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Gaussian Measure of a Small Ball and Capacity in Wiener Space

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Summary. We give asymptotic bounds for the Gaussian measure of a small ball in terms of the hitting probabilities of a suitably chosen infinite dimensional Brownian motion. Our estimates refine earlier works of Erickson [6].

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In honor of Professor M. Csörgő on occasion of his 65th birthday

1. INTRODUCTION

Let $(\mathcal{B}, \|\cdot\|, \mu)$ denote a Gauss space in that $(\mathcal{B}, \|\cdot\|)$ is a Banach space which carries a centered, Gaussian measure μ living on the Borel σ–field of $\mathcal{B}$.

An old problem, which has received much recent attention, is to describe the rate at which the μ–measure of a small balanced ball (in $\mathcal{B}$) goes to zero, as the radius of the ball decreases to zero. To make this precise, suppose p is a continuous semi–norm on $\mathcal{B}$. That is, $p : \mathcal{B} \mapsto \mathbb{R}$ such that

- (i) For all $x, y \in \mathcal{B}$, $p(x+y) \leqslant p(x) + p(y)$;
- (ii) for all $\alpha \in \mathbb{R}$ and all $x \in \mathcal{B}$, $p(\alpha x) = |\alpha| p(x)$;
- (iii) whenever $x \to y$ in $\mathcal{B}$, $p(x-y) \to 0$.

In particular, note that $x \mapsto p(x)$ is a (nonnegative) continuous map from $\mathcal{B}$ into $\mathbb{R}_+$.

The so–called *small ball problem* for μ consists of finding a good approximation to the following:

$$\mu_p(r) \triangleq \mu\big(\omega \in \mathcal{B} : p(\omega) \leqslant r\big), \tag{1.1}$$

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as $r \to 0^+$. A major breakthrough is the recent work of Kuelbs and Li [11] relating μ_p to a combinatorial problem on the reproducing kernel Hilbert space corresponding to μ. The latter route typically leads one to long-standing open problems in functional analysis; cf. [11] for details.

Throughout this paper, we only deal with the case when p is transient. Rather than developing a theory for this, we define transience via the following technical assumption which will prevail throughout the rest of the paper:

Assumption 1.1. We assume the following two conditions:
(a) p is *transient* in the sense that for some $\kappa > 2$,

$$\limsup_{r \to 0^+} r^{-\kappa} \mu_p(r) < \infty; \text{ and}$$

(b) p is *nondegenerate* in the sense that $1 > \mu_p(1) > 0$.

Remark 1.1.1.
(a) Suppose $\dim(\mathcal{B}) = d < \infty$. In other words, $\mathcal{B}$ is finite dimensional with (topological) dimension d. It is easy to see that as $r \to 0^+$, $\mu_p(r) \sim Cr^d$ for some $C > 0$. Therefore, Assumption 1.1(a) says that $d > 2$ in this case. This corresponds to the well–known condition of transience in the classical sense.
(b) When p has rank $\geqslant 3$, p is transient; see [6] for details.
(c) If p is nondegenerate, there exists $c > 0$, such that $1 > \mu_p(c) > 0$. By considering the semi–norm $c^{-1}p$ instead, we see that Assumption 1.1(b) is not an essential restriction.

Motivated by the work of Erickson [6], this paper proposes a different approximation of μ_p. To begin, recall that the triple $(\mathcal{B}, \|\cdot\|, \mu)$ corresponds to a *Wiener space* C. To define it, let $C(\mathcal{B})$ denote the space of all continuous functions $\omega : [0,1] \mapsto \mathcal{B}$ with $\omega(0) = 0$, endowed with the compact open topology. Let $\mathcal{C}$ denote the associated Borel field. For all $\omega \in C(\mathcal{B})$, let $B_t(\omega) = \omega(t)$. It is a well known fact that there exists a probability measure $\mathbb{P}$ on the measure space $(C(\mathcal{B}), \mathcal{C})$ which renders the process B a μ–Brownian motion; cf. Gross [8] or Üstünel [13] for a modern treatment as well as some of the new developments in this area. In particular, we mention the following important properties:

(i) with $\mathbb{P}$–probability one, $t \mapsto B_t$ is continuous;
(ii) B has independent and stationary increments (under $\mathbb{P}$);
(iii) for all $x \in \mathcal{B}^*$, the random variable $\langle x, B_t \rangle$ has a one-dimensional Gaussian distribution with mean 0 and variance $t \int_{\mathcal{B}} \|x\|^2 \mu(dx)$ (under $\mathbb{P}$);
(iv) B is a $\mathcal{B}$–valued diffusion.
(v) $B_0 = 0$, $\mathbb{P}$–almost surely.

We will denote by $\mathbb{E}$ the expectation operator corresponding to the underlying (Gaussian) probability measure $\mathbb{P}$.

It will be convenient to write our results in terms of the μ–*Ornstein-Uhlenbeck process* O given by

$$O_t \triangleq e^{-t/2} B_{e^t}, \qquad t \geqslant 0. \tag{1.2}$$

We note in passing the elementary fact that O is a $\mathcal{B}$–valued stationary diffusion whose stationary measure is μ.

For any $\varkappa > 1$, define,

$$\lambda_p(r; \varkappa) \triangleq \sup \big\{ a > 0 : \mu_p(a) \leqslant \varkappa \mu_p(r) \big\}, \qquad r > 0. \tag{1.3}$$

In the finite–dimensional case, it is possible to show that for any $\varkappa > 0$, as $r \to 0^+$, $\lambda_p(r; \varkappa) \sim \varkappa^{1/d} r$, where $d \geqslant 3$ is the dimension of $\mathcal{B}$. In this connection, see also Remark 1.1.1(a).

The promised correspondence between μ_p and the process O can then be described in terms of λ_p as follows:

Theorem 1.2. *Suppose $p : \mathcal{B} \mapsto \mathbb{R}$ is a nondegenerate, transient semi–norm on $\mathcal{B}$. Then, for all $T > 0$, and all $\varkappa > 1$, there exists a constant $c \in (1, \infty)$ such that for all $r \in \big(0, 1/c\big)$,*

$$\frac{\mu_p(r)}{c\, r^2} \leqslant \mathbb{P}\Big(\inf_{0 \leqslant t \leqslant T} p(O_t) \leqslant r \Big) \leqslant \frac{c\, \mu_p(r)}{\big(\lambda_p(r; \varkappa) - r\big)^2}.$$

In fact, the proof of Theorem 1.2 can be used, with little change, to show the following:

Corollary 1.3. *Suppose $p : \mathcal{B} \to \mathbb{R}$ is a nondegenerate, transient semi–norm on $\mathcal{B}$. Then, for all $\lambda > 0$ and all $\varkappa > 1$, there exists a constant $c \in (1, \infty)$ such that for all $r \in (0, 1/c)$,*

$$\frac{\mu_p(r)}{c\, r^2} \leqslant \int_0^\infty e^{-\lambda T} \mathbb{P}\Big(\inf_{0 \leqslant t \leqslant T} p(O_t) \leqslant r \Big) dT \leqslant \frac{c\, \mu_p(r)}{\big(\lambda_p(r; \varkappa) - r\big)^2}.$$

Remark.

(a) The quantity,

$$\int_0^\infty e^{-\lambda T} \mathbb{P}\Big(\inf_{0 \leqslant t \leqslant T} p(O_t) \leqslant r \Big) dT,$$

is the λ–*capacity* of the "ball" $\big\{ \omega \in \mathcal{B} : \ p(\omega) \leqslant r \big\}$; cf. Üstünel [13] and Fukushima et al. [7] for details. It turns out that under very general conditions, $\lambda_p(r; \varkappa) - r$ has polynomial decay rate; cf. Remark 1.1.1(a) above. Thus, Theorem 1.2 and its variants provide exact (and essentially equivalent) asymptotics between the λ–capacity of a small ball and the small ball probability μ_p given by (1.1).

(b) In infinite dimensions – which is what is of interest here – the methods of Erickson [6] provide an upper bound of $\mu_p\big((1+\varepsilon)r\big)$ for any $\varepsilon > 0$. In such cases, μ_p decays exponentially fast. Therefore, Theorem 1.2 is an essential improvement.

A refinement of Theorem 1.2 is proved in Section 2. Section 3 contains an explicit application. The methods of the latter section can be combined with those of [6] to provide a host of other examples.

2. THE MAIN ESTIMATE

In this section, we provide bounds for the probability that O hits a small ball in terms of the small ball probability μ_p defined by (1.1). The main result of this section is the following probability estimate:

Theorem 2.1. *Suppose p is a continuous, nondegenerate, transient semi–norm on $\mathcal{B}$ in the sense of Assumption 1.1. Suppose further that $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a bounded measurable function satisfying: for some $\varkappa > 1$,*

$$I_g \triangleq \liminf_{r\to 0^+} \frac{g(r)}{\big(\lambda_p(r;\varkappa) - r\big)^2} > 0.$$

Then, there exists a constant $c \in (1,\infty)$ which depends only on $\varkappa$, $\sup_x g(x)$ and I_g, such that for all $r \in \big(0, 1/c\big)$,

$$\frac{\mu_p(r)g(r)}{c\ r^2} \leqslant \mathbb{P}\Big(\inf_{0\leqslant t\leqslant g(r)} p(O_t) \leqslant r\Big) \leqslant \frac{c\ \mu_p(r)g(r)}{\big(\lambda_p(r;\varkappa) - r\big)^2}.$$

Remark 2.1.1.

(a) The special case when $g(r) \equiv T$ immediately yields Theorem 1.2.

(b) Suppose $g \equiv T$ and $\dim(\mathcal{B}) = d < \infty$. This is the finite dimensional version of Theorem 2.1 and is well known. Namely, that in $d > 2$ dimensions, there exists a constant $c \in (1,\infty)$ such that for all r small enough,

$$c^{-1} r^{d-2} \leqslant \mathbb{P}\Big(\inf_{0\leqslant t\leqslant T} p(O_t) \leqslant r\Big) \leqslant c r^{d-2}.$$

(Recall Remark 1.1.1(a) regarding transience in finite dimensions as well as the estimates for λ_p.)

(c) The lower bound in Theorem 2.1 holds as long as g is bounded and measurable. More precisely, the condition that $I_g > 0$ is only needed for the upper bound.

(d) According to Weber [14] (cf. also Albin [1] and Khoshnevisan and Shi [9]), there is a connection between the modulus of continuity of a Gaussian process and its hitting probabilities. In our setting, the Gaussian process is infinite dimensional. In some infinite dimensional cases, such moduli of continuity are found; cf. Csáki and Csörgő [3] and Csáki et al. [4]. While it seems somewhat unlikely, one cannot help but ask if there is a connection between our results and such moduli of continuity in infinite dimensions.

The rest of this section is devoted to the proof of Theorem 2.1.

Lemma 2.2. *Define,*

$$W_t \triangleq \frac{O_t - e^{-t/2} O_0}{\sqrt{1-e^{-t}}}. \tag{2.1}$$

For any fixed $t \geqslant 0$, W_t as an element of $\mathcal{B}$ is an independent copy of O_0, satisfying,

$$O_t = \sqrt{1-e^{-t}} W_t + e^{-t/2} O_0.$$

Proof. By (1.2), $W_t = e^{-t/2}(B_{e^t} - B_1)$ which is independent of $O_0 = B_1$. To verify that W_t is distributed as O_0, it suffices to show that for all $x \in \mathcal{B}^*$,

$$\langle x, O_t \rangle - e^{-t/2} \langle x, O_0 \rangle \overset{(d)}{=} \sqrt{1-e^{-t}} \langle x, O_0 \rangle.$$

This is a finite dimensional result which can be readily verified by checking means and covariances. $\diamondsuit$

Lemma 2.3. *Suppose $r > 0$ and $t \geqslant 0$ are fixed and p satisfies Assumption 1.1. Then uniformly over all $f \in \mathcal{B}$ with $p(f) \leqslant r$,*

$$\mathbb{P}\big(p(O_t) \leqslant r \mid O_0 = f\big) \leqslant \mu_p\left(\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}\; r\right).$$

Proof. Let W_t be as in (2.1). By properties of p,

$$\sqrt{1-e^{-t}} p(W_t) \leqslant p(O_t) + e^{-t/2} p(O_0).$$

Therefore, conditional on $\{p(O_0) \leqslant r\}$, $\{p(O_t) \leqslant r\}$ implies $\{p(W_t) \leqslant cr\}$, where

$$c \triangleq \sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}.$$

Since by Lemma 2.2 $p(W_t) \overset{(d)}{=} p(O_0)$, the result follows. $\diamondsuit$

Lemma 2.4. *Suppose p is a continuous semi–norm on $\mathcal{B}$ for which Assumption 1.1 is verified. Then for any $T > 0$,*

$$\limsup_{r\to 0^+} r^{-2} \int_0^T \mu_p\Big(\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}\; r\Big) dt < \infty.$$

Proof. Note that for all $0 < r < 1$,

$$\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}\; r \leqslant 1, \qquad \text{if and only if} \qquad t \geqslant 2\ln\Big(\frac{1+r^2}{1-r^2}\Big).$$

Therefore, for all $r > 0$ small,

$$\int_0^T \mu_p\Big(\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}\; r\Big) dt \leqslant 2\ln\Big(\frac{1+r^2}{1-r^2}\Big) + \int_{2\ln\big((1+r^2)/(1-r^2)\big)}^T \mu_p\Big(\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}\; r\Big) dt.$$

As $r \to 0^+$, the first term behaves like $4r^2$. On the other hand, for $t \leqslant T$,

$$\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}} \leqslant \frac{c_1}{\sqrt{t}},$$

for some constant c_1. By Assumption 1.1(a), there exist c_2, c_3, $c_4 > 0$, such that for all $r > 0$ small, the second term is bounded above by

$$c_2 \int_{c_3 r^2}^T \Big(\frac{r}{\sqrt{t}}\Big)^{\kappa} dt \leqslant c_4 r^2.$$

This concludes the proof. $\diamondsuit$

Lemma 2.4 is sharp. Indeed, using Assumption 1.1(b) instead of 1.1(a) in the proof of Lemma 2.4, we immediately arrive at the following:

Lemma 2.5. *Under the conditions of Lemma 2.4, for all $T > 0$,*

$$\liminf_{r\to 0^+} r^{-2} \int_0^T \mu_p\Big(\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}\; r\Big) dt > 0.$$

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Fix $\varkappa > 1$ as given. The conditions on g imply the existence of a constant $c \in (1, \infty)$, such that for all $r \in (0, 1)$,

$$c^{-1}\big(\lambda_p(r;\varkappa) - r\big)^2 \leqslant g(r) \leqslant c. \tag{2.2}$$

For all $r \geqslant 0$, define,

$$\tau(r) \triangleq \inf\{s > 0 : \; p(O_s) \leqslant r\}. \tag{2.3}$$

Since p is continuous, $p^{-1}([0,r])$ is closed. Therefore, $\tau(r)$ is a stopping time for the diffusion O. By (2.2) and the stationarity of O, for all $r \in (0,1)$,

$$\begin{aligned} g(r)\mu_p(r) &= \mathbb{E}\Big[\int_0^{g(r)} \mathbb{1}\{p(O_t) \leqslant r\} dt\Big] \\ &= \mathbb{E}\Big[\int_0^{g(r)} \mathbb{1}\{p(O_t) \leqslant r\} dt \; \mathbb{1}\{0 \leqslant \tau(r) < g(r)\}\Big] \\ &\leqslant \mathbb{E}\Big[\int_0^{c} \mathbb{1}\{p(O_t) \leqslant r\} dt \; \mathbb{1}\{0 \leqslant \tau(r) < g(r)\}\Big], \end{aligned} \tag{2.4}$$

where $\mathbb{1}\{\cdots\}$ is the indicator of whatever appears in the brackets. By Lemma 2.3,

$$\begin{aligned} \mathbb{P}\big(p(O_t) \leqslant r, \tau(r) = 0\big) &= \mathbb{P}\big(p(O_t) \leqslant r \mid p(O_0) \leqslant r\big)\mathbb{P}\big(\tau(r) = 0\big) \\ &\leqslant \mu_p\Big(\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}\; r\Big)\mathbb{P}\big(\tau(r) = 0\big). \end{aligned}$$

Therefore,

$$\mathbb{E}\Big[\int_0^c \mathbb{1}\{p(O_t) \leqslant r\} dt \; \mathbb{1}\{\tau(r) = 0\}\Big] \leqslant \mathbb{P}\big(\tau(r) = 0\big) \int_0^c \mu_p\Big(\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}\; r\Big) dt. \tag{2.5}$$

On the other hand, applying the strong Markov property at time $\tau(r)$, we arrive at the following:

$$\begin{aligned} &\mathbb{E}\Big[\int_0^c \mathbb{1}\{p(O_t) \leqslant r\} dt \; \mathbb{1}\{0 < \tau(r) < g(r)\}\Big] \\ &\qquad = \mathbb{E}\Big[\mathbb{1}\{0 < \tau(r) < g(r)\} \int_0^{c-\tau(r)} \mathbb{1}\{p(O_t) \leqslant r\} dt \circ \theta\big(\tau(r)\big)\Big] \\ &\qquad \leqslant \mathbb{P}\big(0 < \tau(r) < g(r)\big) \sup_f \mathbb{E}\Big[\int_0^c \mathbb{1}\{p(O_t) \leqslant r\} dt \;\Big|\; O_0 = f\Big], \end{aligned}$$

where θ is the shift functional on the paths of the diffusion O and the supremum is over all $f \in \mathcal{B}$ with $p(f) \leqslant r$. By Lemma 2.3,

$$\begin{aligned} &\mathbb{E}\Big[\int_0^c \mathbb{1}\{p(O_t) \leqslant r\} dt \; \mathbb{1}\{0 < \tau(r) < g(r)\}\Big] \\ &\qquad \leqslant \mathbb{P}\big(0 < \tau(r) < g(r)\big) \cdot \int_0^c \mu_p\Big(\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}\; r\Big) dt. \end{aligned}$$

Together with (2.5) and (2.4), this proves the following:

$$g(r)\mu_p(r) \leqslant \mathbb{P}\big(\inf_{0\leqslant t\leqslant g(r)} p(O_t)\leqslant r\big)\cdot \int_0^c \mu_p\Bigg(\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}}\; r\Bigg)dt.$$

By Lemma 2.4, for all $r>0$ small, the right hand side is bounded above by a constant multiple of $r^2\mathbb{P}\big(\inf_{0\leqslant t\leqslant g(r)} p(O_t)\leqslant r\big)$. In other words, we have proven the following:

$$\liminf_{r\to 0^+} \frac{r^2}{g(r)\mu_p(r)}\mathbb{P}\big(\inf_{0\leqslant t\leqslant g(r)} p(O_t)\leqslant r\big) > 0. \tag{2.6}$$

This constitutes the first half of Theorem 2.1. We have also verified Remark 2.1.1(c). The second half proceeds along different lines.

Recall the definition of λ_p from (1.3). Since μ_p is a decreasing function, one easily deduces that

(i) $r\mapsto \lambda_p(r;\varkappa)$ is decreasing;

(ii) $\lambda_p(r;\varkappa)\geqslant r$.

Fix $t\geqslant 0$. By Lemma 2.2 (in its notation), using the properties of p,

$$p(O_t)\leqslant \sqrt{1-e^{-t}}p(W_t)+e^{-t/2}p(O_0).$$

(This should be compared to the proof of Lemma 2.3). Observe that conditional on $\big\{p(O_0)\leqslant r\big\}$, $\big\{p(W_t)<1\big\}$ implies the following:

$$p(O_t)\leqslant \sqrt{1-e^{-t}}+r\leqslant t^{1/2}+r.$$

Therefore, for all $t\leqslant \big(\lambda_p(r;\varkappa)-r\big)^2$,

$$\inf_f \mathbb{P}\big(p(O_t)\leqslant \lambda_p(r;\varkappa)\;\big|\; O_0=f\big)\geqslant \mathbb{P}\big(p(W_t)\leqslant 1\big)=\mu_p(1)>0, \tag{2.7}$$

where the infimum is taken over all $f\in\mathcal{B}$ which satisfy $p(f)\leqslant r$. The rest of the proof follows from stationarity and the strong Markov property at time $\tau(r)$, viz.,

$$\begin{aligned}
3g(r)\mu_p\big(\lambda_p(r;\varkappa)\big) &= \mathbb{E}\Big[\int_0^{3g(r)} \mathbb{1}\big\{p(O_t)\leqslant\lambda_p(r;\varkappa)\big\}dt\Big]\\
&\geqslant \mathbb{E}\Big[\int_{\tau(r)}^{3g(r)} \mathbb{1}\big\{p(O_t)\leqslant\lambda_p(r;\varkappa)\big\}dt\;\Big|\; 0\leqslant\tau(r)\leqslant g(r)\Big]\cdot\mathbb{P}\big(0\leqslant\tau(r)\leqslant g(r)\big)\\
&\geqslant \mathbb{E}\Big[\int_0^{2g(r)} \mathbb{1}\big\{p(O_t)\leqslant\lambda_p(r;\varkappa)\big\}dt\circ\theta\big(\tau(r)\big)\;\Big|\; 0\leqslant\tau(r)\leqslant g(r)\Big]\cdot\mathbb{P}\big(0\leqslant\tau(r)\leqslant g(r)\big)\\
&\geqslant \inf_f \mathbb{E}\Big[\int_0^{g(r)} \mathbb{1}\big\{p(O_t)\leqslant\lambda_p(r;\varkappa)\big\}dt\;\Big|\; O_0=f\Big]\cdot\mathbb{P}\big(0\leqslant\tau(r)\leqslant g(r)\big),
\end{aligned}$$

where the infimum is taken over all $f \in \mathcal{B}$ such that $p(f) \leqslant r$ and θ is as before the shift on the paths of O. By (2.2),

$$3g(r)\mu_p\big(\lambda_p(r;\varkappa)\big) \\ \geqslant \inf_f \mathbb{E}\bigg[\int_0^{c^{-1}(\lambda_p(r;\varkappa)-r)^2} \mathbb{1}\{p(O_t)\leqslant r\}dt \;\bigg|\; O_0 = f\bigg]\,\mathbb{P}\big(0\leqslant \tau(r)\leqslant g(r)\big),$$

where the infimum is taken over all $f \in \mathcal{B}$ with $p(f)\leqslant r$. Since $c > 1$, (2.7) implies

$$3g(r)\mu_p\big(\lambda_p(r;\varkappa)\big) \geqslant \mu_p(1)\; c^{-1}\big(\lambda_p(r;\varkappa)-r\big)^2\;\mathbb{P}\big(0\leqslant\tau(r)\leqslant g(r)\big) \\ = \mu_p(1)\; c^{-1}\big(\lambda_p(r;\varkappa)-r\big)^2\;\mathbb{P}\Big(\inf_{0\leqslant t\leqslant g(r)} p(O_t)\leqslant r\Big).$$

By Assumption 1.1(b), $\mu_p(1) > 0$. By (1.3), we have proven the following:

$$\mathbb{P}\Big(\inf_{0\leqslant t\leqslant g(r)} p(O_t)\leqslant r\Big) \leqslant \frac{3c\varkappa}{\mu_p(1)}\frac{\mu_p(r)g(r)}{\big(\lambda_p(r;\varkappa)-r\big)^2}.$$

Together with (2.6), this proves Theorem 2.1. ◊

3. AN APPLICATION

Recall the μ–Brownian motion $B = (B_t;\, t\geqslant 0)$ from Introduction. The goal of this section is to provide some estimates of the escape rates of B when p is transient. In light of Remark 1.1.1, it is not too difficult to convince oneself that under Assumption 1.1, $t \mapsto p(B_t)$ is transient in that $\mathbb{P}$–a.s., $\lim_{t\to\infty} p(B_t) = \infty$. It is the goal of this section to estimate the rate at which this blow-up occurs. We improve results of Erickson [6] concerning the following class of problems.

Assumption 3.1. We assume the existence of constants $r_0 > 0$, $K_0 > 1$, $0 < \alpha \leqslant 2$, $\chi > 0$ and $\beta \in \mathbb{R}$ such that for all $0 < r \leqslant r_0$,

$$K_0^{-1} r^\beta \exp\Big(-\frac{\chi}{r^\alpha}\Big) \leqslant \mu_p(r) \leqslant K_0 r^\beta \exp\Big(-\frac{\chi}{r^\alpha}\Big). \tag{3.1}$$

See Erickson [6], Li [12] and their combined references for many examples of when this assumption is valid.

Note that under Assumption 3.1, $\mathcal{B}$ is forced to be infinite-dimensional; see Remark 1.1.1. Moreover, Assumption 3.1 implies Assumption 1.1.

Our result on escape rates is the following:

Theorem 3.2. *Suppose Assumption 3.1 is satisfied. Consider a measurable nonincreasing function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$. Let*

$$I_1(\psi) \triangleq \int_1^\infty \psi^{\beta-\alpha}(t) \exp\big(-\frac{\chi}{\psi^\alpha(t)}\big)\frac{dt}{t},$$
$$I_2(\psi) \triangleq \int_1^\infty \psi^{\beta-2\alpha-2}(t) \exp\big(-\frac{\chi}{\psi^\alpha(t)}\big)\frac{dt}{t}.$$

Then

$$I_1(\psi) = \infty \quad \Longrightarrow \quad \mathbb{P}(p(B_t) \geqslant t^{1/2}\psi(t), \text{ eventually for all } t \text{ large}) = 0, \tag{3.2}$$
$$I_2(\psi) < \infty \quad \Longrightarrow \quad \mathbb{P}(p(B_t) \geqslant t^{1/2}\psi(t), \text{ eventually for all } t \text{ large}) = 1. \tag{3.3}$$

Proof of (3.2). Assume $I_1(\psi) = \infty$. Define the Erdős sequence, $t_k \triangleq \exp(k/\log k)$. For simplicity, write $\psi_k \triangleq \psi(t_k)$. Furthermore, define the measurable event,

$$E_k \triangleq \{p(B_{t_k}) \leqslant t_k^{1/2}\psi_k\}.$$

According to an argument of Erdős [5], there exists a constant $C > 1$ such that,

$$C^{-1}(\log\log t_k)^{-1/\alpha} \leqslant \psi_k \leqslant C(\log\log t_k)^{-1/\alpha}. \tag{3.4}$$

By Brownian scaling,

$$\mathbb{P}(E_k) = \mu_p(\psi_k).$$

Since $I_1(\psi) = \infty$, (3.4) and Assumption 3.1 together yield:

$$\sum_k \mathbb{P}(E_k) = \infty.$$

Suppose we could show

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \sum_{n=1}^{N-k} \mathbb{P}(E_k \cap E_{n+k})}{\big(\sum_{k=1}^N \mathbb{P}(E_k)\big)^2} < \infty, \tag{3.5}$$

then, according to Kochen and Stone [10], $\mathbb{P}(E_k, \text{ infinitely often}) > 0$. The latter is a tail event. By the classical 0–1 law of Kolmogorov, B_t has a trivial σ–field. Therefore, (3.5) implies $\mathbb{P}(E_k, \text{ i.o.}) = 1$, which yields (3.2).

It remains to prove (3.5). By Anderson's inequality (cf. [2]) and elementary Brownian properties (cf. Introduction),

$$\begin{aligned}\mathbb{P}(E_k \cap E_{n+k}) &= \mathbb{P}(E_k;\ p(B_{t_{n+k}} - B_{t_k} + B_{t_k}) \leqslant t_{n+k}^{1/2}\psi_{n+k}) \\ &\leqslant \mathbb{P}(E_k)\,\mathbb{P}(p(B_{t_{n+k}} - B_{t_k}) \leqslant t_{n+k}^{1/2}\psi_{n+k}) \\ &= \mathbb{P}(E_k)\,\mu_p\Big(\sqrt{\frac{t_{n+k}}{t_{n+k} - t_k}}\psi_{n+k}\Big).\end{aligned}$$

From here, we can apply a classical argument going back at least to Erdős [5], and only provide its outline. Let C_i $(1 \leqslant i \leqslant 9)$ denote some unimportant constants. When $n \leqslant \log k$, $\sqrt{t_{n+k}/(t_{n+k}-t_k)}\psi_{n+k} \leqslant C_1\, n^{-1/2}$, which implies

$$\mathbb{P}(E_k \cap E_{n+k}) \leqslant C_2\, \mathbb{P}(E_k)\, \exp(-C_3\, n^{\alpha/2}). \tag{3.6}$$

For $\log k < n < (\log k)^2$, we have $t_{n+k}/(t_{n+k}-t_k) \leqslant C_4$, which yields

$$\mathbb{P}(E_k \cap E_{n+k}) \leqslant C_5\, \mathbb{P}(E_k)\, \exp(-C_6\, \log k) \leqslant C_5 \mathbb{P}(E_k)\, \exp(-C_6 \sqrt{n}). \tag{3.7}$$

Finally, if $n \geqslant (\log k)^2$, then $t_{n+k}/(t_{n+k}-t_k) \leqslant 1 + C_7\, (\log(n+k))/n$. Hence

$$\begin{aligned}\mathbb{P}(E_k \cap E_{n+k}) &\leqslant C_8\, \mathbb{P}(E_k)\, \psi_{n+k}^{\beta}\, \exp\big(-\frac{\chi}{\psi_{n+k}^{\alpha}}\big)\\ &\leqslant C_9\, \mathbb{P}(E_k)\, \mathbb{P}(E_{n+k}).\end{aligned} \tag{3.8}$$

Combining (3.6)–(3.8) gives (3.5). ♢

To prove the other part of Theorem 3.2, we need a preliminary result.

Lemma 3.3. *Under Assumption 3.1, there exist* $\varkappa > 1$, $r_1 > 0$ *and* $K_1 > 1$ *such that for all* $0 < r \leqslant r_1$,

$$r + K_1^{-1} r^{1+\alpha} \leqslant \lambda_p(r;\varkappa) \leqslant r + K_1 r^{1+\alpha}.$$

Proof. We will prove the upper bound for $\lambda_p(r;\varkappa)$. The lower bound follows from similar arguments. Fix any $\gamma > 0$. Two applications of (3.1) show that for all $0 < r \leqslant r_0$ so small that $r + \gamma r^{1+\alpha} \leqslant r_0$,

$$\begin{aligned}\mu_p(r+\gamma r^{1+\alpha}) &\leqslant K_0 (r+\gamma r^{1+\alpha})^{\beta} \exp\Big(-\frac{\chi}{(r+\gamma r^{1+\alpha})^{\alpha}}\Big)\\ &\leqslant K_0^2 \big[(1+\gamma r_0^{\alpha})^{\beta} \vee 1\big] \mu_p(r) \mathcal{Q}\end{aligned}$$

where,

$$\mathcal{Q} \triangleq \exp\Big(\frac{\chi}{r^{\alpha}} - \frac{\chi}{(r+\gamma r^{1+\alpha})^{\alpha}}\Big).$$

A little calculus shows that whenever $\alpha > 0$, for all $0 < r \leqslant r_0$,

$$1 - \frac{1}{(1+\gamma r^{\alpha})^{\alpha}} \leqslant \big((1+\gamma r_0)^{\alpha-1} \vee 1\big) \alpha\gamma r^{\alpha}.$$

Hence,

$$\mathcal{Q} \leqslant \exp\Big(\chi\alpha\gamma\big(1 \vee (1+\gamma r_0)^{\alpha-1}\big)\Big).$$

Fix any $\varkappa > K_0^2$. Note that for all $\gamma > 0$ small,

$$K_0^2\big[(1+\gamma r_0^\alpha)^\beta \vee 1\big]\exp\Big(\chi\alpha\gamma[1\vee(1+\gamma r_0)^{\alpha-1}]\Big)\leqslant \varkappa.$$

Thus, we have shown that $\mu_p(r+\gamma r^{1+\alpha})\leqslant \varkappa\mu_p(r)$. That is, for any $\varkappa > K_0^2$ and all $\gamma > 0$ small enough, $\lambda_p(r;\varkappa)\geqslant r+\gamma r^{1+\alpha}$. ♢

Corollary 3.4. *Suppose that Assumption 3.1 holds. For $K > 0$, there exists a constant $c > 1$ such that for all $r \in (0, 1/c)$,*

$$\begin{aligned} c^{-1}\, r^{\beta+\alpha-2}\exp\Big(-\frac{\chi}{r^\alpha}\Big) &\leqslant \mathbb{P}\big(p(B_t)\leqslant t^{1/2}r,\ \textit{for some } t\in[1,1+K\,r^\alpha]\big)\\ &\leqslant c\, r^{\beta-\alpha-2}\exp\Big(-\frac{\chi}{r^\alpha}\Big). \end{aligned}$$

We are now ready to complete the proof of Theorem 3.2.

Proof of (3.3). Assume $I_2(\psi)<\infty$. Let t_k and ψ_k be as in the proof of (3.2). Define,

$$\mathcal{U}_k \triangleq \mathbb{P}\big(p(B_t)\leqslant t_{k+1}^{1/2}\psi_k,\ \text{for some } t\in[t_k,t_{k+1}]\big).$$

By Brownian scaling,

$$\mathcal{U}_k \leqslant \mathbb{P}\big(p(B_t)\leqslant t^{1/2}\varphi_k,\ \text{for some } t\in[1,t_{k+1}/t_k]\big),$$

where $\varphi_k \triangleq (t_{k+1}/t_k)^{1/2}\psi_k$. By (3.4), $t_{k+1}/t_k = 1+O(\varphi_k^\alpha)$. Therefore, Corollary 3.4 shows that

$$\mathcal{U}_k \leqslant C_9\,\varphi_k^{\beta-\alpha-2}\exp\Big(-\frac{\chi}{\varphi_k^\alpha}\Big)\leqslant C_{10}\,\psi_k^{\beta-\alpha-2}\exp\Big(-\frac{\chi}{\psi_k^\alpha}\Big),$$

where C_9 and C_{10} are two constants. Since $I_2(\psi)<\infty$, this yields $\sum_k \mathcal{U}_k<\infty$. By the Borel–Cantelli lemma, for all k_0 large enough (random but finite $\mathbb{P}$–a.s.), the following holds $\mathbb{P}$–a.s. for all $k\geqslant k_0$,

$$\inf_{t_k\leqslant t\leqslant t_{k+1}} p(B_t)\geqslant t_{k+1}^{1/2}\psi_k.$$

Take any $k\geqslant k_0$ and $t\in[t_k,t_{k+1}]$. Then, by monotonicity,

$$p(B_t)\geqslant t^{1/2}\psi(t),$$

as desired. ♢

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