Gaussian stationary processes: adaptive wavelet decompositions, discrete approximations and their convergence *†‡

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Abstract

We establish particular wavelet-based decompositions of Gaussian stationary processes in continuous time. These decompositions have a multiscale structure, independent Gaussian random variables in high-frequency terms, and the random coefficients of low-frequency terms approximating the Gaussian stationary process itself. They can also be viewed as extensions of the earlier wavelet-based decompositions of Zhang and Walter (1994) for stationary processes, and Meyer, Sellan and Taqqu (1999) for fractional Brownian motion. Several examples of Gaussian random processes are considered such as the processes with rational spectral densities. An application to simulation is presented where an associated Fast Wavelet Transform-like algorithm plays a key role.

1 Introduction

Consider a real-valued Gaussian stationary process $X = \{X(t)\}_{t\in\mathbb{R}}$ having the integral representation

$$X(t) = \int_{\mathbb{R}} g(t-u)dB(u) = \int_{\mathbb{R}} e^{itx}\widehat{g}(x)d\widehat{B}(x), \tag{1.1}$$

where $g \in L^2(\mathbb{R})$ is a real-valued function, called a *kernel function*, $\widehat{g} \in L^2(\mathbb{R})$ is its Fourier transform defined by convention as

$$\widehat{g}(x) = \int_{\mathbb{R}} e^{-ixu}g(u)du,$$

$\{B(u)\}_{u\in\mathbb{R}}$ is a standard Brownian motion and $\{\widehat{B}(x)\}_{x\in\mathbb{R}} = \{B_1(x) + iB_2(x)\}_{x\in\mathbb{R}}$ is a complex-valued Brownian motion satisfying $B_1(x) = B_1(-x)$, $B_2(x) = -B_2(-x)$, $x \geq 0$, with two independent Brownian motions $\{B_1(x)\}_{x\geq 0}$ and $\{B_2(x)\}_{x\geq 0}$ such that $EB_1(1)^2 = EB_2(1)^2 = (4\pi)^{-1}$. (The latter conditions on $\widehat{B}(x)$ ensure that the second integral in (1.1) is real-valued and has the same covariance structure as the first integral in (1.1).) Many Gaussian stationary processes, especially those of practical interest, can be represented by (1.1). See, for example, Rozanov

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(1967) and others. The covariance function $R(t) = EX(t)X(0)$ of X and its Fourier transform are given by

$$R(t) = (g * g^\vee)(t) = \frac{1}{2\pi}\widehat{|\widehat{g}|^2}(-t), \quad \widehat{R}(x) = |\widehat{g}(x)|^2, \tag{1.2}$$

where $g^\vee(u) = g(-u)$ is the time reversion operation and $*$ stands for convolution. The Fourier transform $\widehat{R}(x)$ is also known as the spectral density of X. Note, however, that the two rightmost expressions in (1.2) are not meaningful for general $g \in L^2(\mathbb{R})$ because the function R may be in neither $L^2(\mathbb{R})$ nor $L^1(\mathbb{R})$.

Under mild assumptions on g and in a special Gaussian case, Theorem 1 of Zhang and Walter (1994) states that a Gaussian process X in (1.1) has a wavelet-based expansion

$$X(t) = \sum_{n=-\infty}^{\infty} a_{J,n}\theta^J(t - 2^{-J}n) + \sum_{j=J}^{\infty}\sum_{n=-\infty}^{\infty} d_{j,n}\Psi^j(t - 2^{-j}n), \tag{1.3}$$

for any $J \in \mathbb{Z}$, with convergence in the $L^2(\Omega)$-sense for each t. Here, $a_J = \{a_{J,n}\}_{n\in\mathbb{Z}}$, $d_j = \{d_{j,n}\}_{j\geq J, n\in\mathbb{Z}}$ are independent $\mathcal{N}(0,1)$ random variables. The functions θ^j and Ψ^j are defined through their Fourier transforms as

$$\widehat{\theta^j}(x) = \widehat{g}(x)\, 2^{-j/2}\, \widehat{\phi}(2^{-j}x), \quad \widehat{\Psi^j}(x) = \widehat{g}(x)\, 2^{-j/2}\, \widehat{\psi}(2^{-j}x), \tag{1.4}$$

where ϕ and ψ are scaling and wavelet functions, respectively, associated with a suitable orthogonal Multiresolution Analysis (MRA, in short). For more information on scaling function, wavelet and MRA, see for example Mallat (1998), Daubechies (1992), or many others. Moreover, the coefficients $a_{j,n}$ and $d_{j,n}$ in (1.3) can be expressed as

$$a_{j,n} = \int_{\mathbb{R}} X(t)\theta_j(t - 2^{-j}n)dt, \quad d_{j,n} = \int_{\mathbb{R}} X(t)\Psi_j(t - 2^{-j}n)dt, \tag{1.5}$$

with the functions θ_j and Ψ_j, "dual" to θ^j and Ψ^j, defined through

$$\widehat{\theta_j}(x) = \overline{\widehat{g}(x)^{-1}}\, 2^{-j/2}\, \widehat{\phi}(2^{-j}x), \quad \widehat{\Psi_j}(x) = \overline{\widehat{g}(x)^{-1}}\, 2^{-j/2}\, \widehat{\psi}(2^{-j}x). \tag{1.6}$$

Zhang and Walter (1994) call (1.3) a Karhunen-Loève-like (KL-like) wavelet-based expansion. It is discussed in several textbooks, for example, Walter and Shen (2001), and Vidakovic (1999). The sum $\sum_n a_{J,n}\theta^J(t - 2^{-J}n)$ in (1.3) is interpreted as an approximation term at scale 2^{-J}, and the sums $\sum_n d_{j,n}\Psi^j(t - 2^{-j}n)$, $j \geq J$, are interpreted as detail terms at finer scales 2^{-j}, $j \geq J$. The KL-like expansion is related to the wavelet-vaguelette expansions of Donoho (1995), the expansions of Benassi and Jaffard (1994), and others, where $J = -\infty$ in (1.3) and hence the first approximation term in (1.3) is absent.

Though the approximation term $\sum_n a_{J,n}\theta^J(t - 2^{-J}n)$ in (1.3) involves independent $\mathcal{N}(0,1)$ random variables $a_{J,n}$ which are convenient to deal with in theory, the term is also unnatural in one important respect. It is customary with wavelet bases that not only an approximation term but also the respective approximation coefficients, the sequence $a_{J,n}$ in this case, approximate the signal at hand. The sequence $a_{J,n}$ does not have this property because it consists of independent random variables and hence cannot approximate a typically dependent stationary process $X(t)$. In this work, we modify the approximation terms as

$$\sum_{n=-\infty}^{\infty} a_{J,n}\theta^J(t - 2^{-J}n) = \sum_{n=-\infty}^{\infty} X_{J,n}\Phi^J(t - 2^{-J}n) \tag{1.7}$$

so that the new approximation coefficients $X_J = \{X_{J,n}\}_{n\in\mathbb{Z}}$ now have this property, namely,

$$2^{J/2} X_{J,[2^J t]} \approx X(t) \tag{1.8}$$

in a suitable sense, as $J \to \infty$, where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

In the relation (1.7) above,

$$\widehat{\Phi}^J(x) = \frac{\widehat{g}(x)}{\widehat{g}_J(2^{-J}x)} 2^{-J/2}\widehat{\phi}(2^{-J}x) = \frac{\widehat{\theta}^J(x)}{\widehat{g}_J(2^{-J}x)} \tag{1.9}$$

with the discrete Fourier transform $\widehat{g}_J(y)$ of a sequence $g_J = \{g_{J,n}\}$. (A discrete Fourier transform of $g = \{g_n\}$ is defined by

$$\widehat{g}(x) = \sum_{n=-\infty}^{\infty} g_n e^{-inx}, \quad x \in \mathbb{R},$$

and is periodic with the period 2π.) The random sequence $X_J = \{X_{J,n}\}$ in (1.7) is defined as

$$\widehat{X}_J(x) = \widehat{g}_J(x)\widehat{a}_J(x) \tag{1.10}$$

in the frequency domain. Moreover, we expect that

$$X_{J,n} = \int_{\mathbb{R}} X(t)\Phi_J(t - 2^{-J}n)dt, \tag{1.11}$$

where

$$\widehat{\Phi}_J(x) = \overline{\left(\frac{\widehat{g}_J(2^{-J}x)}{\widehat{g}(x)}\right)} 2^{-J/2}\widehat{\phi}(2^{-J}x). \tag{1.12}$$

The relation (1.7) can be informally and easily verified by taking the Fourier transform of its two sides.

It is well-known (e.g. Daubechies (1992) in the deterministic context) that (1.8) is a property of the corresponding wavelet basis functions. When

$$G_J(2^{-J}x) := \frac{\widehat{g}_J(2^{-J}x)}{\widehat{g}(x)} \approx 1, \tag{1.13}$$

we have $\widehat{\Phi}_J(x) \approx 2^{-J/2}\widehat{\phi}(2^{-J}x) \approx 2^{-J/2}$ for large J (typically, $\widehat{\phi}(0) = \int_{\mathbb{R}} \phi(t)dt = 1$) and hence, by (1.11), we expect that

$$\begin{aligned} 2^{J/2} X_{J,n} &= \frac{2^{J/2}}{2\pi} \int_{\mathbb{R}} \widehat{X}(x) e^{-ix2^{-J}n} \overline{\widehat{\Phi}_J(x)} dx \\ &\approx \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{X}(x) e^{-ix2^{-J}n} dx = X(2^{-J}n). \end{aligned}$$

The conditions for (1.7) and (1.8) will thus involve the function G_J given in (1.13).

Though the modification (1.7) appears small, it is fundamental and important in several ways, and surprisingly leads to many research questions. For example, the modification allows for several applications such as simulation considered in Section 8 below. The wavelet-based decomposition (1.3) with (1.7) can also be viewed as a generalization to the wide framework of stationary Gaussian processes of a particular wavelet decomposition of fractional Brownian motion established in Sellan (1995), Meyer, Sellan and Taqqu (1999). It thus shows that self-similarity (of fractional Brownian motion, for instance) is not a necessary condition for constructing wavelet decompositions in that fashion. See also Pipiras (2004) who explored a similar decomposition for a non-Gaussian self-similar process called the Rosenblatt process. Didier and Pipiras (2006) study analogous wavelet decompositions for stationary time series in discrete time.

The decomposition (1.3) with (1.7) can be viewed as being more general than (1.3) - becoming (1.3) when $X_j = a_j$ are independent, $\mathcal{N}(0,1)$ random variables. For this reason, both decompositions should be viewed under one framework. This is the view taken in the following definition and in a parallel paper Didier and Pipiras (2006).

Definition 1.1 Decompositions (1.3) and (1.3) with (1.7) will be called *adaptive wavelet decompositions.*

Adaptiveness refers to the fact that the basis functions are chosen based on the dependence structure of the underlying stationary Gaussian process.

The rest of the paper is organized as follows. In Section 2, we briefly introduce a wavelet basis to be used in wavelet-based decompositions. In Section 3, we state the assumptions on the discrete deterministic approximations g_J and the functions g. In Section 4, we consider several examples of Gaussian stationary processes and their discrete approximations. The KL-like wavelet decomposition (1.3) and its modification (1.7) are proved in Section 5. In particular, we reprove the decomposition (1.3) because inaccurate assumptions were used in Zhang and Walter (1994). We show that there is a FWT-like algorithm relating $\{X_{j,n}\}$ across different scales in Section 6. In Sections 7 and 8, we examine convergence of discrete random approximations X_J and illustrate simulation in practice. Finally, in Appendix A, we consider integration of stationary Gaussian processes.

2 Wavelet bases of $L^2(\mathbb{R})$

We specify here a scaling function ϕ and a wavelet ψ which will be used below. There are many choices for these functions. We shall work with particular Meyer wavelets (Meyer (1992), Mallat (1998)) because of their nice theoretical properties. The results of this paper and their proofs rely on specific nice properties of the selected Meyer wavelets. Other wavelet bases could be taken, e.g. the celebrated Daubechies wavelets, and are being currently investigated. Meyer wavelets are also used in Zhang and Walter (1994), Meyer et al. (1999) and others.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz class of $C^\infty(\mathbb{R})$ functions f that decay faster than any polynomial at infinity and so do their derivatives, that is,

$$\lim_{|t|\to\infty} t^m \frac{d^n f(t)}{dt^n} = 0,$$

for any $m, n \geq 1$. We can choose a scaling function $\phi \in \mathcal{S}(\mathbb{R})$ satisfying

$$\widehat{\phi}(x) \in [0,1], \quad \widehat{\phi}(x) = \widehat{\phi}(-x),$$

$$\widehat{\phi}(x) = \begin{cases} 1, & |x| \leq 2\pi/3, \\ 0, & |x| > 4\pi/3, \end{cases} \qquad \widehat{\phi}(x) \text{ decreases on } [0,\infty).$$

The corresponding CMF u has the discrete Fourier transform

$$\widehat{u}(x) = \begin{cases} \sqrt{2}\,\widehat{\phi}(2x), & |x| \leq 2\pi/3, \\ 0, & |x| > 2\pi/3. \end{cases}$$

The wavelet function ψ associated with ϕ is such that $\psi \in \mathcal{S}(\mathbb{R})$ and

$$\widehat{\psi}(x) = \frac{1}{\sqrt{2}}\, \widehat{v}\left(\frac{x}{2}\right) \widehat{\phi}\left(\frac{x}{2}\right) \quad \text{with} \quad \widehat{v}(x) = e^{-ix}\overline{\widehat{u}(x+\pi)}, \tag{2.1}$$

where v is the other CMF. One can verify that, for the Meyer wavelets,

$$\widehat{\psi}(x) = e^{-\frac{ix}{2}} \left(\widehat{\phi}\left(\frac{x}{2}\right)^2 - \widehat{\phi}(x)^2 \right)^{1/2}. \tag{2.2}$$

In particular, $\widehat{\psi}(x) = 0$ for $|x| \leq 2\pi/3$ and $|x| \geq 8\pi/3$. The collection of functions $\phi(t - k), 2^{j/2}\psi(2^j t - k)$, $k \in \mathbb{Z}, j \geq 0$, makes an orthonormal basis of $L^2(\mathbb{R})$.

3 Basis functions and discrete approximations

Let $g \in L^2(\mathbb{R})$ be a kernel function appearing in (1.1), and $g_J = \{g_{J,n}\}_{n\in\mathbb{Z}}$, $J \in \mathbb{Z}$, be sequences of real numbers such that $g_J \in l^2(\mathbb{Z})$. Following Section 1 (see, in particular, (1.13)), we shall think of g_J as a *discrete (deterministic) approximation* of g at scale 2^{-J}.

A discrete approximation $g_J \in l^2(\mathbb{Z})$ induces a *discrete (random) approximation* $X_J = \{X_{J,n}\}$ defined by (1.10), that is,

$$X_{J,n} = \sum_{k=-\infty}^{\infty} g_{J,k} a_{J,n-k} \tag{3.1}$$

in the time domain, or symbolically

$$\widehat{X}_J(x) = \widehat{g}_J(x)\widehat{a}_J(x) \tag{3.2}$$

in the frequency domain, where $a_J = \{a_{J,n}\}$ are independent $\mathcal{N}(0,1)$ random variables (Gaussian white noise). As $J \to \infty$, we expect that $2^{J/2} X_{J,[2^J t]}$ approximates $X(t)$ defined by (1.1). Conversely, we may think that a random discrete approximation X_J of X given by (1.1) can be represented by (3.1) with a sequence g_J. Hence, X_J also induces a deterministic discrete approximation g_J of g.

We will make some of the following assumptions on g and g_J. Let $L^p_{loc}(\mathbb{R})$ consist of functions which are in L^p on any compact interval of $\mathbb{R}$. Set also

$$G_J(x) = \frac{\widehat{g}_J(x)}{\widehat{g}(2^J x)}, \quad x \in \mathbb{R}. \tag{3.3}$$

Note that, with the notation (3.3), expressions (1.9) and (1.12) become

$$\widehat{\Phi}^J(x) = (G_J(2^{-J}x))^{-1}\, 2^{-J/2}\widehat{\phi}(2^{-J}x), \quad \widehat{\Phi}_J(x) = \overline{G_J(2^{-J}x)} 2^{-J/2}\widehat{\phi}(2^{-J}x) \tag{3.4}$$

Assumption 1: Suppose that

$$\widehat{g}^{-1} \in L^2_{loc}(\mathbb{R}). \tag{3.5}$$

Assumption 2: Suppose that, for any $J \in \mathbb{Z}$,

$$G_J,\ G_J^{-1} \in L^2_{loc}(\mathbb{R}). \tag{3.6}$$

Assumption 3: Suppose that, for any $J_0 \in \mathbb{Z}$,

$$\max_{p=-1,1} \max_{k=0,1,2} \sup_{J \geq J_0} \sup_{|x| \leq 4\pi/3} \left| \frac{\partial^k (G_J(x))^p}{\partial x^k} \right| < \infty. \tag{3.7}$$

Assumption 4: Suppose that, for large $|x|$,

$$\left| \frac{\partial^k \widehat{g}(x)}{\partial x^k} \right| \leq \frac{\text{const}}{|x|^{k+1}}, \quad k = 0, 1, 2. \tag{3.8}$$

Assumption 5: Assume that, for large J,

$$|G_J(0) - 1| \leq \text{const } 2^{-J}. \tag{3.9}$$

As explained below, Assumptions 1 and 2 ensure that the basis functions used in decompositions are well-defined. Assumptions 3, 4 and 5 will be used to establish the modification (1.7) and to show that X_J is an approximation sequence for X in the sense of (1.8).

Observe that the functions θ^j and Ψ^j in (1.4) are well-defined pointwise through the inverse Fourier transform since $\widehat{\theta^j}, \widehat{\Psi}^j \in L^1(\mathbb{R})$ for $\widehat{g} \in L^2(\mathbb{R})$. By using Assumptions 1 and 2, the functions θ_j and Ψ_j in (1.6), Φ^j in (1.9) and Φ_j in (1.12) (see also (3.4)) are well-defined pointwise through the inverse Fourier transform as well. Moreover, $\theta_j, \Psi_j, \Phi^j, \Phi_j$ are in $L^2(\mathbb{R})$ because their Fourier transforms are in $L^2(\mathbb{R})$.

Appendix A contains some results on defining integrals $\int X(t)f(t)dt$. See, in particular, the definition of a related function space $L^2_g(\mathbb{R})$ in (A.6) of integrands $f(t)$. Since $\theta_j, \Psi_j \in L^2_g(\mathbb{R})$, the coefficients $a_{j,n}$ and $d_{j,n}$ in (1.5) are well-defined. Using properties of integrals developed in Appendix A, it is easy to see that $a_{j,n}$ and $d_{j,n}$ are independent $\mathcal{N}(0,1)$ random variables. Since $\Phi_j \in L^2_g(\mathbb{R})$, the integral in (1.11) is well-defined as well.

Another consequence of the above assumptions are useful bounds on the functions Φ^j, Ψ^j. We will use these bounds several times below.

Lemma 3.1 *Under Assumptions 3 and 4 above, we have*

$$|2^{-j/2}\Phi_j(2^{-j}u)|, |2^{-j/2}\Phi^j(2^{-j}u)| \leq \frac{C}{1+|u|^2}, \quad u \in \mathbb{R}, \tag{3.10}$$

$$|\Psi^j(2^{-j}u)| \leq \frac{C2^{-j/2}}{1+|u|^2}, \quad u \in \mathbb{R}, \tag{3.11}$$

where a constant C does not depend on $j \geq j_0$, for fixed j_0.

Proof: By definition of Φ^j in (1.9) (see also (3.4)) and after a change of variables, observe that

$$2^{-j/2}\Phi^j(2^{-j}u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iux} (G_j(x))^{-1} \widehat{\phi}(x) dx, \quad u \in \mathbb{R}. \tag{3.12}$$

Since $\text{supp}\{\widehat{\phi}\} \subset \{|x| \leq 4\pi/3\}$, we obtain by Assumption 3 that

$$|2^{-j/2}\Phi^j(2^{-j}u)| \leq C, \quad u \in \mathbb{R}, \tag{3.13}$$

for a constant C which does not depend on $j \geq j_0$, for fixed j_0. Using integration by parts in (3.12) twice and Assumption 3, we have

$$2^{-j/2}\Phi^j(2^{-j}u) = -\frac{1}{2\pi u^2}\int_{\mathbb{R}} e^{iux}\frac{\partial^2}{\partial x^2}\Big((G_j(x))^{-1}\widehat{\phi}(x)\Big)dx, \quad u \in \mathbb{R}.$$

By Assumption 3 and properties of $\widehat{\phi}$, for any $j \geq j_0$,

$$|2^{-j/2}\Phi^j(2^{-j}u)| \leq \frac{C}{|u|^2}\int_{|x|\leq 4\pi/3}\Bigg(\left|\frac{\partial^2}{\partial x^2}(G_j(x))^{-1}\right| + \left|\frac{\partial}{\partial x}(G_j(x))^{-1}\right| + \left|(G_j(x))^{-1}\right|\Bigg)dx \leq \frac{C}{|u|^2}, \quad u \in \mathbb{R}.$$

The bound (3.10) for Φ^j follows from (3.13) and (3.14). The case of Φ_j is proved similarly.

To show the bound (3.11), observe from (1.4) that

$$\Psi^j(2^{-j}u) = \frac{2^{j/2}}{2\pi}\int_{\mathbb{R}} e^{iux}\widehat{g}(2^jx)\widehat{\psi}(x)dx, \quad u \in \mathbb{R}. \tag{3.14}$$

Since $\mathrm{supp}\{\widehat{\psi}\} \subset \{2\pi/3 \leq |x| \leq 8\pi/3\}$, we obtain by Assumption 4 that

$$|\Psi^j(2^{-j}u)| \leq C2^{j/2}\int_{2\pi/3}^{8\pi/3}\frac{dx}{1+2^jx} \leq C'2^{-j/2}, \quad u \in \mathbb{R}, \tag{3.15}$$

for constants C, C' which do not depend on $j \geq j_0$, for fixed j_0. Using integration by parts in (3.14) and Assumption 4, we have

$$\Psi^j(2^{-j}u) = -\frac{2^{j/2}}{2\pi u^2}\int_{\mathbb{R}} e^{iux}\frac{\partial^2}{\partial x^2}\Big(\widehat{g}(2^jx)\widehat{\psi}(x)\Big)dx, \quad u \in \mathbb{R}. \tag{3.16}$$

Hence, by using properties of $\widehat{\psi}$ and Assumption 4, for $j \geq j_0$,

$$\begin{aligned}|\Psi^j(2^{-j}u)| &\leq \frac{C2^{j/2}}{|u|^2}\int_{2\pi/3\leq|x|\leq 8\pi/3}\left(2^{2j}\left|\frac{\partial^2\widehat{g}}{\partial x^2}(2^jx)\right| + 2^j\left|\frac{\partial\widehat{g}}{\partial x}(2^jx)\right| + |\widehat{g}(2^jx)|\right)dx \\ &\leq \frac{C'2^{j/2}}{|u|^2}\int_{2\pi/3}^{8\pi/3}\left(\frac{2^{2j}}{1+2^{3j}x^3} + \frac{2^j}{1+2^{2j}x^2} + \frac{1}{1+2^jx}\right)dx \leq C''\frac{2^{-j/2}}{|u|^2}, \quad u \in \mathbb{R}. \end{aligned} \tag{3.17}$$

The bound (3.11) follows from (3.15) and (3.17). □

4 Examples

We consider here several examples of Gaussian stationary processes together with their possible discrete approximations.

Example 4.1 The Ornstein-Uhlenbeck (OU) process X is perhaps the best-known Gaussian stationary process. It is the only Gaussian stationary process which is Markov. The OU process can be represented by (1.1) with

$$g(t) = \sigma e^{-\lambda t}1_{\{t\geq 0\}}, \quad \widehat{g}(x) = \frac{\sigma}{\lambda + ix}, \tag{4.1}$$

for some $\lambda > 0$ and $\sigma > 0$.

At this point, one can approximate either g or X. We do so for the process X because it has a well-known discrete approximation. Observe from (1.1) and (4.1) that, for $J, n \in \mathbb{Z}$,

$$X(2^{-J}(n+1)) = e^{-\lambda 2^{-J}} X(2^{-J}n) + \sigma \sqrt{\frac{1-e^{-2\lambda 2^{-J}}}{2\lambda}}\, a_{J,n+1},$$

where $\{a_{J,n}\}_{n\in\mathbb{Z}}$ is a Gaussian white noise. Therefore, since we expect $2^{J/2}X_{J,[2^J t]} \approx X(t)$, it appears natural to consider the discrete approximation

$$X_{J,n} = 2^{-J/2}\sigma \sqrt{\frac{1-e^{-2\lambda 2^{-J}}}{2\lambda}}\, (I - e^{-\lambda 2^{-J}}B)^{-1}\, a_{J,n}, \tag{4.2}$$

where B denotes the backshift operator (not to be confused with Bm) and $I = B^0$. In other words, X_J is an AR(1) time series (see Brockwell and Davis (1991)).

In view of (4.1) and (4.2), the deterministic discrete approximations g_J have the discrete Fourier transforms

$$\widehat{g}_J(x) = 2^{-J/2}\sigma \sqrt{\frac{1-e^{-2\lambda 2^{-J}}}{2\lambda}}\, (1 - e^{-\lambda 2^{-J}} e^{-ix})^{-1}. \tag{4.3}$$

Furthermore, $\widehat{g}$ and $\widehat{g}_J$ satisfy Assumptions 1 to 5. Indeed, Assumptions 1 and 2 hold because, for every $J \in \mathbb{Z}$,

$$\widehat{g}^{-1}(x) = \frac{\lambda + ix}{\sigma}, \quad G_J(x) = 2^{J/2}\sqrt{\frac{1-e^{-2\lambda 2^{-J}}}{2\lambda}}\, \frac{2^{-J}\lambda + ix}{1 - e^{-\lambda 2^{-J}} e^{-ix}} \tag{4.4}$$

and G_J^{-1} are continuous functions on $\mathbb{R}$, and thus square-integrable on compact sets.

To show Assumption 3, consider the domain $D^{J_0} = \{z \in \mathbb{C} :\ 0 \leq \text{Re}(z) \leq 2^{-J_0}\lambda, |\text{Im}(z)| \leq 4\pi/3\}$. The functions

$$F(z) = \begin{cases} \frac{z}{1-e^{-z}}, & z \in \mathbb{C}\backslash\{i2k\pi,\ k \in \mathbb{Z}\}, \\ 1, & z = 0, \end{cases}$$

and $F(z)^{-1}$ are holomorphic and different from zero on the open set $D^{J_0}_\epsilon = \{w \in \mathbb{C} :\ \inf_{z\in D^{J_0}} |z - w| < \epsilon\} \supset D^{J_0}$. By setting $z = 2^{-J}\lambda + ix \in D^{J_0}$, we have $G_J(x) = C_J F(z)$ for all $J \geq J_0$ and $|x| \leq 4\pi/3$, where $0 < c_1 \leq C_J \leq c_2 < +\infty$ for some c_1, c_2. Hence, Assumption 3 must hold.

Assumption 4 follows from the relation

$$\frac{\partial^k \widehat{g}(x)}{\partial x^k} = \frac{\sigma(-i)^k k!}{(\lambda + ix)^{k+1}}, \quad k = 0, 1, 2, ...$$

Finally, Assumption 5 is also satisfied because

$$|G_J(0) - 1| = \left| 2^{J/2}\sqrt{\frac{1-e^{-2\lambda 2^{-J}}}{2\lambda}} \left(\frac{2^{-J}\lambda}{1 - e^{-\lambda 2^{-J}}} \right) - 1 \right| = \sqrt{\frac{\lambda 2^{-J}}{1-e^{-\lambda 2^{-J}}}}\ \left| \sqrt{\frac{1+e^{-\lambda 2^{-J}}}{2}} \right.$$

$$\left. - \sqrt{\frac{1-e^{-\lambda 2^{-J}}}{\lambda 2^{-J}}} \right| \leq C_1 \left(\left| \sqrt{\frac{1+e^{-\lambda 2^{-J}}}{2}} - 1 \right| + \left| \sqrt{\frac{1-e^{-\lambda 2^{-J}}}{\lambda 2^{-J}}} - 1 \right| \right) \leq C_2\, 2^{-J}$$

for constants $C_1, C_2 > 0$.

Example 4.2 Consider a Gaussian stationary process (1.1) with a kernel function g having the Fourier transform

$$\widehat{g}(x) = \frac{f(x)}{h(x)}. \tag{4.5}$$

Here,

$$f(x) = \prod_{k \in \mathcal{P}_1} p(a_k, b_k; x)\, p(-a_k, b_k; x) \prod_{m \in \mathcal{P}_2} p(0, c_m; x), \tag{4.6}$$

$$h(x) = \prod_{k \in \mathcal{Q}_1} p(d_k, e_k; x)\, p(-d_k, e_k; x) \prod_{m \in \mathcal{Q}_2} p(0, f_m; x) \tag{4.7}$$

with

$$p(a, b; x) = ix + ia + b, \tag{4.8}$$

where $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1$ and $\mathcal{Q}_2$ are finite sets of indices. It is assumed that polynomials $f(x)$ and $h(x)$ have no common roots, and also that $\forall k \in \mathcal{Q}_1$, $e_k \neq 0$, and $\forall m \in \mathcal{Q}_2$, $f_m \neq 0$. Note that the polynomials f and h are Hermitian symmetric. Hence, $\widehat{g}$ is also Hermitian symmetric and thus g is real-valued. Kernel functions $\widehat{g}$ as in (4.5) correspond to rational spectral densities (Rozanov (1967)).

To define a discrete approximation $\widehat{g}_J$ of $\widehat{g}$, consider first $p(a, b; x)$, which is a "building block" of f in (4.6) and h in (4.7). Define a discrete approximation of $p(a, b; x)$ as

$$p_J(a, b; x) = 2^J \left(1 - e^{-2^{-J}b - 2^{-J}ia - ix}\right) \tag{4.9}$$

and also, in analogy to (3.3), set

$$P_J(x) = \frac{p_J(a, b; x)}{p(a, b; 2^J x)} = \frac{1 - e^{-2^{-J}b - 2^{-J}ia - ix}}{ix + 2^{-J}ia + 2^{-J}b}. \tag{4.10}$$

The form (4.9) ensures that $p_J(a, b; x) p_J(-a, b; x)$ and $p_J(0, b; x)$ are Hermitian symmetric functions. Define now a discrete approximation $\widehat{g}_J$ of $\widehat{g}$ by (4.5), where p's in (4.6) and (4.7) are replaced by p_J's. The function G_J is then given by (4.5), where p's in (4.6) and (4.7) are replaced by P_J's.

We shall now verify that $\widehat{g}$ and $\widehat{G}_J$ satisfy Assumptions 1-5. Assumptions 1 and 2 are satisfied because the "building blocks" p^{-1}, P_J and P_J^{-1} for $\widehat{g}^{-1}$, G_J and G_J^{-1} are continuous functions on the real line. To show Assumption 3, it is enough to prove (3.7) for the function P_J. Similarly to the case of the Ornstein-Uhlenbeck process, we are interested in the behavior of F and F^{-1} for $z = i(x + 2^{-J}a) + 2^{-J}b$, where $|x| \leq 4\pi/3$ and $J \geq J_0$. So, define the set

$$D^{J_0} = \left\{ z \in \mathbb{C} : \ 0 \leq \Re(z) \leq 2^{-J_0}b, |\Im(z)| \leq \frac{4\pi}{3} + \Re(z)\left|\frac{a}{b}\right| \right\},$$

and note that $z = i(x + 2^{-J}a) + 2^{-J}b \in D^{J_0}$ when $|x| \leq 4\pi/3$ and $J \geq J_0$. Also, consider the set $D^{J_0}_\epsilon = \{w \in \mathbb{C} : \ \inf_{z \in D^{J_0}} |z - w| < \epsilon\}$. The functions F and F^{-1} are holomorphic on $D^{J_0}_\epsilon \supset D^{J_0}$ for small enough ϵ, and thus Assumption 3 holds.

Consider now Assumption 4. The condition (3.8) is satisfied for $k = 0$ by the definition of $\widehat{g}$ and the implicit assumption $\widehat{g} \in L^2(\mathbb{R})$ (that is, the polynomial h has a higher degree than the polynomial f). When $k = 1$, note that

$$\frac{\partial \widehat{g}(x)}{\partial x} = \frac{f'(x)}{h(x)} - \frac{f(x)h'(x)}{(h(x))^2}$$

and the condition (3.8) follows since the difference between the degrees of $f'(x)$ and $h(x)$, and those of $f(x)h'(x)$ and $(h(x))^2$ increased by 1. The case $k = 2$ can be argued in a similar way.

To show (3.9) in Assumption 5, it is enough to prove it for

$$P_J(0) = \frac{1 - e^{-2^{-J}b - 2^{-J}ia}}{2^{-J}ia + 2^{-J}b}.$$

This can be done by using standard properties of exponentials and using their Taylor expansions.

Finally, let us note that the discrete approximations g_J based on (4.9) correspond to ARMA time series X_J (Brockwell and Davis (1991)).

5 Adaptive wavelet decompositions

We first reestablish the decomposition (1.3) of Zhang and Walter (1994) by providing a more rigorous proof.

Theorem 5.1 *(Zhang and Walter (1994)) Let X be a Gaussian stationary process given by (1.1). Suppose that Assumptions 1 and 2 of Section 3 hold. Then, with the notation of Section 1, the process X admits the following wavelet-based decomposition: for any $J \in \mathbb{Z}$,*

$$X(t) = \sum_{n=-\infty}^{\infty} a_{J,n}\theta^J(t - 2^{-J}n) + \sum_{j=J}^{\infty}\sum_{n=-\infty}^{\infty} d_{j,n}\Psi^j(t - 2^{-j}n) \quad (5.1)$$

$$= \sum_{j=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} d_{j,n}\Psi^j(t - 2^{-j}n), \quad (5.2)$$

with the convergence in the $L^2(\Omega)$-sense for each t, and independent $\mathcal{N}(0,1)$ random variables $a_{J,n}, d_{j,n}$ that are expressed through (1.5).

Proof: (Zhang and Walter (1994)) Under Assumptions 1 and 2, the basis functions θ^J and Ψ^j in (5.1) and (5.2) are well-defined pointwise (Section 3). The coefficients $a_{j,n}$, $d_{j,n}$ are well-defined, independent $\mathcal{N}(0,1)$ random variables (Section 3). Except for more rigor, the rest of the proof follows that of Zhang and Walter (1994). Since the proof is short, we provide it to the reader's convenience.

Observe that

$$E\Bigg(X(t) - \sum_{n=-N_1}^{N_2} a_{J,n}\theta^J(t - 2^{-J}n) - \sum_{j=J}^{K}\sum_{n=-M_1}^{M_2} d_{j,n}\Psi^j(t - 2^{-j}n)\Bigg)^2$$

$$= E\Bigg(X(t)^2 - 2\sum_{n=-N_1}^{N_2} X(t)a_{J,n}\theta^J(t - 2^{-J}n) - 2\sum_{j=J}^{K}\sum_{n=-M_1}^{M_2} X(t)d_{j,n}\Psi^j(t - 2^{-j}n)$$

$$+\Bigg(\sum_{n=-N_1}^{N_2} a_{J,n}\theta^J(t - 2^{-J}n) + \sum_{j=J}^{K}\sum_{n=-M_1}^{M_2} d_{J,n}\Psi^j(t - 2^{-j}n)\Bigg)^2\Bigg). \quad (5.3)$$

By using Appendix A and the definition of function θ^j (Sections 1 and 3), we have

$$EX(t)a_{J,n} = EX(t)\int_{\mathbb{R}} X(s)\theta_J(s - 2^{-J}n)ds = \frac{1}{2\pi}\int_{\mathbb{R}} e^{itx}|\widehat{g}(x)|^2\widehat{\theta_J(\cdot - 2^{-J}n)}(x)dx$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(t-2^{-J}n)x}|\widehat{g}(x)|^2\widehat{\theta}_J(x)dx = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(t-2^{-J}n)x}2^{-J/2}\widehat{g}(x)\widehat{\phi}_J(2^{-J}x)dx = \theta^J(t - 2^{-J}n). \quad (5.4)$$

Similarly, we have

$$EX(t)d_{j,n} = \Psi^j(t - 2^{-j}n). \tag{5.5}$$

Using (5.4), (5.5) and independence of $a_{J,n}$, $d_{j,n}$, relation (5.3) becomes

$$R(0) - \sum_{n=-N_1}^{N_2} \theta^J(t - 2^{-J}n)^2 - \sum_{j=J}^{K} \sum_{n=-M_1}^{M_2} (\Psi^j(t - 2^{-j}n))^2. \tag{5.6}$$

Observe from the definition of θ^J that

$$\theta^J(t - 2^{-J}n) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(t-2^{-J}n)x} 2^{-J/2} \widehat{g}(x) \widehat{\phi}_J(2^{-J}x) dx$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(x) 2^{J/2} (\phi(2^J \widehat{(\cdot + t)} - n))(x) dx = \frac{1}{2\pi} \int_{\mathbb{R}} g(u) 2^{J/2} \phi(n - 2^J(u+t)) du, \tag{5.7}$$

and similarly

$$\Psi^j(t - 2^{-j}n) = \frac{1}{2\pi} \int_{\mathbb{R}} g(u) 2^{j/2} \psi(n - 2^j(u+t)) du. \tag{5.8}$$

Since the collection of functions $2^{J/2}\phi(n - 2^J(u+t))$, $2^{j/2}\phi(n - 2^j(u+t))$, $j \geq J$, $n \in \mathbb{Z}$, makes an orthonormal basis of $L^2(\mathbb{R})$ for any $t \in \mathbb{R}$, and since $R(0) = \int_{\mathbb{R}} |g(t)|^2 dt$, we obtain from (5.7) and (5.8) that relation (5.6) converges to 0 as N_i, M_i $(i = 1, 2)$ and K approach infinity. □

In the next result, we modify the approximation term in the decomposition (5.1) according to (1.7).

Theorem 5.2 *Let X be a Gaussian stationary process given by (1.1). Suppose that Assumptions 1 and 2 of Section 3 hold. Then, with the notation of Section 1, the process X admits the following wavelet-based decomposition: for any $J \in \mathbb{Z}$,*

$$X(t) = \sum_{n=-\infty}^{\infty} X_{J,n} \Phi^J(t - 2^{-J}n) + \sum_{j=J}^{\infty} \sum_{n=-\infty}^{\infty} d_{j,n} \Psi^j(t - 2^{-j}n). \tag{5.9}$$

The convergence in (5.9) is in the $L^2(\Omega)$-sense for each t under Assumption 3, and it is almost sure, uniform over compact intervals of t under Assumptions 3 and 4. The sequence $X_J = \{X_{J,n}\}_{n\in\mathbb{Z}}$ is defined by either (1.11) or (3.1).

Proof: We first argue that the definitions (1.11) and (3.1) of X_j are equivalent. By using Appendix A, observe that, for $X_{J,n}$ defined by (1.11) and $a_{J,n}$ defined by (1.5),

$$E\Big(X_{J,n} - \sum_{k=-N_1}^{N_2} g_{J,k} a_{J,n-k}\Big)^2 = E\bigg(\int_{\mathbb{R}} X(t) \bigg(\Phi_J(t - 2^{-J}n) - \sum_{k=-N_1}^{N_2} g_{J,k} \theta_J(t - 2^{-J}(n-k))\bigg) dt\bigg)^2$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \Big| \widehat{g}_J(2^{-J}x) - \sum_{k=-N_1}^{N_2} g_{J,k} e^{ix2^{-J}k} \Big|^2 2^{-J} \left| \widehat{\phi}(2^{-J}x) \right|^2 dx \longrightarrow 0,$$

as $N_i \to \infty$ $(i = 1, 2)$, since $\sum_k g_{J,k} e^{ixk}$ converges to $\widehat{g}_J(x)$ in $L^2(-\pi, \pi)$ and $\widehat{\phi}$ has a compact support.

To show (5.9), we start with (5.1) and modify its first sum as (1.7), that is,

$$\sum_{n=-\infty}^{\infty} a_{J,n}\theta^J(t-2^{-J}n) = \sum_{n=-\infty}^{\infty} X_{J,n}\Phi^J(t-2^{-J}n). \tag{5.10}$$

We first show that, under Assumption 3, the R.H.S. converges in the $L^2(\Omega)$-sense for fixed t and, under Assumptions 3 and 4, almost surely, uniformly over compacts of t. Observe that, by Lemma 3.1, $|\Phi^J(t-2^{-J}n)| \le C/(1+|t-2^{-J}n|^2)$ and, by Lemma 3 in Meyer et al. (1999), $|X_{J,n}| \le A\sqrt{\log(2+|n|)}$ a.s., where a random variable A does not depend on n. The almost sure convergence uniformly on compacts $t \in K$ follows since

$$\sup_{t\in K}\sum_{n=-\infty}^{\infty} |X_{J,n}||\Phi^J(t-2^{-J}n)| \le A \sup_{t\in K}\sum_{n=-\infty}^{\infty} \frac{\sqrt{\log(2+|n|)}}{1+|t-2^{-J}n|^2} < \infty \quad \text{a.s.}$$

For the convergence in $L^2(\Omega)$, observe that, for fixed t,

$$E\left(\sum_{n=-\infty}^{\infty} |X_{J,n}||\Phi^J(t-2^{-J}n)|\right)^2 \le CE\sum_{n=-\infty}^{\infty}\frac{|X_{J,n}|^2}{1+|n|^2}\sum_{n=-\infty}^{\infty}\frac{1}{1+|n|^2} < \infty.$$

We shall now prove the equality in (5.10). Observe that, for each u,

$$\theta^J(u) = \sum_{k=-\infty}^{\infty} g_{J,k}\Phi^J(u-2^{-J}k). \tag{5.11}$$

Indeed, arguing as above,

$$F_m(u) = \sum_{k=-m}^{m} g_{J,k}\Phi^J(u-2^{-J}k) \longrightarrow F(u) = \sum_{k=-\infty}^{\infty} g_{J,k}\Phi^J(u-2^{-J}k). \tag{5.12}$$

pointwise, and

$$\widehat{F}_m(x) = \left(\sum_{k=-m}^{m} g_{J,k}e^{-i2^{-J}kx}\right)\frac{\widehat{g}(x)}{\widehat{g}_J(2^{-J}x)}2^{-J/2}\widehat{\phi}(2^{-J}x) \longrightarrow \widehat{\theta}^J(x) \tag{5.13}$$

in $L^2(\mathbb{R})$, since $\sum_{k=-m}^{m} g_{J,k}e^{-ikx}$ converges to $\widehat{g}_J(x)$ in $L^2(-\pi,\pi)$, and $\widehat{g}(x)/\widehat{g}_J(2^{-J}x)$ is bounded by Assumption 3 on the compact support of $\widehat{\phi}(2^{-J}x)$. Hence, $F_m \to \theta^J$ in $L^2(\mathbb{R})$ and $\theta^J = F$ a.e. Since both F and θ^J are continuous, we obtain (5.11).

Set now, for $m \ge 1$,

$$a_{J,n}^{(m)} = \begin{cases} a_{J,n}, & |n| \le m, \\ 0, & |n| > m, \end{cases} \qquad X_{J,n}^{(m)} = \sum_{k=-\infty}^{\infty} g_{J,k}a_{J,n-k}^{(m)}.$$

By using (5.11), we obtain that

$$\sum_{n=-\infty}^{\infty} a_{J,n}^{(m)}\theta^J(t-2^{-J}n) = \sum_{n=-\infty}^{\infty} X_{J,n}^{(m)}\Phi^J(t-2^{-J}n). \tag{5.14}$$

The L.H.S. of (5.14) converges in $L^2(\Omega)$ to the L.H.S. of (5.10) (and, in fact, also almost surely by the Three Series Theorem). Let us show that the R.H.S. of (5.14) converges to the R.H.S. of (5.10). We want to argue next that

$$\sup_{m\geq 1} |X_{J,n}^{(m)}| \leq A\sqrt{\log(2+|n|)} \quad \text{a.s.} \tag{5.15}$$

for a random variable A which only depends on J. By using the Lévy-Octaviani inequality (e.g. Proposition 1.1.1 in Kwapień and Woyczyński (1992)), we have

$$P\left(\sup_{m=1,\dots,M} |X_{J,n}^{(m)}| > a\right) \leq 2P\Big(|X_{J,n}^{(M)}| > a\Big), \tag{5.16}$$

for any $a > 0$ and $M \geq 1$. By the Three Series Theorem, $X_{J,n}^{(M)} \to X_{J,n}$ almost surely, as $M \to \infty$. Hence, passing to the limit with M in (5.16), we have

$$P\left(\sup_{m\geq 1} |X_{J,n}^{(m)}| > a\right) \leq 2P(|X_{J,n}| > a).$$

The bound (5.15) now follows as in the proof of Lemma 3 in Meyer et al. (1999). By using Lemma 3.1 and the bound (5.15), the R.H.S. of (5.14) converges a.s. to the R.H.S. of (5.10).

It is left to show that the second term in (5.9) converges almost surely and uniformly on compacts. By Lemma 3 in Meyer et al. (1999),

$$|d_{j,n}| \leq A\sqrt{\log(2+|j|)}\sqrt{\log(2+|n|)} \quad \text{a.s.},$$

where a random variable A does not depend on j, n. By Lemma 3.1, we have

$$|\Psi^j(t - 2^{-j}n)| = |\Psi^j(2^{-j}(2^j t - n))| \leq \frac{C2^{-j/2}}{1+|2^j t - n|^2},$$

for $j \geq J$. Then, as in the proof of Theorem 2 in Meyer et al. (1999),

$$\begin{aligned}\sum_{j=J}^{\infty}\sum_{n=-\infty}^{\infty} |d_{j,n}||\Psi^j(t-2^{-j}n)| &\leq A' \sum_{j=J}^{\infty} 2^{-j/2}\sqrt{\log(2+|j|)} \sum_{n=-\infty}^{\infty} \frac{\sqrt{\log(2+|n|)}}{1+|2^j t - n|^2} \\ &\leq A'' \sum_{j=J}^{\infty} 2^{-j/2}\sqrt{\log(2+|j|)}\sqrt{\log(2+|2^j t|)} < \infty\end{aligned}$$

a.s. uniformly over compact intervals of t. □

6 FWT-like algorithm

We show here that discrete approximation sequences X_j are related across different scales by a FWT-like algorithm.

Proposition 6.1 *Let X_j and d_j be the sequences appearing in (5.9), and let u and v denote the CMFs associated with the orthogonal Meyer MRA. Then, under Assumptions 1–4 of Section 3:*

(i) (Reconstruction step)

$$X_{j+1} = u_j * \uparrow_2 X_j + v_j * \uparrow_2 d_j, \tag{6.1}$$

where the filters u_j and v_j are defined through their discrete Fourier transforms

$$\widehat{u}_j(x) = \frac{\widehat{g}_{j+1}(x)}{\widehat{g}_j(2x)}\widehat{u}(x), \quad \widehat{v}_j(x) = \widehat{g}_{j+1}(x)\widehat{v}(x); \tag{6.2}$$

(ii) (Decomposition step)

$$X_j = \downarrow_2 (\overline{u}_j^d * X_{j+1}), \quad d_j = \downarrow_2 (\overline{v}_j^d * X_{j+1}), \tag{6.3}$$

where $\overline{x}$ stand for the time reversal of a sequence x, and the filters u_j^d and v_j^d are defined through their discrete Fourier transforms by

$$\widehat{u}_j^d(x) = \overline{\left(\frac{\widehat{g}_j(2x)}{\widehat{g}_{j+1}(x)}\right)}\widehat{u}(x), \quad \widehat{v}_j^d(x) = \overline{\left(\frac{1}{\widehat{g}_{j+1}(x)}\right)}\widehat{v}(x). \tag{6.4}$$

The convergence in (6.1) and (6.3) is in the $L^2(\Omega)$-sense, and also absolute almost surely.

PROOF: Observe first that the filters u_j, v_j, u_j^d, v_j^d are well-defined since $\widehat{u}_j, \widehat{v}_j, \widehat{u}_j^d, \widehat{v}_j^d \in L^2(-\pi,\pi)$. The latter follows by writing

$$\widehat{u}_j(x) = G_{j+1}(x)\Big(G_j(2x)\Big)^{-1}\widehat{u}(x), \quad \widehat{v}_j(x) = G_{j+1}(x)\widehat{g}(2^{j+1}x)\widehat{v}(x),$$

$$\widehat{u}_j^d(x) = \overline{G_{j+1}(x)^{-1}}\ \overline{G_j(2x)}\widehat{u}(x), \quad \widehat{v}_j^d(x) = \overline{G_{j+1}(x)^{-1}}\ \overline{\widehat{g}(2^{j+1}x)^{-1}}\widehat{v}(x) \tag{6.5}$$

(see (3.3)), and using Assumptions 1 and 3.

(i) To show (6.1), we need to prove

$$X_{j+1,n} = \sum_{k=-\infty}^{\infty} X_{j,k}u_{j,n-2k} + \sum_{k=-\infty}^{\infty} d_{j,k}v_{j,n-2k}. \tag{6.6}$$

We first prove the convergence in (6.6) in the $L^2(\Omega)$-sense. Observe that, by using (1.11), (1.5) and Appendix A,

$$E\Bigg(X_{j+1,n} - \Bigg(\sum_{k=-K}^{K}\Big(X_{j,k}u_{j,n-2k} + d_{j,k}v_{j,n-2k}\Big)\Bigg)\Bigg)^2$$

$$= E\Bigg(\int_{\mathbb{R}} X(t)\Bigg(\Phi_{j+1}(t-2^{-j-1}n) - \sum_{k=-K}^{K}\Phi_j(t-2^{-j}k)u_{j,n-2k} - \sum_{k=-K}^{K}\Psi_j(t-2^{-j}k)v_{j,n-2k}\Bigg)dt\Bigg)^2$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{g}(x)|^2 \Bigg| e^{-i2^{-j-1}nx}\widehat{\Phi}_{j+1}(x) - \widehat{\Phi}_j(x)\sum_{k=-K}^{K} e^{-i2^{-j}kx}u_{j,n-2k} - \widehat{\Psi}_j(x)\sum_{k=-K}^{K} e^{-i2^{-j}kx}v_{j,n-2k}\Bigg|^2 dx$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \Bigg| e^{-i2^{-j-1}nx}\overline{\widehat{g}_{j+1}(2^{-j-1}x)}2^{-(j+1)/2}\widehat{\phi}(2^{-j-1}x) - \overline{\widehat{g}_j(2^{-j}x)}2^{-j/2}\widehat{\phi}(2^{-j}x)\cdot$$

$$\cdot\sum_{k=-K}^{K} e^{-i2^{-j}kx}u_{j,n-2k} - 2^{-j/2}\widehat{\psi}(2^{-j}x)\sum_{k=-K}^{K} e^{-i2^{-j}kx}v_{j,n-2k}\Bigg|^2 dx.$$

Hence, it is sufficient to prove that

$$\overline{\widehat{g}_j(2^{-j}x)}2^{-j/2}\widehat{\phi}(2^{-j}x)\sum_{k=-\infty}^{\infty} e^{-i2^{-j}kx}u_{j,n-2k} + 2^{-j/2}\widehat{\psi}(2^{-j}x)\sum_{k=-\infty}^{\infty} e^{-i2^{-j}kx}v_{j,n-2k}$$

$$= e^{-i2^{-j-1}nx}\overline{\widehat{g}_{j+1}(2^{-j-1}x)}2^{-(j+1)/2}\widehat{\phi}(2^{-j-1}x) \tag{6.7}$$

with the convergence in $L^2(\mathbb{R})$. We only consider the case $n = 2p$ (the case $n = 2p + 1$ may be treated in an analogous fashion). Then, relation (6.7) becomes

$$\overline{\widehat{g}_j(2^{-j}x)}\ 2^{-j/2}\ \widehat{\phi}(2^{-j}x) \sum_{m=-\infty}^{\infty} e^{i2^{-j}mx} u_{j,2m} + 2^{-j/2}\widehat{\psi}(2^{-j}x) \sum_{m=-\infty}^{\infty} e^{i2^{-j}mx} v_{j,2m}$$
$$= \overline{\widehat{g}_{j+1}(2^{-j-1}x)}\ 2^{-(j+1)/2}\ \widehat{\phi}(2^{-j-1}x). \tag{6.8}$$

The L.H.S. of (6.8) is

$$\overline{\widehat{g}_j(2^{-j}x)}\ 2^{-j/2}\ \widehat{\phi}(2^{-j}x)\frac{\overline{\widehat{u}_j(2^{-j-1}x)} + \overline{\widehat{u}_j(2^{-j-1}x+\pi)}}{2} + 2^{-j/2}\widehat{\psi}(2^{-j}x)\frac{\overline{\widehat{v}_j(2^{-j-1}x)} + \overline{\widehat{v}_j(2^{-j-1}x+\pi)}}{2}$$
$$= 2^{-1}2^{-j/2}\ \widehat{\phi}(2^{-j}x)\left(\overline{\widehat{g}_{j+1}(2^{-j-1}x)}\ \overline{\widehat{u}(2^{-j-1}x)} + \overline{\widehat{g}_{j+1}(2^{-j-1}x+\pi)}\ \overline{\widehat{u}(2^{-j-1}x+\pi)}\right)$$
$$+ 2^{-1}2^{-j/2}\ \widehat{\psi}(2^{-j}x)\left(\overline{\widehat{g}_{j+1}(2^{-j-1}x)}\ \overline{\widehat{v}(2^{-j-1}x)} + \overline{\widehat{g}_{j+1}(2^{-j-1}x+\pi)}\ \overline{\widehat{v}(2^{-j-1}x+\pi)}\right)$$
$$= \overline{\widehat{g}_{j+1}(2^{-j-1}x)}\Big(2^{-j/2}\ \widehat{\phi}(2^{-j}x)\frac{\overline{\widehat{u}(2^{-j-1}x)} + \overline{\widehat{u}(2^{-j-1}x+\pi)}}{2}$$
$$+ 2^{-j/2}\widehat{\psi}(2^{-j}x)\frac{\overline{\widehat{v}(2^{-j-1}x)} + \overline{\widehat{v}(2^{-j-1}x+\pi)}}{2}\Big)$$
$$+ 2^{-1}2^{-j/2}\Big(\widehat{\phi}(2^{-j}x)\overline{\widehat{u}(2^{-j-1}x+\pi)} + \widehat{\psi}(2^{-j}x)\overline{\widehat{v}(2^{-j-1}x+\pi)}\Big)\cdot$$
$$\cdot\Big(\widehat{g}_{j+1}(2^{-j-1}x+\pi) - \widehat{g}_{j+1}(2^{-j-1}x)\Big).$$

This is also R.H.S. of (6.8) since

$$2^{-j/2}\ \widehat{\phi}(2^{-j}x)\frac{\overline{\widehat{u}(2^{-j-1}x)} + \overline{\widehat{u}(2^{-j-1}x+\pi)}}{2} + 2^{-j/2}\widehat{\psi}(2^{-j}x)\frac{\overline{\widehat{v}(2^{-j-1}x)} + \overline{\widehat{v}(2^{-j-1}x+\pi)}}{2}$$
$$= 2^{-(j+1)/2}\widehat{\phi}(2^{-j-1}x)$$

(this is the Fourier transform of the last relation in the proof of Theorem 7.7 in Mallat (1998)) and, with $y = 2^{-j-1}x$,

$$\widehat{\phi}(2^{-j}x)\overline{\widehat{u}(2^{-j-1}x+\pi)} + \widehat{\psi}(2^{-j}x)\overline{\widehat{v}(2^{-j-1}x+\pi)} = \widehat{\phi}(2y)\overline{\widehat{u}(y+\pi)} + \widehat{\psi}(2y)\overline{\widehat{v}(y+\pi)}$$
$$= 2^{-1/2}\widehat{\phi}(y)\Big(\widehat{u}(y)\overline{\widehat{u}(y+\pi)} + \widehat{v}(y)\overline{\widehat{v}(y+\pi)}\Big) = 0,$$

where we used the facts $\widehat{\phi}(2y) = 2^{-1/2}\widehat{\phi}(y)\widehat{u}(y)$ ((7.30) in Mallat (1998)), $\widehat{\psi}(2y) = 2^{-1/2}\widehat{\phi}(y)\widehat{v}(y)$ ((7.57) in Mallat (1998)) and $\widehat{u}(y)\overline{\widehat{u}(y+\pi)} + \widehat{v}(y)\overline{\widehat{v}(y+\pi)} = 0$ (Theorem 7.8 in Mallat (1998)).

We now show that the convergence in (6.6) is also absolute almost surely. By using Assumptions 3, 4, and integration by parts twice, we may conclude that $|u_{j,k}|, |v_{j,k}| \le C(1 + |k|^2)^{-1}$, $k \in \mathbb{Z}$. By Lemma 3 in Meyer et al. (1999), $|X_{j,k}|, |d_{j,k}| \le A\sqrt{\log(2 + |k|)}$ a.s., where a random variable A does not depend on k. The absolute convergence a.s. now follows.

(ii) The proof of (6.3) follows by similar arguments. We need to prove that

$$X_{j,n} = \sum_{k=-\infty}^{\infty} X_{j+1,k} u^d_{j,k-2n} \tag{6.9}$$

and

$$d_{j,n} = \sum_{k=-\infty}^{\infty} X_{j+1,k} v^d_{j,k-2n}. \tag{6.10}$$

To show (6.9) with convergence in the $L^2(\Omega)$-sense, it suffices to prove that

$$e^{-i2^{-j}nx}\overline{\widehat{g}_j(2^{-j}x)}2^{-j/2}\widehat{\phi}(2^{-j}x)$$

$$= \overline{\widehat{g}_{j+1}(2^{-(j+1)}x)}2^{-(j+1)/2}\widehat{\phi}(2^{-(j+1)}x) \sum_{k=-\infty}^{\infty} e^{-ik2^{-(j+1)}x} u^d_{j,k-2n}. \tag{6.11}$$

But the R.H.S. of (6.11) is

$$\overline{\widehat{g}_{j+1}(2^{-(j+1)}x)}2^{-(j+1)/2}\widehat{\phi}(2^{-(j+1)}x) \sum_{m=-\infty}^{\infty} e^{-i(2n+m)2^{-(j+1)}x} u^d_{j,m}$$

$$= \overline{\widehat{g}_{j+1}(2^{-(j+1)}x)}2^{-(j+1)/2}\widehat{\phi}(2^{-(j+1)}x)e^{-in2^jx}\widehat{u}^d_j(2^{-(j+1)}x)$$

$$= 2^{-(j+1)/2}\widehat{\phi}(2^{-(j+1)}x)e^{-in2^jx}\overline{\widehat{g}_j(2^{-j}x)}\widehat{u}(2^{-(j+1)}x), \tag{6.12}$$

which is also the L.H.S. of (6.11) by using $\widehat{\phi}(2y) = 2^{-1/2}\widehat{\phi}(y)\widehat{u}(y)$. The proof of the equality (6.10) in the $L^2(\Omega)$-sense is similar. The absolute almost surely convergence of (6.9) and (6.10) may be deduced by arguments analogous to those for the absolute almost surely convergence of (6.6). □

7 Convergence of random discrete approximations

We will also assume the following:

Assumption 6: Suppose that there are $\beta \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0, 1]$ such that, for any compact K,

$$\left| X(t) - X(s) - X^{(1)}(s)(t-s) - \ldots - X^{(\beta)}(s)\frac{(t-s)^\beta}{\beta!} \right| \le A|t-s|^{\beta+\alpha} \quad \text{for all } t \in \mathbb{R},\ s \in K, \text{ a.s.,} \tag{7.1}$$

where a random variable A depends only on K. (As usual, $f^{(k)}$ denotes the kth derivative of f.)

Note that (7.1) implies, for some random variable B,

$$|X(t) - X(s)| \le B|t-s|^\gamma \quad \text{for all } t \in \mathbb{R},\ s \in K, \quad \text{with } \gamma = 1 \wedge (\beta + \alpha). \tag{7.2}$$

Condition (7.1) in Assumption 6 is satisfied by many Gaussian stationary processes. It follows, in particular, from the two conditions:

$$X^{(\beta)} \quad \text{is } \alpha\text{-Hölder} \quad \text{a.s.} \tag{7.3}$$

and

$$|X(t)| \le C(1 + |t|)^{\beta+\alpha} \quad \text{a.s.} \tag{7.4}$$

By Theorem and a discussion on pp. 181-182 in Cramér and Leadbetter (1967), (7.3) follows from

$$\int_0^\infty x^{2\beta+2\alpha}\log(1+x)|\widehat{g}(x)|^2 dx < \infty. \tag{7.5}$$

There is also an equivalent condition in terms of the autocovariance function of a stationary Gaussian process.

Condition (7.4) is always satisfied for stationary Gaussian processes that are bounded on compact intervals, such as for those satisfying (7.3). In fact, a stronger condition holds:

$$|X(t)| \le C\sqrt{\log(2+|t|)} \quad \text{a.s.}, \tag{7.6}$$

where C is a random variable. To see this, note that the discrete-time sequence $X_k = \sup_{t\in[k,k+1)}|X(t)|$ is stationary. Moreover, by Theorem 2 in Lifshits (1995), p. 142, for some $m \in \mathbb{R}$ and $\sigma > 0$,

$$P(X_0 \ge m + \tau) \le 2(1 - \Phi(\tau/\sigma)), \quad \tau > 0,$$

where Φ is the distribution function of standard normal law. In other words, the right tail of the distribution function of X_n decays at least as fast as that of the distribution function of standard normal law. The bound (7.6) can then be obtained as (3.15) in Lemma 3 of Meyer et al. (1999).

We shall need some assumptions stronger than parts of Assumptions 3 and 5.

Assumption 3*: Suppose that, for any $J_0 \in \mathbb{Z}$,

$$\max_{k=0,1,\dots,\beta+[\alpha]+2}\ \sup_{J\ge J_0}\ \sup_{|x|\le 4\pi/3}\left|\frac{\partial^k G_J(x)}{\partial x^k}\right| < \infty. \tag{7.7}$$

Assumption 5*: Assume that, for large J,

$$|G_J(0) - 1| \le \text{const}\ 2^{-(\beta+1)J}. \tag{7.8}$$

As in Lemma 3.1, under Assumption 3*, we have

$$|2^{-j/2}\Phi_j(2^{-j}u)| \le \frac{C}{1+|u|^{\beta+[\alpha]+2}}, \quad u \in \mathbb{R}, \tag{7.9}$$

where a constant C does not depend on $j \ge j_0$, for fixed j_0.

In addition, we will suppose the following:

Assumption 7: If $\beta \ge 1$ in Assumption 6, suppose that

$$G_J^{(n)}(0) = \frac{\partial^n G_J}{\partial x^n}(0) = 0, \quad n = 1,\dots,\beta. \tag{7.10}$$

The next result establishes convergence of random discrete approximations.

Proposition 7.1 *Under Assumptions 2,3,5 of Section 3 and Assumption 6 above, we have*

$$\sup_{t\in K}|2^{J/2}X_{J,[2^Jt]} - X(t)| \le A_1\, 2^{-J\gamma} \quad \text{a.s.}, \tag{7.11}$$

where K is a compact interval and A_1 is a random variable that does not depend on J. If, in addition, Assumptions 3,5* and 7 above hold, then*

$$\sup_{t\in K}|2^{J/2}X_{J,[2^Jt]} - X([2^Jt]2^{-J})| \le A_2\, 2^{-J(\beta+\alpha)} \quad \text{a.s.}, \tag{7.12}$$

where a random variable A_2 does not depend on J.

Proof: Suppose without loss of generality that $K = [0,1]$. In view of Assumption 6, it is enough to show (7.12) or that

$$\sup_{k=0,\dots,2^J} \left| 2^{J/2} X_{J,k} - X(k2^{-J}) \right| \le A\, 2^{-J(\beta+\alpha)} \quad \text{a.s.}$$

Note by Assumption 7 and the properties of the scaling function ϕ that

$$\int_{\mathbb{R}} u^n \Phi_J(u) du = (-i)^{-n} \widehat{\Phi}_J^{(n)}(0) = (-i)^{-n} \frac{\partial^n}{\partial x^n} \Big(\overline{G_J(x)} 2^{-J/2} \widehat{\phi}(2^{-J}x) \Big) \Big|_{x=0} = 0,$$

for $n = 1, \dots, \beta$. By using Appendix A, Assumptions 5* and 6 (with (7.9)), we have

$$|2^{J/2} X_{J,k} - X(k2^{-J})| \le 2^{J/2} \int_{\mathbb{R}} \left| X(t) - X(k2^{-J}) - \ldots - X^{(\beta)}(k2^{-J}) \frac{(t-k2^{-J})^\beta}{\beta!} \right| |\Phi_J(t-k2^{-J})| dt$$

$$+ X(k2^{-J}) |G_J(0) - 1| \le A 2^{J/2} \int_{\mathbb{R}} |t - k2^{-J}|^{\beta+\alpha} |\Phi_J(t-k2^{-J})| dt + B 2^{-(\beta+1)J}$$

$$= A 2^{J/2} \int_{\mathbb{R}} |u|^{\beta+\alpha} |\Phi_J(u)| du + B 2^{-(\beta+1)J} \quad (\text{setting } u = 2^{-J}v)$$

$$= A' 2^{-J(\beta+\alpha)} \int_{\mathbb{R}} |v|^{\beta+\alpha} |2^{-J/2} \Phi_J(2^{-J}v)| dv \le A'' 2^{-J(\beta+\alpha)} \int_{\mathbb{R}} \frac{|v|^{\beta+\alpha}}{1+|v|^{\beta+[\alpha]+2}} dv = A''' 2^{-J(\beta+\alpha)}. \quad \square$$

According to Proposition 7.1, the discrete approximations $2^{J/2} X_{J,[2^J t]}$ converge to the process $X(t)$. Note also that, when $\beta \ge 1$, the convergence is faster on the dyadics than on the whole interval. An interesting question is whether the faster convergence rate $\beta + \alpha$ can be obtained on an interval for some other approximation based on $X_{J,[2^J t]}$.

For a function f, defined on either $\mathbb{R}$ or $\mathbb{Z}$, consider the operator

$$(\Delta_h^p f)(a) = \sum_{k=0}^{p} \binom{p}{k} (-1)^{p-k} f(a+kh), \quad p \in \mathbb{N},$$

where a, h are in either $\mathbb{R}$ or $\mathbb{Z}$, respectively. When $f = f_k$ is a function on $\mathbb{Z}$, we write

$$\Delta^p f_k = (\Delta_1^p f)(k) \quad \text{and} \quad \Delta^p f = (\Delta_1^p f)(0).$$

In view of the condition (7.1), to obtain the faster rate $\beta + \alpha$ on a whole interval, it is natural to try an approximation which includes the terms mimicking the β derivatives in (7.1). Thus, for $\beta \ge 1$, consider the approximations

$$\widehat{X}_{\beta,J}(t) = 2^{J/2} X_{J,[2^J t]} + 2^{J/2} \sum_{p=1}^{\beta} \frac{\Delta^p X_{J,[2^J t]}}{2^{-Jp}} \frac{(t - [2^J t]2^{-J})^p}{p!}, \tag{7.13}$$

with the idea that $2^{J/2} \Delta^p X_{J,[2^J t]} \approx X^{(p)}(t) 2^{-Jp}$ for large J. For example, when $\beta = 1$,

$$\widehat{X}_{1,J}(t) = 2^{J/2} X_{J,[2^J t]} + 2^{J/2} \frac{X_{J,[2^J t]+1} - X_{J,[2^J t]}}{2^{-J}} (t - [2^J t]2^{-J}).$$

When $\beta = 0$, we get $\widehat{X}_{0,J}(t) = 2^{J/2} X_{J,[2^J t]}$.

Although intuitive, the approximation $\widehat{X}_{\beta,J}$ in (7.13) may not converge to $X(t)$ at the faster rate $\beta+\alpha$ on compact intervals (see Remark 7.1 below). It turns out, though, that a modification of (7.13) does attain that rate. In order to build such approximation, we will make use of Lemma 7.1 below, stated without proof. Observe from (7.12) that replacing $2^{J/2}\Delta^p X_{J,[2^J t]}$ by $(\Delta^p_{2^{-J}} X)([2^J t]/2^J)$ in the approximation (7.13) makes an error of the desired faster rate $\alpha+\beta$. Lemma 7.1 shows that, after suitable correction, $(\Delta^p_{2^{-J}} X)([2^J t]/2^J)$ approximates $X^{(p)}([2^J t]/2^J)$ (and then $X^{(p)}(t)$) at the desired rate $\alpha+\beta$. This correction needs to be taken into account when considering a modification to $\widehat{X}_{\beta,J}$.

For any $x \in \mathbb{R}$, define the function s_x on $\mathbb{Z}$ by

$$s_x(k) = x + k, \quad k \in \mathbb{Z}.$$

Note that $\Delta^p s_0^j = \sum_{k=0}^p \binom{p}{k}(-1)^{p-k}k^j$, $j \in \mathbb{N}$ (recall from above that $\Delta^p s_0^j = (\Delta_1^p s_0^j)(0)$).

Lemma 7.1 *Let $\beta \in \mathbb{N}$, $\alpha \in (0,1)$ and $G \subseteq \mathbb{R}$ be an open interval. If $f : G \to \mathbb{R}$ is a Lipschitz function of order $\beta+\alpha$ in the sense of (7.1), then, for $\mathbb{N} \ni p \le \beta$, $a \in G$, we have*

$$\frac{\Delta_h^p f(a)}{h^p} - f^{(p)}(a) = \sum_{j=1}^{\beta-p} \frac{f^{(p+j)}(a)}{(p+j)!}\, h^j\, \Delta^p s_0^{p+j} + O(h^{\beta+\alpha-p}), \tag{7.14}$$

as $h \to 0$.

Define the approximation function $\widetilde{X}_{\beta,J}(t)$ by

$$\widetilde{X}_{\beta,J}(t) = \widetilde{X}_{(0),J} + \sum_{p=1}^{\beta} \frac{\widetilde{X}_{(p),J}(t)}{p!}\,(t-[2^J t]2^{-J})^p, \tag{7.15}$$

where

$$\widetilde{X}_{(0),J} := 2^{J/2} X_{J,[2^J t]}, \quad \widetilde{X}_{(\beta),J} := \frac{2^{J/2}\Delta^\beta X_{J,[2^J t]}}{2^{-J\beta}}$$

and

$$\widetilde{X}_{(p),J} := \frac{2^{J/2}\Delta^p X_{J,[2^J t]}}{2^{-J}} + \sum_{j=1}^{\beta-p} \frac{\widetilde{X}_{(p+j),J}}{(p+j)!}\, 2^{-Jj}\, \Delta^p s_0^{p+j}, \quad p = 1,2,...,\beta-1.$$

Proposition 7.2 *Under stronger assumptions of Proposition 7.1, we have*

$$\sup_{t\in K} |\widetilde{X}_{\beta,J}(t) - X(t)| \le A 2^{-J(\beta+\alpha)} \quad \text{a.s.,} \tag{7.16}$$

where K is a compact interval and A is random variable that does not depend on J.

Proof: If the relation

$$\widetilde{X}_{(p),J} - X^{(p)}([2^J t]2^{-J}) = O(2^{-J(\beta+\alpha-p)}) \tag{7.17}$$

holds for $p = 0,1,2,...,\beta$, then, by Assumption 6,

$$|\widetilde{X}_{\beta,J}(t) - X(t)| \le \left| \widetilde{X}_{\beta,J}(t) - X([2^J t]2^{-J}) - \sum_{p=1}^{\beta} \frac{X^{(p)}([2^J t]2^{-J})}{p!}\,(t-[2^J t]2^{-J})^p \right|$$

$$+\left|X(t)-X([2^Jt]2^{-J})-\sum_{p=1}^{\beta}\frac{X^{(p)}([2^Jt]2^{-J})}{p!}\,(t-[2^Jt]2^{-J})^p\right| = O(2^{-J(\beta+\alpha)}),$$

which proves (7.16).

Relation (7.17) holds for $p = 0$ by Proposition 7.1. To show (7.17) for $\beta \geq 1$, we argue by backward induction. For $p = \beta$, by Proposition 7.1 and Lemma 7.1, we have

$$|\widetilde{X}_{(\beta),J} - X^{(\beta)}([2^Jt]2^{-J})| \leq \left|\frac{2^{J/2}\Delta^\beta X_{J,[2^Jt]}}{2^{-J\beta}} - \frac{\Delta^\beta X([2^Jt]2^{-J})}{2^{-J\beta}}\right|$$

$$+\left|\frac{\Delta^\beta X([2^Jt]2^{-J})}{2^{-J\beta}} - X^{(\beta)}([2^Jt]2^{-J})\right| = O(2^{-J(\beta+\alpha-\beta)}).$$

Assume by induction that (7.17) holds for $p+1, ..., \beta-1, \beta$ (with $p \geq 1$). Then, by Proposition 7.1 and Lemma 7.1, we obtain that

$$|\widetilde{X}_{(p),J} - X^{(p)}([2^Jt]2^{-J})| \leq \left|\frac{2^{J/2}\Delta^p X_{J,[2^Jt]}}{2^{-Jp}} - \frac{\Delta^p X([2^Jt]2^{-J})}{2^{-Jp}}\right|$$

$$+\left|\frac{\Delta^p X([2^Jt]2^{-J})}{2^{-Jp}} - X^{(p)}([2^Jt]2^{-J}) - \sum_{j=1}^{\beta-p}\frac{X^{(p+j)}([2^Jt]2^{-J})}{(p+j)!}2^{-Jj}\Delta^p s_0^{p+j}\right|$$

$$+\left|\sum_{j=1}^{\beta-p}\frac{X^{(p+j)}([2^Jt]2^{-J})}{(p+j)!}2^{-Jj}\Delta^p s_0^{p+j} - \sum_{j=1}^{\beta-p}\frac{\widetilde{X}_{(p+j),J}}{(p+j)!}2^{-Jj}\Delta^p s_0^{p+j}\right| = O(2^{-J(\beta+\alpha-p)}). \quad \square$$

Remark 7.1 When $\beta = 2$, the approximation $\widetilde{X}_{\beta,J}$ becomes

$$\widetilde{X}_{2,J} = 2^{J/2}X_{J,[2^Jt]} + 2^{J/2}\left(\frac{X_{J,[2^Jt]+1} - X_{J,[2^Jt]}}{2^{-J}} + \frac{X_{J,[2^Jt]+2} - 2X_{J,[2^Jt]+1} + X_{J,[2^Jt]}}{2\,2^{-J}}\right)(t-[2^Jt]2^{-J})$$

$$+2^{J/2}\frac{X_{J,[2^Jt]+2} - 2X_{J,[2^Jt]+1} + X_{J,[2^Jt]}}{(2^{-J})^2}\,(t-[2^Jt]2^{-J})^2. \tag{7.18}$$

Compare (7.18) with the approximations $\widehat{X}_{2,J}$ given in (7.13). Observe that, if $\widehat{X}_{2,J}$ also converges to X at the rate $2+\alpha$, then

$$O(2^{-J(2+\alpha)}) = \widetilde{X}_{2,J} - \widehat{X}_{2,J}$$
$$= 2^{J/2}\frac{X_{J,[2^Jt]+2} - 2X_{J,[2^Jt]+1} + X_{J,[2^Jt]}}{2\,2^{-J}}\,(t-[2^Jt]2^{-J})$$

or, by using (7.12) in Proposition 7.1,

$$O(2^{-J(2+\alpha)}) = \frac{X(([2^Jt]+2)2^{-J}) - 2X(([2^Jt]+1)2^{-J}) + X([2^Jt]2^{-J})}{2^{-J}}\,(t-[2^Jt]2^{-J})$$

or, by using Taylor expansions,

$$O(2^{-J(2+\alpha)}) = (2X''(t_1) - X''(t_2))2^{-J}\,(t-[2^Jt]2^{-J}),$$

with $t_1 = t_1(J)$ and $t_2 = t_2(J)$ that are close to t. The last relation may not be satisfied under our assumptions, showing that one cannot expect $\widehat{X}_{2,J}$ to converge to X at the rate $2+\alpha$.

Although the approximations $\widetilde{X}_{\beta,J}$ converge to X at the faster rate $\beta+\alpha$, these approximations do not necessarily have continuous paths. Indeed, it can be easily verified that $\widetilde{X}_{\beta,J}$ is continuous when $\beta = 1, 2$ but not so when $\beta = 3$. For a fixed $\beta \geq 2$, it may be desirable to have not only a continuous but also a $C^{\beta-1}$ approximation $\overline{X}_{\beta,J}$. Moreover, in analogy to (7.15), in order to have the faster convergence, we would expect the p-th derivative of the approximation $\overline{X}_{\beta,J}$ at $[2^J t]2^{-J}$ to approximate the p-th derivative of the process X at t.

We generally found such $C^{\beta-1}$ approximations difficult to construct. One difficulty is the following. As in (7.15), we may seek an approximation $\overline{X}_{\beta,J}$ which is a polynomial of order β on an interval $([2^J t]2^{-J}, [2^J t]2^{-J} + 1)$. Since $\overline{X}_{\beta,J}$ is globally $C^{\beta-1}$, we would require its derivatives $\overline{X}^{p}_{\beta,J}$, $p = 0, 1, ..., \beta-1$, to be equal to prescribed values at the endpoints $[2^J t]2^{-J}$ and $[2^J t]2^{-J}+1$. Requiring this yields 2β equations that a polynomial $\overline{X}_{\beta,J}$ must satisfy. Since a polynomial of order β has only $\beta + 1$ coefficients, this is not possible in general. Despite this difficulty, we have found the following general scheme to yield $C^{\beta-1}$ approximations, at least for the first several values of $\beta \geq 2$.

To construct a C^1 approximation $\overline{X}_{2,J}$, we could require first that its derivative

$$
\begin{aligned}
2^{-J/2}\overline{X}^{(1)}_{2,J}(t) &= 2^{-J/2}\widehat{X}_{1,J}(t) \text{ based on the sequence } \frac{\Delta X_{J,[2^J t]}}{2^{-J}} \\
&= \frac{\Delta X_{J,[2^J t]}}{2^{-J}} + \frac{1}{2^{-J}}\left(\frac{\Delta X_{J,[2^J t]+1}}{2^{-J}} - \frac{\Delta X_{J,[2^J t]}}{2^{-J}}\right)(t - [2^J t]2^{-J}) \\
&= \frac{\Delta X_{J,[2^J t]}}{2^{-J}} + \frac{\Delta^2 X_{J,[2^J t]}}{(2^{-J})^2}(t - [2^J t]2^{-J}). \qquad (7.19)
\end{aligned}
$$

Observe that, by construction using continuous approximation $\widehat{X}_{1,J}$, $\overline{X}^{(1)}_{2,J}$ is continuous. Moreover, $\overline{X}^{(1)}_{2,J}$ approximates $X^{(1)}(t)$, and $\overline{X}^{(2)}_{2,J}$ on the interval $([2^J t]2^{-J}, [2^J t]2^{-J} + 1)$ approximates $X^{(2)}(t)$. Integrating (7.19) and requiring it to be continuous yields the following approximation

$$
\begin{aligned}
2^{-J/2}\overline{X}_{2,J}(t) = \frac{X_{J,[2^J t]+1} + X_{J,[2^J t]}}{2} &+ \frac{\Delta X_{J,[2^J t]}}{2^{-J}}(t - [2^J t]2^{-J}) \\
&+ \frac{\Delta^2 X_{J,[2^J t]}}{(2^{-J})^2}(t - [2^J t]2^{-J})^2. \qquad (7.20)
\end{aligned}
$$

Note that $\overline{X}_{2,J}$ differs from $\widehat{X}_{2,J}$ by the constant term.

Similarly, to construct a C^2 approximation $\overline{X}_{3,J}$, we could require that

$$
\overline{X}^{(1)}_{3,J}(t) = \overline{X}_{2,J}(t) \text{ based on the sequence } \frac{\Delta X_{J,[2^J t]}}{2^{-J}}.
$$

Integrating the resulting expression and requiring it to be continuous yields

$$
\begin{aligned}
\overline{X}_{3,J}(t) = \frac{1}{6}X_{J,[2^J t]+2} + \frac{4}{6}X_{J,[2^J t]+1} + \frac{1}{6}X_{J,[2^J t]} + \frac{1}{2}\frac{\Delta_2 X_{J,[2^J t]}}{2^{-J}}(t - [2^J t]2^{-J}) \\
+ \frac{1}{2}\frac{\Delta^2 X_{J,[2^J t]}}{(2^{-J})^2}(t - [2^J t]2^{-J})^2 + \frac{1}{6}\frac{\Delta^3 X_{J,[2^J t]}}{(2^{-J})^3}(t - [2^J t]2^{-J})^3 \qquad (7.21)
\end{aligned}
$$

(the subindex 2 in $\Delta_2 X_{J,[2^J t]}$ is not a typo). The approximation (7.21) is C^2 and its derivatives of orders $p = 0, 1, 2, 3$ approximate those of the process X.

We expect that the above scheme yields $C^{\beta-1}$ approximations $\overline{X}_{\beta,J}$ for any $\beta \geq 2$. However, as explained in Remark 7.2 below, we cannot expect these approximations to convergence at the faster rate $\beta+\alpha$. This is perhaps not surprising, because the discontinuous approximations in (7.15) are already nontrivial.

Remark 7.2 One cannot expect the approximation $\overline{X}_{2,J}$ in (7.21) to converge to X at the faster rate $2+\alpha$. Indeed, if this rate were achieved, we would have (see (7.18))

$$O(2^{-J(2+\alpha)}) = \widetilde{X}_{2,J}(t) - \overline{X}_{2,J}(t)$$

$$= X_{J,[2^Jt]+1} - X_{J,[2^Jt]} - \frac{X_{J,[2^Jt]+2} - 2X_{J,[2^Jt]+1} + X_{J,[2^Jt]}}{2\, 2^{-J}}\,(t-[2^Jt]2^{-J})$$

or, by using (7.11),

$$O(2^{-J(2+\alpha)}) = X(([2^Jt]+1)2^{-J}) - X([2^Jt]2^{-J})$$

$$-\frac{X(([2^Jt]+2)2^{-J}) - 2X(([2^Jt]+1)2^{-J}) + X([2^Jt]2^{-J})}{2^{-J}}\,(t-[2^Jt]2^{-J})$$

or, by Taylor expansions,

$$O(2^{-J(2+\alpha)}) = X'([2^Jt]2^{-J})2^{-J} + \frac{1}{2}X''(t_1)2^{-2J} - X''(t_2)2^{-J}\,(t-[2^Jt]2^{-J}),$$

with $t_1 = t_1(J)$ and $t_2 = t_2(J)$ close to t, or, by expanding $X'([2^Jt]2^{-J})$ further,

$$O(2^{-J(2+\alpha)}) = X'(t)2^{-J} + \frac{1}{2}X''(t_1)2^{-2J} - X''(t_2)2^{-J}\,(t-[2^Jt]2^{-J}).$$

This relation may not be satisfied under our assumptions on X.

8 Simulation: the case of the OU process

We will illustrate here how the results of Sections 6 and 7 can be used to simulate a stationary process X. We consider only the case of the OU process in Example 4.1. Recall from that example that the discrete approximations taken for the OU process are

$$\widehat{g}_J(x) = 2^{-J/2}\sigma\sqrt{\frac{1-e^{-2\lambda 2^{-J}}}{2\lambda}}\,(1-e^{-\lambda 2^{-J}}e^{-ix})^{-1} \tag{8.1}$$

and the corresponding discrete random approximations X_J are suitable AR(1) time series in (4.2). With the choice (8.1) of approximations, observe that the filters u_j and v_j used in reconstruction (6.1) become

$$\widehat{u}_j(x) = \frac{2^{-1/2}}{\sqrt{1+e^{-2\lambda 2^{-(j+1)}}}}(1+e^{-\lambda 2^{-(j+1)}}e^{-ix})\widehat{u}(x), \tag{8.2}$$

$$\widehat{v}_j(x) = 2^{-(j+1)/2}\sigma\sqrt{\frac{1-e^{-2\lambda 2^{-(j+1)}}}{2\lambda}}\,(1-e^{-\lambda 2^{-(j+1)}}e^{-ix})^{-1}\widehat{v}(x). \tag{8.3}$$

Suppose one wants to simulate the OU process on the interval $[0,1]$. The idea is to begin by generating a discrete approximation X_0 at scale 2^0. This step is easy as X_0 is an AR(1) time

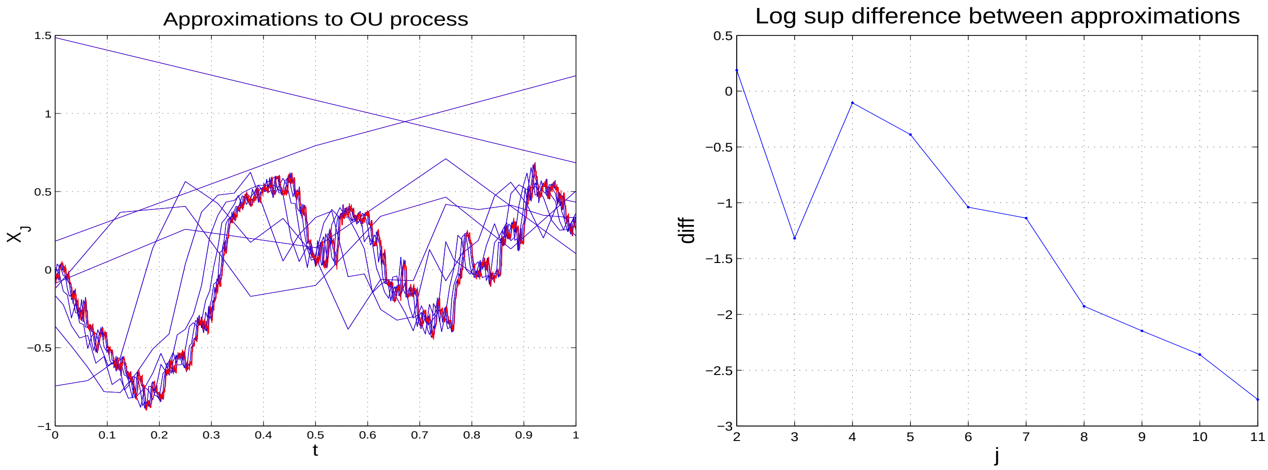

Figure 1: Approximations X_J and the logarithms of their sup differences.

series. Then, substituting X_0 into (6.1), one may get the approximation X_1, and continuing recursively from X_1 now, the approximation X_J for arbitrary fixed $J \geq 1$. Note that applying (6.1) recursively each time essentially involves just simulating independent $\mathcal{N}(0,1)$ random variables and computing filters u_j and v_j. Proposition 7.1 ensures that the properly normalized X_J approximate the OU process uniformly over $[0,1]$ and exponentially fast in J.

We illustrate this in Figure 1 for the OU process with $\lambda = 1$, $\sigma = 1$. The plot on the left depicts the consecutive approximations X_j from X_0 at scale 2^0 to X_J at the finest scale 2^{-J} with $J = 11$. In the right plot, we present the sup-differences between consecutive approximations X_{j-1} and X_j, $j = 2, \dots, 11$, on the log scale. The decay in that plot confirms that normalized approximations X_J converge to the OU process exponentially fast in J.

Several comments should be made on how approximations X_j are obtained in Figure 1. Though theoretically unjustified, we use not Meyer but the celebrated Daubechies CMFs with $N = 8$ zero moments. The advantage of these CMFs is that they have finite length (equal to $2N$). In particular, the filters u_j in (8.3) are then also finite (of length $2N + 2$) for any j. The filters v_j, however, are not finite and are truncated in practice, disregarding those elements that are smaller than a prescribed level $\delta = 10^{-10}$. Let us also note that applications of (6.1) involve more elements of X_j than those plotted in Figure 1. This is achieved by taking the initial approximation X_0 of suitable length. Some indication on how this is done, can be seen from the analogous simulation of fractional Brownian motion in Pipiras (2005).

Finally, let us indicate another interesting feature of the above simulation. Focus on the filters v_j defined by (8.3). They have infinite length and are truncated in practice. It may seem from the definition (8.3) that v_j have to be taken of very long length as j increases because the elements of the filter

$$(1 - e^{-\lambda 2^{-(j+1)}} e^{-ix})^{-1} = \sum_{k=0}^{\infty} e^{-\lambda 2^{-(j+1)} k} e^{-ixk}$$

decay extremely slowly for larger j. In fact, the opposite turns out to be true. As j increases, the filters v_j can essentially be taken of finite length $2N - 2$, and things get even better for larger j in a way!

To explain why this happens, recall (e.g. Mallat (1998), Theorem 7.4) that N zero moments translates into the factorization

$$\widehat{v}(x) = (1 - e^{-ix})^N \widehat{v}_{0,N}(x), \tag{8.4}$$

where, in the case of Daubechies CMF v, the filters $v_{0,N}$ have also finite length. An explanation follows by observing that

$$\frac{1 - e^{-ix}}{1 - e^{-\lambda 2^{-(j+1)}} e^{-ix}} = \sum_{k=0}^{\infty} a_k^{(j)} e^{-ixk} \to 1,$$

or $a_0^{(j)} \to 1$, $a_0^{(k)} \to 0$, $k \geq 1$, as $j \to \infty$. More precisely,

$$\frac{1 - e^{-ix}}{1 - e^{-\lambda 2^{-(j+1)}} e^{-ix}} - 1 = \frac{-e^{-ix}(1 - e^{-\lambda 2^{-(j+1)}})}{1 - e^{-\lambda 2^{-(j+1)}} e^{-ix}} = -(1 - e^{-\lambda 2^{-(j+1)}}) \sum_{k=1}^{\infty} e^{-\lambda 2^{-(j+1)}(k-1)} e^{-ixk},$$

so that the elements $a_k^{(j)}$, $k \geq 1$, are bounded by $1 - e^{-\lambda 2^{-(j+1)}} \leq \lambda 2^{-(j+1)} \to 0$, as $j \to \infty$.

A On the integration of stationary Gaussian processes

Let $\{X(t)\}_{t \in \mathbb{R}}$ be a Gaussian stationary process given by (1.1). We define here the integral

$$\int_{\mathbb{R}} X(t) f(t) dt, \tag{A.1}$$

for suitable functions f and state its properties as used throughout the paper. No proofs will be given as most of them are standard. Our strategy will be to define (A.1) both pathwise and as an $L^2(\Omega)$ limit and to show that the two definitions coincide in relevant cases. In the pathwise case, the integral (A.1) will be denoted by $\mathcal{I}_\omega(f)$ (i.e. defined ω-wise), and, in the $L^2(\Omega)$ case, it will be denoted by $\mathcal{I}_2(f)$.

For simplicity, we assume that the sample paths of X are continuous. Path continuity is not a stringent assumption since, by Belayev's alternative (Belayev (1960)), either the sample paths of a Gaussian stationary process are continuous or very badly-behaved in the sense of possessing discontinuities of the second type.

Assume first that $f(t) = \sum_{i=1}^n f_i 1_{[a_i, b_i)}(t)$ is a step function. For such function, the stochastic integral (A.1) may be defined pathwise as the ordinary Riemann integral

$$\mathcal{I}_\omega\Big(\sum_{i=1}^n f_i 1_{[a_i,b_i]}\Big) = \sum_{i=1}^n \int_{a_i}^{b_i} X(t) dt. \tag{A.2}$$

Lemma A.1 *The integral (A.2) has the following properties: for step functions f, f_1 and f_2, and with the notation $\mathcal{I}(f) = \mathcal{I}_\omega(f)$:*

(P1) $\mathcal{I}(f)$ is a Gaussian random variable with mean zero.

(P2) The following moment formulae hold:

$$E\mathcal{I}(f)^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{g}(x)|^2 |\widehat{f}(x)|^2 dx; \tag{A.3}$$

$$E\Big[\mathcal{I}(f_1)\mathcal{I}(f_2)\Big] = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{g}(x)|^2 \widehat{f}_1(x) \overline{\widehat{f}_2(x)} dx; \tag{A.4}$$

$$E\Big[\mathcal{I}(f)X(t)\Big] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itx} |\widehat{g}(x)|^2 \widehat{f}(x) dx. \tag{A.5}$$

(P3) For real c and d, $\mathcal{I}(cf_1 + df_2) = c\mathcal{I}(f_1) + d\mathcal{I}(f_2)$.

An extension of the integral (A.1) to more general functions f can be achieved by an argument of approximation in $L^2(\Omega)$. Consider the space of deterministic functions

$$L_g^2 := \Big\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |\widehat{f}(x)|^2 |\widehat{g}(x)|^2 dx < \infty \Big\} \tag{A.6}$$

with the inner product

$$\langle f_1, f_2 \rangle_{L_g^2} := \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f_1}(x) \overline{\widehat{f_2}(x)} |\widehat{g}(x)|^2 dx. \tag{A.7}$$

Denote also

$$\mathcal{I}_X^s = \Big\{ \mathcal{I}_\omega(f) : \; f \text{ is a step function} \Big\}, \tag{A.8}$$

equipped with the ordinary $L^2(\Omega)$ inner product

$$E\mathcal{I}_\omega(f_1)\mathcal{I}_\omega(f_2). \tag{A.9}$$

The space $\mathcal{I}_X^s$ and the restriction of L_g^2 to step functions are isometric since, for elementary functions f_1 and f_2,

$$E\mathcal{I}_\omega(f_1)\mathcal{I}_\omega(f_2) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f_1}(x) \overline{\widehat{f_2}(x)} |\widehat{g}(x)|^2 dx = \langle f_1, f_2 \rangle_{L_g^2}. \tag{A.10}$$

Thus, a natural way to define the integral $\mathcal{I}_2$ for a given $f \in L_g^2$ is to take a sequence of step functions l_n that approximate f in the L_g^2 norm, and set $\mathcal{I}_2(f)$ as the corresponding $L^2(\Omega)$ limit of $\mathcal{I}_\omega(l_n)$. The following result can be proved as Lemma 5.1 in Pipiras and Taqqu (2000).

Lemma A.2 *For every function $f \in L_g^2(\mathbb{R})$, there is a sequence $\{l_n\}$ of step functions such that $\|f - l_n\|_{L_g^2} \longrightarrow 0$.*

Given $f \in L_g^2$, we may use Lemma A.2 to define (A.1) as

$$\mathcal{I}_2(f) = \lim(L^2(\Omega))\mathcal{I}_\omega(l_n), \tag{A.11}$$

where $\{l_n\}$ is a sequence of step functions such that $\|f - l_n\|_{L_g^2} \to 0$. This definition does not depend on the approximating sequence of f. The integral $\mathcal{I}_2(f)$ has the following properties.

Theorem A.1 *The map $\mathcal{I}_2 : f \longrightarrow \mathcal{I}_2(f)$ defined by (A.11) is an isometry between the spaces L_g^2 and $\mathcal{I}_X = \{\mathcal{I}_2(f) : f \in L_g^2\}$. Moreover, $\mathcal{I}_2(f) = \mathcal{I}_\omega(f)$ a.s. for step functions f, and the integral $\mathcal{I}_2(f)$ satisfies the properties (P1), (P2) and (P3) of Lemma A.1 with $\mathcal{I}(f) = \mathcal{I}_2(f)$ and $f, f_1, f_2 \in L_g^2$.*

It is possible to define (A.1) also pathwise for more general integrand functions. As discussed in Section 7, for a Gaussian stationary process $\{X(t)\}_{t\in\mathbb{R}}$, we have, almost surely,

$$|X(t)| \le C\sqrt{\log(2 + |t|)}, \quad t \in \mathbb{R}, \tag{A.12}$$

where C is a random variable. Consider the space

$$\mathcal{L} := \{ f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} \sqrt{\log(2 + |t|)} |f(t)| dt < \infty \}. \tag{A.13}$$

For $f \in \mathcal{L}$, in view of (A.12) we may define

$$\mathcal{I}_\omega(f) = \int_{\mathbb{R}} X(t) f(t) dt$$

pathwise as an improper Riemann integral. One can show that $\mathcal{I}_2(f) = \mathcal{I}_\omega(f)$ a.s. for $f \in \mathcal{L} \cap L_g^2$.

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