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GEGENBAUER TRANSFORMS VIA THE RADON TRANSFORM

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ABSTRACT

Use is made of the Radon transform on even dimensional spaces and Gegenbauer functions of the second kind to obtain a general Gegenbauer transform pair. In the two-dimensional limit the pair reduces to a Tchebycheff transform pair.

1. Introduction.

Gegenbauer polynomials of the first kind appear in a natural way when studying the Radon transform of functions which have certain spherical symmetry. We shall make use of this property of the Radon transform to obtain a new Gegenbauer transform pair. Although the final result does not contain Gegenbauer functions of the second kind, these functions are important in the derivation and their use here supplements the informative recent study of these particular special functions by Durand [1] and by Durand, Fishbane, and Simmons [2].

The work which follows serves a threefold purpose. First, we are able to demonstrate an important use of the Radon transform as a tool. Second, more insight is obtained regarding the use of Gegenbauer functions of the second kind. Finally, we are able to derive a set of equations which constitute a useful Gegenbauer transform pair which has a fundamental connection to the dimensionality of the space $\mathbb{R}^n$.

2. The Radon Transform.

Let $x = (x_1, x_2, \cdots, x_n)$ be a point in $\mathbb{R}^n$ $(n \geq 2)$ and let $F \in \mathcal{S}$ be a function of the n real variables $x_1, x_2, \cdots, x_n$. The properties of the space $\mathcal{S}$, which consists of all rapidly decreasing C^∞ functions on $\mathbb{R}^n$, are developed by Schwartz [3]. The reason for working in a space with such nice properties will be clear when it becomes necessary to make changes in the order of integration and to perform repeated integrations by parts.

Given $F \in \mathcal{S}$, the Radon transform of F is given by [4],

(1) $$f(\xi,p) = \int F(x)\ \delta(p - \xi\cdot x)\ dx\ ,$$

where p is real, ξ is an arbitrary unit vector in $\mathbb{R}^n$, $\xi\cdot x = \sum_{\kappa=1}^{n} \xi_\kappa x_\kappa$,

δ is the Dirac delta function, $dx = dx_1\, dx_2 \cdots dx_n$, and the integral is over the entire space. It is important to observe that the symmetry condition

(2) $$f(\xi,-p) = f(-\xi,p)$$

follows directly from the definition (1).

Following the initial work by Radon [5], many of the technical properties of the Radon transform were worked out by several authors [4, 6-10]. Among other things these authors develop a formal expression for inversion of the transform, valid for functions in $\mathcal{S}$, and it turns out that the inversion formula for even n is considerably more complicated than the formula for odd n. There is a Hilbert transform associated with the even case which remains unevaluated for the most general functions. Our concern here is with this even n case exclusively and involves defining F in such a fashion that it is possible to perform the Hilbert transform.

3. Decomposition of F

We consider those functions F which may be decomposed either as

(3) $$F(x) = G_\ell(r)\, S_{\ell m}(\hat{x}) \ ,$$

or as a linear combination of terms of this form. Here, $x \in \mathbb{R}^n$, $r = |x|$, $\hat{x} = x/r$, and the doubly subscripted $S_{\ell m}(\hat{x})$ is a real spherical (or surface) harmonic [11,12] of degree ℓ which comes from an orthonormal set with $N(n,\ell)$ members. That is, members of the set $\{S_{\ell 1}(\hat{x}), S_{\ell 2}(\hat{x}), \cdots , S_{\ell N(n,\ell)}(\hat{x})\}$ satisfy

(4) $$\int_\Omega S_{\ell m}(\hat{x})\, S_{\ell' m'}(\hat{x})\, d\hat{x} = \delta_{\ell \ell'}\, \delta_{m m'}$$

where $d\hat{x}$ is understood to be the surface element in hyperspherical polar coordinates, and $\int_\Omega$ designates an integral over the unit sphere. A very useful property of the $S_{\ell m}(\hat{x})$ is that they satisfy the symmetry condition

$$(5) \qquad S_{\ell m}(-\hat{x}) = (-1)^{\ell} S_{\ell m}(\hat{x}) .$$

Further properties, including an explicit expression for $N(n,\ell)$, can be found in Hochstadt [12].

4. Radon Transform of the Decomposed Function

When the Radon transform (1) is applied to (3) the result is

$$(6) \qquad f(\xi,p) = \int G_{\ell}(r)\, S_{\ell m}(\hat{x})\, \delta(p - \xi \cdot x)\, dx .$$

Without loss of generality we may assume that $p \geq 0$ since one may always calculate $f(\xi,-p)$ from (2). If we convert (6) to spherical coordinates $(dx \rightarrow r^{n-1}\, dr\, d\hat{x})$ and observe that $\delta(p - r\xi\cdot\hat{x}) = \delta(\frac{p}{\xi\cdot\hat{x}} - r)/|\xi\cdot\hat{x}|$ for purposes of doing the r integration we obtain

$$(7) \qquad f(\xi,p) = \int_\Omega S_{\ell m}(\hat{x}) \left(\frac{p}{\xi\cdot\hat{x}}\right)^{n-1} G_{\ell}(p/\xi\cdot\hat{x}) \frac{d\hat{x}}{|\xi\cdot\hat{x}|} .$$

Application of the Hecke-Funk theorem [12] yields

$$(8) \qquad f(\xi,p) = \frac{\omega_{n-1}\, S_{\ell m}(\xi)}{C_{\ell}^{\nu}(1)} \int_0^1 \left(\frac{p}{t}\right)^{n-1} G_{\ell}\left(\frac{p}{t}\right) C_{\ell}^{\nu}(t)\, (1-t^2)^{\nu-\frac{1}{2}}\, \frac{dt}{t} ,$$

where $C_{\ell}^{\nu}(t)$ is a Gegenbauer polynomial of the first kind, ω_n is the surface area of a unit sphere in $\mathbb{R}^n$, and $\nu = \frac{1}{2}(n-2)$. The ratio $\omega_{n-1}/C_{\ell}^{\nu}(1)$

may be written in terms of Gamma functions,

$$M_\ell^\nu = \omega_{n-1} / C_\ell^\nu(1) = \frac{(4\pi)^\nu \, \Gamma(\ell+1) \, \Gamma(\nu)}{\Gamma(\ell+2\nu)} \quad . \tag{9}$$

Equation (8) may be converted to the desired form by making the change of variables $r = p/t$,

$$f(\xi,p) = M_\ell^\nu \, S_{\ell m}(\xi) \int_p^\infty r^{2\nu} \, G_\ell(r) \, C_\ell^\nu(\tfrac{p}{r}) \, [1 - (\tfrac{p}{r})^2]^{\nu-\frac{1}{2}} \, dr \quad . \tag{10}$$

It will be especially useful to write this equation as

$$f(\xi,p) = g_\ell(p) \, S_{\ell m}(\xi), \tag{11}$$

where

$$g_\ell(p) = M_\ell^\nu \int_p^\infty r^{2\nu} \, G_\ell(r) \, C_\ell^\nu(\tfrac{p}{r}) \, [1 - (\tfrac{p}{r})^2]^{\nu-\frac{1}{2}} \, dr \; . \tag{12}$$

The symmetry conditions on f and $S_{\ell m}$ yield the defining equation for $g_\ell(-p)$,

$$g_\ell(-p) = (-1)^\ell \, g_\ell(p) \; . \tag{13}$$

5. The Inversion

We now turn our attention to inverting the Radon transform when F is given by (3) and f is given by (11). The inversion may be written as an integration over a unit sphere in ξ space [9]

$$F(x) = \int_\Omega f^*(\xi, \xi\cdot x) \, d\hat{\xi} \; , \tag{14}$$

where

(15) $$f^{*}(\xi,\xi\cdot x) = \mathrm{T}\, f(\xi,p) \ .$$

(Keep in mind that ξ is already a unit vector. We have used the notation $\hat{\xi}$ in (14) to emphasize that the integration is over the unit sphere.) For even n the operator T is defined by

(16) $$f^{*}(\xi,t) = \frac{(-1)^{n/2}}{2(2\pi)^{n-1}} \ \mathrm{H}\{(\frac{\partial}{\partial p})^{n-1} f(\xi,p)\} \ ,$$

and H designates the Hilbert transform

(17) $$\mathrm{H}\{q(p)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{q(p)}{p-t}\, dp \ .$$

After inserting the decompositions for F and f in (14) we have

(18) $$G_{\ell}(r)\, S_{\ell m}(\hat{x}) = \int_{\Omega} S_{\ell m}(\xi)\, g_{\ell}^{*}(r\xi\cdot\hat{x})\, d\hat{\xi}$$

$$= M_{\ell}^{\nu}\, S_{\ell m}(\hat{x}) \int_{-1}^{1} g_{\ell}^{*}(rt)\, C_{\ell}^{\nu}(t)\, (1-t^{2})^{\nu-\frac{1}{2}}\, dt \ ,$$

where the second step was obtained by applying the Hecke-Funk theorem again [12].

By inspection of (18) it is clear that

(19) $$G_{\ell}(r) = M_{\ell}^{\nu} \int_{-1}^{1} g_{\ell}^{*}(rt)\, C_{\ell}^{\nu}(t)\, (1-t^{2})^{\nu-\frac{1}{2}}\, dt$$

and g_{ℓ}^{*} must be calculated from $g_{\ell}^{*} = \mathrm{T}\, g_{\ell}$.

6. Improvement on the Inversion Formula

For even n $(2,4,6,\cdots)$ and $\nu = \tfrac{1}{2}(n-2)$ it is possible to modify (19) considerably by actually doing the t integration. Explicitly, $g_\ell^*(rt)$ is given by

$$g_\ell^*(rt) = \frac{(-1)^{n/2}}{2(2\pi)^{n-1}} \frac{1}{\pi} \int_{-\infty}^{\infty} g_\ell^{(n-1)}(p)\ (p - rt)^{-1}\ dp \quad , \tag{20}$$

where $g_\ell^{(n-1)}(p) = \left(\frac{d}{dp}\right)^{n-1} g_\ell(p)$. If (20) is substituted into (19) and the order of integration reversed, the equation for $G_\ell(r)$ becomes

$$G_\ell(r) = \frac{(-1)^{n/2}}{2(2\pi)^{n-1}} \frac{M_\ell^\nu}{\pi r} \int_{-\infty}^{\infty} g_\ell^{(n-1)}(p)\ I_\ell^\nu(\tfrac{p}{r})\ dp \quad , \tag{21}$$

where

$$I_\ell^\nu(\tfrac{p}{r}) = \int_{-1}^{1} C_\ell^\nu(t)\ (\tfrac{p}{r} - t)^{-1}\ (1 - t^2)^{\nu-\frac{1}{2}}\ dt \quad . \tag{22}$$

At this point it is clearly desirable to require that $r > 0$. The $r = 0$ case may be done separately starting with (19). The integration in (22) may be taken over four separate regions, $\int_{-\infty}^{-r} + \int_{-r}^{0} + \int_{0}^{r} + \int_{r}^{\infty}$. If we observe that $I_\ell^\nu(-\tfrac{p}{r}) = (-1)^{\ell+1}\ I_\ell^\nu(\tfrac{p}{r})$ then by a change of variable $p \to -p$ over the negative p region in (21) it follows that

$$G_\ell(r) = \frac{(-1)^{n/2}}{(2\pi)^{n-1}} \frac{M_\ell^\nu}{\pi r} \left\{ \int_{0}^{r} g_\ell^{(n-1)}(p)\ I_\ell^\nu(\tfrac{p}{r})\ dp + \int_{r}^{\infty} g_\ell^{(n-1)}(p)\ I_\ell^\nu(\tfrac{p}{r})\ dp \right\} . \tag{23}$$

The reason for writing $G_\ell(r)$ in this form is to enable us to evaluate the I_ℓ^ν integrals.

Notice that in (23) the $\int_0^r$ integral forces $\frac{p}{r} \leq 1$ and the $\int_r^\infty$ integral forces $\frac{p}{r} \geq 1$. For convenience we momentarily designate

$$\frac{p}{r} = x \quad \text{if} \quad \frac{p}{r} < 1 \quad ,$$

$$\frac{p}{r} = z \quad \text{if} \quad \frac{p}{r} > 1 \quad .$$

(Unlike prior usage of x , here x is a real variable rather than a vector.) This establishes contact with the usage of x and z in the Appendix and in [1,2] where the Gegenbauer functions of the second kind D_λ^α are discussed.

From (A-1) and (A-3) we immediately obtain

$$(24) \qquad I_\ell^\nu(x) = \pi \, (1 - x^2)^{\nu-\frac{1}{2}} \, D_\ell^\nu(x)$$

and

$$(25) \qquad I_\ell^\nu(z) = 2\pi \, e^{-i\pi\nu} \, (z^2 - 1)^{\nu-\frac{1}{2}} \, D_\ell^\nu(z) \; .$$

These results, combined with (23) give

$$(26) \qquad G_\ell(r) = \frac{M_\ell^\nu}{(2\pi)^{n-1} \, r} \Big\{ (-1)^{n/2} \int_0^r g_\ell^{(n-1)}(p) \, D_\ell^\nu(x) \, (1 - x^2)^{\nu-\frac{1}{2}} \, dp$$

$$-2 \int_r^\infty g_\ell^{(n-1)}(p) \, D_\ell^\nu(z) \, (z^2 - 1)^{\nu-\frac{1}{2}} \, dp \Big\} \; .$$

By use of (A-4) and (A-14) the two integrals may be combined, and after some simplification we find

(27) $$G_\ell(r) = \frac{\Gamma(\ell+1)\ 2^{-\nu}\ \pi^{-\nu-1}}{\Gamma(\ell+2\nu)\ r} \int_0^\infty g_\ell^{(n-1)}(p)\ E^\nu_{\ell+2\nu-1}(\tfrac{p}{r})\ dp$$

$$- \frac{\Gamma(\ell+1)\ \Gamma(\nu)}{2\pi^{\nu+1}\ \Gamma(\ell+2\nu)\ r} \int_r^\infty g_\ell^{(n-1)}(p)\ C^\nu_\ell(\tfrac{p}{r})\ [(\tfrac{p}{r})^2 - 1]^{\nu-\frac{1}{2}}\ dp\ .$$

The $\int_0^\infty$ integral can be shown to vanish. To see this, first perform $n-1$ integrations by parts to obtain an integral of the form

$$\int_0^\infty g_\ell(p)\ Q_{\ell-2}(\tfrac{p}{r})\ dp\ ,$$

where $Q_{\ell-2}(z)$ is a polynomial of degree $\ell-2$, and $Q_{\ell-2}(-z) = (-1)^\ell\ Q_{\ell-2}(z)$. (The integrated parts always vanish by symmetry.) Next, make use of (12) to replace $g_\ell(p)$. This leads to an integral of the form

$$\int_0^\infty dp\ Q_{\ell-2}(\tfrac{p}{r}) \int_p^\infty dt\ t^{2\nu}\ G_\ell(t)\ C^\nu_\ell(\tfrac{p}{t})\ [1 - (\tfrac{p}{t})^2]^{\nu-\frac{1}{2}}\ .$$

A change in the order of integration over the indicated region of the pt plane leads to

$$\int_0^\infty dt\ t^{2\nu}\ G_\ell(t) \int_0^t dp\ Q_{\ell-2}(\tfrac{p}{r})\ C^\nu_\ell(\tfrac{p}{t})\ [1 - (\tfrac{p}{t})^2]^{\nu-\frac{1}{2}}\ .$$

Now the p integration can be shown to vanish. If the variable change $p = yt$ is made, and the symmetry of the functions in the integrand taken into account the p integral becomes (aside from a constant factor)

$$\int_{-1}^{1} Q_{\ell-2}(\tfrac{ty}{r})\ C_\ell^\nu(y)\ (1-y^2)^{\nu-\frac{1}{2}}\ dy \quad .$$

Since $Q_{\ell-2}$ is a polynomial of degree $\ell-2$ in y it follows by orthogonality that this integral vanishes.

Hence we finally have the desired result, which consists of the Gegenbauer transform pair,

$$G_\ell(r) = \frac{-\Gamma(\ell+1)\ \Gamma(\nu)}{2\ \pi^{\nu+1}\ \Gamma(\ell+2\nu)\ r} \int_r^\infty g_\ell^{(n-1)}(p)\ C_\ell^\nu(\tfrac{p}{r})\ [(\tfrac{p}{r})^2-1]^{\nu-\frac{1}{2}}\ dp \tag{28}$$

and

$$g_\ell(p) = \frac{(4\pi)^\nu\ \Gamma(\ell+1)\ \Gamma(\nu)}{\Gamma(\ell+2\nu)} \int_p^\infty r^{2\nu}\ G_\ell(r)\ C_\ell^\nu(\tfrac{p}{r})\ [1-(\tfrac{p}{r})^2]^{\nu-\frac{1}{2}}\ dr\ , \tag{29}$$

where $\nu=\frac{1}{2}(n-2)$ and the dimensionality n is even ($n=2,\ 4,\ 6,\ \cdots$).

7. Limiting Case $n=2$

It is especially interesting to examine the $n=2$ limiting case of the above transform pair since that corresponds to the Radon transform on a plane. The result is straightforward if one first multiplies by ν/ν and then lets $\nu\to 0$. The result is the Tchebycheff transform pair [13]

$$G_\ell(r) = \frac{-1}{\pi r} \int_r^\infty g_\ell'(p)\ T_\ell(\tfrac{p}{r})\ [(\tfrac{p}{r})^2-1]^{-\frac{1}{2}}\ dp \tag{30}$$

and

$$g_\ell(p) = 2\int_p^\infty G_\ell(r)\ T_\ell(\tfrac{p}{r})\ [1-(\tfrac{p}{r})^2]^{-\frac{1}{2}}\ dr\ . \tag{31}$$

APPENDIX

In this appendix we collect several formulas which are needed in the preceding work. Some of these are included for convenience and may be found in [2]. Others, notably those involving the Tchebycheff functions, do not seem to be available in the standard sources. Our notation and conventions conform to that used by Durand, Fishbane, and Simmons [2], since their treatment of the Gegenbauer functions is the best available source for the type of results needed here. These authors derive many properties of the Gegenbauer functions of the first kind $C_\lambda^\alpha(z)$ and second kind $D_\lambda^\alpha(z)$ for general values of α, λ, and z. Our concern here is primarily with the restricted case where both α and λ are nonnegative integers (designated by writing $\alpha = \nu$ and $\lambda = \ell$) and z is real. We use x (in place of z) to emphasize that the argument lies on the interval $[-1,+1]$ or $[0,1]$ and z whenever the argument is complex or greater than unity.

For integral λ and $Re\,\alpha > -\frac{1}{2}$, D_λ^α and C_λ^α are related by [2]

(A-1) $$D_\ell^\alpha(z) = e^{i\pi\alpha}\,(z^2-1)^{\frac{1}{2}-\alpha}\,\frac{1}{2\pi}\int_{-1}^{1} C_\ell^\alpha(t)\,(z-t)^{-1}\,(1-t^2)^{\alpha-\frac{1}{2}}\,dt \;.$$

To obtain $D_\ell^\alpha(x)$ we make use of the general prescription

(A-2) $$D_\lambda^\alpha(x) = \lim_{\varepsilon\to 0^+}\frac{e^{-i\pi\alpha}}{i}\left[e^{i\pi\alpha}\,D_\lambda^\alpha(x+i\varepsilon) - e^{-i\pi\alpha}\,D_\lambda^\alpha(x-i\varepsilon)\right]$$

$$(z^2-1) \to \begin{cases} (1-x^2)\,e^{i\pi} & \text{for } (x+i\varepsilon) \\ (1-x^2)\,e^{-i\pi} & \text{for } (x-i\varepsilon) \end{cases} \;.$$

This yields

$$\text{(A-3)} \qquad D_\ell^\alpha(x) = (1-x^2)^{\frac{1}{2}-\alpha} \frac{1}{\pi} \int_{-1}^{1} C_\ell^\alpha(t)\,(x-t)^{-1}\,(1-t^2)^{\alpha-\frac{1}{2}}\,dt \quad .$$

For integral $\alpha = \nu$ the following relation holds

$$\text{(A-4)} \qquad (z^2-1)^{\nu-\frac{1}{2}} D_\ell^\nu(z) = \tfrac{1}{2}(z^2-1)^{\nu-\frac{1}{2}} C_\ell^\nu(z) - \frac{2^{-\nu}}{\Gamma(\nu)} E^\nu_{\ell+2\nu-1}(z) \quad ,$$

where $E^\nu_{\ell+2\nu-1}(z)$ is a polynomial of degree $\ell+2\nu-1$. These polynomials can be expressed in terms of associated Legendre functions of the first kind,

$$\text{(A-5)} \qquad E^\nu_{\ell+2\nu-1}(z) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (z^2-1)^{\frac{1}{2}(\nu-\frac{1}{2})} P^{\nu-\frac{1}{2}}_{\ell+\nu-\frac{1}{2}}(z).$$

Explicitly, in terms of Tchebycheff polynomials of the first kind T_ℓ and second kind U_ℓ , (The argument may be either z or x.)

$$E^0_{\ell-1} = \frac{1}{\ell} U_{\ell-1}$$

$$\text{(A-6)} \qquad E^1_{\ell+1} = T_{\ell+1}$$

$$E^2_{\ell+3} = \tfrac{1}{2}[(\ell+1)\,T_{\ell+3} - (\ell+3)\,T_{\ell+1}] \,.$$

In general,

$$\text{(A-7)} \qquad E^\nu_{\ell+2\nu-1} = 2^{1-\nu} \sum_{k=0}^{\nu-1} (-1)^k \binom{\nu-1}{k} \frac{(\ell+\nu-k-1)!\,(\ell+2\nu-1)!}{\ell!\,(\ell+2\nu-k-1)!} T_{\ell+2\nu-2k-1} \,,$$

where we have used standard symbols for factorals and binomial coefficients.

The E's satisfy the recursion relation

(A-8) $$E^{\nu}_{\ell+2\nu-1}(z) = (\ell+1)\, z\, E^{\nu-1}_{\ell+2\nu-2}(z) - (\ell+2\nu-2)\, E^{\nu-1}_{\ell+2\nu-3}(z)$$

and the symmetry property

(A-9) $$E^{\nu}_{\ell+2\nu-1}(-z) = (-1)^{\ell+1}\, E^{\nu}_{\ell+2\nu-1}(z) .$$

Explicit results for the functions D^{ν}_{ℓ} may be written conveniently in terms of the function $V_{\ell}(z)$ where

(A-10) $$V_{\ell}(z) = T_{\ell}(z) - (z^2 - 1)^{\frac{1}{2}}\, U_{\ell-1}(z) ,$$

with recursion relation

(A-11) $$V_{\ell+2}(z) = 2\, z\, V_{\ell+1}(z) - V_{\ell}(z) .$$

We have

$$D^{0}_{\ell}(z) = \lim_{\alpha\to 0} \frac{1}{\alpha} D^{\alpha}_{\ell}(z) = \frac{1}{\ell} V_{\ell}(z)$$

(A-12) $$D^{1}_{\ell}(z) = -\tfrac{1}{2}(z^2 - 1)^{-\frac{1}{2}}\, V_{\ell+1}(z)$$

$$D^{2}_{\ell}(z) = -\tfrac{1}{4}(z^2 - 1)^{-3/2}\, \tfrac{1}{2}[(\ell+1)V_{\ell+3}(z) - (\ell+3)V_{\ell+1}(z)] .$$

In general,

(A-13) $$D^{\nu}_{\ell}(z) = \frac{-(z^2 - 1)^{\frac{1}{2}-\nu}}{2^{2\nu-1}\,\Gamma(\nu)} \sum_{k=0}^{\nu-1} (-1)^k \binom{\nu-1}{k} \frac{(\ell+\nu-k-1)!\;(\ell+2\nu-1)!}{\ell!\;(\ell+2\nu-k-1)!}\, V_{\ell+2\nu-2k-1}(z).$$

By application of (A-2) and (A-4),

$$\text{(A-14)} \qquad D_\ell^\nu(x) = \frac{(-1)^{\nu+1}\,(1-x^2)^{\frac{1}{2}-\nu}}{2^{\nu-1}\,\Gamma(\nu)}\, E_{\ell+2\nu-1}^{\nu}(x) \ .$$

The Tchebycheff expansion for the C_ℓ^ν is given by

$$C_\ell^0(z) = \operatorname*{Lim}_{\alpha\to 0} \frac{1}{\alpha} C_\ell^\alpha(z) = \frac{2}{\ell} T_\ell(z)$$

$$\text{(A-15)} \qquad C_\ell^1(z) = U_\ell(z)$$

$$C_\ell^2(z) = \tfrac{1}{4}(z^2-1)^{-1}\,[(\ell+1)U_{\ell+2}(z) - (\ell+3)U_\ell(z)] \ .$$

In general,

$$\text{(A-16)} \qquad C_\ell^\nu(z) = \frac{(z^2-1)^{1-\nu}}{4^{\nu-1}\,\Gamma(\nu)} \sum_{k=0}^{\nu-1} (-1)^k \binom{\nu-1}{k} \frac{(\ell+\nu-k-1)!\ (\ell+2\nu-1)!}{\ell!\ (\ell+2\nu-k-1)!} U_{\ell+2\nu-2k-2}(z) .$$

These results also hold for $z \to x$.

For future reference, we examine another form for D_ℓ^ν which is especially valuable for large values of ℓ or z. We first define $\zeta = (z^2-1)^{\frac{1}{2}}$ and observe that

$$(z+\zeta)^{-1} = z - \zeta \qquad \text{and} \qquad (z+\zeta)^2 - 1 = 2\zeta(z+\zeta) \ .$$

In terms of these variables we have

$$D_\ell^0(z) = \frac{1}{\ell}(z+\zeta)^{-\ell}$$

$$\text{(A-17)} \qquad D_\ell^1(z) = \frac{-1}{2\zeta}(z+\zeta)^{-\ell-1}$$

$$D_\ell^2(z) = \frac{\ell}{4\zeta^2}(z+\zeta)^{-\ell-2}\left\{1 + \frac{z+2\zeta}{\ell\zeta}\right\} \ .$$

Higher terms have the form

$$(\text{A-18}) \qquad D_\ell^\nu(z) = \frac{(-1)^\nu \, \ell^{\nu-1} (z+\zeta)^{-\ell-\nu}}{2^\nu \, \Gamma(\nu) \, \zeta^\nu} \left\{ 1 + \frac{\nu(\nu-1)}{2\ell} \cdot \frac{z+2\zeta}{\zeta} + O(\ell^{-2}) \right\} .$$

Or, in general,

$$(\text{A-19}) \qquad D_\ell^\nu(z) = \frac{(-1)^\nu \, \ell^{\nu-1} (z+\zeta)^{-\ell-\nu}}{2^\nu \, \Gamma(\nu) \, \zeta^\nu} (2\ell\zeta)^{1-\nu} \sum_{k=0}^{\nu-1} (-1)^k \binom{\nu-1}{k} \frac{(\ell+k)! \, (\ell+2\nu-1)!}{\ell! \, (\ell+\nu+k)!} (z+\zeta)^{\nu-1-2k} .$$

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