, chemical, and biological sciences. They are considered in the description of the irreversible deposition of colloidal particles on a substrate (see the survey [1] and the special volume [20]), hard sphere interactions in point processes (see [26] and [7], Section 4.8), adsorption modelling (see [4] and the surveys [9, 25]) and also in the modelling of communication and reservation protocols (see [5, 6]).

The extensive body of experimental results related to the large scale behavior of packing numbers stands in sharp contrast with the limited collection of rigorous mathematical results, especially in $d \geq 2$. The main obstacle to a rigorous mathematical treatment of the packing process is that the short range interactions of arriving particles create long range spatial dependence, thus turning N_λ into a sum of spatially correlated random variables. Equilibrium systems (where particles are allowed to depart as well as arrive) present a different set of mathematical challenges and are not considered here.

In the case where $d = 1$ and $S = [-1/2, 1/2]$, a famous result of Rényi [21] shows that *jamming limit*, defined as $\lim_{\lambda\to\infty} \lambda^{-1}\mathbb{E}\, N_\lambda$, exists as an integral which evaluates to roughly 0.748; also in this case, Mackenzie [11] shows that $\lim_{\lambda\to\infty} \lambda^{-1}\mathrm{Var}N_\lambda$ exists as an integral which evaluates to roughly 0.03815. Dvoretzky and Robbins [8] show that the jamming numbers N_λ are asymptotically normal as $\lambda \to \infty$, but their techniques do not address the case $d > 1$.

The above results were established in the 1960s and progress in extending them rigorously to higher dimensions has been slow until recently. Penrose [12] establishes the existence of a jamming limit for any $d \geq 1$ and any choice of S, and also [13] obtains a CLT for a related model (monolayer ballistic deposition with a rolling mechanism) but comments in [13] that 'Except in the case $d = 1$ ... a CLT for infinite-input continuum RSA remains elusive.'

In the present work we show for any d and any S that $\lambda^{-1}\mathrm{Var}N_\lambda$ converges to a positive limit and that N_λ satisfies a central limit theorem, i.e., the fluctuations of the random variable N_λ are indeed Gaussian in the large λ limit. This puts the recent experimental results and Monte Carlo simulations of Quintanilla and Torquato [22] and Torquato (ch. 11.4 of [26]) on rigorous footing.

We also provide a bound on the rate of convergence to the normal, and on the rate of convergence of $\lambda^{-1}\mathbb{E}\, N_\lambda$ to the jamming limit.

Throughout $\mathcal{N}(0,1)$ denotes a mean zero normal random variable with variance one.

Theorem 1.1 *Let $\mathcal{S}_\lambda$ be as above and put $N_\lambda := N_\lambda(\mathcal{S}_\lambda)$. There are constants $\mu := \mu(S,d) \in (0,\infty)$ and $\sigma^2 := \sigma^2(S,d) \in (0,\infty)$ such that as $\lambda \to \infty$ we have*

$$|\lambda^{-1}\mathbb{E}\, N_\lambda - \mu| = O(\lambda^{-1/d}) \tag{1.1}$$

and $\lambda^{-1}\mathrm{Var} N_\lambda \to \sigma^2$ with

$$\sup_{t\in\mathbb{R}} \left| P\left[\frac{N_\lambda - \mathbb{E}\, N_\lambda}{(\mathrm{Var} N_\lambda)^{1/2}} \leq t \right] - P[\mathcal{N}(0,1) \leq t] \right| = O((\log \lambda)^{3d} \lambda^{-1/2}). \tag{1.2}$$

The process of accepted solids in Q_λ induces a natural random point measure ν_λ on $[0,1]^d$ given by

$$\nu_\lambda := \sum_{i=1}^{\infty} \delta_{\lambda^{-1/d} U_{i,\lambda}} \mathbf{1}_{\{S_{i,\lambda} \text{is accepted}\}} \tag{1.3}$$

where δ_x stands for the unit point mass at x. It also induces a natural random volume measure ν'_λ on $\mathbb{R}^d$, normalized to have the same total measure as ν_λ, defined for all Borel $A \subseteq \mathbb{R}^d$ by

$$\nu'_\lambda(A) := \frac{\lambda}{|S|} \left| A \cap \left(\bigcup [\lambda^{-1/d} S_{i,\lambda} : i \geq 1, S_{i,\lambda} \text{ is accepted}] \right) \right| \tag{1.4}$$

where $|\cdot|$ denotes Lebesgue measure and $\lambda^{-1/d}A := \{\lambda^{-1/d}x : x \in A\}$. The measure ν'_λ is not necessarily supported by Q_1 due to boundary effects, but for sufficiently large λ it is supported by Q_1^+, where we set $Q_1^+ := [-1,2)^d$ (a fattened version of Q_1).

Let $\bar{\nu}_\lambda := \nu_\lambda - \mathbb{E}[\nu_\lambda]$ and $\bar{\nu}'_\lambda := \nu'_\lambda - \mathbb{E}[\nu'_\lambda]$. Let $\mathcal{R}(Q_1^+)$ denote the class of bounded, almost everywhere continuous functions on Q_1^+. For $f \in \mathcal{R}(Q_1^+)$ and μ a signed measure on $\mathbb{R}^d$ with finite total mass, let $\langle f, \mu \rangle := \int_{\mathbb{R}} f d\mu$. The following theorem provides the limit theory (law of large numbers and central limit theorems) for the integrals of test functions $f \in \mathcal{R}(Q_1^+)$ against the random point measure ν_λ and the random volume measure ν'_λ induced by the packing process. In particular, it shows that the finite dimensional distributions of the centered packing point measures $(\bar{\nu}_\lambda)_\lambda$ converge to those of a certain mean zero generalized Gaussian field, namely white noise on Q_1 with variance σ^2 per unit volume, and likewise for the centered packing volume measures $(\bar{\nu}'_\lambda)_\lambda$.

Theorem 1.2 *Let μ and σ^2 be as in Theorem 1.1. Then for any f, g in $\mathcal{R}(Q_1^+)$,*

$$\lim_{\lambda\to\infty} \lambda^{-1}\mathbb{E}\,[\langle f, \nu_\lambda \rangle] = \mu \int_{[0,1]^d} f(x) dx$$

and

$$\lim_{\lambda\to\infty} \lambda^{-1}\mathrm{Cov}(\langle f,\nu_\lambda\rangle,\langle g,\nu_\lambda\rangle) = \sigma^2 \int_{[0,1]^d} f(x)g(x)dx.$$

Also, the finite-dimensional distributions of the random field $(\lambda^{-1/2}\langle f,\bar{\nu}_\lambda\rangle, f \in \mathcal{R}(Q_1^+))$ *converge as* $\lambda \to \infty$ *to those of a mean zero generalized Gaussian field with covariance kernel*

$$(f,g) \mapsto \sigma^2 \int_{[0,1]^d} f(x)g(x)dx, \quad f,g \in \mathcal{R}(Q_1^+).$$

Moreover, the same conclusions hold with ν_λ *and* $\bar{\nu}_\lambda$ *replaced by* ν'_λ *and* $\bar{\nu}'_\lambda$ *respectively.*

The remainder of this paper is organized as follows. Section 2 provides the general limit theory (weak law of large numbers and a central limit theorem) for spatial measures which satisfy a weak dependency condition termed stabilization and which are defined in terms of point sets in $\mathbb{R}^d \times \mathbb{R}_+$. Section 3 shows that the correlations of packing status of solids decay exponentially with the distance between them, thus showing that the packing measures ν_λ satisfy the stabilization criteria of the general results in Section 2. Finally, Section 4 shows that the convexity hypothesis implies a non-zero limiting variance σ^2.

Remarks.

1. *Finite input.* Let $\tau \in (0,\infty)$ and let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. Inputting only the first $\lceil \lambda\tau \rceil$ solids of the sequence $\mathcal{S}_\lambda$ yields RSA packing of the cube Q_λ with *finite input.* The finite-input packing number, i.e., the total number of solids accepted from $S_{1,\lambda}, S_{2,\lambda}, ..., S_{\lceil \tau\lambda \rceil,\lambda}$, is asymptotically normal as $\lambda \to \infty$ with τ fixed. This is proved in [17], and extended in [3] to the case where the spatial coordinates come from a non-homogeneous point process. Packing measures induced by RSA packing with finite input have finite dimensional distributions converging to those of a mean zero generalized Gaussian field with a covariance structure depending upon the underlying density of points [2], [3].

2. *Stabilization.* One might expect that the restriction of the packing measure ν_λ or ν'_λ to a localized region of space depends only on incoming particles with 'nearby' spatial locations, in some well-defined sense. This local dependency property is denoted *stabilization*; when the region of spatial dependency has a diameter with an exponentially decaying tail, it is called *exponential stabilization.* These notions are spelt out in general terms in Section 2. Theorem 2.1 provides a general spatial limit theory for exponentially stabilizing measures; this is an infinite-input analog to known results [3, 14, 15, 16] for the finite-input setting, and is of independent interest.

A form of stabilization for infinite input RSA was proved in [12], but without any tail bounds. *Exponential* stabilization in the infinite input setting is perhaps not surprising, but it has been challenging to rigorously establish this key localization feature. In Section 3, we show that infinite-input packing measures stabilize exponentially, so that the general results of Section 2 are applicable to these measures.

3. *Related models* in the literature (see e.g. [17]) include cooperative sequential adsorption, RSA with solids of random size or shape, ballistic deposition with a rolling mechanism, and spatial birth-growth models. For all of these, limit theorems in the finite-input setting are discussed in [13]. It seems likely that these can be extended to the infinite-input setting using the methods of this paper, although we do not discuss any of them in detail. Nor do we consider non-homogeneous point processes as input.

4. *Rates of convergence.* Even in $d = 1$, the rate given by Theorem 1.1 is new. Quintanilla and Torquato [22] use Monte Carlo simulations to predict convergence of the distribution function for N_λ to that of a normal, but they do not obtain rates. Penrose and Yukich [19] obtain rates of approximation to the normal for RSA packing with finite (Poisson) input.

5. *Numerical values.* We do not provide any new analytical methods for computing numerical values of μ and σ^2 when $d \geq 1$. For $1 \leq d \leq 6$, Torquato et al. [27] employ numerical and theoretical methods to estimate μ and other structural characteristics for infinite input RSA packing.

6. *Jamming variability.* A significant amount of work is needed (see Section 4) to show that the limiting variance σ^2 in Theorems 1.1 and 1.2 is non-zero, and we prove this using the following notions.

Given $L > 0$, we shall us say that a point set $\eta \subset \mathbb{R}^d \setminus [0, L]^d$ is *admissible* if the translates of S centered at the points of η are non-overlapping. Given such an η, let $N[[0, L]^d|\eta]$ denote the (random) number of solids from the sequence $\mathcal{S}_{L^d}$ which are packed in $[0, L]^d$ given the *pre-packed configuration* η. In other words, $N[[0, L]^d|\eta]$ arises as the number of solids packed in $[0, L]^d$ in the course of the usual infinite input packing process subject to the additional rule that an incoming solid is discarded should it overlap any solid centered at a point of η. Say that the convex body S has *jamming variability* if there exists a $L > 0$ such that $\inf_\eta \mathrm{Var} N[[0, L]^d|\eta] > 0$ with the infimum taken over admissible point sets $\eta \subset \mathbb{R}^d \setminus [0, L]^d$.

In Proposition 4.1 we shall show that each bounded convex body $S \subset \mathbb{R}^d$ with non-empty interior has jamming variability; here we use the assumed convexity of S.

2 Terminology, auxiliary results

Let $\mathbb{R}_+ := [0, \infty)$. Given a point $(x_1, \ldots, x_d, t) = (x, t) \in \mathbb{R}^d \times \mathbb{R}_+$, the first d coordinates of the point will be interpreted as spatial components with the $(d+1)$-st regarded as a time mark. We shall need to consider point sets which are finite in the spatial directions and locally finite in the time direction, formally defined as follows.

Definition 2.1 *A point set $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$ is* temporally locally finite *(or TLF for short) if $\mathcal{X} \cap (\mathbb{R}^d \times [0, t])$ is finite for all $t > 0$.*

In this section we adapt the general results and terminology from [3, 15, 16, 19] on limit theory for stabilizing spatial measures defined in terms of finite point sets in $\mathbb{R}^d$, to the setting of spatial measures defined in terms of TLF point sets in $\mathbb{R}^d \times \mathbb{R}_+$ (typically obtained as Poisson processes). In subsequent sections, we show that these general results can be applied to obtain the limit theorems for RSA described in Section 1.

For $x \in \mathbb{R}^d$ and $r > 0$, let $B_r(x)$ denote the Euclidean ball centered at x of radius r. We abbreviate $B_r(\mathbf{0})$ by B_r. Given $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, $a > 0$ and $y \in \mathbb{R}^d$, we let $y + a\mathcal{X} := \{(y + ax, t) : (x, t) \in \mathcal{X}\}$; in other words, scalar multiplication and translation on $\mathbb{R}^d \times \mathbb{R}_+$ act only on the spatial components. For $A \subset \mathbb{R}^d$ we write $y + aA$ for $\{y + ax : x \in A\}$; also, we write ∂A for the boundary of A, and write A_+ for $A \times \mathbb{R}_+$. Let $|\cdot|$ denote the Euclidean norm, and for nonempty subsets A, A' of $\mathbb{R}^d$, set $D_2(A, A') := \inf\{|x - y| : x \in A, y \in A'\}$.

Let $\xi(\mathcal{X}, A)$ be an $\mathbb{R}_+$-valued function defined for all pairs $(\mathcal{X}, A)$, where $\mathcal{X}$ is a TLF subset of $\mathbb{R}^d \times \mathbb{R}_+$ and A is a Borel subset of $\mathbb{R}^d$. *Assume throughout this section that ξ satisfies the following criteria*:

1. $\xi(\cdot, A)$ is measurable for each Borel A,
2. $\xi(\mathcal{X}, \cdot)$ is a finite measure on $\mathbb{R}^d$ for each TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$,
3. ξ is *translation invariant*, that is $\xi(i + \mathcal{X}, i + A) = \xi(\mathcal{X}, A)$ for all $i \in \mathbb{Z}^d$, all TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, and all Borel $A \subseteq \mathbb{R}^d$,
4. ξ is *uniformly locally bounded* (or just *bounded* for short) in the sense that there is a finite constant $||\xi||_\infty$ such that for all TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}^+$ we have

$$\xi(\mathcal{X}, [0, 1]^d) \leq ||\xi||_\infty. \tag{2.1}$$

5. ξ is *locally supported*, i.e. there exists a constant ρ such that $\xi(\mathcal{X}, A) = 0$ whenever $D_2(\mathcal{X}, A) > \rho$.

Note that if $\xi(\mathcal{X}, \cdot)$ is a point measure supported by the points of $\mathcal{X}$, then ξ is locally supported (in fact, in this case we can set $\rho = 0$).

For all $\lambda > 0$, let $\mathcal{P}_\lambda$ denote a homogeneous Poisson point process in $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\lambda dx \times ds$, with dx denoting Lebesgue measure on $\mathbb{R}^d$ and ds Lebesgue measure on $\mathbb{R}_+$. We put $\mathcal{P} := \mathcal{P}_1$. Here and henceforth we shall assume that the point processes, random fields, and random variables considered in this paper are all defined on a common underlying probability space $(\Omega, \mathcal{F}, P)$.

Thermodynamic limits and central limit theorems for functionals in geometric probability are often proved by showing that the functionals satisfy a type of local spatial dependence known as stabilization [3, 14, 15, 16, 17, 18, 24] and that will be our goal here as well. First, we adapt the definitions in [3, 14, 15] to the context of measures defined in terms of TLF point sets in $\mathbb{R}^d$. Recall that Q_λ denotes the cube $[0, \lambda^{1/d})^d$.

Definition 2.2 *We say ξ is* homogeneously stabilizing *if there exists an almost surely finite random variable R' (a* radius of homogeneous stabilization *for ξ) such that for all TLF $\mathcal{X} \subset (\mathbb{R}^d \backslash B_{R'})_+$ we have*

$$\xi((\mathcal{P} \cap (B_{R'})_+) \cup \mathcal{X}, Q_1) = \xi(\mathcal{P} \cap (B_{R'})_+, Q_1). \tag{2.2}$$

We say ξ is exponentially stabilizing *if (i) it is homogeneously stabilizing and R' can be chosen so that $\limsup_{L\to\infty} L^{-1} \log P[R' > L] < 0$, and (ii) for all $\lambda \geq 1$ and all $i \in \mathbb{Z}^d$, there exists a random variable $R := R^\xi(i, \lambda)$ (a* radius of stabilization *for ξ at i with respect to $\mathcal{P}$ in $(Q_\lambda)_+$) such that for all TLF $\mathcal{X} \subset [Q_\lambda \ \backslash B_R(i)]_+$, and all Borel $A \subseteq Q_1$, we have*

$$\xi\left((\mathcal{P} \cap [B_R(i) \cap Q_\lambda]_+) \cup \mathcal{X}, i + A\right) = \xi\left(\mathcal{P} \cap [B_R(i) \cap Q_\lambda]_+, i + A\right) \tag{2.3}$$

and moreover the tail probability $\tau(L)$ defined for $L > 0$ by

$$\tau(L) := \sup_{\lambda \geq 1,\ i \in \mathbb{Z}^d} P[R^\xi(i, \lambda) > L] \tag{2.4}$$

satisfies $\limsup_{L\to\infty} L^{-1} \log \tau(L) < 0$.

Loosely speaking, $R := R^\xi(i, \lambda)$ is a radius of stabilization if the ξ-measure on $i + Q_1$ is unaffected by changes to the Poisson points outside $B_R(i)$ (but inside Q_λ). When ξ is homogeneously stabilizing, the limit

$$\xi(\mathcal{P}, i + Q_1) := \lim_{r\to\infty} \xi\left(\mathcal{P} \cap (B_r(i))_+, i + Q_1\right)$$

exists almost surely for all $i \in \mathbb{Z}^d$. The random variables $(\xi(\mathcal{P}, i+Q_1), i \in \mathbb{Z}^d)$ form a stationary random field.

Given ξ, for all $\lambda > 0$, all TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, and all Borel $A \subset \mathbb{R}^d$ we let $\xi_\lambda(\mathcal{X}, A) := \xi(\lambda^{1/d}\mathcal{X}, \lambda^{1/d}A)$. Define the random measure μ_λ^ξ on $\mathbb{R}^d$ by

$$\mu_\lambda^\xi(\,\cdot\,) := \xi_\lambda(\mathcal{P}_\lambda \cap Q_1, \cdot) \tag{2.5}$$

and the centered version $\overline{\mu}_\lambda^\xi := \mu_\lambda^\xi - \mathbb{E}\,[\mu_\lambda^\xi]$. By the assumed locally supported property of ξ, μ_λ is supported by the fattened cube $Q_1^+ := [-1,2)^d$ for large enough λ.

If ξ is stabilizing, define $\mu(\xi) := \mathbb{E}\,[\xi(\mathcal{P}, Q_1)]$ and and if ξ is exponentially stabilizing, define

$$\sigma^2(\xi) := \sum_{i \in \mathbb{Z}^d} \mathrm{Cov}\,[\xi(\mathcal{P}, Q_1),\ \xi(\mathcal{P}, i+Q_1)]\,,$$

where the sum can be shown to converge absolutely by exponential stabilization and (2.1). The following general theorem provides laws of large numbers and normal approximation results for $\langle f, \mu_\lambda^\xi \rangle$, suitably scaled and centered, for $f \in \mathcal{R}(Q_1^+)$. This set of results for measures determined by TLF point sets is similar to previously known results for measures determined by finite point sets (Theorem 2.1 of [18], Theorems 2.1 and 2.3 of [3], and Corollary 2.4 of [19]).

Theorem 2.1 *Suppose that ξ is exponentially stabilizing. Then as $\lambda \to \infty$, for f and g in $\mathcal{R}(Q_1^+)$ we have*

$$\lim_{\lambda\to\infty} \lambda^{-1}\mathbb{E}\,[\langle f, \mu_\lambda^\xi \rangle] = \mu(\xi) \int_{[0,1]^d} f(x)dx \tag{2.6}$$

and

$$\lim_{\lambda\to\infty} \lambda^{-1}\mathrm{Cov}[\langle f, \mu_\lambda^\xi \rangle, \langle g, \mu_\lambda^\xi \rangle] = \sigma^2(\xi) \int_{[0,1]^d} f(x)g(x)dx. \tag{2.7}$$

Also,

$$|\lambda^{-1}\mathbb{E}\,[\mu_\lambda^\xi(Q_1^+)] - \mu(\xi)| = O(\lambda^{-1/d}). \tag{2.8}$$

Moreover, if $\sigma^2(\xi) > 0$ then

$$\sup_{t\in\mathbb{R}} \left| P\left[\frac{\mu_\lambda^\xi(Q_1^+) - \mathbb{E}\,[\mu_\lambda^\xi(Q_1^+)]}{(\mathrm{Var}[\mu_\lambda^\xi(Q_1^+)])^{1/2}} \le t \right] - P[\mathcal{N}(0,1) \le t] \right| = O((\log\lambda)^{3d}\lambda^{-1/2}) \tag{2.9}$$

and the finite-dimensional distributions of the random field $(\lambda^{-1/2}\langle f, \bar{\mu}_\lambda^\xi \rangle, f \in \mathcal{R}(Q_1^+))$ converge as $\lambda \to \infty$ to those of a mean zero generalized Gaussian field with covariance kernel

$$(f,g) \mapsto \sigma^2(\xi) \int_{[0,1]^d} f(x)g(x)dx, \quad f, g \in \mathcal{R}(Q_1^+).$$

We shall use Theorem 2.1 to prove the results on RSA described in Section 1. It seems likely that Theorem 2.1 can also be applied to obtain similar results for the related models listed in Remark 3 of Section 1. For some of these, certain generalizations of Theorem 2.1 may be needed; for example, in some cases one may need to allow for the Poisson points to carry independent identically distributed random marks, and in others the boundedness condition (2.1) may need to be relaxed to a moments condition.

By appropriate discretization, the proof of the weak law of large numbers (2.6, 2.8) follows from a modification of methods in [15] whereas the proof of (2.7, 2.9) follows a discretized version of the methods in [3], [16], [19]. We refer to the extended version of this paper [23] for complete details.

3 Proof of stabilization for packing

In this section, we show that the random packing measures ν_λ and ν'_λ described in Section 1 can each be expressed in terms of a suitably defined measure-valued functional ξ of TLF point sets in $\mathbb{R}^d \times \mathbb{R}_+$, of the general type considered in Section 2, applied to a Poisson point process in space-time. Then we show that in both cases the appropriate choice of ξ satisfies the exponential stabilization condition described in Definition 2.2, so that Theorem 2.1 is applicable to this choice of ξ. We defer to the next section the proof that in both cases the appropriate choice of ξ satisfies $\sigma^2(\xi) > 0$.

Throughout we let d_S stand for the diameter of S. In our proofs, *we shall assume that* $2d_S < 1$. This assumption entails no loss of generality, since once Theorems 1.1 and 1.2 hold under this assumption, the results follow for general S by obvious scaling arguments.

Let us say that two points (x,t) and (y,u) in $\mathbb{R}^d \times \mathbb{R}_+$ are *adjacent* if $(x+S) \cap (y+S) \neq \emptyset$. Given TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, let us first list the points of $\mathcal{X}$ in order of increasing time-marks using the lexicographic ordering on $\mathbb{R}^d$ as a tie-breaker in the case of any pairs of points of $\mathcal{X}$ with equal time-marks. Then consider the points of $\mathcal{X}$ in the order of the list; let the first point in the list be accepted, and let each subsequent point be accepted if it is not adjacent to any previously accepted point of $\mathcal{X}$; otherwise let it be rejected. We call this the *usual rule* for packing points of $\mathcal{X}$, since it corresponds to the packing rule of Section 1 with the input ordering determined by time-marks. Let $\mathcal{A}(\mathcal{X})$ denote the subset of $\mathcal{X}$ consisting of all accepted points when the points of $\mathcal{X}$ are packed according to the usual rule.

We consider two specific measure-valued functionals ξ^* and ξ' on TLF point sets in $\mathbb{R}^d \times \mathbb{R}_+$, of the general type considered in Section 2, which are defined as follows.

For any bounded Borel $A \subset \mathbb{R}^d$, recall that $A_+ := A \times \mathbb{R}_+$. For any TLF point set $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$ let $\xi^*(\mathcal{X}, A)$ be the number of points of $\mathcal{A}(\mathcal{X})$ which lie in A_+, and with $|\cdot|$ denoting Lebesgue measure, let

$$\xi'(\mathcal{X}, A) := |S|^{-1} \left| A \cap \left(\bigcup_{(x,t) \in \mathcal{A}(\mathcal{X})} (x + S) \right) \right|.$$

Then ξ^* and ξ' are clearly translation invariant, and are bounded (i.e., satisfy (2.1)), since only a bounded number of solids can be packed in any fixed cube.

Recall that $\mathcal{P}_\lambda$ denotes a homogeneous Poisson point process of intensity λ on $\mathbb{R}^d \times \mathbb{R}_+$, and $\mathcal{P} = \mathcal{P}_1$. Assume $\mathcal{P}_\lambda$ is obtained from $\mathcal{P}$ by $\mathcal{P}_\lambda := \lambda^{-1/d}\mathcal{P}$. For all $\lambda > 0$, recall the definition of ξ_λ in Section 2, and define the random measures

$$\mu_\lambda^{\xi^*}(\,\cdot\,) := \xi_\lambda^*(\mathcal{P}_\lambda \cap (Q_1)_+, \cdot) \quad \text{and} \quad \mu_\lambda^{\xi'}(\,\cdot\,) := \xi_\lambda'(\mathcal{P}_\lambda \cap (Q_1)_+, \cdot).$$

Let $N_\lambda^{\xi^*}$ denote the total mass of $\mu_\lambda^{\xi^*}$, i.e.

$$N_\lambda^{\xi^*} := \mu_\lambda^{\xi^*}(\mathcal{P}_\lambda \cap [0,1]_+^d, [0,1]^d).$$

Then $\mu_\lambda^{\xi^*}$ and $\mu_\lambda^{\xi'}$ are the *random packing point measure* and the *random packing volume measure*, respectively, corresponding to the random sequential adsorption process obtained by taking the spatial locations of the points of $\mathcal{P} \cap Q_\lambda$, in order of increasing time-mark, as the input sequence. Since these spatial locations are independent and uniformly distributed on Q_λ, we have the distributional equalities

$$\mu_\lambda^{\xi^*} \stackrel{\mathcal{D}}{=} \nu_\lambda, \quad \mu_\lambda^{\xi'} \stackrel{\mathcal{D}}{=} \nu_\lambda', \quad \text{and} \quad N_\lambda^{\xi^*} \stackrel{\mathcal{D}}{=} N_\lambda, \tag{3.1}$$

where the measures ν_λ and ν_λ' are given in (1.3) and (1.4) and the jamming number N_λ is also given in Section 1.

We show in Lemmas 3.5 and 3.7 below that both ξ^* and ξ' are exponentially stabilizing, and therefore we can apply Theorem 2.1 to either of these choices of ξ. To proceed with the proof of exponential stabilization, consider a partition of $\mathbb{R}^d$ into translates of the unit cube $C := Q_1 = [0,1)^d$. It is convenient to index these translates as C_i, $i := (i_1, \ldots, i_d) \in \mathbb{Z}^d$, with $C_i := (i_1, \ldots, i_d) + C$. We shall write $C_i^+ := \bigcup_{j \in \mathbb{Z}^d,\, ||i-j||_\infty \le 1} C_j$, that is to say C_i^+ is the union of C_i and its neighboring cubes. We also consider the *moat* $\Delta C_i := C_i^+ \setminus C_i$.

We need further terminology. Given TLF $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, and given $A \subset \mathbb{R}^d$, we say that $\mathcal{X}$ *fully packs* the region A if every point in A_+ is adjacent to at least one point of $\mathcal{A}(\mathcal{X})$. For $t > 0$, we say $\mathcal{X}$ *fully packs* A *by time* t if $\mathcal{X} \cap (\mathbb{R}^d \times [0,t])$ fully packs A. Given $B \subseteq \mathbb{R}^d$, we say that a finite point configuration $\mathcal{X} \subset (B \cap C_i^+)_+$ is *maximal* or *strongly saturates* the cube C_i in B if for each TLF external configuration $\mathcal{Y} \subset (B \setminus C_i^+)_+$, $\mathcal{X} \cup \mathcal{Y}$ fully packs the region $B \cap C_i$ (the existence of maximal configurations is guaranteed by Lemmas 3.1 and 3.3 below).

We shall be interested in strong saturation of C_i in B when $B = \mathbb{R}^d$ or when $B = Q_\lambda$. The reason for our interest is this: If we knew that there was a constant $\tau < \infty$ such that $\mathcal{P} \cap (C_{\mathbf{0}}^+ \times [0, \tau])$ strongly saturated $C_{\mathbf{0}}$ in $\mathbb{R}^d$ almost surely, then points in $\mathcal{P}$ with time marks exceeding τ would have no bearing on the packing status of points in $\mathcal{P} \cap (C_{\mathbf{0}})_+$. Thus, to check stabilization of ξ at $\mathbf{0}$ it would be enough to replace $\mathcal{P}$ by the Poisson point process $\mathcal{P} \cap (\mathbb{R}^d \times [0, \tau])$, and follow the stabilization arguments for packing with finite Poisson input (section four of [17]). While clearly no such constant τ exists, we shall show in Lemma 3.3 that a finite random τ exists.

We say that $\mathcal{X}$ *locally strongly saturates* C_i if for each $\eta \subseteq \mathcal{X} \cap (\Delta C_i)_+$, the point set $(\mathcal{X} \cap (C_i)_+) \cup \eta$ fully packs C_i. The following lemma shows that local strong saturation implies strong saturation.

Lemma 3.1 *Suppose $\mathcal{X} \subset (C_i^+)_+$ is TLF and locally strongly saturates C_i. Then for any $B \subseteq \mathbb{R}^d$ with $C_i \subseteq B$, $\mathcal{X} \cap B$ strongly saturates C_i in B.*

Proof. Let $\mathcal{Y} \subset (B \setminus C_i^+)_+$ be TLF. Let $\eta := \mathcal{A}((\mathcal{X} \cap B_+) \cup \mathcal{Y}) \cap (\Delta C_i)_+$. We claim that

$$\mathcal{A}((\mathcal{X} \cap B_+) \cup \mathcal{Y}) \cap (C_i^+)_+ = \mathcal{A}((\mathcal{X} \cap (C_i)_+) \cup \eta). \tag{3.2}$$

Indeed, considering each point of $(\mathcal{X} \cap (C_i)_+) \cup \eta$ in the usual temporal order, we see that the decision on whether to accept is the same for these points whether we are applying the usual packing rule to $(\mathcal{X} \cap B_+) \cup \mathcal{Y}$ or to $(\mathcal{X} \cap (C_i)_+) \cup \eta$.

Since we assume $\mathcal{X}$ locally strongly saturates C_i, $(\mathcal{X} \cap (C_i)_+) \cup \eta$ fully packs C_i, and so by (3.2), $(\mathcal{X} \cap B_+) \cup \mathcal{Y}$ fully packs C_i. □

We will use one more auxiliary lemma.

Lemma 3.2 *With probability 1, $\mathcal{P}$ has the property that for any $\eta \subseteq \mathcal{P} \cap (\Delta C_{\mathbf{0}})_+$, there exists $T < \infty$ such that the point set $(\mathcal{P} \cap (C_{\mathbf{0}})_+) \cup \eta$ fully packs $C_{\mathbf{0}}$ by time T.*

Proof. Suppose that for each rational hypercube Q contained in $C_{\mathbf{0}}$, $\mathcal{P} \cap Q_+ \neq \emptyset$; this event has probability 1.

Take $\eta \subset \mathcal{P} \cap (\Delta C_{\mathbf{0}})_+$. Let $\mathcal{A} := \mathcal{A}((\mathcal{P} \cap (C_{\mathbf{0}})_+) \cup \eta)$. Clearly $\mathcal{A}$ is finite. Let V be the set of $x \in C_{\mathbf{0}}$ such that $(x, 0)$ does not lie adjacent to any point of $\mathcal{A}$. Then V is open in $C_{\mathbf{0}}$ (because we assume S is closed) and if it is non-empty, it contains a rational cube contained in $C_{\mathbf{0}}$ so that V_+ contains a point of $\mathcal{P} \cap (C_{\mathbf{0}})_+$. But then this point should have been accepted so there is a contradiction. Hence V is empty and since $\mathcal{A}$ is finite this shows that $C_{\mathbf{0}}$ is fully packed within a finite time. □

For $i \in \mathbb{Z}^d$, let $T_i := T_i(\mathcal{P})$ denote the time till local strong saturation, defined to be the smallest $t \in [0, \infty]$ such that C_i is locally strongly saturated by the point set $(\mathcal{P} \cap (C_i^+)_+) \cap (\mathbb{R}^d \times [0, t])$ (and set $T_i = \infty$ if no such t exists). Clearly, $T_i,\ i \in \mathbb{Z}^d$, are identically distributed random variables depending only on $\mathcal{P} \cap (C_i^+)_+$. In particular, $(T_i,\ i \in \mathbb{Z}^d)$ forms a 2-dependent random field, meaning that T_i is independent of $(T_j,\ \|j - i\|_\infty > 2)$ for each $i \in \mathbb{Z}^d$. We can now prove the key result that $T_{\mathbf{0}}$ is almost surely finite.

Lemma 3.3 *It is the case that* $P[T_{\mathbf{0}} = \infty] = 0$.

Proof. Suppose that $T_{\mathbf{0}} = \infty$. Then for each positive integer τ there exists $\eta_\tau \subseteq \mathcal{P} \cap (\Delta C_{\mathbf{0}})_+$ such that $(\mathcal{P} \cap (C_{\mathbf{0}})_+) \cup \eta_\tau$ does not fully pack $C_{\mathbf{0}}$ by time τ.

Assume $\mathcal{P} \cap (\Delta C_{\mathbf{0}})_+$ is locally finite (this happens almost surely). Then $\mathcal{P} \cap (\Delta C_{\mathbf{0}} \times [0, 1])$ is finite so that we can take a strictly increasing subsequence $(\tau_n^{(1)})_{n \ge 1}$ of the integers τ, along which $\eta_{\tau_n^{(1)}} \cap (\Delta C_{\mathbf{0}} \times [0, 1])$ is the same for all n. Then we can take a further strictly increasing subsequence $(\tau_n^{(2)})_{n \ge 1}$ of $(\tau_n^{(1)})_{n \ge 1}$ along which $\eta_{\tau_n^{(2)}} \cap (\Delta C_{\mathbf{0}} \times [0, 2])$ is the same for all n. Repeating this procedure and using Cantor's diagonal argument, i.e. taking $\tau_n := \tau_n^{(n)}$ for all n, we obtain a subsequence τ_n tending to infinity, and a limit set $\eta \subset (\Delta C_{\mathbf{0}} \times \mathbb{R}_+)$, such that for all k, it is the case that

$$\eta_{\tau_n} \cap (\Delta C_{\mathbf{0}} \times [0, k]) = \eta \cap (\Delta C_{\mathbf{0}} \times [0, k]) \tag{3.3}$$

for all but finitely many n.

Let $k > 0$, and choose n to be large enough so that $\tau_n \ge k$ and such that (3.3) holds. Then the point set $(\mathcal{P} \cap (C_{\mathbf{0}})_+) \cup \eta_{\tau_n}$ does not yet fully pack $C_{\mathbf{0}}$ by time τ_n, and therefore $(\mathcal{P} \cap (C_{\mathbf{0}})_+) \cup \eta$ does not yet fully pack $C_{\mathbf{0}}$ by time k.

Since $(\mathcal{P} \cap (C_{\mathbf{0}})_+) \cup \eta$ does not yet fully pack $C_{\mathbf{0}}$ by time k for any k, we are in the complement of the event described in Lemma 3.2. Thus by that result, the event $\{T_{\mathbf{0}} = \infty\}$ is contained in an event of probability zero, which completes the proof of Lemma 3.3. □

Using Lemma 3.3, we can now prove that ξ^* and ξ', defined at the start of this section, satisfy the first part of exponential stabilization (exponential decay of the tail of R'). To this end, consider the following $\{0,1\}$-valued random field $(\pi_i, i \in \mathbb{Z}^d)$ on $(\Omega, \mathcal{F}, P)$:

$$\pi_i := \begin{cases} 1, & \text{if } T_i \leq T^*, \\ 0, & \text{otherwise}, \end{cases}$$

where T^* is a constant to be specified below. Clearly the field $(\pi_i)_{i\in\mathbb{Z}^d}$ inherits the 2-dependence property of $(T_i)_{i\in\mathbb{Z}^d}$. We shall use the following auxiliary lemma showing that if T^* is chosen so that $P[\pi_i = 0]$ is small enough then the probabilities of observing long paths of zeros in π_i decay exponentially in the sense made precise below.

Given $L > 0$, let $E_1(L)$ be the event that there is a path of zeros from some site $i \in \{-1,0,1\}^d$ to the complement of $B_{L/2-\sqrt{d}}$ in the random field $(\pi_i, i \in \mathbb{Z}^d)$. More formally, $E_1(L)$ is the event that there exists a sequence $i_0, i_1, i_2, \ldots, i_n$, such that (a) $i_0 \in \{-1,0,1\}^d$, and (b) $i_n \notin B_{L/2-2\sqrt{d}}$, and (c) for $j = 1, \ldots, n$, $i_j \in \mathbb{Z}^d$ and $\|i_j - i_{j-1}\|_\infty = 1$ and $\pi_{i_j} = 0$.

Lemma 3.4 *There exists $\delta^* > 0$ such that if T^* is chosen large enough so that $P[\pi_i = 0] \leq \delta^*$, the probability of event $E_1(L)$ decays exponentially in L, i.e. $limsup_{L\to\infty} L^{-1} \log P[E_1(L)] < 0$.*

Proof. This is a direct consequence of the product measure domination result of Liggett, Schonmann and Stacey ((7.65) of [10]), combined with the exponential decay of the cluster radius in the subcritical regime of Bernoulli percolation, see e.g. Section 5.2 of [10]. Alternatively, the lemma can be proved directly by a standard path-counting argument. □

Lemma 3.5 *There exists a positive constant K_1 such that for either $\xi = \xi^*$ or $\xi = \xi'$, there is a stabilization radius R' as described in Definition 2.2, satisfying*

$$P\left[R' > L\right] \leq K_1 \exp(-L/K_1), \qquad \forall L > 0.$$

Proof. Choose $\delta^* > 0$ as given by Lemma 3.4. Using Lemma 3.3, take $T^* > 0$ such that $P[T_{\mathbf{0}} \geq T^*] \leq \delta^*$.

Let us say that the cube C_i is T^*-*saturated* iff $T_i \leq T^*$, that is to say iff $\pi_i = 1$. By Lemma 3.1, if C_i is T^*-saturated then for any $B \subseteq \mathbb{R}^d$ with $C_i \subseteq B$, $\mathcal{P} \cap ([C_i^+ \cap B] \times [0, T^*])$ strongly saturates C_i in B.

We declare a point $(x,t) \in \mathcal{P} \cap (C_i)_+$ to be *causally relevant* if either

- $\pi_i = 0$,

- or $\pi_i = 1$ and $t \leq T^*$.

Otherwise the point $x \in \mathcal{P} \cap (C_i)_+$ is declared *causally irrelevant.*

We now argue as follows, directly adapting the oriented percolation technique introduced in section four of [17]. We convert the collection of points $\mathcal{P}$ (in $\mathbb{R}^d \times \mathbb{R}_+$) into a directed graph by providing a directed connection from (y,s) to (x,t) whenever $|y-x| \leq 2d_S$ and $s < t$ and, moreover, both (x,t) and (y,s) are causally relevant. By the *causal cluster* $\mathrm{Cl}[(x,t);\mathcal{P}]$ of $(x,t) \in \mathcal{P}$ we understand the set of all causally relevant points (y,s) of $\mathcal{P}$ such that there is a directed path from (y,s) to (x,t) (referred to as a *causal chain for* (x,t) in the sequel). Necessarily the points in the causal cluster for (x,t) have time mark at most t.

For each $(x,t) \in \mathcal{P}$ we define the *causal cube cluster* of (x,t) in $\mathbb{R}^d$ by

$$\bar{\mathrm{Cl}}[(x,t);\mathcal{P}] := \bigcup [C_j^+ : \ (C_j)_+ \cap \mathrm{Cl}[(x,t);\mathcal{P}] \neq \emptyset]$$

and for each $i \in \mathbb{Z}^d$ we define its *causal cube cluster* as the union of clusters given by

$$\bar{\mathrm{Cl}}[i;\mathcal{P}] := \bigcup_{(x,t)\in\mathcal{P}\cap(C_i^+)_+} \bar{\mathrm{Cl}}[(x,t);\mathcal{P}]. \tag{3.4}$$

The significance of causal cube clusters is as follows. First, we assert that the packing status of a given point (x,t) is unaffected by changes to $\mathcal{P}$ outside $\bar{\mathrm{Cl}}[(x,t);\mathcal{P}]$. Indeed, viewing the directed connections as potential direct interactions between overlapping solids in the course of the sequential packing process, we can repeat the corresponding argument from Lemma 4.1 in [17], adding the extra observation that causally irrelevant points will not be accepted regardless of the outside packing configuration and hence do not have to be taken into account. Similarly, the packing status of the totality of points falling within distance d_S of the cube C_i can only be affected by the status of points falling in the causal cube cluster $\bar{\mathrm{Cl}}[i;\mathcal{P}]$. Consequently, we see that for either $\xi = \xi^*$ or $\xi = \xi'$, we can define a radius of stabilization by

$$R' := \mathrm{diam}(\bar{\mathrm{Cl}}[\mathbf{0};\mathcal{P}]). \tag{3.5}$$

We need to show that R' is almost surely finite and has an exponentially decaying tail.

For $L > 0$, let $E_1(L)$ be as in Lemma 3.4. For $i \in \mathbb{Z}^d$, let $E_2(L,i)$ be the event that there exists $(x,t) \in \mathcal{P} \cap (C_i)_+$, such that $t \leq T^*$ and there exists a causal chain for (x,t) which starts at some point of $\mathcal{P} \setminus (B_{L-2\sqrt{d}})_+$. Define the event

$$E_2(L) := \bigcup \left[E_2(L,i) : i \in \mathbb{Z}^d, C_i \cap B_{L/2} \neq \emptyset \right].$$

Then we assert that the event $\{R' > L\}$ is contained in $E_1(L) \cup E_2(L)$. Indeed, if $E_2(L)$ does not occur, then for any causal chain for any $(x,t) \in \mathcal{P} \cap (C_{\mathbf{0}}^+)_+$ starting outside $(B_{L-2\sqrt{d}})_+$, all points in the causal chain of (x,t) lying inside $(B_{L/2})_+$ must have time-coordinate greater than T^*; if also $E_1(L)$ does not occur, at least one of these points must lie in a cube which is T^*-saturated, and therefore be causally irrelevant, so in fact there is no causal chain for any $(x,t) \in \mathcal{P} \cap (C_{\mathbf{0}}^+)_+$ starting outside $(B_{L-2\sqrt{d}})_+$. Hence, $\bar{\mathrm{C}}\mathrm{l}[\mathbf{0}, \mathcal{P}] \subseteq B_L$, so that $R' \leq L$.

By the choice of δ^* and T^* as in Lemma 3.4 we have exponential decay of $P[E_1(L)]$. Since T^* is fixed, we can use the methods of [17] for finite (Poisson) input packing, in particular the argument leading to Lemma 4.2 in [17], to see that there is a constant K_3 such that $P[E_2(L,i)] \leq K_3 \exp(-L/K_3)$ for all $i \in \mathbb{Z}^d \cap B_{L/2}$. Since the number of such i is only $O(L^d)$, we see that $P[E_2(L)]$ also decays exponentially in L, and hence so does $P[E_1(L)] + P[E_2(L)]$. Since the event $\{R' > L\}$ is contained in $E_1(L) \cup E_2(L)$, Lemma 3.5 is proved. □

To finish checking that ξ^* and ξ' satisfy the conditions for Theorem 2.1, we consider strong saturation, not only of unit cubes but of cubes of slightly less than unit size. Let Q_ζ^+ denote the cube $[-\zeta^{1/d}, 2\zeta^{1/d})^d$, i.e. the cube of side $3\zeta^{1/d}$ concentric with Q_ζ. Let us say that Q_ζ is *locally strongly saturated* by a finite point set $\mathcal{X} \subset (Q_\zeta^+)_+$ if for every $\eta \subseteq \mathcal{X} \cap (Q_\zeta^+ \setminus Q_\zeta)_+$, the point set $(\mathcal{X} \cap (Q_\zeta)_+) \cup \eta$ fully packs Q_ζ.

Lemma 3.6 *Given $\delta > 0$, there exist constants $\varepsilon > 0$ and $t_0 < \infty$ such that for all $\zeta \in [1-\varepsilon, 1]$,*

$$P[\mathcal{P} \cap (Q_\zeta^+ \times [0, t_0]) \text{ locally strongly saturates } Q_\zeta] > 1 - \delta. \tag{3.6}$$

Proof. By Lemma 3.3, we can choose t_0 such that $\mathcal{P} \cap (Q_1^+ \times [0, t_0])$ locally strongly saturates Q_1, with probability at least $1 - \delta/2$. Having chosen t_0 in this way, we can then choose ε, with $2d_S < (1-\varepsilon)^{1/d}$, so that for any $\zeta \in [1-\varepsilon, 1]$,

$$P[\mathcal{P} \cap ((Q_1 \setminus Q_\zeta) \times [0, t_0]) \neq \emptyset] < \delta/2.$$

For $\zeta < 1$ with $2d_S < \zeta^{1/d}$, if $\mathcal{P} \cap (Q_1^+ \times [0, t_0])$ strongly saturates Q_1, and $\mathcal{P} \cap ((Q_1 \setminus Q_\zeta) \times [0, t_0])$ is empty, then $\mathcal{P} \cap (Q_\zeta^+ \times [0, t_0])$ strongly saturates Q_ζ. Hence, the preceding probability estimates complete the proof. □

Lemma 3.7 *There exists a positive constant K_4 such that for either $\xi = \xi^*$ or $\xi = \xi'$, there is a family of stabilization radii $R(i, \lambda) := R^\xi(i, \lambda)$, defined for $\lambda \geq 1$ and $i \in \mathbb{Z}^d$ as described in*

Definition 2.2, which satisfy

$$\sup_{\lambda \geq 1, i \in \mathbb{Z}^d} P\left[R(i,\lambda) > L\right] \leq K_4 \exp(-L/K_4). \tag{3.7}$$

Proof. First let us restrict attention to λ with $\lambda^{1/d} \in \mathbb{N}$. Adapting notation from the proof of Lemma 3.5, for $(x,t) \in \mathcal{P} \cap (Q_\lambda)_+$ we let $\mathrm{Cl}[(x,t); \mathcal{P} \cap (Q_\lambda)_+]$ denote the set of all causally relevant points (y,s) of $\mathcal{P} \cap (Q_\lambda)_+$ such that there is a directed path from (y,s) to (x,t), with all points in the path lying inside $(Q_\lambda)_+$. Then define the causal cube cluster in Q_λ for (x,t) by

$$\bar{\mathrm{Cl}}[(x,t); \mathcal{P} \cap (Q_\lambda)_+] := \bigcup [C_j^+ \cap Q_\lambda : (C_j)_+ \cap \mathrm{Cl}[(x,t); \mathcal{P} \cap (Q_\lambda)_+] \neq \emptyset]$$

and and for $i \in \mathbb{Z}^d$ by

$$\bar{\mathrm{Cl}}[i; \mathcal{P} \cap (Q_\lambda)_+] := \bigcup_{(x,t) \in \mathcal{P} \cap (Q_\lambda \cap C_i^+)_+} \bar{\mathrm{Cl}}[(x,t); \mathcal{P} \cap (Q_\lambda)_+].$$

Define

$$R(i,\lambda) := \mathrm{diam}(\bar{\mathrm{Cl}}[i; \mathcal{P} \cap (Q_\lambda)_+]), \quad \lambda^{1/d} \in \mathbb{N}. \tag{3.8}$$

Then for $i \in \mathbb{Z}^d$, the packing statuses of points of $\mathcal{P} \cap (C_i^+ \cap Q_\lambda)_+$ are unaffected by changes to $\mathcal{P} \cap (Q_\lambda)_+$ in the region $(Q_\lambda \setminus B_{R(i,\lambda)}(i))_+$, by the same argument as in proof of Lemma 3.5. Here we are using the fact that $\lambda^{1/d} \in \mathbb{Z}$, and that if $C_i \subset Q_\lambda$ is T^*-saturated then C_i is strongly saturated in Q_λ by $\mathcal{P} \cap (Q_\lambda \times [0, T^*])$ (Lemma 3.1). Thus, $R(i,\lambda)$ serves as a radius of stabilization in the sense of Definition 2.2 (for either ξ^* or ξ'). Moreover, $\bar{\mathrm{Cl}}[i; \mathcal{P} \cap (Q_\lambda)_+] \subseteq \bar{\mathrm{Cl}}[i; \mathcal{P}]$, and so with K_1 in as in proof of Lemma 3.5 we have $P[R(i,\lambda) > L] \leq K_1 \exp(-L/K_1)$, uniformly over i, λ with $\lambda^{1/d} \in \mathbb{N}$.

Now suppose $\lambda^{1/d} \notin \mathbb{N}$. In this case, instead of dividing Q_λ into cubes of side 1, some of which would not fit exactly, we divide Q_λ into cubes of side slightly less than 1, which do fit exactly, and repeat the above argument.

More precisely, we modify the proof of Lemma 3.5. With δ_2 as in that proof, we use Lemma 3.6 to choose constants $\varepsilon > 0$ and $T^* < \infty$ (with $\max(2d_S, 1/2) < (1-\varepsilon)^{1/d}$) in such a way that for any $\zeta \in [1-\varepsilon, 1]$ we have

$$P[\mathcal{P} \cap (Q_\zeta^+ \times [0, T^*]) \text{ locally strongly saturates } Q_\zeta] > 1 - \delta_2.$$

With ε thus fixed, for all large enough λ we can choose $\zeta = \zeta(\lambda) \in [1-\varepsilon, 1]$ in such a way that $\lambda^{1/d}/\zeta^{1/d}$ is an integer. Partitioning $\mathbb{R}^d$ into cubes C_i' of volume ζ, we can then follow the argument already given for the case $\lambda^{1/d} \in \mathbb{N}$, using the fact that each of the unit cubes $i + Q_1$, for which we need to check conditions in Theorem 2.1, is contained in the union of at most 2^d cubes in the partition $\{C'_j\}$. □

4 Proof of volume order variance growth

At the end of this section, we complete the proofs of Theorems 1.1 and 1.2. First, we need to show that the limiting variance $\sigma^2(S,d)$ is non-zero for all d and all S. This is achieved by Proposition 4.1 and Lemma 4.1 below. The first of these results establishes that any convex $S \subseteq \mathbb{R}^d$ with nonempty interior satisfies jamming variability (as defined in remark 6, Section 1), and the second establishes that this is sufficient to guarantee that $\sigma^2(S,d) > 0$. Recall from (3.1) that we can work just as well with $N_\lambda^{\xi^*}$ as with N_λ.

Proposition 4.1 *The convex body S has jamming variability.*

Proof. Given S, for all $x \in \mathbb{R}^d$ define

$$\|x\| := \sup\{a \geq 0 : (x + aS) \cap aS = \emptyset\}.$$

It is straightforward to verify that $\|\cdot\|$ is a norm on $\mathbb{R}^d$, using the convexity of S to verify the triangle inequality. For nonempty $A \subset \mathbb{R}^d$, and $x \in \mathbb{R}^d$, write $D(x, A)$ for $\inf\{\|x - y\| : y \in A\}$. By our earlier assumption that $2d_S < 1$ we have $\|x\| > \|x\|_\infty$ for all $x \in \mathbb{R}^d$.

For $\mathcal{L} \subset \mathbb{R}^d$, we shall say $\mathcal{L}$ is *packed* if $\|x - y\| \geq 1$ for all $x \in \mathcal{L}$, $y \in \mathcal{L}$, and that $\mathcal{L}$ is *maximally packed* if it is packed and

$$D(w, \mathcal{L}) < 1, \quad \forall w \in \mathbb{R}^d. \tag{4.1}$$

We shall say $\mathcal{L}$ is a *periodic set* if for all $x \in \mathcal{L}$ and $z \in \mathbb{Z}^d$ we have $x + z \in \mathcal{L}$.

Let $\mathcal{L}$ be a maximally packed periodic subset of $\mathbb{R}^d$ (it is not hard to see that such an $\mathcal{L}$ exists). Then the function $x \mapsto D(x, \mathcal{L})$ is a continuous function on $\mathbb{R}^d$ that is periodic (i.e., $D(x, \mathcal{L}) = D(x + z, \mathcal{L})$ for all $x \in \mathbb{R}^d, z \in \mathbb{Z}^d$). Hence the range of this function is the continuous image of the compact torus $\mathbb{R}^d/\mathbb{Z}^d$, and so is compact. Hence by (4.1) we have

$$\beta := \sup\{D(w, \mathcal{L}) : w \in \mathbb{R}^d\} < 1.$$

Then for $x \in \mathbb{R}^d$ and $\alpha > 0$, by scaling

$$D(x, \alpha\mathcal{L}) = \alpha D(\alpha^{-1}x, \mathcal{L}) \leq \alpha\beta. \tag{4.2}$$

Choose $\delta > 0$ such that $\beta(1+6\delta) < 1-2\delta$. For $i = 1,2$, let $\mathcal{L}_i := (1+3i\delta)\mathcal{L}$. By (4.2) and the choice of δ we have for all $x \in \mathbb{R}^d$ and $i = 1,2$ that

$$D(x, \mathcal{L}_i) < 1 - 2\delta. \tag{4.3}$$

Let c_1 denote the number of points of $\mathcal{L}$ in $[0,1)^d$. Denote by $\mathrm{Box}(L)$ the hypercube $[-L/2, L/2]^d$. For $i = 1,2$, let $n_i(L)$ denote the number of points of $\mathcal{L}_i$ in $\mathrm{Box}(L-4)$. Then as $L \to \infty$, for $i = 1,2$ we have

$$n_i(L) \sim c_1(1+3\delta i)^{-d} L^d. \tag{4.4}$$

Let $n_3(L)$ denote the maximum integer m such that there exists a packed subset of $\mathrm{Box}(L) \setminus \mathrm{Box}(L-6)$ with m elements. Then there is a finite constant c_2 such that for all $L \geq 6$ we have

$$n_3(L) \leq c_2 L^{d-1}. \tag{4.5}$$

By (4.4) and (4.5), we can choose L_0 such that for $L \geq L_0$ we have

$$n_3(L) < n_1(L) - n_2(L). \tag{4.6}$$

For $x \in \mathbb{R}^d$ and $r > 0$, set $\tilde{B}_r(x) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$ (a ball of radius r using the norm $\|\cdot\|$). For bounded $A \subset \mathbb{R}^d$, let $T(A)$ denote the time of the first Poisson arrival in A, i.e. set

$$T(A) := \inf\{t : \mathcal{P} \cap (A \times \{t\}) \neq \emptyset\},$$

with the convention that the infimum of the empty set is ∞. Fix $L \geq L_0$, and for $i = 1,2$ define the event E_i by

$$E_i := \left\{ \max\{T(\tilde{B}_\delta(x)) : x \in \mathcal{L}_i \cap \mathrm{Box}(L-4)\} < T\left(\mathrm{Box}(L) \setminus \cup_{x \in \mathcal{L}_i \cap \mathrm{Box}(L-4)} \tilde{B}_\delta(x)\right)\right\}.$$

Let $i = 1$ or $i = 2$. If y, y' are distinct points of $\mathcal{L}_i$ then $\|y - y'\| \geq 1 + 3\delta$. Hence, if also $w \in \tilde{B}_\delta(y)$ and $w' \in \tilde{B}_\delta(y')$, then $\|w - w'\| \geq 1 + \delta$ by the triangle inequality. Moreover, for $x \in \mathbb{R}^d$, by (4.3) and the triangle inequality we can find $y = y(x) \in \mathcal{L}_i$ such that $\|x - w\| \leq 1 - \delta$ for all $w \in \tilde{B}_\delta(y)$. Hence, if E_i occurs then the set of accepted points (i.e., centroids of accepted shapes) of the infinite input packing process on $\mathrm{Box}(L)$ induced by $\mathcal{P}$ with arbitrary external pre-packed configuration η in $\mathbb{R}^d \setminus \mathrm{Box}(L)$, includes one point from each $\tilde{B}_\delta(x), x \in \mathcal{L}_i \cap \mathrm{Box}(L-4)$, and also contains no other points from $\mathrm{Box}(L-6)$.

Thus for any pre-packed configuration η in $\mathbb{R}^d \setminus \mathrm{Box}(L)$, if E_1 occurs the number of accepted points in $\mathrm{Box}(L)$ is at least $n_1(L)$, and if E_2 occurs the number of accepted points is at most

$n_2(L)+n_3(L)$. Also, the probabilities $P[E_1]$ and $P[E_2]$ are strictly positive and do not depend on η. By (4.6), it follows that there is a constant $\varepsilon > 0$, independent of η, such that $\mathrm{Var}[N^{\xi^*}_{L^d}(\mathrm{Box}(L))|\eta] \geq \varepsilon$. Thus we have established the required jamming variability. $\square$

Lemma 4.1 *It is the case that* $\liminf_{\lambda\to\infty} \lambda^{-1}\mathrm{Var}[N^{\xi^*}_\lambda] > 0$.

Proof. By Proposition 4.1, there exists $L > 0$ such that $\inf_\eta \mathrm{Var} N[[0,L]^d|\eta] > 0$, where the infimum is over all admissible $\eta \subset \mathbb{R}^d \setminus [0,L]^d$. We consider λ with $\lambda^{1/d}/(L+4) \in \mathbb{N}$. We subdivide the cube Q_λ into $n(\lambda) := \lambda/(L+4)^d$ equal-sized sub-cubes $\tilde{C}_{1,\lambda}, \tilde{C}_{2,\lambda}, \ldots, \tilde{C}_{n(\lambda),\lambda}$ arising as translates of $\mathrm{Box}(L+4)$ centered at $x_{1,\lambda}, \ldots, x_{n(\lambda),\lambda}$ respectively. For $1 \leq i \leq n(\lambda)$, let $\tilde{C}^-_{i,\lambda}$ be the translate of $\mathrm{Box}(L)$ centered at $x_{i,\lambda}$, and let $M_{i,\lambda}$ be the translate of $\mathrm{Box}(L+2) \setminus \mathrm{Box}(L)$ centered at $x_{i,\lambda}$ (a 'moat' around $\tilde{C}^-_{i,\lambda}$).

Using terminology from Section 3, let $F_{i,\lambda}$ be the event that the point set $\mathcal{P} \cap (M_{i,\lambda})_+$ fully packs $M_{i,\lambda}$ by time 1, and let $G_{i,\lambda}$ be the event that $\mathcal{P} \cap ((\tilde{C}_{i,\lambda} \setminus M_{i,\lambda}) \times [0,1])$ is empty. Let $E_{i,\lambda} := F_{i,\lambda} \cap G_{i,\lambda}$. Then $p := P[E_{i,\lambda}]$ satisfies $p > 0$, and does not depend on i or λ.

Observing that the events $E_{i,\lambda}, 1 \leq i \leq n(\lambda)$, are independent (the cubes $\tilde{C}_i$ are disjoint), denote the (random) set of indices for which $E_{i,\lambda}$ occurs by $I(\lambda) := \{i_1, ..., i_{K(\lambda)}\}$. Then $\mathbb{E}\,[K(\lambda)] = pn(\lambda)$. Conditional on the event $E_{i,\lambda}$, the packing process inside $\tilde{C}^-_{i,\lambda}$ has a particularly simple form - before time 1 there are no points in $\tilde{C}^-_{i,\lambda}$, and after that time the newly arriving solids centered in $\tilde{C}^-_{i,\lambda}$ undergo the packing process according to the usual rules with the additional restriction that a solid overlapping another one packed in $M_{i,\lambda}$ before time 1 is rejected. Note that for $i \in I(\lambda)$, no new solids are accepted in $M_{i,\lambda}$ after time 1 and, moreover, the acceptance times of solids accepted in $M_{i,\lambda}$ before time 1 have no influence on the behavior of the packing process in $\tilde{C}^-_{i,\lambda}$ after time 1; only their spatial locations matter. For a configuration η of accepted points (only spatial locations taken into account) in $M_{i,\lambda}$, the process described above will be referred to as packing in $\tilde{C}^-_{i,\lambda}$ in the presence of the *pre-packed configuration* η.

Let $\mathcal{M}_\lambda$ be the sigma-algebra generated by the points of $\mathcal{P} \cap (Q_\lambda \times [0,1])$, i.e. the Poisson arrivals up to time 1. Event $E_{i,\lambda}$ is $\mathcal{M}_\lambda$-measurable, for each i.

By the conditional variance formula we have

$$\begin{aligned}
\mathrm{Var}\left[N^{\xi^*}_\lambda\right] &= \mathbb{E}\left[\mathrm{Var}\left(N^{\xi^*}_\lambda|\mathcal{M}_\lambda\right)\right] + \mathrm{Var}\left[\mathbb{E}\left(N^{\xi^*}_\lambda|\mathcal{M}_\lambda\right)\right] \\
&\geq \mathbb{E}\left[\mathrm{Var}\left(N^{\xi^*}_\lambda \middle| \mathcal{M}_\lambda\right)\right] \\
&= \mathbb{E}\,\mathrm{Var}\left[\sum_{i\in I(\lambda)} N^{\xi^*}_\lambda[\tilde{C}^-_{i,\lambda}] + \left(N^{\xi^*}_\lambda - \sum_{i\in I(\lambda)} N^{\xi^*}_\lambda[\tilde{C}^-_{i,\lambda}]\right) \,\middle|\, \mathcal{M}_\lambda\right],
\end{aligned}$$

where we set $N_\lambda^{\xi^*}[\tilde{C}_{k,\lambda}^-] := \xi^*(\mathcal{P} \cap Q_\lambda, \tilde{C}_{k,\lambda}^-)$, the number of solids packed in $\tilde{C}_{k,\lambda}^-$. Conditionally on $\mathcal{M}_\lambda$, the packing processes after time 1 over different sub-cubes $\tilde{C}_{i,\lambda}^-, i \in I(\lambda)$, are independent of each other and of the packing process after time 1 in $Q_\lambda \setminus \cup_{i \in I(\lambda)} \tilde{C}_{i,\lambda}^-$. Hence,

$$\mathrm{Var}\left[N_\lambda^{\xi^*}\right] \geq \mathbb{E} \sum_{i \in I(\lambda)} \left[\mathrm{Var}[N_\lambda^{\xi^*}[\tilde{C}_{i,\lambda}^-] \mid \mathcal{M}_\lambda]\right] \geq \mathbb{E}\,[K] \inf_\eta \mathrm{Var} N[[0,L]^d|\eta],$$

where the infimum is taken over all admissible configurations η outside $[0,L]^d$, and where $N[[0,L]^d|\eta]$ stands for number of solids packed in $[0,L]^d$ in the presence of the pre-packed configuration η. By Proposition 4.1, this infimum is strictly positive, and Lemma 4.1 follows. □

Proof of Theorems 1.1 and 1.2. Let ξ be ξ^* as defined in Section 3. Then Lemmas 3.5 and 3.7 show that $\xi = \xi^*$ satisfies the exponential stabilization conditions in Theorem 2.1, so it satisfies the conclusions (2.6), (2.7) and (2.8) of that result. The conclusion (2.8) gives us (1.1) of Theorem 1.1. Also, by putting $f \equiv g \equiv 1$ on Q_1^+ and using (2.7), we obtain the variance convergence $\lambda^{-1}\mathrm{Var} N_\lambda \to \sigma^2$ asserted in Theorem 1.1. By Lemma 4.1, we may therefore deduce that $\sigma^2 > 0$. Hence we may apply the last part of Theorem 2.1 to obtain the rest of the conclusions in Theorem 1.2 as they pertain to ν_λ; also the conclusion (2.9) of Theorem 2.1 gives us (1.2).

To get the same results for ν', we argue similarly with $\xi = \xi'$. We need to check that the limiting means and variances are the same, i.e. $\mu(\xi') = \mu(\xi^*)$ and $\sigma^2(\xi') = \sigma^2(\xi^*)$. To see this, note that if $f \equiv 1$ on Q_1^+, then $\langle f, \mu_\lambda^{\xi'} \rangle = \langle f, \mu_\lambda^{\xi^*} \rangle$ so application of (2.6) to this choice of f yields

$$\mu(\xi') = \lim_{\lambda \to \infty} \lambda^{-1} \mathbb{E}\,[\langle f, \mu_\lambda^{\xi'} \rangle] = \lim_{\lambda \to \infty} \lambda^{-1} \mathbb{E}\,[\langle f, \mu_\lambda^{\xi^*} \rangle] = \mu(\xi^*)$$

and a similar argument using (2.7) shows that $\sigma^2(\xi') = \sigma^2(\xi^*)$. □

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