General Closed-Form PML Constitutive Tensors to Match Arbitrary Bianisotropic and Dispersive Linear Media

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***Abstract*— The perfectly matched layer (PML) constitutive tensors that match more general linear media presenting bianisotropic and dispersive behavior are obtained for single interface problems and for two-dimensional (2-D) and three-dimensional (3-D) corner regions. The derivation is based on the analytic continuation of Maxwell's equations to a complex variables domain. The formulation is Maxwellian so that it is equally applicable to the finite-difference time-domain (FDTD) or finite-element (FEM) methods. It recovers, as special cases, previous anisotropic media formulations, and dispersive media formulations.**

***Index Terms*— Bianisotropic media, dispersive media, FDTD, FEM, perfectly matched layer.**

I. Introduction

The perfectly matched layer (PML) absorbing boundary condition (ABC) was first developed for planar interfaces in isotropic and nondispersive media [1]. Apart from the related work on the nonplanar quasi-PML [2]–[4], recent work on the PML has been focused on its extension to 1) more general geometries (cylindrical [5]–[8], spherical [5], [6], and conformal [9] interfaces) and 2) to more general media (dispersive [10], [11], and anisotropic [12], [13] media).

The extension of the PML to more general media in [10]–[13] has been carried out by two basic approaches. The first approach [10], [13] constitutes writing the Maxwell's curl equations (ME's) in terms of $\overline{D}$ and $\overline{B}$ also, instead of $\overline{E}$ and $\overline{H}$ only (thereby producing a material-independent PML, or MIPML) and then modifying them by introducing the matched conductivities and field-splitting directly on $\overline{D}$ and $\overline{B}$ as is done for $\overline{E}$ and $\overline{H}$ in the original PML (isotropic, nondispersive media). The second approach [11], [12] constitutes determining the reflection coefficients for the interface in terms of the unknown PML constitutive parameters and analytically deriving the resultant PML constitutive parameters that produce a matched interface.

Of these two approaches, the MIPML is more powerful in the following sense. First, the costly analytical effort to derive the actual constitutive parameters inside the PML from the reflection coefficient analysis is avoided. Second, and more importantly, the extension to corner regions (which play a pivotal role on the algorithm) is possible.

A disadvantage of MIPML, however, is the need to work with the $\overline{D}$, $\overline{B}$, $\overline{E}$, and $\overline{H}$ fields, as opposed to only the $\overline{E}$ and $\overline{H}$ fields in the original PML-FDTD formulation. This, in principle, doubles the memory requirements. In addition, the MIPML is a non-Maxwellian formulation (i.e., the resultant fields inside the PML do not obey the Maxwell's equations) and no information is available on the *actual* constitutive tensors inside the PML region. This is a serious drawback for methods other than the finite-difference time-domain (FDTD), such as the finite-element method (FEM).

As a result, it is of interest to develop a scheme endowed with the attractive features of the two approaches outlined above, minus their disadvantages. Such scheme would have the following characteristics: 1) general, i.e., applicable to materials with electric and/or magnetic anisotropy (or even bianisotropy), and/or dispersion; 2) Maxwellian, and therefore suitable for the FEM, and serving as a physical basis for possible engineered artificial materials [14]; 3) conceptually simple, resulting in little analytical effort to derive the PML constitutive parameters to match these general media even at corner regions; and 4) amenable to be formulated in terms of $\overline{E}$ and $\overline{H}$ only.

The derivation of the PML for nonplanar geometries in [5], [6], [9] was carried out through an analytic continuation (complex stretching approach [15]) of ME's to a complex variables spatial domain. In [16], it was described how the complex stretching approach can be applied to obtain PML for dispersive and anisotropic media in a straightforward manner. However, the formulation was non-Maxwellian and no PML constitutive parameter was obtained. In this work, we will show that it is possible to further extend this approach to obtain Maxwellian PML's for an arbitrary bianisotropic and dispersive media, exhibiting the attractive characteristics described on the previous paragraph. This shows that the conjecture [12], [13] that no matching conditions could be obtained for arbitrary anisotropic media in three dimensions (3-D) is false.

Throughout this work, the $e^{-i\omega t}$ time convention is used.

II. Formulation

The PML was shown to be equivalent to an analytic continuation of ME's to a complex variables spatial domain

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[5], [15]. Closed-form solutions for the fields inside a PML region can be easily obtained by noting that closed-form solutions that already exist for Maxwellian fields in real space map directly to the complex space through this analytic continuation. The general effect of this analytic continuation is to alter the eigenfunctions of ME's so that propagating modes are mapped continuously to exponentially decaying modes, allowing for the reflectionless absorption of electromagnetic waves. This concept is also applicable to other linear wave phenomena (scalar [17], elastic [18]) to achieve a PML ABC in such cases. It can also be extended to other coordinate systems [5], [6], [9]. In Cartesian coordinates, this analytic continuation is expressed by the following transformation:

$$u \rightarrow \tilde{u} = \int_0^u s_u(u')\, du' \tag{1}$$

where $s_u(u)$ are the complex stretching variables [15] and u stands for x, y, z. This transformation can also be expressed in terms of a dyadic function $\overline{\overline{\Gamma}}$ such that

$$\overline{r} \rightarrow \tilde{\overline{r}} = \overline{\overline{\Gamma}} \cdot \overline{r} \tag{2}$$

$$\overline{\overline{\Gamma}} = \hat{x}\hat{x}\left(\frac{\tilde{x}}{x}\right) + \hat{y}\hat{y}\left(\frac{\tilde{y}}{y}\right) + \hat{z}\hat{z}\left(\frac{\tilde{z}}{z}\right). \tag{3}$$

The Maxwell's equations in complex space reads as usual but with the nabla operator changed as

$$\begin{aligned}\nabla \rightarrow \tilde{\nabla} &= \hat{x}\,\frac{\partial}{\partial \tilde{x}} + \hat{y}\,\frac{\partial}{\partial \tilde{y}} + \hat{z}\,\frac{\partial}{\partial \tilde{z}} \\ &= \hat{x}\,\frac{1}{s_x}\,\frac{\partial}{\partial x} + \hat{y}\,\frac{1}{s_y}\,\frac{\partial}{\partial y} + \hat{z}\,\frac{1}{s_z}\,\frac{\partial}{\partial z}.\end{aligned} \tag{4}$$

Or simply,

$$\tilde{\nabla} = \overline{\overline{S}} \cdot \nabla \tag{5}$$

with

$$\overline{\overline{S}} = \hat{x}\hat{x}\left(\frac{1}{s_x}\right) + \hat{y}\hat{y}\left(\frac{1}{s_y}\right) + \hat{z}\hat{z}\left(\frac{1}{s_z}\right). \tag{6}$$

Since $s_u(u)$ and $\partial/\partial u'$ commute for $u \neq u'$, and $\overline{\overline{S}}$ is a diagonal tensor, the following identity can be verified for the Cartesian PML and any vector function $\overline{a}(\overline{r})$ in Cartesian coordinates

$$\nabla \times (\overline{\overline{S}}^{-1} \cdot \overline{a}) = (\det \overline{\overline{S}})^{-1}\overline{\overline{S}} \cdot (\overline{\overline{S}} \cdot \nabla) \times \overline{a} \tag{7}$$

where $\det \overline{\overline{S}} = (s_x s_y s_z)^{-1}$. The dyadic $\overline{\overline{\Gamma}}$ also satisfies a similar equation.

In a bianisotropic and dispersive media, the ME's are

$$\nabla \times \overline{E}(\overline{r}) = i\omega \overline{B}(\overline{r}) \tag{8a}$$

$$\nabla \times \overline{H}(\overline{r}) = -i\omega \overline{D}(\overline{r}) \tag{8b}$$

with

$$\overline{D}(\overline{r}) = \overline{\overline{\epsilon}}(\omega) \cdot \overline{E}(\overline{r}) + \overline{\overline{\xi}}(\omega) \cdot \overline{H}(\overline{r}) \tag{8c}$$

$$\overline{B}(\overline{r}) = \overline{\overline{\zeta}}(\omega) \cdot \overline{E}(\overline{r}) + \overline{\overline{\mu}}(\omega) \cdot \overline{H}(\overline{r}). \tag{8d}$$

The PML in complex space for such a media is obtained by just keeping the *same* constitutive parameters *everywhere* and enforcing the complex stretching on the PML region [16]. Inside the complex space PML, the modified ME's then simply reads

$$\tilde{\nabla} \times \overline{E}^c(\tilde{\overline{r}}) = i\omega \overline{B}^c(\tilde{\overline{r}}) \tag{9a}$$

$$\tilde{\nabla} \times \overline{H}^c(\tilde{\overline{r}}) = -i\omega \overline{D}^c(\tilde{\overline{r}}) \tag{9b}$$

with

$$\overline{D}^c(\tilde{\overline{r}}) = \overline{\overline{\epsilon}}(\omega) \cdot \overline{E}^c(\tilde{\overline{r}}) + \overline{\overline{\xi}}(\omega) \cdot \overline{H}^c(\tilde{\overline{r}}) \tag{9c}$$

$$\overline{B}^c(\tilde{\overline{r}}) = \overline{\overline{\zeta}}(\omega) \cdot \overline{E}^c(\tilde{\overline{r}}) + \overline{\overline{\mu}}(\omega) \cdot \overline{H}^c(\tilde{\overline{r}}). \tag{9d}$$

The subscript c indicates that the fields in (9) are not Maxwellian. Using (2) and (5) we can recast (9a) and (9b) in the real space domain

$$(\overline{\overline{S}} \cdot \nabla) \times \overline{E}^c(\overline{\overline{\Gamma}} \cdot \overline{r}) = i\omega \overline{B}^c(\overline{\overline{\Gamma}} \cdot \overline{r}) \tag{10a}$$

$$(\overline{\overline{S}} \cdot \nabla) \times \overline{H}^c(\overline{\overline{\Gamma}} \cdot \overline{r}) = -i\omega \overline{D}^c(\overline{\overline{\Gamma}} \cdot \overline{r}). \tag{10b}$$

Using (7), we write (10a) and (10b) as

$$\nabla \times [\overline{\overline{S}}^{-1} \cdot \overline{E}^c(\overline{\overline{\Gamma}} \cdot \overline{r})] = i\omega(\det \overline{\overline{S}})^{-1}\overline{\overline{S}} \cdot \overline{B}^c(\overline{\overline{\Gamma}} \cdot \overline{r}) \tag{11a}$$

$$\nabla \times [\overline{\overline{S}}^{-1} \cdot \overline{H}^c(\overline{\overline{\Gamma}} \cdot \overline{r})] = -i\omega(\det \overline{\overline{S}})^{-1}\overline{\overline{S}} \cdot \overline{D}^c(\overline{\overline{\Gamma}} \cdot \overline{r}). \tag{11b}$$

Introducing a new set of fields defined as

$$\overline{E}^a(\overline{r}) = \overline{\overline{S}}^{-1} \cdot \overline{E}^c(\overline{\overline{\Gamma}} \cdot \overline{r}) \tag{12a}$$

$$\overline{H}^a(\overline{r}) = \overline{\overline{S}}^{-1} \cdot \overline{H}^c(\overline{\overline{\Gamma}} \cdot \overline{r}) \tag{12b}$$

$$\overline{D}^a(\overline{r}) = (\det \overline{\overline{S}})^{-1}\overline{\overline{S}} \cdot \overline{D}^c(\overline{\overline{\Gamma}} \cdot \overline{r}) \tag{12c}$$

$$\overline{B}^a(\overline{r}) = (\det \overline{\overline{S}})^{-1}\overline{\overline{S}} \cdot \overline{B}^c(\overline{\overline{\Gamma}} \cdot \overline{r}) \tag{12d}$$

and substituting back in (11), we have

$$\nabla \times \overline{E}^a(\overline{r}) = i\omega \overline{B}^a(\overline{r}) \tag{13a}$$

$$\nabla \times \overline{H}^a(\overline{r}) = -i\omega \overline{D}^a(\overline{r}) \tag{13b}$$

with

$$\begin{aligned}\overline{D}^a(\overline{r}) = &\{(\det \overline{\overline{S}})^{-1}[\overline{\overline{S}} \cdot \overline{\overline{\epsilon}}(\omega) \cdot \overline{\overline{S}}]\} \cdot \overline{E}^a(\overline{r}) \\ &+ \{(\det \overline{\overline{S}})^{-1}[\overline{\overline{S}} \cdot \overline{\overline{\xi}}(\omega) \cdot \overline{\overline{S}}]\} \cdot \overline{H}^a(\overline{r})\end{aligned} \tag{13c}$$

$$\begin{aligned}\overline{B}^a(\overline{r}) = &\{(\det \overline{\overline{S}})^{-1}[\overline{\overline{S}} \cdot \overline{\overline{\zeta}}(\omega) \cdot \overline{\overline{S}}]\} \cdot \overline{E}^a(\overline{r}) \\ &+ \{(\det \overline{\overline{S}})^{-1}[\overline{\overline{S}} \cdot \overline{\overline{\mu}}(\omega) \cdot \overline{\overline{S}}]\} \cdot \overline{H}^a(\overline{r}).\end{aligned} \tag{13d}$$

Therefore, the fields $\overline{E}^a$, $\overline{H}^a$, $\overline{D}^a$, $\overline{B}^a$ obey the ME's. They also coincide with the original fields $\overline{E}$, $\overline{H}$, $\overline{D}$, $\overline{B}$ inside the physical domain (non-PML region), since $\overline{\overline{\Gamma}} = \overline{\overline{S}} = \overline{\overline{I}}$ there. Furthermore, from (12), it is seen that they present the same attenuative behavior of $\overline{E}^c$, $\overline{H}^c$, $\overline{D}^c$, $\overline{B}^c$ inside the PML, while retaining the perfect matching conditions. The significance of this result is that, given a general medium $\overline{\overline{\epsilon}}(\omega)$, $\overline{\overline{\xi}}(\omega)$, $\overline{\overline{\zeta}}(\omega)$, $\overline{\overline{\mu}}(\omega)$, a Maxwellian PML can be obtained with bianisotropic constitutive parameters given as

$$\overline{\overline{\epsilon}}_{\text{PML}} = (\det \overline{\overline{S}})^{-1}[\overline{\overline{S}} \cdot \overline{\overline{\epsilon}}(\omega) \cdot \overline{\overline{S}}] \tag{14a}$$

$$\overline{\overline{\mu}}_{\text{PML}} = (\det \overline{\overline{S}})^{-1}[\overline{\overline{S}} \cdot \overline{\overline{\mu}}(\omega) \cdot \overline{\overline{S}}] \tag{14b}$$

$$\overline{\overline{\zeta}}_{\text{PML}} = (\det \overline{\overline{S}})^{-1}[\overline{\overline{S}} \cdot \overline{\overline{\zeta}}(\omega) \cdot \overline{\overline{S}}] \tag{14c}$$

$$\overline{\overline{\xi}}_{\text{PML}} = (\det \overline{\overline{S}})^{-1}[\overline{\overline{S}} \cdot \overline{\overline{\xi}}(\omega) \cdot \overline{\overline{S}}]. \tag{14d}$$

These formulas give directly the bianisotropic constitutive parameters that have to be present both in single interface problems (with two of the stretching variables in (6) set equal to unity) and two-dimensional (2-D) or 3-D corner interfaces (with one single or none of the stretching variables set equal to unity).

The field transformations in (12) are similar to the ones carried out in [19] for fields subject to affine transformations that change the metric of the space. This is because (2) bears a formal resemblance to the expression of an affine transformation. On this respect, we should note that $\overline{\overline{\Gamma}}(\overline{r})$ is a function of position, and therefore, (2) defines a *nonlinear* transformation on $\overline{r}$. Moreover, it always preserves the orthogonality of the metric, since $\overline{\overline{\Gamma}}$ is diagonal (also true for PML's in other orthogonal coordinates systems [6], [9]). In the Fourier domain, the PML is to be viewed as a *complex* mapping (stretching) of the metric.

III. Concluding Remarks

A general formulation is presented to extend the PML to arbitrary linear media presenting bianisotropy and/or dispersion. It is based on the analytic continuation of Maxwell's equations to a complex variables domain. It is shown that, in these media, a Maxwellian PML formulation is also possible. This formulation is such that the PML can be realized as a medium with suitable defined bianisotropic constitutive tensors. From the original media tensors $\overline{\overline{\epsilon}}(\omega)$, $\overline{\overline{\xi}}(\omega)$, $\overline{\overline{\zeta}}(\omega)$, $\overline{\overline{\mu}}(\omega)$, a set of perfectly matched tensors $\overline{\overline{\epsilon}}_{\rm PML}(\omega)$, $\overline{\overline{\xi}}_{\rm PML}(\omega)$, $\overline{\overline{\zeta}}_{\rm PML}(\omega)$, $\overline{\overline{\mu}}_{\rm PML}(\omega)$ is obtained in a simple and systematic way. A limitation of this approach, however, is that it does not apply to nonlinear media. For a recent (non-Maxwellian) extension of the PML to nonlinear media, see [20].

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