The Annals of Mathematical Statistics
1970, Vol. 41, No. 4, 1260–1272

GAUSSIAN PROCESSES WITH STATIONARY INCREMENTS: LOCAL TIMES AND SAMPLE FUNCTION PROPERTIES[1]

BY SIMEON M. BERMAN

New York University

0. Abstract. Let $X(t)$, $0 \leqq t \leqq 1$, be separable measurable Gaussian process with mean 0, stationary increments, and $\sigma^2(t) = E(X(t)-X(0))^2$. If $\sigma^2(t) \sim C|t|^\alpha$, $t \to 0$, for some α, $0 < \alpha < 2$, then the Hausdorff dimension of $\{s: X(t) = X(s)\}$ is equal to $1-(\alpha/2)$ for almost all t, almost surely. Under further variations and refinements of this condition there is a jointly continuous local time for almost every sample function. This extends the author's previous results for stationary Gaussian processes and for continuity in the space variable alone. The result on joint continuity of the local time is used to prove that the sample function has an "approximate derivative" of infinite magnitude at each point (and so is nowhere differentiable); and that the set of values in the range of at most countable multiplicity is nowhere dense in the range.

1. Introduction; properties of local times. Let $x(t)$, $0 \leqq t \leqq 1$, be a real-valued Borel function, and μ the linear Borel measure. For every pair of linear Borel sets A, I, where I is a subset of [0, 1], define

$$\nu(A, I) = \mu[x^{-1}(A) \cap I]; \tag{1.1}$$

for each I, $\nu(\cdot, I)$ is the "occupation time distribution" of $x(t)$, $t \in I$. If, for fixed I, it is absolutely continuous with respect to Borel measure, then its derivative $\phi(x, I)$ is called the local time of $x(\cdot)$ relative to I. In this case we say that the local time exists relative to I. Put $\phi(x, t) = \phi(x, [0, t])$, $0 \leqq t \leqq 1$; then $\phi(x, t)$ exists if $\phi(x, 1)$ does, and for each s and t, $s < t$, we have $\phi(x, s) \leqq \phi(x, t)$, for almost all x. It follows from the definition that ϕ satisfies the equation

$$\nu(A, I) = \int_A \phi(x, I)\, dx. \tag{1.2}$$

This is analogous to the equation defining conditional probability: ϕ is like the conditional probability of sets I given the sigma-field of sets A. With the assistance of this analogy, we prove:

LEMMA 1.1. *If the local time exists relative to* [0, 1], *then there is a version of the local time* $\phi(x, t)$, $-\infty < x < \infty$, $0 \leqq t \leqq 1$, *such that*

(i) *For each* x, $\phi(x, I)$ *is a measure on the Borel subsets of* [0, 1];

(ii) *For every subinterval* J *of* [0, 1] *with rational endpoints, we have* $\phi(x, J) = 0$ *if* x *does not belong to the closure of the range of* $x(t)$, $t \in J$.

This version is called *regular*.

Received July 22, 1969.

[1] This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the National Science Foundation Grant NSF GP-11460.

PROOF. The proof of (i) is very similar to that of the existence of a regular version of conditional probability; the equation (1.2) is repeatedly used [5] page 26. For the proof of (ii) let ϕ be a version having the property (i). For each J with rational endpoints, and $n \geqq 1$, the set $\{x: \phi(x, J) \geqq 1/n$ and x is outside the closure of the range of $x(t), t \in J\}$ has Borel measure 0; indeed, if not, then $x(t)$ would spend positive time outside its range. The union of these sets over all $n \geqq 1$ and intervals J with rational endpoints, which we denote by N, also has measure 0. Define a new version of the local time as $\phi(x, B), x \notin N$, and equal to 0, $x \in N$, for all B. This version inherits property (i), and, in addition, has property (ii).

LEMMA 1.2. *If there is a version of ϕ such that $\phi(x, J)$ is continuous in x for each subinterval J with rational endpoints, then there is a regular version with the same property.*

PROOF. If s and r are rationals, $s < r$, then $\phi(x, s) \leqq \phi(x, r)$ for almost all x; thus, by the assumed continuity in x, the inequality holds for all x; therefore, for each x, $\phi(x, t)$ is nondecreasing on the rationals t in $[0, 1]$. Let $\phi^*(x, t)$ be the extension of ϕ by right continuity to all t, $0 \leqq t \leqq 1$. As in the proof of the existence of a regular conditional probability, $\phi^*(x, t)$ is a version of the local time for each t; furthermore, as a monotone function of t, it defines a measure $\phi^*(x, I)$ for each x. Since $\phi^*(x, J) = \phi(x, J)$ for all J with rational endpoints, the former is also continuous in x for each such J. In accordance with the proof of Lemma 1.1, $\phi^*(x, J) = 0$ for almost all x outside the closure of the range of $x(t), t \in J$, for each J. Since $\phi^*(x, J)$ is continuous it must vanish *everywhere* outside the closure, for every J; therefore, ϕ^* is regular.

LEMMA 1.3. *If a version of the local time is jointly continuous, then it is regular.*

The proof is similar to the previous one.

LEMMA 1.4. *If there is a version of the local time $\phi(x, t)$ such that*

$$\int_{-\infty}^{\infty} \phi^2(x, 1)\, dx < \infty$$

then every version has this property; in particular, there is a regular version having it.

PROOF. Two versions of $\phi(x, 1)$ agree almost everywhere in x.

LEMMA 1.5. *Let $x(\cdot)$ be continuous, and ϕ a regular version of the local time; then, for each x, the measure $\phi(x, \cdot)$ (on $[0, 1]$) has support contained in $\{t: x(t) = x\}$ or else is equal to* 0.

PROOF. The set $\{t: x(t) \neq x\}$ is the union of the open sets

$$I_n = \{t: |x(t) - x| > 1/n\}, \qquad n \geqq 1.$$

Each of these is a countable union of open intervals J with rational endpoints. By continuity, $x(\cdot)$ is bounded away from x on J; therefore, by the regularity of ϕ, $\phi(x, J)$ must vanish. By countable additivity, $\phi(x, I_n)$ must also vanish, as must $\phi(x, \{t: x(t) \neq x\})$.

For a regular version ϕ and each x, consider the product measure $\phi(x,\cdot)\times\phi(x,\cdot)$ on the unit square; it is defined for rectangles $I\times J$ as $\phi(x,I)\,\phi(x,J)$. If $g(s,t)$ is a nonnegative Borel function on the square, then the integral

$$\int_0^1\int_0^1 g(s,t)\,\phi(x,ds)\,\phi(x,dt)$$

is Borel measurable in x.

LEMMA 1.6. *Put* $H(s,t)=\int_{-\infty}^{\infty}\phi(x,s)\,\phi(x,t)\,dx$; *then, for any nonnegative Borel function* $g(s,t)$, *we have*

$$\int_0^1\int_0^1 g(s,t)H(ds,dt)=\int_{-\infty}^{\infty}\int_0^1\int_0^1 g(s,t)\,\phi(x,ds)\,\phi(x,dt)\,dx.$$

PROOF. Since the equation holds for products of indicator functions of pairs of Borel sets, it holds by approximation for all g.

For fixed I the Fourier–Stieltjes transform of $\nu(\cdot,I)$ is equal to

$$f(u,I)=\int_I e^{iuX(t)}\,dt,\qquad -\infty<u<\infty, \tag{1.3}$$

[1]; we also put $f(u,t)=f(u,[0,t])$.

Let $X(t,\omega)$, $0\leqq t\leqq 1$, $\omega\in\Omega$, be a separable measurable stochastic process on some probability space Ω. For each ω, let ν be the occupation time distribution of $X(\cdot,\omega)$, defined by (1.1). Properties of ν holding for almost all ω will be said to hold almost surely; therefore, we shall suppress the argument ω in ν, ϕ, f and X, writing the last as $X(t)$.

2. Dimension of the x-values of the sample functions of a Gaussian process. We recall that a linear set is said to have Hausdorff dimension less than or equal to γ, where $\gamma>0$, if, for every $\gamma'>\gamma$ and every n, there is a covering by open sets I_{nk}, $k\geqq 1$, such that diameter $(I_{nk})\leqq 1/n$ for all k, and $\lim_{n\to\infty}\sum_k|\text{diameter}\,(I_{nk})|^{\gamma'}<\infty$. The dimension is equal to γ if it is less than or equal to γ but not to $\gamma'<\gamma$. In [2] it was shown that if X is a Gaussian process such that

$$\int_0^1\int_0^1 (E(X(s)-X(t))^2)^{-\frac{1}{2}}\,ds\,dt<\infty,$$

then for almost all t the set $\{s: X(s)=X(t)\}$ is infinite almost surely. Now we get a more exact estimate of the size of this set for a class of processes with stationary increments. Analogous theorems for the set of zeros of a stable process are well known; for example, see [3] or [9].

THEOREM 2.1. *Let* $X(t)$, $0\leqq t\leqq 1$, *be a separable measurable Gaussian process with stationary increments such that* $EX(t)\equiv 0$ *and* $\sigma^2(t)=E(X(t)-X(0))^2$ *is continuous and positive on* $(0,1]$, *and*

$$\sigma^2(t)\sim C\,|t|^{\alpha},\qquad t\to 0, \tag{2.1}$$

for some constant C, *and some* α, $0<\alpha<2$. *Then, for almost every* t, *the dimension of the set*

$$\{s: X(s)=X(t)\} \tag{2.2}$$

is equal to $1-(\alpha/2)$, *almost surely.*

PROOF. The proof consists of two parts: First it is shown that $1-\alpha/2$ is an upper bound on the dimension; then, it is shown that it is also a lower bound.

Upper bound. Let I_{nk} be the interval $((k-1)2^{-n}, k2^{-n})$, $k = 1, \cdots, 2^n$. For each x, the probability that $X(k2^{-n}) = x$ for some $1 \leqq k \leqq 2^n$, $n \geqq 1$, is equal to 0; therefore, the open intervals I_{nk} for which $X(t) = x$ for some $t \in I_{nk}$ form a covering of the set of x-values almost surely.

Under the hypothesis (2.1), for any $\beta < \alpha/2$, $X(t)$ satisfies a uniform Hölder condition of order β almost surely: there exists a constant D—independent of the sample function—and a positive random variable δ such that

$$|X(t)-X(t')| \leqq D|t-t'|^{\beta}, \qquad |t-t'| \leqq \delta, \qquad t, t' \in [0, 1].$$

[7] page 519. For $\gamma > 0$, the sum of the γth powers of the lengths of the covering sets in the nth cover is $\sum_{k=1}^{2^n} (|\mu(I_{nk})|^{\gamma}: X(t) = x \text{ for some } t \in I_{nk}) = 2^{-n\gamma} \cdot \#$ (intervals I_{nk}: $X(t) = x$ for some $t \in I_{nk}$). Under the stated Hölder condition if n is large enough, then $X(t)$ assumes the value x in I_{nk} only if $X(k2^{-n})$ falls within $D2^{-n\beta}$ units of x; therefore, the sum above is, for large n, at most $2^{-n\gamma} \cdot \#$(indices k: $|X(k2^{-n})-x| \leqq D2^{-n\beta}$, $1 \leqq k \leqq 2^n$). By Fatou's theorem, the lim inf of this expression is finite almost surely if the lim inf of the expected value is: it will be shown that

$$\liminf_{n\to\infty} 2^{-n\gamma} \textstyle\sum_{k=1}^{2^n} P[|X(k2^{-n})-x| \leqq D2^{-n\beta}] < \infty, \qquad \gamma > 1-\beta.$$

We may assume $X(0) = 0$ because there is no loss in replacing $X(t)$ by $X(t)-X(0)$. By the elementary estimate

$$P[|X(k2^{-n})-x| \leqq u] \leqq P[|X(k2^{-n})| \leqq u] \leqq (2/\pi)^{\frac{1}{2}} u/\sigma(k2^{-n}), \qquad u > 0,$$

we find that the lim inf above is not more than

$$\begin{aligned}\liminf_{n\to\infty} 2^{-n(\gamma+\beta-1)} &\textstyle\sum_{k=1}^{2^n} (2/\pi)^{\frac{1}{2}}/\sigma(k2^{-n})2^n \\ &= D \cdot \liminf_{n\to\infty} 2^{-n(\gamma+\beta-1)}(2/\pi)^{\frac{1}{2}} \textstyle\int_0^1 ds/\sigma(s).\end{aligned}$$

This is equal to 0 for $\gamma > 1-\beta$ because the integral is finite under the hypothesis (2.1); thus, the dimension of $\{t: X(t) = x\}$ is not more than $1-\alpha/2$, almost surely. Since the process is measurable, Fubini's theorem implies that the dimension of $\{t: X(t) = x\}$ is not more than $1-\alpha/2$ for almost all x, almost surely.

Under the hypothesis on $\sigma^2(t)$, we have

$$\int_0^1 \int_0^1 \frac{ds\,dt}{\sigma(s-t)} < \infty; \tag{2.3}$$

therefore, the local time exists almost surely, as proved in [1]. It follows from the definition of the local time that the pre-image of a set of linear measure 0 in the range is a set of similar measure in the domain; therefore, from the above result about the dimension of $\{t: X(t) = x\}$ for almost all x we infer a similar statement about the set (2.2) for almost every t.

Lower bound. Under (2.3) it is shown in [1] that there is a version of the local time such that $\int_{-\infty}^{\infty} \phi^2(x, 1)\,dx < \infty$, almost surely; therefore, by Lemma 1.4, it may be assumed that ϕ is also regular. For each sample function let $H(s, t)$ be defined as in Lemma 1.6; then, by the formula (1.3), we have

$$H(s,t) = \int_{-\infty}^{\infty} f(u,s)\bar{f}(u,t)\,du = \int_{-\infty}^{\infty}\int_0^t\int_0^s e^{iu(X(s')-X(t'))}\,ds'\,dt'\,du. \tag{2.4}$$

If $g(s, t)$ is a nonnegative Borel function, then

$$E\{\int_0^1\int_0^1 g(s,t)H(ds,dt)\} = (2\pi)^{\frac{1}{2}}\int_0^1\int_0^1 g(s,t)\sigma^{-1}(t-s)\,ds\,dt; \tag{2.5}$$

this follows from (2.4) by a standard approximation, Fubini's theorem, and the inversion formula for the transform of the Gaussian density. From (2.5) and (2.1) it follows that $E\{\int_0^1\int_0^1 |s-t|^{-\beta} H(ds, dt)\} < \infty$, $\beta < 1-(\alpha/2)$; therefore, by Fubini's theorem, $\int_0^1\int_0^1 |s-t|^{-\beta} H(ds, dt) < \infty$, *almost surely*, and, by Lemma 1.6, $\int_0^1\int_0^1 |s-t|^{-\beta} \phi(x, ds)\, \phi(x, dt) < \infty$, *for almost all* x, *almost surely*. This implies that

$$\int_0^1\int_0^1 |s'-t'|^{-\beta}\phi(X(t),ds')\phi(X(t),dt') < \infty \tag{2.6}$$

for almost all $t \in [0, 1]$, *almost surely*;

indeed, by the remarks following (2.3), a property holding almost everywhere in the range holds in the same manner in the domain.

Under the condition that (2.3) is finite, $\phi(X(t), 1)$ is positive for almost all t in [0, 1], almost surely [2]; therefore, by the regularity of ϕ, the measure $\phi(X(t), ds)$ is positive on [0, 1]. By Lemma 1.5, the support of $\phi(X(t), ds)$ is contained in the set (2.2). We have shown that for almost all t, there is a positive Borel measure ϕ on (2.2) for which the "energy integral" (2.6) is finite. Under (2.1) the set (2.2) is closed because $X(t)$ is continuous almost surely. Frostman's theorem [6] implies that the dimension of the set is at least β. Since β is an arbitrary number smaller than $1-(\alpha/2)$, it follows that the dimension is at least $1-(\alpha/2)$.

3. Some properties of functions with jointly continuous local times. The first result is that a Borel function with a jointly continuous local time $\phi(x, t)$, $-\infty < x < \infty$, $0 \leqq t \leqq 1$ is not only non-differentiable at every point, but also the magnitude of the difference quotient is arbitrarily large on a set of density 1 near each point. (This result applies to a large class of stochastic processes previously considered by Trotter [12], Boylan [4], Ray [10], and others.)

LEMMA 3.1. *If* $\phi(x, t)$ *is jointly continuous for* $-\infty < x < \infty$, $0 \leqq t \leqq 1$, *then for every* t *in* [0, 1] *and every* $M > 0$,

$$\lim_{\varepsilon\to 0}\frac{\mu[s: |s-t| < \varepsilon, |x(t)-x(s)| < M|t-s|]}{\mu[s: |s-t| \leqq \varepsilon]} = 0.$$

PROOF. The above ratio is at most equal to $(1/2\varepsilon)\mu[s: |s-t| \leqq \varepsilon, |x(t)-x(s)| \leqq M\varepsilon]$, which, by the definition of ϕ, is

$$(1/2\varepsilon)\int_{x(t)-M\varepsilon}^{x(t)+M\varepsilon}[\phi(x,t+\varepsilon)-\phi(x,t-\varepsilon)]\,dx.$$

This converges to 0 with ε because $\phi(x, t)$ is jointly continuous and monotonic in t.

LEMMA 3.2. *If* $x(t)$, $0 \leqq t \leqq 1$, *is continuous, and its local time exists and is jointly continuous, then the set* $\{x$: *The pre-image of* $\{x\}$ *is countable*$\}$ *is nowhere dense in the range of* $x(t)$.

PROOF. It suffices to show that $\phi(x, 1) = 0$ if the pre-image of $\{x\}$ is countable; indeed, the set of zeros of $\phi(x, 1)$ is nowhere dense in the range of $x(t)$ [1]. By Lemma 1.3, ϕ is regular. Let x be a point such that $\{t: x(t) = x\}$ is countable; then, by Lemma 1.5, the measure $\phi(x, dt)$ has support contained in this countable set or else has none. Since $\phi(x, t)$ is continuous in t, the support is empty; thus $\phi(x, 1) = 0$.

4. Gaussian processes with jointly continuous local times: the first class. Put $\phi_{nk}(x) = \phi(x, I_{nk})$ and $f_{nk}(u) = f(u, I_{nk})$, where I_{nk} is the dyadic interval $((k-1)2^{-n}, k2^{-n})$, $k = 1, \cdots, 2^n$, $n \geqq 1$. Under the following conditions there is a jointly continuous local time:

LEMMA 4.1. *If, for some* $\varepsilon > 0$,

$$\int_{-\infty}^{\infty} |u|^{1+\varepsilon} |f(u, J)|^2 \, du < \infty, \tag{4.1}$$

for every subinterval J of $[0, 1]$ with rational endpoints;

$$\liminf_{n\to\infty} \sum_{k=1}^{2^n} \int_{-\infty}^{\infty} (1+|u|^{1+\varepsilon}) |f_{nk}(u)|^2 \, du = 0, \tag{4.2}$$

then $\phi(x, t)$ *is jointly continuous.*

PROOF. Under the hypothesis (4.1) $|f(u, J)|$ is integrable, and so $\phi(x, J)$ exists and is continuous, for all J with rational endpoints [1]; thus, by Lemma 1.2, there is a regular version with the same property; therefore, this regular version of $\phi(x, t)$ is continuous in x for each rational t, and nondecreasing in t for each x. In order to prove joint continuity it suffices to prove continuity in t for each x: $\phi(x, t)$ must have no jumps in t for each x.

Since f_{nk} is absolutely integrable, ϕ_{nk} may be represented by the inversion integral:

$$\begin{aligned}\phi_{nk}^2(x) &= \left|(1/2\pi) \int_{-\infty}^{\infty} e^{-iux} f_{nk}(u) \, du\right|^2 \\ &\leqq (1/2\pi)^2 \int_{-\infty}^{\infty} du/(1+|u|^{1+\varepsilon}) \cdot \int_{-\infty}^{\infty} (1+|u|^{1+\varepsilon}) |f_{nk}(u)|^2 \, du\end{aligned}$$

(Cauchy–Schwarz inequality). Sum over k:

$$\sum_{k=1}^{2^n} \phi_{nk}^2(x) \leqq \text{constant} \cdot \sum_{k=1}^{2^n} \int_{-\infty}^{\infty} (1+|u|^{1+\varepsilon}) |f_{nk}(u)|^2 \, du.$$

Under the hypothesis (4.2) it follows that $\liminf_{n\to\infty} \sum_{k=1}^{2^n} \phi_{nk}^2(x) = 0$, *for all* x; therefore, $\phi(x, t)$ has no jumps for each x, and so it is jointly continuous.

THEOREM 4.1. *Let* $X(t)$, $0 \leqq t \leqq 1$, *be a separable, measurable Gaussian process with mean* 0, *stationary increments, and* $\sigma^2(t) = E|X(t) - X(0)|^2$ *continuous and satisfying*

$$\int_0^1 \int_0^1 [\sigma(s-t)]^{-2-\varepsilon} \, ds \, dt < \infty \tag{4.3}$$

for some $\varepsilon > 0$; *then the local time exists and is jointly continuous, almost surely.*

PROOF. Under the hypothesis (4.3), the conditions (4.1) and (4.2) are fulfilled; this is proved in [2], Lemmas 2.1 and 5.2.

In [1] it was shown that under the condition (4.3) the local time is continuous in x for each t; thus, if the sample functions are continuous, the values in the range of X of finite multiplicity form a set of Category I. We now get a stronger result from Theorem 4.1 and Lemma 3.2. Theorem 3.1 and Lemma 3.1 also provide a more general version of the result in [2] on nondifferentiability.

5. Joint continuity of a stochastic process of two parameters. We prove a general result giving conditions in terms of moments that a stochastic process $X(s, t)$ have jointly continuous sample functions. We make the usual assumption that the process is separable and stochastically continuous, so that the pairs of dyadic fractions form a separability sequence. The following is a generalization of Kolmogoroff's continuity condition to processes of two variables.

THEOREM 5.1. *If there are positive constants r, C and ε such that*

$$E[X(s+h,\tau)-X(s,\tau)|^r \leqq C|h|^{1+\varepsilon}, \qquad s, s+h \in [0,1], \text{ and } \tau = 0, 1; \tag{5.1}$$

$$E[X(\sigma,t+h)-X(\sigma,t)|^r \leqq C|h|^{1+\varepsilon}, \qquad t, t+h \in [0,1], \text{ and } \sigma = 0, 1; \tag{5.2}$$

$$E[X(s+h,t+h')-X(s+h,t)-X(s,t+h')+X(s,t)|^r \leqq C|hh'|^{1+\varepsilon}, \qquad s, s+h, t, t+h' \in [0,1], \tag{5.3}$$

then $X(s, t)$, $0 \leqq s, t \leqq 1$, is almost surely continuous.

PROOF. We expand X in a double Schauder series and show that it converges uniformly on the unit square almost surely. Since the series converges to X at all dyadic pairs, and since the latter form a separability sequence, it follows that the series coincides with X throughout the unit square; therefore, X is continuous almost surely.

Recall the Schauder functions [11]:

$$\chi_1(t) = 1, \qquad 0 \leqq t \leqq 1,$$

$$\chi_{2^n+k}(t) = 2^{n/2}, \qquad t \in \left[\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right),$$

$$= -2^{n/2}, \qquad t \in \left[\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right)$$

$$= 0, \qquad \text{elsewhere}, \quad k = 1, \cdots, 2^n, \quad n \geqq 0;$$

and $\phi_n(t) = \int_0^t \chi_n(s)\,ds$, $n \geqq 1$. $\phi_{2^n+k}(t)$ increases linearly from 0 to $(\frac{1}{2})2^{-n/2}$ on $[(2k-2)2^{-(n+1)}, (2k-1)2^{-(n+1)}]$; and decreases linearly from $(\frac{1}{2})2^{-n/2}$ to 0 on $[(2k-1)2^{-(n+1)}, 2k \cdot 2^{-(n+1)}]$; and is equal to 0 elsewhere. It follows that for every n and t,

$$\textstyle\sum_{k=1}^{2^n} \phi_{2^n+k}(t) \leqq (\frac{1}{2})2^{-n/2}. \tag{5.4}$$

Under the conditions (5.1) and (5.2) the processes $X(s, 0)$, $0 \leqq s \leqq 1$, and $X(0, t)$, $0 \leqq t \leqq 1$, are almost surely continuous [7] page 519. Put $\xi(s, t) = X(s, t) - X(s, 0) - X(0, t) + X(0, 0)$; then ξ is jointly continuous if and only if X is. Since $\xi(s, 0) \equiv 0$ we may, for fixed s, expand $\xi(s, t)$ in a formal Schauder series in t:

$$\xi(s,t) = \phi_1(t)\int_0^1 \chi_1(u)\xi(s, du) + \sum_{n=0}^{\infty}\sum_{k=1}^{2^n} \phi_{2^n+k}(t)\int_0^1 \chi_{2^n+k}(u)\xi(s, du).$$

Here we have used the differential notation $\int_b^a \xi(s, du) = \xi(s, b) - \xi(s, a)$. Since $\xi(0, t) \equiv 0$, it follows that $\xi(0, du) \equiv 0$, and so $\xi(s, du)$ has a formal Schauder expansion in s. The double expansion for $\xi(s, t)$ is the sum of four terms:

$$\begin{aligned}(5.5)\qquad & \phi_1(s)\phi_1(t)\int_0^1\int_0^1 \chi_1(u)\chi_1(v)\xi(du, dv)\\ &+\phi_1(s)\sum_{n=0}^{\infty}\sum_{k=1}^{2^n}\phi_{2^n+k}(t)\int_0^1\int_0^1 \chi_1(u)\chi_{2^n+k}(v)\xi(du, dv)\\ &+\phi_1(t)\sum_{n=0}^{\infty}\sum_{k=1}^{2^n}\phi_{2^n+k}(s)\int_0^1\int_0^1 \chi_{2^n+k}(u)\chi_1(v)\xi(du, dv)\\ &+\sum_{m,n=0}^{\infty}\sum_{j=1}^{2^m}\sum_{k=1}^{2^n}\phi_{2^m+j}(s)\phi_{2^n+k}(t)\\ &\qquad\cdot\int_0^1\int_0^1 \chi_{2^m+j}(u)\chi_{2^n+k}(v)\xi(du, dv).\end{aligned}$$

The first term is $st \cdot \xi(1, 1)$, which is continuous. The proofs of the uniform convergence of the sums in the last three terms are similar, so that we record only that for the last term, which involves the doubly infinite series.

Note that $\xi(ds, dt) = X(ds, dt)$, so that the double integral in a typical term of the double series is a sum of four second order differences of the general form

$$2^{(m+n)/2}\left[X\left(\frac{2j-1}{2^{m+1}}, \frac{2k-1}{2^{n+1}}\right) - X\left(\frac{2j-1}{2^{m+1}}, \frac{2k-2}{2^{n+1}}\right) - X\left(\frac{2j-2}{2^{m+1}}, \frac{2k-1}{2^{n+1}}\right)\right.$$
$$\left. + X\left(\frac{2j-2}{2^{m+1}}, \frac{2k-2}{2^{n+1}}\right)\right] = 2^{(m+n)/2}\xi_{2^m+j,\, 2^n+k},$$

with the variations that j and k may be replaced by $j+1$ and $k+1$, respectively. We shall show that the tail of the double series

$$(5.6)\qquad \sum_{m,n} 2^{(m+n)/2}\left[\sum_{k,j}\phi_{2^m+j}(s)\phi_{2^n+k}(t)\xi_{2^m+j, 2^n+k}\right]$$

is almost surely majorized by a series of positive constants independent of s and t. Pick $\delta > 0$. By the Chebychev inequality and by (5.3), we have

$$\begin{aligned}\Pr\left[|\xi_{2^m+j,\, 2^n+k}| \geqq (mn)^{-1-\delta}\right] &\leqq (mn)^{r(1+\delta)}E\,|\xi_{2^m+j,\, 2^n+k}|^r\\ &\leqq C(mn)^{r(1+\delta)}2^{-[(m+1)+(n+1)](1+\varepsilon)};\end{aligned}$$

therefore, the series

$$\sum_{m,n}\sum_{j,k}\Pr\left[|\xi_{2^m+j,\, 2^n+k}| > (mn)^{-1-\delta}\right]$$

converges; hence, by the Borel–Cantelli lemma, only finitely many of the events

$$|\xi_{2^m+j,\, 2^n+k}| > (mn)^{-1-\delta}, \qquad j = 1, \cdots, 2^m, \quad k = 1, \cdots, 2^n \quad m \geqq 1, \quad n \geqq 1,$$

occur, almost surely. From this and (5.4) we conclude that the (m, n)th term of the series (5.6) is asymptotically, uniformly (in s, t) dominated by $(mn)^{-1-\delta}$.

6. Several more classes of Gaussian processes with jointly continuous local times. Let $X(t)$, $0 \leqq t \leqq 1$, be a separable measurable Gaussian process. In [1] it was shown that for a certain class of such processes which are stationary, and whose correlation function satisfies $1-r(t) \sim C|t|^\alpha$, $t \to 0$, for some α, $1 \leqq \alpha < 2$, and several other conditions, there is a version of the local time continuous in x almost surely, for each t. This was done in the following way. Under the given conditions, the integral

$$\psi(x,t) = (1/2\pi)\int_{-\infty}^{\infty} e^{-iux}\left(\int_0^t e^{iuX(s)}\,ds\right)du$$

exists as a quadratic mean limit for every (x, t); furthermore, for each t, a separable version of $\psi(x, t)$, $-\infty < x < \infty$, has continuous sample functions; finally, it is shown that this version of ψ is also a version of the local time.

We shall extend this result in two directions: stationarity will be replaced by the more general assumption of stationary increments, and the continuity of the local time in x by the conclusion that it is jointly continuous in (x, t).

LEMMA 6.1. *Let $X(t)$, $0 \leqq t \leqq 1$, be a Gaussian process with mean 0 and stationary increments, and with $\sigma^2(t) = E(X(t)-X(0))^2$. If there exist numbers $\delta > 0$, $\varepsilon > 0$ and an integer $N > 1$ such that for all even $n \geqq N$:*

The determinant of the covariance matrix of

$$\frac{X(t_j)-X(t_{j-1})}{\sigma(t_j-t_{j-1})}, \qquad j = 1,\cdots,n, \tag{6.1}$$

is bounded away from 0 on the subset of the unit cube

$$\{(t_1,\cdots,t_n)\colon 0 = t_0 \leqq t_1 \leqq \cdots \leqq t_n \leqq 1\};$$

and

$$\int\cdots\int_{\{0=t_0\leqq t_1\leqq\cdots\leqq t_n\leqq t\}} \left[\prod_{j=1}^n \sigma(t_j-t_{j-1})\right]^{-1-2\delta}\prod_{j=1}^n dt_j = 0(t^{1+\varepsilon}), \qquad t\to 0, \tag{6.2}$$

then

(a) *$\psi(x, t)$ exists as a quadratic mean limit for each (x, t); and*

(b) *Every separable version of $\psi(x, t)$, $-\infty < x < \infty$, $0 \leqq t \leqq 1$, is jointly continuous almost surely.*

PROOF. There is no loss in assuming that $X(0) = 0$ (cf. proof of Theorem 2.1). By the argument in [1] page 286, the existence of the quadratic mean limit follows from the finiteness of

$$\int_0^1\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ee^{iuX(s)+ivX(t)}\,du\,dv\,ds\,dt,$$

which follows from the calculations below.

In order to prove (b), we apply Theorem 5.1 to $\psi(x, t)$, $-\infty < x < \infty$, $0 \leqq t \leqq 1$. (The restriction of the first parameter to [0, 1] in the statement of that theorem is removable.) The features of the confirmation of the conditions (5.1) and (5.2) are contained in that of (5.3) so that we present only the latter. By a slight modification in the calculation in [1], the nth moment of $\psi(x+h, t+h')-\psi(x+h, t)-\psi(x, t+h')+\psi(x, t)$ is dominated by

$$(6.3) \qquad h^{n\delta}\pi^{-n}\int_t^{t+h'}\cdots\int_t^{t+h'}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^n |u_j|^\delta \cdot E[\exp(i\sum_{j=1}^n u_j X(t_j))]\prod_{j=1}^n du_j \prod_{j=1}^n dt_j;$$

furthermore, under the first condition of the lemma, this expression is at most a constant multiple of

$$h^{n\delta}\int\cdots\int_{\{0=t_0\leqq t_1\leqq\cdots\leqq t_n\leqq h'\}}[\prod_{j=1}^n \sigma(t_j-t_{j-1})]^{1-2\delta}\prod_{j=1}^n dt_j.$$

Under the second condition of the lemma, this is at most a constant multiple of $(hh')^{1+\varepsilon}$ for some $\varepsilon>0$ is n is large enough; therefore, the condition (5.3) is satisfied.

LEMMA 6.2. *Under the conclusion of Lemma* 6.1, *there is a version of the local time which is jointly continuous, almost surely.*

PROOF. By a slight extension of the reasoning in [1] it can be shown that if a separable version of $\psi(x, t)$ has jointly continuous sample functions, then it serves as the local time for $X(t)$ almost surely.

In the following theorems we describe classes of Gaussian processes with stationary increments which satisfy the conditions of Lemma 6.1.

THEOREM 6.1. *If $\sigma^2(t)$ is continuous and concave for $0\leqq t\leqq 1$, then the local time exists and is jointly continuous almost surely.*

PROOF. By an adaptation of a result of Marcus [8] to the case of stationary increments, we get

$$\Pr[|X(t_j)-X(t_{j-1})|\leqq x, j=1,\cdots,n] \leqq \prod_{j=1}^n (2/\pi)^{\frac{1}{2}}\int_0^{2^{1/2}x/\sigma(t_j-t_{j-1})} e^{-\frac{1}{2}y^2}\,dy, \qquad x>0.$$

Divide both sides of this inequality by x^n, and let $x\to 0$: then the limit of the left-hand side is $(2\pi)^{-n/2}$ multiplied by the reciprocal of the square root of the determinant of the covariance matrix of $X(t_j)-X(t_{j-1})$, $j=1,\cdots,n$. The limit of the right-hand side is $(2/\pi^{\frac{1}{2}})^n/\prod_{j=1}^n\sigma(t_j-t_{j-1})$; thus,

$$\det[E(X(t_i)-X(t_{i-1}))(X(t_j)-X(t_{j-1}))]\geqq 2^{-3n}\prod_{j=1}^n\sigma^2(t_j-t_{j-1});$$

finally, from this we obtain

$$\det\left[\frac{E(X(t_i)-X(t_{i-1}))\cdot(X(t_j)-X(t_{j-1}))}{\sigma(t_i-t_{i-1})\cdot\sigma(t_j-t_{j-1})}\right]\geqq 2^{-3n};$$

therefore, the first condition of Lemma 6.1 is satisfied.

Since $\sigma^2(t)$ is concave, $\sigma^2(t)/t$ decreases; therefore, $\sigma^2(t)\geqq t\sigma^2(1)$, and so $\sigma(t)\geqq t^{\frac{1}{2}}\sigma(1)$; thus the second condition of the lemma is satisfied for all n sufficiently large, e.g. $n\geqq 4$.

THEOREM 6.2. *If* $\sigma^2(t)$ *satisfies:*

(a) *The form* $\sigma^2(t_i)+\sigma^2(t_j)-\sigma^2(t_i-t_j)$, $i, j=1,\cdots,n$, *is strictly positive for* $t_1,\cdots,t_n \in [0, 1]$ *and every n, that is, the process has no singularities.*

(b) $\sigma^2(t)$ *is twice continuously differentiable on* $(0, t)$.

(c) *There exists a constant* $C > 0$ *such that*

$$\sigma^2(t) \sim Ct, \qquad t\downarrow 0. \tag{6.4}$$

(d) *For* $s < t < s' < t'$, $0 < s < t' < 1$, *and for either or both* $t-s \to 0$, $t'-s' \to 0$, *we have*

$$\frac{\int_s^t \int_{s'}^{t'} (\sigma^2)''(u-v)\,du\,dv}{[(t-s)(t'-s')]^{\frac{1}{2}}} \to 0, \tag{6.5}$$

then the local time is jointly continuous.

PROOF. The condition (6.4) is certainly sufficient for the second condition of Lemma 6.1, e.g. for $n \geqq 4$.

Under the differentiability assumptions on $\sigma^2(t)$ we have

$$\frac{E(X(t_i)-X(t_{i-1}))(X(t_j)-X(t_{j-1}))}{\sigma(t_i-t_{i-1})\cdot\sigma(t_j-t_{j-1})} = \frac{1}{2}\frac{\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}(\sigma^2)''(u-v)\,du\,dv}{\sigma(t_i-t_{i-1})\sigma(t_j-t_{j-1})}, \qquad i<j. \tag{6.6}$$

Put $M = \{(t_1,\cdots,t_n)\colon\ 0 = t_0 \leqq t_1 < \cdots < t_n \leqq 1\}$, and M' its closure; then a point $(t_1,\cdots,t_n)$ belongs to $M'-M$ if there is strict equality among at least two t_i's, $1 \leqq i \leqq n$. Since the process has no singularities, the determinant of the covariance matrix (6.6) is positive on M; we shall show that it is bounded away from 0 on M so that it is strictly positive on M'.

Under the conditions (6.4) and (6.5), the correlation (6.6) converges to 0 if either $t_i-t_{i-1} \to 0$ or $t_j-t_{j-1} \to 0$ or both; therefore, the distribution of the standardized differences (6.1) converges to a nonsingular limiting Gaussian distribution because the ith random variable is asymptotically independent of all others as $t_i-t_{i-1} \to 0$; hence, the corresponding determinant converges to a positive limit and so is bounded away from 0.

THEOREM 6.3. *Suppose that* $\sigma^2(t)$ *satisfies conditions* (a) *and* (b) *above, and for some* α, $1 < \alpha < 2$, *and some* $C > 0$,

$$\sigma^2(t) \sim Ct^\alpha, \qquad t\downarrow 0, \tag{6.7}$$

$$(\sigma^2)''(t) \sim \alpha(\alpha-1)Ct^{\alpha-2}, \qquad t\downarrow 0; \tag{6.8}$$

then the local time is jointly continuous.

PROOF. The condition (6.7) is sufficient for the second condition of Lemma 6.1; indeed, if $\delta < (1/\alpha)-(\frac{1}{2})$, then (6.2) holds for all n sufficiently large.

In order to prove the nonsingularity of the limiting distribution of the random variables (6.1) as a point $(t_1,\cdots,t_n)$ moves toward the boundary of the set M

defined in the previous proof, it is sufficient, by the reasoning in [1], to consider three different cases:

(i) t_i and t_{i-1} are bounded away from t_j and t_{j-1} and either $t_i - t_{i-1} \to 0$ or $t_j - t_{j-1} \to 0$ or both; in this case the integrand on the right-hand side of (6.6) is bounded, and the right-hand side converges to 0 because $\sigma(t) \sim Ct^{\alpha/2}$.

(ii) $t_i - t_{i-1} \to 0$, $t_{j-1} - t_i \to 0$, but $t_j - t_{j-1}$ is bounded away from 0. Under (6.8), and by virtue of (6.6), the correlation of the increments is not more than a constant multiple of

$$\int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (u-v)^{\alpha-2}\, du\, dv \cdot (t_i - t_{i-1})^{-\alpha/2}$$

$$= (\alpha-1)^{-1} \int_{t_{j-1}}^{t_j} [(u-t_{i-1})^{\alpha-1} - (u-t_i)^{\alpha-1}]\, du \cdot (t_i - t_{i-1})^{-\alpha/2}$$

$$\leqq \frac{(t_i - t_{i-1})^{-\alpha/2}}{\alpha(\alpha-1)} [|(t_j - t_{i-1})^\alpha - (t_j - t_i)^\alpha| + |(t_{j-1} - t_{i-1})^\alpha - (t_{j-1} - t_i)^\alpha|],$$

which converges to 0 because $\alpha > 1$, and $(\alpha/2) < 1$.

(iii) $t_i - t_{i-1} \to 0$, $t_j - t_{j-1} \to 0$, $t_{j-1} - t_i \to 0$. By (6.6), (6.7) and (6.8), the correlation is asymptotic to

$$\frac{[(t_j - t_{i-1})^\alpha - (t_{j-1} - t_{i-1})^\alpha - (t_j - t_i)^\alpha + (t_{j-1} - t_i)^\alpha]}{2|t_j - t_{j-1}|^{\alpha/2} |t_i - t_{i-1}|^{\alpha/2}}.$$

This is representable in the spectral form

$$\frac{\int_{-\infty}^{\infty} (e^{i\lambda t_j} - e^{i\lambda t_{j-1}})(e^{-i\lambda t_i} - e^{-i\lambda t_{i-1}}) |\lambda|^{-\alpha-1}\, d\lambda}{[\int_{-\infty}^{\infty} |1 - e^{i\lambda(t_j - t_{j-1})}|^2 |\lambda|^{-\alpha-1}\, d\lambda \cdot \int_{-\infty}^{\infty} |1 - e^{i\lambda(t_i - t_{i-1})}|^2 |\lambda|^{-\alpha-1}\, d\lambda]^{\frac{1}{2}}}.$$

The same argument used in [1] shows that the limiting correlation matrix is non-singular. This completes the proof.

An example of a Gaussian process with stationary increments satisfying the conditions of these theorems is the one with $E(X(t) - X(0))^2 = |t|^\alpha$, with $\alpha = 1$ for Theorems 6.1 and 6.2 (Brownian motion), and with $1 < \alpha < 2$ for Theorem 6.3. The case $0 < \alpha < 1$ is covered by Theorem 4.1. By Lemma 2.1 the sample functions are nowhere differentiable; the result is new for $1 < \alpha < 2$ because α was restricted to $(0, 1)$ in [2].

7. Correction of a previous result. Theorem 2.1 of [1] is incorrect as stated. Its conclusion holds if the hypothesis is strengthened to include continuity in x of the local time relative to every subinterval of $[0, 1]$ with rational endpoints. Lemma 1.2 above is used in the proof. The hypothesis of Theorem 7.1 must be strengthened in the same way. The main results of [1] on Gaussian processes are valid as given because the continuity condition holds for every interval (with rational endpoints).

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