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# GAUSSIAN PROCESSES, MOVING AVERAGES AND QUICK DETECTION PROBLEMS[1]

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In this paper, we are interested in moving averages of the type $\int_0^t f(t-s)\,dX(s)$, where $X(t)$ is a Wiener process and $\int_0^\infty f^2(t)\,dt < \infty$. By a suitable choice of the weighting function $f$, such processes can be used to detect a change in the drift of $X(t)$. First passage times of these moving-average processes and more general Gaussian processes are studied. Limit theorems for Gaussian processes and Gaussian sequences which include these moving-average processes and their discrete-time analogs as special cases are also proved.

**1. Introduction.** In a continuous production process, samples of fixed size are taken at regular intervals of time and a statistic $X_n$ is computed from the $n$th sample, $n = 1, 2, \cdots$. In [6], we have considered process inspection schemes based on moving averages of the type $\sum_{i=1}^n c_{n-i} X_i$, where $(c_i)$ is a suitably chosen sequence of weights. Unlike weighted sums of the form $\sum_{i=1}^n a_i X_i$, the moving averages $Y_n = \sum_{i=1}^n c_{n-i} X_i$ do not have a Markovian or martingale structure, and the exact performance of the process inspection schemes based on $Y_n$ is difficult to analyze. In the case where the $X_n$'s are normal, the particular Gaussian structure of the sequence $Y_n$ has enabled us to find sharp bounds and to study the asymptotic behavior of the average run length (i.e., the expected number of articles sampled before action is taken when the quality of the output has remained at a constant level), and numerical comparisons with the average run length of the Shewhart chart have also been given in [6].

In Section 2, we shall apply the continuous-time moving average analogs to detect a change in the drift of a Wiener process. The average run length of such procedures is studied. In Section 3, we shall consider the first passage times of more general continuous-time Gaussian processes. Sections 4 and 5 are devoted to limit theorems for Gaussian sequences and Gaussian processes which include $\sum_{i=1}^n c_{n-i} X_i$ and their continuous-time analogs as special cases.

**2. Continuous-time moving-average analogs and their applications to quick detection procedures.** Let $X(t)$, $t \geqq 0$, be a Wiener process with $EX(t) = \theta t$, where $\theta$ may be increasing over time. We say that a disorder has occurred if $\theta$ exceeds a certain value $\theta_0$, in which case corrective action should be taken. We shall assume for simplicity that $\theta_0 = 0$, for otherwise we can consider $X(t) - \theta_0 t$

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instead of $X(t)$. The longer action is delayed, the more serious the consequences of the occurrence of the disorder would be, and so we want a procedure which can detect quickly the presence of a disorder. The procedure proposed by Page [10] for discrete-time problems can be extended to the present situation: Take corrective action as soon as $X(t) - \min_{0 \leqq s \leqq t} X(s) \geqq h$, some preassigned number. Shiryaev [13] has proposed another procedure based on a process derived from the trajectory of $X(t)$ via a stochastic differential equation. In this section, we shall study a class of detection procedures which use continuous-time analogs of moving averages of the type $\sum_{i=1}^{n} c_{n-i} X_i$. Let $f$ be a nonnegative, nonincreasing function on $[0, \infty)$ such that $\infty > \int_0^\infty f(t)\, dt > 0$. We shall assert that a disorder has occurred and take corrective action as soon as $\int_0^t f(t-s)\, dX(s) \geqq c$, where $c$ is a suitably chosen constant.

To evaluate the performance of our detection procedures, let us consider the continuous-time analog of the average run length, i.e., the expected duration $E_\theta T$ before action is taken when $\theta$ has remained at a constant level. For $\theta > 0$, it is desired that $E_\theta T$ be small, while for $\theta \leqq 0$, we want $E_\theta T$ to be large. Define $T = \inf\{t\colon \int_0^t f(t-s)\, dX(s) \geqq c\}$, where the constant $c$ is so chosen that $E_0 T = M$, some large preassigned number. This guarantees that the expected duration before a false alarm is at least $M$.

A convenient choice of the weighting function $f$ is the following: $f(t) = 1$ for $0 \leqq t \leqq \alpha$ and $f(t) = 0$ for $t > \alpha$. The process $\int_0^t f(t-s)\, dX(s)$ then reduces to $X(t) - X(t-\alpha)$ for $t \geqq \alpha$ (and to $X(t)$ for $t < \alpha$). Letting $T_\alpha(c) = \inf\{t \geqq \alpha\colon X(t) - X(t-\alpha) \geqq c\}$, we have $E_\theta T_\alpha(c) = E_0 T_\alpha(c - \theta\alpha)$. Let $W(t)$ be the standard Wiener process and define $T(x) = \inf\{t \geqq 0\colon W(t+1) - W(t) \geqq x\}$. Then it is easy to see that

$$E_0 T_\alpha(x) = \alpha + \alpha E T(x\alpha^{-\frac{1}{2}})\,. \tag{1}$$

The distribution of $T(x)$ is given by Shepp [12] who has proved that for $n = 1, 2, \cdots$,

$$P[T(x) > n] = \int_{-\infty}^{x} \int \cdots_{D} \int \det \varphi(y_i - y_{j+1} + x)\, dy_2 \cdots dy_{n+1}\, du \tag{2}$$

where $D = \{x - u < y_2 < y_3 < \cdots < y_{n+1}\}$ and the determinant is of size $(n+1) \times (n+1)$, $0 \leqq i, j \leqq n$ with $y_0 = 0$, $y_1 = x - u$. A similar formula for $P[T(x) > n + \theta]$ with $0 < \theta < 1$ is also given in [12].

Since for large $n$, the expression on the right-hand side of (2) is not suited for either numerical calculation or asymptotic evaluation, the following upper and lower bounds on $ET(x)$ are given below:

$$\{1 - \Phi^2(x) + \varphi(x) \textstyle\int_{-\infty}^{x} \Phi(u)\, du\}^{-1} - 1 \leqq ET(x) \leqq \{\Phi^2(x) - \varphi(x) \textstyle\int_{-\infty}^{x} \Phi(u)\, du\}/\lambda(x) \tag{3}$$

where we set $\Psi(x) = \int_{-\infty}^{x} \Phi(u)\, du$ and

$$\begin{aligned}\lambda(x) = {} & (1 - \Phi(x))(\Phi^2(x) - \varphi(x)\Psi(x)) + \varphi(x)\Phi(x)\Psi(x) - \varphi^2(x) \textstyle\int_{-\infty}^{x} \Psi(u)\, du \\ & - \textstyle\int_{-\infty}^{x} \int_{-\infty}^{x} \varphi(2x - z)\varphi(u + z - x)\Phi(z)\, dz\, du \\ & + \textstyle\int_{-\infty}^{x} \int_{-\infty}^{u} \varphi(x - u + z)\varphi(x + u - z)\Phi(z)\, dz\, du\,.\end{aligned}$$

To prove (3), we shall use the following lemma.

LEMMA 1. *Let* $S(t) = W(t+1) - W(t)$, $t \geqq 0$. *Then for any* $\beta > 2$,

$$P[\max_{0 \leqq t \leqq \beta-1} S(t) < x, \max_{\beta-1 < t \leqq \beta} S(t) \geqq x] \geqq \lambda(x) P[\max_{0 \leqq t \leqq \beta-2} S(t) < x] .$$

PROOF. $S(t)$ is a stationary Gaussian process with covariance $ES(\tau)\, S(t) = \max(1 - |t-\tau|, 0)$. Let $I = [0, \beta-2]$, $J_1 = [\beta-2, \beta-1]$, $J_2 = (\beta-1, \beta]$, $J = J_1 \cup J_2$. Suppose $t_1, \cdots, t_k \in I$, $t_{k+1}, \cdots, t_{k+m} \in J$ and $\tau \in J_2$ such that $\tau > \max\{t_1, \cdots, t_{k+m}\}$. Then

$$\begin{aligned} &P[S(t_1) \leqq x, \cdots, S(t_{k+m}) \leqq x, S(\tau) > x] \\ &\qquad \geqq P[S(t_1) \leqq x, \cdots, S(t_k) \leqq x] \\ &\qquad\qquad \times P[S(t_{k+1}) \leqq x, \cdots, S(t_{k+m}) \leqq x, S(\tau) > x] . \end{aligned} \tag{4}$$

To prove (4), it is well known that if $g(y_1, \cdots, y_n)$ is the density function of the multivariate normal distribution with means 0, variances 1 and correlation matrix $(\lambda_{ij})$, then

$$\frac{\partial g}{\partial \lambda_{ij}} = \frac{\partial^2 g}{\partial y_i \partial y_j}, \qquad i \neq j .$$

(cf. [14]). Using this, it can be shown that $P[S(t_1) \leqq x, \cdots, S(t_{k+m}) \leqq x, S(\tau) > x]$ is a non-decreasing function of the correlation coefficient $\lambda_{ij}$ between $S(t_i)$ and $S(t_j)$ for $1 \leqq i < j \leqq k+m$. The inequality (4) follows easily from this fact.

Let $D$ be a countable dense subset of $[0, \beta]$ and let $D_n \uparrow D$ as $n \uparrow \infty$. Then using (4), we obtain

$$\begin{aligned} &P[\max_{0 \leqq t \leqq \beta-1} S(t) < x, \max_{\beta-1 < t \leqq \beta} S(t) \geqq x] \\ &\quad \geqq \lim_{n\to\infty} \lim_{m\to\infty} P[\max_{t \in D_m \cap [0,\beta-1]} S(t) \leqq x, \max_{t \in D_n \cap J_2} S(t) > x] \\ &\quad \geqq \lim_{n\to\infty} \lim_{m\to\infty} P[\max_{t \in D_m \cap I} S(t) \leqq x] \\ &\qquad\quad \times P[\max_{t \in D_m \cap J_1} S(t) \leqq x, \max_{t \in D_n \cap J_2} S(t) > x] \\ &\quad \geqq P[\max_{t \in I} S(t) \leqq x] P[\max_{t \in J_1} S(t) \leqq x, \max_{t \in J_2} S(t) > x] \\ &\quad = P[\max_{t \in I} S(t) < x] P[\max_{\beta-2 \leqq t \leqq \beta-1} S(t) < x, \max_{\beta-1 < t \leqq \beta} S(t) \geqq x] . \end{aligned} \tag{5}$$

Since $S(t)$ is stationary Gaussian, we have

$$\begin{aligned} &P[\max_{\beta-2 \leqq t \leqq \beta-1} S(t) < x, \max_{\beta-1 < t \leqq \beta} S(t) \geqq x] \\ &\qquad = P[\max_{0 \leqq t \leqq 1} S(t) < x, \max_{1 < t \leqq 2} S(t) \geqq x] \\ &\qquad = P[1 < T(x) \leqq 2] = P[T(x) > 1] - P[T(x) < 2] = \lambda(x) . \end{aligned} \tag{6}$$

The last relation in (6) follows from (2). Using (5) and (6), we obtain the desired conclusion. □

To prove (3), we note that $\sum_{n=1}^{\infty} P[T(x) > n] \leqq ET(x) \leqq \sum_{n=0}^{\infty} P[T(x) > n]$. It is well known that if $Y_1, Y_2, \cdots, Y_k$ has a multivariate normal distribution with $EY_i = 0$ and $EY_i^2 = 1$, then $P[Y_1 < c, \cdots, Y_k < c_k]$ is a non-decreasing function of the correlation coefficient $r_{ij}$ between $Y_i$ and $Y_j$ for $1 \leqq i < j \leqq n$.

From this it easily follows that

$$(7)\qquad P[T(x) > n] \geqq P[\max_{0\leqq t\leqq 1} S(t) < x] \cdots P[\max_{n-1\leqq t\leqq n} S(t) < x] \\ = (P[T(x) > 1])^n = \{\Phi^2(x) - \varphi(x)\Psi(x)\}^n .$$

The last relation in (7) follows from (2). Using (7), we easily obtain the lower bound in (3).

To obtain the upper bound in (3), we let $a_0 = 1$ and $a_n = P[\max_{0\leqq t\leqq n} S(t) < x] = P[T(x) > n]$ for $n \geqq 1$. It follows from Lemma 1 that for $n \geqq 3$,

$$(8)\qquad a_n = a_{n-1} - P[\max_{0\leqq t\leqq n-1} S(t) < x,\ \max_{n-1<t\leqq n} S(t) \geqq x] \\ \leqq a_{n-1} - a_{n-2}\lambda(x) .$$

Summing over (8) for $n \geqq 3$, we have

$$ET(x) \leqq 1 + (a_2/\lambda(x)) = P[T(x) > 1]/\lambda(x)$$

and so we obtain the upper bound in (3).

It is easy to see that as $x \to \infty$, $\Psi(x) \sim x$. Therefore the lower bound in (3) is asymptotic to $(x\varphi(x))^{-1}$. Also it can be shown that $\lambda(x) \sim x\varphi(x)$ as $x \to \infty$. Hence it follows from (3) that

$$(9)\qquad ET(x) \sim (2\pi)^{\frac{1}{2}} x^{-1} \exp(x^2/2) \qquad \text{as } x \to \infty .$$

Compare this result with the discrete-time relation:

$$(10)\qquad EN(x) = (1 - \Phi(x))^{-1} \sim (2\pi)^{\frac{1}{2}} x \exp(x^2/2) \qquad \text{as } x \to \infty$$

where $N(x) = \inf\{n \geqq 1 : S(n) \geqq x\}$.

Another interesting choice of the weighting function $f$ is the negative exponential function $f(t) = e^{-\alpha t}$, $t \geqq 0$, $\alpha > 0$. If $EX(t) = \theta t$, then the process $\int_0^t f(t-s)\,dX(s)$ has the same distribution as the process $V_0(t) + \theta(1 - e^{-\alpha t})/\alpha$, where $V_x(t)$ denotes the Ornstein–Uhlenbeck process (with infinitesimal generator defined by $\frac{1}{2}(d^2/dx^2) - \alpha x(d/dx)$) starting at $V_x(0) = x$. Let $\tau(c) = \inf\{t \geqq 0 : \int_0^t f(t-s)\,dX(s) \geqq c\}$, $\tau_x(c) = \inf\{t \geqq 0 : V_x(t) \geqq c\}$. Since the process $V_0(t) - (\theta/\alpha)e^{-\alpha t}$ has the same distribution as the process $V_x(t)$ with $x = -\theta/\alpha$, it follows that

$$(11)\qquad E_\theta \tau(c) = E\tau_x(c + x) , \qquad \text{where } x = -\theta/\alpha .$$

LEMMA 2. *For $b < x < h$, define $\tau_x(h)$ as before and define $\tau_x(h, b) = \inf\{t \geqq 0 : V_x(t) \notin (b, h)\}$. Then*

$$(12)\qquad E\tau_x(h, b) = 2(\pi/\alpha)^{\frac{1}{2}}\left(\int_b^h e^{\alpha y^2}\,dy\right)^{-1}\left\{\left(\int_b^x e^{\alpha y^2}\,dy\right)\left(\int_x^h \Phi((2\alpha)^{\frac{1}{2}}y)e^{\alpha y^2}\,dy\right) \\ - \left(\int_x^h e^{\alpha y^2}\,dy\right)\left(\int_b^x \Phi((2\alpha)^{\frac{1}{2}}y)e^{\alpha y^2}\,dy\right)\right\} ;$$

$$(13)\qquad E\tau_x(h) = 2(\pi/\alpha)^{\frac{1}{2}} \int_x^h \Phi((2\alpha)^{\frac{1}{2}}y)e^{\alpha y^2}\,dy .$$

PROOF. Darling and Siegert [2] have found the Laplace transform of the first passage distribution for the Ornstein–Uhlenbeck process. Instead of using their result which involves the Weber functions, we give here a simple martingale derivation of (12) and (13). We shall make use of the fact that $\{s(V_x(t)), t \geqq 0\}$ and $\{\int_0^{V_x(t)} m(y)\, ds(y) - t, t \geqq 0\}$ are martingales (cf. [5]), where $s(z) = \int_0^z e^{\alpha y^2}\, dy$ is the scale function and $m(z) = 2\int_0^z e^{-\alpha y^2}\, dy = 2(\pi/\alpha)^{\frac{1}{2}}\{\Phi((2\alpha)^{\frac{1}{2}}z) - \frac{1}{2}\}$ defines the speed measure of the Ornstein–Uhlenbeck process $V_x(t)$. (We use the convention that $\int_a^b = -\int_b^a$ if $b < a$.) Letting $K = 2(\pi/\alpha)^{\frac{1}{2}}$, it then follows that $\{K\int_x^{V_x(t)} \Phi((2\alpha)^{\frac{1}{2}}y)e^{\alpha y^2}\, dy - t, t \geqq 0\}$ is also a martingale. From this we obtain

$$(14)\qquad \begin{aligned} E\tau_x(h, b) &= KE\int_x^{V_x(\tau_x(h,b))} \Phi((2\alpha)^{\frac{1}{2}}y)e^{\alpha y^2}\, dy \\ &= KP[V_x(\tau_x(h, b)) = h]\int_x^h \Phi((2\alpha)^{\frac{1}{2}}y)e^{\alpha y^2}\, dy \\ &\quad - KP[V_x(\tau_x(h, b)) = b]\int_b^x \Phi((2\alpha)^{\frac{1}{2}}y)e^{\alpha y^2}\, dy\, . \end{aligned}$$

We also note that

$$(15)\qquad \begin{aligned} P[V_x(\tau_x(h, b)) = h] &= (s(x) - s(b))/(s(h) - s(b)) \\ &= \left(\int_b^x e^{\alpha y^2}\, dy\right)/\left(\int_b^h e^{\alpha y^2}\, dy\right) . \end{aligned}$$

The relation (12) then follows from (14) and (15). Letting $b \to -\infty$ in (12), we obtain (13). □

Now let $V(t)$ be the stationary Ornstein–Uhlenbeck process which is stationary Gaussian with $EV(t) = 0$ and $\mathrm{Cov}\,(V(s), V(t)) = (2\alpha)^{-1}\exp(-\alpha|s - t|)$, i.e., $V(t) = V_0(t) + e^{-\alpha t}Z$, where $Z$ is $N(0, (2\alpha)^{-1})$ and is independent of the process $V_0(t)$. Let $\tau^*(c) = \inf\{t \geqq 0 : V(t) \geqq c\}$. Then it follows from (13) that

$$\begin{aligned} E\tau^*(c) &= (\alpha/\pi)^{\frac{1}{2}}\int_{-\infty}^c e^{-\alpha x^2}\, E\tau_x(c)\, dx \\ &= \alpha^{-1}(2\pi)^{\frac{1}{2}}\int_{-\infty}^{(2\alpha)^{\frac{1}{2}}c} \Phi^2(z)\exp(z^2/2)\, dz\, . \end{aligned}$$

From this it is easy to see that if $U(t)$ is the stationary Ornstein–Uhlenbeck process with $EU(t) = 0$ and $\mathrm{Cov}\,(U(s), U(t)) = \rho\exp(-\alpha|s - t|)$ and if $T_U(c) = \inf\{t \geqq 0 : U(t) \geqq c\}$, then

$$(16)\qquad ET_U(c) = \alpha^{-1}(2\pi)^{\frac{1}{2}}\int_{-\infty}^{c/\rho^{\frac{1}{2}}} \Phi^2(z)\exp(z^2/2)\, dz\, .$$

It then follows from (16) that as $c \to \infty$,

$$(17)\qquad ET_U(c) \sim \alpha^{-1}(2\pi)^{\frac{1}{2}}(c/\rho^{\frac{1}{2}})^{-1}\exp(c^2/2\rho)\, .$$

On the other hand, letting $N_U(c) = \inf\{n \geqq 1 : U(n) \geqq c\}$, we have proved in [6] that

$$(18)\qquad EN_U(c) \sim (2\pi)^{\frac{1}{2}}(c/\rho^{\frac{1}{2}})\exp(c^2/2\rho) \qquad \text{as } c \to \infty\, .$$

We conclude this section with some remarks on the moving-average process $Y(t) = \int_0^t f(t - s)\, dW(s)$, where $0 < \int_0^\infty f^2(t)\, dt < \infty$. This process is nonstationary Gaussian, unlike the stationary moving-average process of the form $\int_{-\infty}^\infty f(t - s)\, d\xi(s)$ considered in [3], where $\xi(s)$ is a process with orthogonal increments. We now list some properties of the process $Y(t)$ below:

(A) $EY(t) = 0$, $EY(t)Y(t + s) = R(t, t + s) = \int_0^t f(u)f(s + u)\, du$ for $t, s \geqq 0$, and the covariance function $R(s, t)$ is positive definite if $\inf_{x \in (0,\delta)} |f(x)| > 0$ for some $\delta > 0$.

(B) If $f$ is continuous, then with probability 1, $Y(t)$ has continuous sample paths.

(C) If $f$ is continuously differentiable and $f(0) = 1$, then the Gaussian measure on the space of continuous functions on $[0, T]$ induced by the process $\{Y(t), 0 \leqq t \leqq T\}$ is equivalent to the Wiener measure (i.e., both measures have the same sets of measure 0).

Property (C) above follows from a result of Shepp ([11] pages 322–323), noting that the covariance function $R(s, t)$ in the present case is positive definite by property (A). By making use of the Radon–Nikodym derivative with respect to the Wiener measure, Shepp [11] has computed the first passage probability $P[T(x) > t \mid S(0) = a]$ for $t \leqq 1$, where $S(t)$ is as defined in Lemma 1 and $T(x)$ is the first time the process $S(t)$ hits $x$.

**3. First passage times for Gaussian processes.** The asymptotic behavior of the mean first passage times for the processes $S(t)$ and $V(t)$ considered in the preceding section is now generalized in the following theorem, a discrete-time version of which is proved in [6].

THEOREM 1. *Let $Y(t)$, $t \geqq 0$, be a real-valued separable Gaussian process with $EY(t) = 0$ and $\lim_{t\to\infty} EY^2(t) = \sigma^2 > 0$. For any real number c, define $T(c) = \inf\{t \geqq 0\colon Y(t) \geqq c\}$. Let $R(s, t) = EY(s)Y(t)$.*

(i) *If $\lim_{\alpha\to\infty} \sup_{t-s\geqq\alpha, s\geqq s_0} R(s, t) \leqq 0$, then for $\nu = 1, 2, \cdots$, $ET^\nu(c) < \infty$ and*

$$\eta > 1/(2\sigma^2) \Rightarrow ET^\nu(c) = o(\exp(\nu\eta c^2)) \qquad as\ c \to \infty\,. \tag{19}$$

(ii) *If $R(s, t) \geqq 0$ and there exists a continuous non-decreasing function $\Psi$ on $[0, \beta]$ such that $\int_1^\infty \Psi(\beta e^{-u^2})\,du < \infty$ and $E(Y(t) - Y(s))^2 \leqq \Psi^2(|t - s|)$ for $|t - s| \leqq \beta$, then for $\nu = 1, 2, \cdots$,*

$$\eta < 1/(2\sigma^2) \Rightarrow \lim_{c\to\infty} (\exp(\nu\eta c^2))/ET^\nu(c) = 0\,. \tag{20}$$

PROOF. (i) follows from the corresponding discrete-time result in [6] since $T(c) \leqq \inf\{n \geqq 1 : Y(n) \geqq c\}$. To prove (ii), choose $\delta_1 > 1$, $\delta_2 > 0$ such that $\eta < \frac{1}{2}(\delta_1\sigma + \delta_2)^{-2}$, and pick $s_1 \geqq 0$ such that $EY^2(s) \leqq \delta_1^2\sigma^2$ for $s \geqq s_1$. Let $I_0 = [0, s_1]$, $I_n = [s_1 + (n-1)\beta, s_1 + n\beta]$ for $n \geqq 1$, and $Z_n = \sup_{t\in I_n} Y(t)$. Define $N(c) = \inf\{n \geqq 0\colon Z_n \geqq c\}$. Clearly $T(c) \geqq \beta(N(c) - 1)$ and so we need only prove that

$$\lim_{c\to\infty} (\exp(\nu\eta c^2))/EN^\nu(c) = 0\,. \tag{21}$$

As in the proof of the lower bound of (3), it can be shown that

$$P[N(c) > n] \geqq P[Z_0 < c] \cdots P[Z_n < c]\,. \tag{22}$$

Choose an integer $p > e$ such that $4\int_1^\infty \Psi(\beta p^{-u^2})\,du < \delta_2$. By Fernique's lemma (cf. [4], [7]), it follows that for $x \geqq (1 + 4\log p)^{\frac{1}{2}}$ and $n = 1, 2, \cdots$,

$$\begin{aligned} &P[Z_n \geqq x(\delta_1\sigma + \delta_2)] \\ &\quad \leqq P[\sup_{t\in I_n} |Y(t)| \geqq x\{\sup_{t\in I_n} R^{\frac{1}{2}}(t, t) + 4\textstyle\int_1^\infty \Psi(\beta p^{-u^2})\,du\}] \\ &\quad \leqq 4p^2 \textstyle\int_x^\infty e^{-u^2/2}\,du\,. \end{aligned}$$

Therefore from (22), we obtain

(23) $$P[N(c) > n] \geqq P[Z_0 < c]\{1 - 4p^2 \int_{c(\delta_1\sigma+\delta_2)^{-1}}^{\infty} \exp(-u^2/2)\, du\}^n .$$

Since $\lim_{c\to\infty} P[Z_0 < c] = 1$ by Fernique's lemma and $\eta < \frac{1}{2}(\delta_1\sigma + \delta_2)^{-2}$, (21) follows easily from (23). □

Let us now consider the moving-average process $Y(t) = \int_0^t f(t-s)\, dW(s)$, where $\int_0^\infty f^2(t)\, dt = \sigma^2 \in (0, \infty)$. Define $T(c)$ as in Theorem 1. Then $|R(t, t+\alpha)| \leqq |\int_0^\infty f(u)f(\alpha+u)\, du| \leqq \{\int_0^\infty f^2(u)\, du\}^{\frac{1}{2}}\{\int_\alpha^\infty f^2(u)\, du\}^{\frac{1}{2}}$, and so $\lim_{\alpha\to\infty} \sup_{|t-s|\geqq\alpha} |R(s, t)| = 0$. Hence by Theorem 1, $ET^\nu(c) < \infty$ for $\nu = 1, 2, \cdots$ and (19) holds. Now assume that $f \geqq 0$ a.e. and that

(24) $\exists$ a continuous non-decreasing function $\Psi$ on $[0, \beta]$ such that $\int_1^\infty \Psi(\beta e^{-u^2})\, du < \infty$ and for all $t \geqq 0$, $0 \leqq x \leqq \beta$, $\int_t^{t+x} f^2(u)\, du + 2\int_0^t f(u)(f(u) - f(u+x))\, du \leqq \Psi^2(x)$.

Then by Theorem 1, (20) holds. A sufficient condition to guarantee (24) is that $f$ is bounded and $\int_0^\infty \{f(u) - f(u+x)\}^2\, du = 0(|\log x|^{-2-\delta})$ as $x \downarrow 0$ for some $\delta > 0$. It is easy to see that the following three interesting choices of $f$ all satisfy this condition:

(a) $f(u) = 1$ for $0 \leqq u \leqq \alpha$, $f(u) = 0$ for $u > \alpha$;
(b) $f(u) = \rho e^{-\alpha u}$ with $\rho > 0$, $\alpha > 0$;
(c) $f(u) = (1+u)^{-\alpha}$ with $\alpha > \frac{1}{2}$.

**4. Analogs of the law of the iterated logarithm.** Let $Y(t)$, $t \geqq 0$, be a real-valued separable Gaussian process with $EY(t) = 0$ and $\lim_{t\to\infty} EY^2(t) = \sigma^2 > 0$. Let $R(s, t) = EY(s)Y(t)$. Nisio [9] has proved that if $\lim_{\alpha\to\infty} \sup_{|t-s|\geqq\alpha} R(s, t) \leqq 0$, then

(25) $$\liminf_{T\to\infty} \{(2\sigma^2 \log T)^{-\frac{1}{2}} \sup_{0\leqq t\leqq T} Y(t)\} \geqq 1 \quad \text{a.e.}$$

(Actually Nisio has only considered the case $EY^2(t) = \sigma^2$ for all $t$, but a trivial modification of her argument proves (25) with $\lim_{t\to\infty} EY^2(t) = \sigma^2$.) In particular, (25) holds for the moving-average process $Y(t) = \int_0^t f(t-s)\, dW(s)$ where $\int_0^\infty f^2(u)\, du = \sigma^2$. Furthermore, if there exists a continuous non-decreasing function on $[0, \beta]$ such that $\int_1^\infty \Psi(\beta e^{-u^2})\, du < \infty$ and $E(Y(t) - Y(s))^2 \leqq \Psi^2(|t-s|)$ for $|t-s| \leqq \beta$, then

(26) $$\limsup_{T\to\infty} \{(2\sigma^2 \log T)^{-\frac{1}{2}} \sup_{0\leqq t\leqq T} |Y(t)|\} \leqq 1 \quad \text{a.e.}$$

(cf. [7], [9]). Therefore if $\int_0^\infty f^2(u)\, du = \sigma^2$ and $f$ satisfies (24), then

(27) $$\lim_{T\to\infty} \{(2\log T)^{-\frac{1}{2}} \sup_{0\leqq t\leqq T} |\textstyle\int_0^t f(t-s)\, dW(s)|\} = \lim_{T\to\infty} \{(2\log T)^{-\frac{1}{2}} \sup_{0\leqq t\leqq T} \textstyle\int_0^t f(t-s)\, dW(s)\} = \sigma \quad \text{a.e.}$$

The following theorem gives the discrete-time analog of Nisio's result. It can be proved by using Nisio's methods [9].

THEOREM 2. *Let $Y_1, Y_2, \cdots$ be a real-valued Gaussian sequence with $EY_i = 0$,*

$EY_iY_j = r_{ij}$ *such that* $\lim_{i\to\infty} r_{ii} = \sigma^2 > 0$. *If* $\lim_{n\to\infty} \sup_{j-i\geqq n, i\geqq i_0} r_{ij} \leqq 0$, *then*

$$\lim_{N\to\infty} \{(2\log N)^{-\frac{1}{2}} \max_{0\leqq n\leqq N} |Y_n|\} = \lim_{N\to\infty} \{(2\log N)^{-\frac{1}{2}} \max_{0\leqq n\leqq N} Y_n\} = \sigma \quad \text{a.e.} \tag{28}$$

As an application of Theorem 2, we obtain

$$\limsup_{n\to\infty} (2\log n)^{-\frac{1}{2}} \sum_{i=1}^n c_{n-i}X_i = (\sum_0^\infty c_i^2)^{\frac{1}{2}} \quad \text{a.e.} \tag{29}$$

where $X_1, X_2, \cdots$ are i.i.d. $N(0, 1)$ random variables and $(c_n)$ is a sequence of real numbers such that $\sum c_n^2 < \infty$. This result can also be proved by a direct application of the Borel–Cantelli Lemma without making use of Theorem 2 (cf. [1]).

**5. Upper and lower class boundaries.** Suppose $Z_1, Z_2, \cdots$ are i.i.d. $N(0, \sigma^2)$ random variables. Let $(b_n)$ be an increasing sequence of positive numbers. Then it is easy to see that

$$P[Z_n \geqq b_n \text{ i.o.}] = 1 \quad \text{or} \quad 0 \quad \text{according as} \quad \sum b_n^{-1}\exp(-b_n^2/2\sigma^2) = \infty \quad \text{or} \quad < \infty\,. \tag{30}$$

We shall call the sequence $(b_n)$ an *upper class boundary* if the series in (30) converges, and say that $(b_n)$ belongs to the *lower class* if the series $= \infty$.

Now consider a real-valued Gaussian sequence $Y_1, Y_2, \cdots$ with $EY_i = 0$, $EY_iY_j = r_{ij}$ such that $\lim_{n\to\infty} r_{nn} = \sigma^2$ and $\lim_{n\to\infty} \sup_{|j-i|\geqq n} r_{ij} \leqq 0$. Theorem 2 suggests that the fluctuation behavior of the sequence $Y_n$ resembles that of the sequence $Z_n$, and so it is natural to ask if (30) still holds if $Z_n$ is replaced by $Y_n$. In general, (30) may fail to hold for the sequence $Y_n$, for example, when $r_{ij} = 0$ for $i \neq j$ and $r_{nn}$ converges to $\sigma^2$ very slowly. However, under mild conditions on the rate of convergence of $r_{nn}$ to $\sigma^2$ and that of $\sup_{|j-i|\geqq n} r_{ij}^+$ to 0 (see Corollary 2 below), (30) can be extended to the sequence $Y_n$.

Given a real-valued Guassian sequence $Y_1, Y_2, \cdots$, there exists a double array $(a_{ni}: n \geqq 1, 1 \leqq i \leqq n)$ of real numbers such that $Y_i = \sum_{i=1}^n a_{ni}X_i$, where $X_1, X_2, \cdots$ are i.i.d. $N(0, 1)$ random variables. To be slightly more general, let us consider representations of the form $Y_n = \sum_{i=-\infty}^n a_{ni}X_i$, where $\cdots, X_{-1}, X_0, X_1, X_2, \cdots$ are i.i.d. standard normal. The following theorem gives conditions on the double array $(a_{ni}: n \geqq 1, -\infty < i \leqq n)$ so that (30) can be extended to the sequence $Y_n$.

THEOREM 3. *Let* $\cdots, X_{-1}, X_0, X_1, X_2, \cdots$ *be* i.i.d. *normal random variables such that* $EX_0 = 0$, $EX_0^2 = 1$. *Suppose* $\sigma > 0$ *and* $(a_{ni}: n \geqq 1, n \geqq i > -\infty)$ *is a double array of real numbers satisfying*

$$\sup_{n>k} \sum_{i=-\infty}^{n-k} a_{ni}^2 = O((\log k)^{-2}) \qquad \textit{as } k \to \infty\,; \tag{31}$$

$$\sup_{n>k} |\sigma^2 - \sum_{i=n-k}^n a_{ni}^2| = O((\log k)^{-1}) \qquad \textit{as } k \to \infty\,. \tag{32}$$

*Let the sequence* $(b_n, n \geqq 1)$ *of positive numbers be ultimately non-decreasing. Then* $P[\sum_{i=-\infty}^n a_{ni}X_i \geqq b_n \text{ i.o.}] = 1$ *or* 0 *according as* $\sum b_n^{-1}\exp(-b_n^2/2\sigma^2) = \infty$ *or* $< \infty$.

PROOF. Let $Y_n = \sum_{i=-\infty}^{n} a_{ni} X_i$, $\sigma_n{}^2 = \sum_{i=-\infty}^{n} a_{ni}^2$. Suppose $\sum b_n{}^{-1} \exp(-b_n{}^2/2\sigma^2) < \infty$. Then conditions (31) and (32) imply that $|\sigma^2 - \sigma_n{}^2| \log n = O(1)$ as $n \to \infty$, and therefore $\sum b_n{}^{-1} \exp(-b_n{}^2/2\sigma_n{}^2) < \infty$. From this, it follows that $\sum P[Y_n \geqq b_n] < \infty$, and so by the Borel–Cantelli lemma, $P[Y_n \geqq b_n \text{ i.o.}] = 0$.

Now assume that $\sum b_n{}^{-1} \exp(-b_n{}^2/2\sigma^2) = \infty$. We shall prove that $P[Y_n \geqq b_n \text{ i.o.}] = 1$. Since by (28), $P[Y_n < 2\sigma(\log n)^{\frac{1}{2}}$ for all large $n] = 1$, we can assume that $b_n \leqq 2\sigma(\log n)^{\frac{1}{2}}$ for all large $n$. Let $\gamma > 1$, $0 < \eta < 1$, $0 < \xi < 1$ such that $\gamma(1 - \eta^2\xi) < 1$. Without loss of generality, we can assume that $b_n \geqq \sigma(2\xi \log n)^{\frac{1}{2}}$ for all large $n$. To see this, let $m_1 < m_2 < \cdots$ be the set of positive integers where $b_{m_i} < \sigma(2\xi \log m_i)^{\frac{1}{2}}$, and suppose that this set is infinite. Since the sequence $(b_n)$ is non-decreasing, $b_n < \sigma(2\xi \log m_i)^{\frac{1}{2}}$ for $n \leqq m_i$. Define $\bar{b}_n = \sigma(2\xi \log m_i)^{\frac{1}{2}}$ if $m_{i-1} < n \leqq m_i$. Then $\sum \bar{b}_n{}^{-1} \exp(-\bar{b}_n{}^2/2\sigma^2) \geqq (2\xi\sigma^2)^{-\frac{1}{2}} \sum (m_i - m_{i-1})/m_i = \infty$. Also if $P[Y_n \geqq \bar{b}_n \text{ i.o.}] = 1$, then $P[Y_n \geqq b_n \text{ i.o.}] = 1$. Hence we shall assume below that for all large $n$,

$$\sigma(2\xi \log n)^{\frac{1}{2}} \leqq b_n \leqq 2\sigma(\log n)^{\frac{1}{2}} . \tag{33}$$

Let $n_k = [k^\gamma]$ for $k \geqq 1$, $Y_{n,j} = \sum_{i=j}^{n} a_{ni} X_i$, $\sigma_{n,j}^2 = \sum_{i=j}^{n} a_{ni}^2$ for $n > j$. By (31), there exists $d > 1$ such that

$$\sup_{n>k} \sum_{i=-\infty}^{n-k} a_{ni}^2 \leqq d^2(\log k)^{-2} , \qquad k > e . \tag{34}$$

Choose $\varepsilon > 0$ such that

$$\varepsilon^2(\gamma - 1)^2 > 8d^2\sigma^2\gamma^2 . \tag{35}$$

For $n_k \leqq m < n_{k+1}$, define

$$A_{m,k} = [b_m + \varepsilon b_m{}^{-1} < Y_{m,n_{k-1}} < b_m + 2\varepsilon b_m{}^{-1}]$$
$$A_k = \textstyle\bigcup_{m=n_k}^{n_{k+1}-1} A_{m,k} .$$

It is easy to see from (32) that as $k \to \infty$,

$$\sup\{|\sigma^2 - \sigma_{m,n_{k-1}}^2| : n_k \leqq m < n_{k+1}\} = O((\log k)^{-1}) . \tag{36}$$

From (33) and (36), we obtain that for $n_k \leqq m < n_{k+1}$,

$$PA_{m,k} \geqq Cb_m{}^{-1} \exp(-b_m{}^2/2\sigma^2) \tag{37}$$

where $C$ is a positive constant.

We now show that for all $k$ large,

$$PA_k \geqq C_1 \textstyle\sum_{m=n_k}^{n_{k+1}-1} b_m{}^{-1} \exp(-b_m{}^2/2\sigma^2) \tag{38}$$

where $C_1$ is a positive constant. Let $N$ be a positive integer (to be chosen later) and for $n_k \leqq m < n_{k+1}$, define

$$A_{m,k}^{(N)} = A_{m,k} - A_{m,k} \cap (\textstyle\bigcup_{\rho=m+N}^{n_{k+1}-1} A_{\rho,k}) .$$

Then since $A_{m_1,k}^{(N)} \cap A_{m_2,k}^{(N)} = \varnothing$ if $|m_1 - m_2| \geqq N$, it follows that

$$\begin{aligned} P(A_k) &\geqq \frac{1}{N} \textstyle\sum_{m=n_k}^{n_{k+1}-1} P(A_{m,k}^{(N)}) \\ &\geqq \frac{1}{N} \textstyle\sum_{m=n_k}^{n_{k+1}-1} \{PA_{m,k} - \sum_{\rho=m+N}^{n_{k+1}-1} P(A_{m,k} \cap A_{\rho,k})\} . \end{aligned} \tag{39}$$

For $n_k \leqq m,\ m + N \leqq \rho < n_{k+1}$, define

$$V_{\rho,m} = \sum_{i=n_{k-1}}^{m} a_{\rho i} X_i\,, \qquad v_{\rho,m}^2 = \sum_{i=n_{k-1}}^{m} a_{\rho i}^2\,. \tag{40}$$

We note that by (34), $v_{\rho,m}^2 \leqq \sum_{i=-\infty}^{m} a_{\rho i}^2 \leqq d^2(\log N)^{-2}$. Letting $\lambda_{\rho m}$ denote the correlation coefficient between $V_{\rho,m}$ and $Y_{m,n_{k-1}}$, the conditional distribution of $V_{\rho,m}$ given $Y_{m,n_{k-1}} = y$ is a normal distribution with mean $y\lambda_{\rho m} v_{\rho,m}/\sigma_{m,n_{k-1}}$ and variance $v_{\rho,m}^2(1 - \lambda_{\rho m}^2)$. We note that

$$(A_{m,k} \cap A_{\rho,k}) \subset (A_{m,k} \cap B_{\rho,m}) \cup (A_{m,k} \cap D_{\rho,m}) \tag{41}$$

where

$$B_{\rho,m} = [|V_{\rho,m} - \lambda_{\rho m} v_{\rho,m} Y_{m,n_{k-1}}/\sigma_{m,n_{k-1}}| > (b_\rho/\sigma) v_{\rho,m}(1 - \lambda_{\rho m}^2)^{\frac{1}{2}}]$$
$$D_{\rho,m} = [Y_{\rho,m+1} > b_\rho + \varepsilon b_\rho^{-1} - |\lambda_{\rho m}| v_{\rho,m}(b_m + 2\varepsilon b_m^{-1})/\sigma_{m,n_{k-1}} - (b_\rho/\sigma) v_{\rho,m}(1 - \lambda_{\rho m}^2)^{\frac{1}{2}}]\,.$$

Since $b_n \uparrow \infty$ and $|\lambda_{\rho m}| \leqq 1$, it follows from (36) that we can choose $k_0$ such that for $k \geqq k_0$ and $\rho > m \geqq n_k$, $D_{\rho,m} \subset [Y_{\rho,m+1} > b_\rho - 3(b_\rho/\sigma)v_{\rho,m}]$. Let $\delta \in (\eta, 1)$. Since $v_{\rho,m} \leqq d(\log N)^{-1}$ and $|\sigma^2 - \sum_{i=m+1}^{\rho} a_{\rho i}^2| \leqq$ constant $(\log N)^{-1}$ for $\rho \geqq m + N$, we can choose $N$ sufficiently large such that for $k \geqq k_0$ and $\rho \geqq m + N > m \geqq n_k$, we have

$$\begin{aligned} PD_{\rho,m} &\leqq P[Y_{\rho,m+1} > \delta b_\rho] = 1 - \Phi(\delta b_\rho (\textstyle\sum_{i=m+1}^{\rho} a_{\rho i}^2)^{-\frac{1}{2}}) \\ &\leqq \exp(-\eta^2 b_\rho^2/2\sigma^2) \leqq \exp(-\eta^2 \xi \log \rho)\,. \end{aligned} \tag{42}$$

The last inequality above follows from (33). Recalling that $\eta^2\xi\gamma > \gamma - 1$, we can pick $k_1 \geqq k_0$ such that for $k \geqq k_1$, $(n_{k+1} - n_k)\exp(-\eta^2\xi\gamma \log k) \leqq \frac{1}{3}$. By the independence of $A_{m,k}$ and $D_{\rho,m}$, we then obtain from (42) that for $k \geqq k_1$ and $n_k \leqq m < n_{k+1}$,

$$\begin{aligned} \textstyle\sum_{\rho=m+N}^{n_{k+1}-1} P(A_{m,k} \cap D_{\rho,m}) &= PA_{m,k} \textstyle\sum_{\rho=m+N}^{n_{k+1}-1} PD_{\rho,m} \\ &\leqq (PA_{m,k})(n_{k+1} - n_k)\exp(-\eta^2\xi\gamma \log k) \\ &\leqq \tfrac{1}{3} PA_{m,k}\,. \end{aligned} \tag{43}$$

Letting $f(y)$ denote the density function of $Y_{m,n_{k-1}}$, we have

$$\begin{aligned} P(A_{m,k} \cap B_{\rho,m}) &= \textstyle\int_{b_m + \varepsilon b_m^{-1}}^{b_m + 2\varepsilon b_m^{-1}} P[B_{\rho,m} \mid Y_{m,n_{k-1}} = y] f(y)\, dy \\ &= 2(1 - \Phi(b_\rho/\sigma)) PA_{m,k}\,. \end{aligned} \tag{44}$$

Hence for $k \geqq k_2 \geqq k_1$ and $n_k \leqq m < n_{k+1}$, we have from (44) that

$$\textstyle\sum_{\rho=m+N}^{n_{k+1}-1} P(A_{m,k} \cap B_{\rho,m}) \leqq PA_{m,k} \sum_{\rho=m+N}^{n_{k+1}-1} \exp(-\eta^2 b_\rho^2/2\sigma^2) \leqq \frac{1}{3} PA_{m,k}\,. \tag{45}$$

It therefore follows from (39), (41), (43) and (45) that for $k \geqq k_2$,

$$PA_k \geqq (3N)^{-1} \textstyle\sum_{m=n_k}^{n_{k+1}-1} PA_{m,k}\,. \tag{46}$$

Using (37) and (46), we obtain (38). From (38), it follows that $\sum PA_k = \infty$, and so $\sum PA_{2k} = \infty$ or $\sum PA_{2k+1} = \infty$. Now $\{A_{2k}: k = 1, 2, \cdots\}$ is independent, and so is the family $\{A_{2k+1}: k = 1, 2, \cdots\}$. Hence either $P[A_{2k}$ i.o.$] = 1$ or $P[A_{2k+1}$ i.o.$] = 1$. Therefore $P[A_k$ i.o.$] = 1$.

For $n_k \leqq m < n_{k+1}$, let $Z_{m,k} = \sum_{i=-\infty}^{n_{k-1}-1} a_{mi} X_i$. By (34), $EZ_{m,k}^2 \leqq d^2(\log(n_k - n_{k-1}))^{-2} \sim d^2(\gamma - 1)^{-2}(\log k)^{-2}$ as $k \to \infty$. Using this fact, together with (33) and (35), it is easy to check that $\sum_{k=1}^{\infty} \sum_{m=n_k}^{n_{k+1}-1} P[|Z_{m,k}| \geqq \varepsilon b_m^{-1}] < \infty$. Therefore $P[|Z_{m,k}| < \varepsilon b_m^{-1}$ for all $n_k \leqq m < n_{k+1}$, for all large $k] = 1$. □

COROLLARY 1. *Suppose* $(f(n), n \geqq 0)$ *is a sequence of real numbers such that* $\sum_{i=0}^{\infty} f^2(i) = \sigma^2 > 0$ *and* $\sum_{i=n}^{\infty} f^2(i) = O((\log n)^{-2})$ *as* $n \to \infty$. *Let* $(b_n)$ *be an ultimately non-decreasing sequence of positive numbers, and let* $\cdots, X_{-1}, X_0, X_1, X_2, \cdots$ *be* i.i.d. *normal random variables with* $EX_0 = 0$, $EX_0^2 = 1$. *If* $\sum b_n^{-1} \exp(-b_n^2/2\sigma^2) < \infty$, *then*

$$P[\sum_{i=-\infty}^{n} f(n-i) X_i \geqq b_n \text{ i.o.}] = P[\sum_{i=1}^{n} f(n-i) X_i \geqq b_n \text{ i.o.}] = 0 . \tag{47}$$

*If* $\sum b_n^{-1} \exp(-b_n^2/2\sigma^2) = \infty$, *then*

$$P[\sum_{i=-\infty}^{n} f(n-i) X_i \geqq b_n \text{ i.o.}] = P[\sum_{i=1}^{n} f(n-i) X_i \geqq b_n \text{ i.o.}] = 1 . \tag{48}$$

PROOF. Let $Y_n = \sum_{i=-\infty}^{n} f(n-i) X_i = \sum_{i=-\infty}^{n} a_{ni} X_i$, where $a_{ni} = f(n-i)$; and let $Z_n = \sum_{i=1}^{n} f(n-i) X_i = \sum_{i=-\infty}^{n} b_{ni} X_i$ where $b_{ni} = 0$ if $i \leqq 0$ and $b_{ni} = f(n-i)$ if $1 \leqq i \leqq n$. It is clear that the double arrays $(a_{ni})$ and $(b_{ni})$ satisfy conditions (31) and (32), and so (47) and (48) follow from Theorem 3. □

COROLLARY 2. *Let* $Y_1, Y_2, \cdots$ *be a real-valued Gaussian sequence with* $EY_i = 0$, $EY_i Y_j = r_{ij}$. *Let* $\sigma > 0$, *and let* $(b_n)$ *be an ultimately non-decreasing sequence of positive numbers. Suppose*

$$|\sigma^2 - r_{nn}| = O((\log n)^{-1}) \qquad \textit{as } n \to \infty , \tag{49}$$

$$\limsup_{n\to\infty} \{(\log n)^2 \sup_{j-i \geqq n, i \geqq i_0} r_{ij}\} \leqq 0 . \tag{50}$$

*Then* $P[Y_n \geqq b_n \text{ i.o.}] = 1$ *or* $0$ *according as* $\sum b_n^{-1} \exp(-b_n^2/2\sigma^2) = \infty$ *or* $< \infty$.

PROOF. Suppose $\sum b_n^{-1} \exp(-b_n^2/2\sigma^2) < \infty$. Then condition (49) implies that $\sum b_n^{-1} \exp(-b_n^2/2r_{nn}) < \infty$, and so $\sum P[Y_n \geqq b_n] < \infty$. Therefore by the Borel–Cantelli lemma, $P[Y_n \geqq b_n \text{ i.o.}] = 0$.

Now assume that $\sum b_n^{-1} \exp(-b_n^2/2\sigma^2) = \infty$. As in the proof of Theorem 3, we can assume that for all large $n$, $\sigma(\log n)^{\frac{1}{2}} \leqq b_n \leqq 2\sigma(\log n)^{\frac{1}{2}}$. Let $\tilde{Y}_n = (\sigma^2/r_{nn})^{\frac{1}{2}} Y_n$, $\tilde{b}_n = (\sigma^2/r_{nn})^{\frac{1}{2}} b_n$. Then $\sum \tilde{b}_n^{-1} \exp(-\tilde{b}_n^2/2\sigma^2) = \infty$, and Var $\tilde{Y}_n = \sigma^2$. Define $f(n) = c$ for $0 \leqq n < e$ and $f(n) = c\{n(\log n)^3\}^{-\frac{1}{2}}$ for $n > e$, where $c > 0$ is so chosen that $\sum_{n=0}^{\infty} f^2(n) = \sigma^2$. Let $\cdots, X_{-1}, X_0, X_1, X_2, \cdots$ be i.i.d. normal random variables with $EX_0 = 0$, $EX_0^2 = 1$, and let $Z_n = \sum_{i=-\infty}^{n} f(n-i) X_i$, $n \geqq 1$. Since $\sum_{i=n}^{\infty} f^2(i) = O((\log n)^{-2})$, it follows from Corollary 1 that

$$P[Z_n \geqq \tilde{b}_n \text{ i.o.}] = 1 . \tag{51}$$

The sequence $Z_1, Z_2, \cdots$ is stationary Gaussian with $EZ_i = 0$, $EZ_i^2 = \sigma^2$ and

$$\begin{aligned} \operatorname{Cov}(Z_i, Z_{i+n}) &= \sum_{j=0}^{\infty} f(j) f(j+n) \\ &\geqq c^2(1 + o(1)) \sum_{j > n \log n} \{j (\log j)^3\}^{-1} \\ &= \tfrac{1}{2} c^2 (1 + o(1)) (\log n)^{-2} . \end{aligned} \tag{52}$$

Therefore by (50), we can choose $n \geqq 1$ such that

$$\text{(53)} \qquad \operatorname{Cov}(\tilde{Y}_i, \tilde{Y}_j) \leqq \operatorname{Cov}(Z_i, Z_j) \quad \text{if } i \geqq i_0 \text{ and } j - i \geqq n .$$

Since $\operatorname{Var} \tilde{Y}_i = \sigma^2 = \operatorname{Var} Z_i$, it follows (53) that for $j = i_0, \cdots, i_0 + n - 1$, the sequence $(\tilde{Y}_{j+in}, i = 1, 2, \cdots)$ is stochastically larger than the sequence $(Z_{j+in}, i = 1, 2, \cdots)$, and hence (51) implies that $P[\tilde{Y}_n \geqq \tilde{b}_n \text{ i.o.}] = 1$, or equivalently, $P[Y_n \geqq b_n \text{ i.o.}] = 1$. □

As an application of Corollary 2, consider the Ornstein–Uhlenbeck process $Y(t) = 2^{\frac{1}{2}} \int_0^t e^{-(t-s)} dW(s)$, and let $b(t)$ be an ultimately non-decreasing positive function on $[0, \infty)$. Since $\operatorname{Cov}(Y(m), Y(n)) = e^{-|m-n|} - e^{-(m+n)}$, it follows that the conditions of Corollary 2 are satisfied. Therefore

$$\text{(54)} \qquad P[Y(n) < b(n) \text{ for all large } n] = 0 \text{ or } 1 \text{ according as}$$
$$\sum (b(n))^{-1} \exp(-b^2(n)/2) = \infty \quad \text{or} \quad < \infty ,$$

or equivalently, according as $\int (b(t))^{-1} \exp(-b^2(t)/2)\, dt = \infty$ or $< \infty$.

On the other hand, it is well known (cf. [8]) that

$$\text{(55)} \qquad P[Y(t) < b(t) \text{ for all large } t] = 0 \text{ or } 1 \text{ according as}$$
$$\int b(t) \exp(-b^2(t)/2)\, dt = \infty \quad \text{or} \quad < \infty .$$

This gives us an example that a different upper and lower class boundary classification may arise when a continuous-time process is restricted to integer time points.

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