GENERAL ELEPHANTS OF THREE-FOLD DIVISORIAL CONTRACTIONS

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The theory of minimal models has enriched the study of higher-dimensional algebraic geometry; see [10] and [12]. For a variety with mild singularities, this theory produces another variety which possesses good properties, after finite elementary transformations called divisorial contractions and flips. Since Mori completed this program in dimension three in [17], it has become desirable to study three-folds explicitly. This paper aims to complete the explicit study of three-fold divisorial contractions whose exceptional divisors contract to Gorenstein points, after the papers [7] and [8].

Reid pointed out that general elements in the anti-canonical systems of three-folds have at worst Du Val singularities in appropriate situations involving contractions of extremal faces. This has become known as the general elephant conjecture. The papers [19], [21] and [22] support it with affirmative answers for Fano three-folds with singularities, and this approach has settled the problem of the existence of three-fold flips in [11] and [17]. Our main theorem is that this conjecture holds for our divisorial contractions:

Theorem. *Let $f\colon (Y \supset E) \to (X \ni P)$ be a germ of a three-fold divisorial contraction whose exceptional divisor E contracts to a Gorenstein point P. Then a general element in the anti-canonical system $|-K_Y|$ has at worst Du Val singularities.*

The statement is analogous to the flipping case, and we can start from a similar position after taking the intersection C of E and the birational transform of a general hyperplane section on X, because the first cohomology of C vanishes. There is, however, a crucial difference between our curve C and that in the flipping case. For any flipping curve in a three-fold, the first cohomology of any closed subscheme supported on this curve always vanishes, but this does not hold for our C. Nevertheless, provided that P is Gorenstein, the numerical information on f obtained in [7] allows us to analyse the local structure of $C \subset Y$ delicately. It encodes the behaviour of the global sections in $|-K_Y|$ and leads to our main theorem.

In fact, we prove more than the theorem in this paper. Let S be a surface on Y defined by a general element in $|-K_Y|$, and S_X its birational transform on X. The theorem guarantees that S and S_X have only Du Val singularities, and the induced morphism $S \to S_X$ factors through the minimal resolution of S_X. We prove the theorem at the same time as providing information on the partial resolution $S \to S_X$; see Theorem 1.8. In some special cases, to be precise in types O and I in Theorem 1.4, the birational transform of a general hyperplane section on X gives

a Du Val section S. On the other hand, in the remaining cases, we can obtain the strong version of the general elephant conjecture, which asserts that the type of any Du Val singularity $Q \in S$ is the same as that of a general Du Val section of a germ $Q \in Y$. In most cases this is proved by a direct search for S, but in some exceptional cases, where $-K_Y$ is linearly equivalent to the sum of E and a Cartier divisor L, we obtain the desired S by showing that E has a Du Val singularity of the required type at any $Q \in S$, and that L moves in a free linear system.

The theorem is applied to classify divisorial contractions. We describe the contractions to cA_n points in Theorem 1.13, following on from the smooth and the cA_1 cases treated in [7] and [8]. We expect that as the singularity at P worsens, there are fewer divisorial contractions over P, because the choice of coordinates at P is more restricted. However, since the defining equation of X at P becomes much more complicated, their explicit study is not necessarily simpler. Instead of complete descriptions in the remaining cD_n and cE_n cases, we give restrictions on the possible divisorial contractions by deriving an upper bound on discrepancies in Corollary 1.15. Now we have sufficient tools to classify divisorial contractions also in the cD_n and cE_n cases, once $P \in X$ has been given explicitly. What we should do is basically to compare discrepancies as in [9] or Lemma 6.1, using the enormous amount of information given by the singular Riemann–Roch technique and the general elephant theorem concerning the multiplicities along E of surfaces inside X with special directions. The connectedness lemma of Shokurov can be applied occasionally as in [3, Theorem 3.10] or [4, Theorem 3.6].

This paper is constructed as follows. In Section 1, we state our theorems precisely. Section 2 is devoted to preparing basic numerical techniques. After recalling the results for divisorial contractions to Gorenstein points in [7], we introduce the fundamental set-up to analyse $C \subset Y$, following [17]. The quite delicate local investigation of $C \subset Y$ is presented in Section 3. Using this, in Section 4 we prove the main theorem, the existence of Du Val sections. In Section 5, we restrict the possible divisorial contractions, focusing on the types of singularities on Du Val sections. This provides an upper-bound on discrepancies in the cD_n and cE_n cases. Finally, in Section 6 we give an explicit description of divisorial contractions in the cA_n case.

1. STATEMENTS

We work over the complex number field $\mathbb{C}$.

Divisorial contractions play a major role in the minimal model program. This program was formulated to generalise the theory of minimal models of surfaces to higher-dimensional varieties. In order to avoid problems caused by the existence of small contractions, this program has to work in a category of varieties with mild singularities; see [10] and [12]. Before we define a divisorial contraction, we have to introduce the class of terminal singularities. Let X be a normal variety. We say that X has at worst *terminal singularities* if X is $\mathbb{Q}$-Gorenstein and if every exceptional divisor has positive coefficient in the discrepancy divisor $K_Y - f^* K_X$ for a resolution of singularities $f \colon Y \to X$.

The minimal model program works in the category of $\mathbb{Q}$-factorial normal varieties with terminal singularities; for such a variety, it produces a good variety after finite elementary transformations called divisorial contractions and flips. Since the main theorem is implied by the corresponding theorem in the analytic category, we define

a divisorial contraction in a general sense. From now on, we work in the analytic category by relating to the algebraic category via the algebraisation theorems of Artin in [1] and [2].

Definition 1.1. Let $f\colon Y \to X$ be a proper morphism with connected fibres between normal varieties with at worst terminal singularities. We say that f is a *divisorial contraction* if the exceptional locus of f is a prime divisor and $-K_Y$ is f-ample.

Our study of three-fold divisorial contractions relies on the classification of three-fold terminal singularities, which should be recalled. Let $P \in X$ be a three-fold germ. We say that P is a *cDV* (*compound Du Val*) *point* if a general hyperplane section has at worst a Du Val singularity at P. The singularity P is said to be cA_n, cD_n, cE_n (*compound* A_n, D_n, E_n) according to the type of the Du Val singularity on a general hyperplane section. For convenience, we say that a smooth point on a surface or a three-fold is A_0, respectively cA_0. It is a result of Reid that three-fold Gorenstein terminal singularities are characterised as isolated cDV points:

Theorem 1.2 ([18, Theorem 1.1]). *Let $P \in X$ be a three-fold germ. P is a Gorenstein terminal singularity if and only if P is an isolated cDV point.*

We then consider a three-fold non-Gorenstein terminal singularity $P \in X$. Let r be the local Gorenstein index of $P \in X$, that is, the smallest positive integer such that rK_X is Cartier at P. Take the index-one cover $\pi\colon (X^\sharp \ni P^\sharp) \to (X \ni P)$, which is a cyclic μ_r-cover. Fix a character generating $\mathrm{Hom}(\mu_r, \mathbb{C}^\times) = \mathbb{Z}/(r)$ and define the weight modulo r for any semi-invariant function on $X^\sharp$ with respect to this character.

Theorem 1.3 ([16]). *There exists a μ_r-equivariant identification*

$$P^\sharp \in X^\sharp \cong o \in (\phi = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4},$$

where x_1, x_2, x_3, x_4 and ϕ are μ_r-semi-invariant, and $\phi = x_4$ if $P^\sharp \in X^\sharp$ is a smooth point. The weights of x_1, x_2, x_3, x_4 and ϕ satisfy one of the following:

(i) $\mathrm{wt}(x_1, x_2, x_3, x_4; \phi) = (1, -1, b, 0; 0)$, *where b is co-prime to r.*

(ii) $r = 4$ *and* $\mathrm{wt}(x_1, x_2, x_3, x_4; \phi) = (1, 3, 3, 2; 2)$.

Mori gave more precise description of ϕ and weights in [16], and the classification is completed by [13, Theorem 6.4].

According to the above classification, any three-fold terminal singularity $P \in X$ has a small deformation to a basket of terminal quotient singularities P_i. The P_i are called *fictitious singularities* in the sense of Reid in [20]. Note that if P is of type (i) in Theorem 1.3, any local index at P_i equals that at P, and if P is of type (ii), one of P_i has local index 2 and all the others have local index 4.

We return to the study of three-fold divisorial contractions. Let

$$f\colon (Y \supset E) \to (X \ni P)$$

be a germ of a three-fold divisorial contraction whose exceptional divisor E contracts to a point P. There are two ways to study f, namely, one starting from X, and the other from Y. By the former approach, Mori classified them in the case where Y is smooth in [15], and Cutkosky extended this result to the case where Y is Gorenstein in [5]. On the other hand, by the latter approach, Kawamata proved that f is a certain weighted blow-up when P is a terminal quotient singularity in

[9], and Corti proved that f is the blow-up when P is an ordinary double point in [3, Theorem 3.10].

Now is the time that we state our theorems. We start with a numerical classification of f. From now on, we assume that P is a Gorenstein point. As in Section 2, set $K_Y = f^*K_X + aE$, and let $I := \{Q : \text{type } \frac{1}{r_Q}(1, -1, \overline{av_Q})\}$ with $v_Q \le r_Q/2$ be the basket of fictitious singularities from non-Gorenstein singularities on Y. We also set $J := \{(r_Q, v_Q)\}_{Q \in I}$ and $1 + d(-1) := \dim \mathcal{O}_X/f_*\mathcal{O}_Y(-2E)$. The following theorem is the same as [7, Theorem 4.5], except for the statement that $d(-1) \ne 0$ in type O. This additional statement is due to Theorem 2.7.

Theorem 1.4. *f belongs to exactly one of the types in the following table. If f is of type IV, then f is the usual blow-up along a smooth point P.*

type	$1 + d(-1)$	J	a
O	≥ 2		1
I	1	$\{(7, 3)\}$ *or* $\{(3, 1), (5, 2)\}$	2
IIa	2	$\{(r, 2)\}$	$4/rE^3 = 2$ *or* 4
IIb	2	$\{(r_1, 1), (r_2, 1)\}$	$(r_1 + r_2)/r_1 r_2 E^3 \ge 2$
III	3	$\{(r, 1)\}$	$(1 + r)/rE^3 \ge 2$
IV	4	$\emptyset$	2

For convenience, we divide type IIb into two types according to the number of non-Gorenstein points on Y. We say that f is of type IIb$^\vee$ or IIb$^{\vee\vee}$ if f is of type II with one, respectively two non-Gorenstein points on Y.

Remark 1.5. If f is of type IIb$^\vee$, then $J = \{(r, 1), (r, 1)\}$ or $\{(2, 1), (4, 1)\}$, and $a = 2$, respectively 3. By [16], the unique non-Gorenstein point $Q \in Y$ can be described as one of the following equations $\phi = 0$, using semi-invariant local coordinates y_1, y_2, y_3, y_4 of the index-one cover $Q^\sharp \in Y^\sharp$ with weights $w = \text{wt}(y_1, y_2, y_3, y_4)$.

- (i) $J = \{(r, 1), (r, 1)\}$.
    - (a) $\phi = y_1 y_2 + g(y_3^r, y_4)$ and $w = (1, -1, 2, 0)$. $g(0, y_4)$ has order 2. The general Du Val section of a germ $Q \in Y$ is A_{2r-1}.
    - (b) $r = 3$, $\phi = y_4^2 + \phi_3(y_1, y_2, y_3) + \phi_{\ge 4}(y_1, y_2, y_3)$ and $w = (1, -1, 2, 0)$. ϕ_3 is $y_1^3 + y_2^3 + y_3^3$, $y_1^3 + y_2 y_3^2$, or $y_1^3 + y_2^3$, and $\phi_{\ge 4}$ has order ≥ 4. The general Du Val section of a germ $Q \in Y$ is E_6.
- (ii) $J = \{(2, 1), (4, 1)\}$. $\phi = y_1^2 + y_2^2 + g(y_3^2, y_4)$ and $w = (1, 3, 3, 2)$. $g(0, y_4)$ has order 3. The general Du Val section of a germ $Q \in Y$ is D_5.

The following definition is due to Reid:

Definition 1.6 ([20])**.** The general elements in the anti-canonical system are called *general elephants*.

In [20] Reid proposed the general elephant conjecture. It is that general elephants of three-folds should have at worst Du Val singularities in appropriate situations involving the contractions of extremal faces. Our main theorem is that this conjecture holds for our divisorial contractions:

Theorem 1.7. *A general elephant of Y has at worst Du Val singularities.*

Let S be a general elephant of Y, and S_X its birational transform on X. By Theorem 1.7, S and S_X have at worst Du Val singularities and the induced morphism $S \to S_X$ factors through the minimal resolution of S_X. We obtain Theorem 1.7 at the same time as providing the following information on the partial resolution $S \to S_X$. Theorems 1.7 and 1.8 are implied by Theorems 2.7, 4.2 and 4.4 and the results in Section 5.

Theorem 1.8.   (i) *Assume that f is of type O or I. Then S_X is a general elephant of X. Moreover, if f is of type I, then P is cE_7 or cE_8.*

(ii) *Assume that f is of type II, III or IV. Then possible types of Du Val singularities on S and S_X are as in the following table. The type of any Du Val singularity $Q \in S$ is the same as that of a general elephant of a germ $Q \in Y$. Moreover, the partial resolution $S \to S_X$ has irreducible exceptional locus except in the case IIa when S_X is of the alternative type D_{r+1}, or in the case $IIb^\vee$ in the second line of the table with S_X of type D_{2r+1}.*

type of f	type of S_X	type of S
IIa	D_r or D_{r+1}	A_{r-1}
$IIb^\vee$, $J = \{(r,1),(r,1)\}$	D_{2r} or D_{2r+1}	A_{2r-1}
$IIb^\vee$, $J = \{(3,1),(3,1)\}$	E_7	E_6
$IIb^\vee$, $J = \{(2,1),(4,1)\}$	E_6	D_5
$IIb^{\vee\vee}$	$A_{r_1+r_2-1}$	A_{r_1-1} and A_{r_2-1}
III	A_r	A_{r-1}
IV	smooth	smooth

We have examples of type I with $J = \{(3,1),(5,2)\}$ in Example 5.3, but I do not know whether type I with $J = \{(7,3)\}$ happens or not. If f is of type I with $J = \{(7,3)\}$, then P has to be cE_7 by Remark 5.2.

The main theorem can be applied to classify divisorial contraction. We classify them according the type of P. Divisorial contractions to smooth or cA_1 points are completely described in the papers [7] and [8]:

Theorem 1.9 ([7]). *Assume that P is smooth. Then f is a weighted blow-up. More precisely, we can take local coordinates x_1, x_2, x_3 at P and co-prime positive integers s and t, such that f is the weighted blow-up of X with weights $\mathrm{wt}(x_1,x_2,x_3) = (1,s,t)$. Moreover, any such f is a divisorial contraction.*

Remark 1.10. We may assume that $s \le t$. The type of f is $IIb^{\vee\vee}$ if $s > 1$, III if $s = 1$ and $t > 1$, and IV if $t = 1$.

Theorem 1.11 ([8]). *Assume that P is cA_1. Then f is a weighted blow-up. More precisely, under a suitable identification*

$$P \in X \cong o \in (x_1x_2 + x_3^2 + x_4^N = 0) \subset \mathbb{C}^4_{x_1x_2x_3x_4},$$

f is the weighted blow-up with one of the following weights:

(i) $\mathrm{wt}(x_1,x_2,x_3,x_4) = (s, 2t-s, t, 1)$, *where s,t are co-prime positive integers such that $s \le t \le N/2$.*

(ii) $N = 3$ *and* $\mathrm{wt}(x_1,x_2,x_3,x_4) = (1,5,3,2)$.

Moreover, any such f is a divisorial contraction.

Remark 1.12. In (i), the type of f is IIb$^{\vee\vee}$ if $s > 1$, III if $s = 1$ and $t > 1$, and O if $t = 1$. In (ii), the type of f is IIa.

In Section 6 of this paper, we describe divisorial contractions to cA_n points with $n \geq 2$:

Theorem 1.13. *Assume that P is cA_n for some $n \geq 2$. Then one of the following holds*:

(i) *Under a suitable identification*

$$P \in X \cong o \in (x_1 x_2 + g(x_3, x_4) = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4},$$

f is the weighted blow-up with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r_1, r_2, a, 1)$, where a divides $r_1 + r_2$ and is co-prime to r_1 and r_2, g has weighted order $r_1 + r_2$ with weights $\mathrm{wt}(x_3, x_4) = (a, 1)$, and the monomial $x_3^{(r_1+r_2)/a}$ appears in g with non-zero coefficient. Moreover, any such f is a divisorial contraction.

(ii) *P is a cA_2 point isomorphic to*

$$o \in (x_1 x_2 + x_3^3 + g_{\geq 4}(x_3, x_4) = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4},$$

where $g_{\geq 4}$ has total order ≥ 4 in x_3, x_4, Y has exactly one non-Gorenstein point Q, isomorphic to $o \in (y_1^2 + y_2^2 + y_3^2 + y_4^3 = 0)$ in the quotient space $\mathbb{C}^4_{y_1 y_2 y_3 y_4}/\frac{1}{4}(1, 3, 3, 2)$, and $K_Y = f^* K_X + 3E$. Moreover, such an example exists in Example 6.8.

Remark 1.14. In (i), we may assume that $r_1 \leq r_2$. The type of f is IIb$^{\vee\vee}$ if $r_1 > 1$ and $a > 1$, III if $r_1 = 1$ and $a > 1$, and O if $a = 1$. In (ii), the type of f is IIb$^\vee$ with $J = \{(2, 1), (4, 1)\}$.

We note that Corti obtained the result in the case where P is an ordinary double point in [3, Theorem 3.10], and that Corti and Mella obtained the result in the case where P is a cA_2 point isomorphic to $o \in (x_1 x_2 + x_3^3 + x_4^3 = 0) \subset \mathbb{C}^4$ in [4, Theorem 3.6].

On the other hand, in the remaining cD_n and cE_n cases we derive an upper-bound on the discrepancy a. The next corollary follows straightforwardly from Theorem 1.8:

Corollary 1.15. *Assume that P is cD_n or cE_n. Then f is of type O, I, IIa or IIb$^\vee$ in Theorem 1.4, and the discrepancy $a \leq 4$.*

We have examples of type O, I, IIb$^\vee$, where $a = 1, 2, 3$, in Examples 2.3, 5.3, 5.4 and 5.5. However I do not know whether type IIa happens or not. If $a = 4$ in the cD_n or cE_n case, then $r = 5$ by Theorem 3.5(iii) and P has to be cD_4, cD_5 or cD_6.

2. Basic numerical results

Let $f \colon (Y \supset E) \to (X \ni P)$ be a germ of a three-fold divisorial contraction whose exceptional divisor E contracts to a Gorenstein point P. Remark that f can be always extended to a morphism between projective varieties by the algebraisation theorem in [1] and [2].

Set $K_Y = f^* K_X + aE$, and let r be the global Gorenstein index of Y. Note that a and r are co-prime by [7, Lemma 4.3]. We take an integer e such that $ae \equiv 1$ modulo r. Let $I := \{Q : \text{type } \frac{1}{r_Q}(1, -1, b_Q)\}$ be the basket of fictitious singularities from non-Gorenstein singularities on Y. Then $(\mathcal{O}_{Y_Q}(E_Q))_Q \cong (\mathcal{O}_{Y_Q}(eK_{Y_Q}))_Q$, where

(Y_Q, E_Q) is the deformed pair at Q from (Y, E). We note that b_Q is co-prime to r_Q and that e is also co-prime to r_Q since $ae \equiv 1$ modulo r. Hence $v_Q := \overline{eb_Q}$ is co-prime to r_Q. Here $\bar{\ }$ denotes the smallest residue modulo r_Q, that is, $\overline{j} := j - \lfloor \frac{j}{r_Q} \rfloor r_Q$, where $\lfloor \ \rfloor$ denotes the round-down, that is, $\lfloor j \rfloor := \max\{k \in \mathbb{Z} \mid k \leq j\}$. Replacing b_Q with $r_Q - b_Q$ if necessary, we may assume that $v_Q \leq r_Q/2$. With this description, $r = 1$ if I is empty, and otherwise r is the lowest common multiple of $\{r_Q\}_{Q \in I}$. Set $J := \{(r_Q, v_Q)\}_{Q \in I}$. We moreover define $d(i) := \dim f_* \mathcal{O}_Y(iE)/f_* \mathcal{O}_Y((i-1)E)$. Note that $d(i) = 0$ for $i \geq 1$ and $d(0) = 1$.

Remark 2.1. If P is a singular point, then $a < \max\{r_Q\}$ unless $a \neq 1$. This is by an analogue to the proof of [8, Lemma 6.10], because there exists a valuation with centre P the discrepancy of which with respect to K_X is 1 by [14].

We have a rough numerical classification of f by [7, Theorem 4.5], which is the same as Theorem 1.4 except for the statement that $d(-1) \neq 0$ in type O. We quote this classification by the label [7, Theorem 4.5] until the additional statement is deduced from Theorem 2.7.

Remark 2.2. The possible values of J can be restricted even if f is of type O, according to the proof of [7, Theorem 4.5]. In fact, J is one of
$\{(7, 3)\}, \{(8, 3)\}, \{(2, 1), (5, 2)\}, \{(3, 1), (5, 2)\}, \{(4, 1), (5, 2)\}, \{(2, 1), (7, 2)\},$
$\{(2, 1), (2, 1), (r_3, 1)\}, \{(2, 1), (3, 1), (3, 1)\}, \{(2, 1), (3, 1), (4, 1)\},$
$\{(2, 1), (3, 1), (5, 1)\}, \{(r, 2)\}, \{(r_1, 1), (r_2, 1)\}, \{(r, 1)\}$ and $\emptyset$.
In particular, the number of the fictitious singularities is ≤ 3.

Example 2.3. The weighted blow-up of the cD_4 singularity $o \in (x_1^2 + x_2^3 + x_3^3 + x_4^6 = 0) \subset \mathbb{C}^4$ with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 2, 2, 1)$ in [7, Example 4.6] is an example for which J is $\{(2, 1), (2, 1), (2, 1)\}$.

The numerical classification comes from the singular Riemann–Roch formula [20, Theorem 10.2] and a relative vanishing theorem [10, Theorem 1-2-5] with the exact sequences

$$0 \to \mathcal{O}_Y((i-1)E) \to \mathcal{O}_Y(iE) \to \mathcal{Q}_i \to 0.$$

The sheaves $\mathcal{Q}_i$ are S_2 by [12, Proposition 5.26], and are reflexive since they are locally free on the restriction $E^o \subseteq E$ of the Gorenstein locus of Y. Thus, since E is Cohen–Macaulay, we have

(2.1) $$\mathcal{Q}_i = i_*(\mathcal{O}_Y(iE) \otimes \mathcal{O}_E|_{E^o}),$$

where i is the induced map $E^o \hookrightarrow E$.

The singular Riemann–Roch formula implies

(2.2) $$\chi(\mathcal{Q}_i) = \frac{1}{12}\{2(3i^2 - 3i + 1) - 3(2i - 1)a + a^2\}E^3$$
$$+ \frac{1}{12}E \cdot c_2(Y) + A_i - A_{i-1},$$
$$A_i := \sum_{Q \in I}\left(-\overline{ie}\frac{r_Q^2 - 1}{12r_Q} + \sum_{j=1}^{\overline{ie}-1}\frac{\overline{jb_Q}(r_Q - \overline{jb_Q})}{2r_Q}\right).$$

On the other hand a relative vanishing theorem implies

(2.3) $$R^j f_* \mathcal{O}_Y(iE) = 0 \qquad (i \leq a, \ j \geq 1).$$

We summarise the formulae in [7, Section 4] obtained from the above equalities:

Proposition 2.4. (i) $rE^3 \in \mathbb{Z}_{>0}$.

(ii) $d(i) = \chi(\mathcal{Q}_i)$ *for* $i \leq a$.

(iii) *If f is of type IIb or III and we set $r_1 = 1$, $r_2 = r$ if f is of type III, then*

$$d(-i) = 1 + \left\lfloor \frac{i}{r_1} \right\rfloor \qquad (0 \leq i < \min\{r_2, a\}).$$

(iv) *Set* $B_i := \sum_{Q \in I} \frac{\overline{iv_Q}(r_Q - \overline{iv_Q})}{2r_Q}$. *Then*

$$\chi(\mathcal{Q}_{-i}) - \chi(\mathcal{Q}_{i+1}) = \left(i + \frac{1}{2}\right) a E^3 + B_{i+1} - B_i.$$

Remark 2.5. The values of $d(-i) = \chi(\mathcal{Q}_{-i})$ below are used later:

(i) (type IIa) By direct calculation using (2.2), we obtain $d(-1) = 1$ and $d(-2) \geq 2$ if $a = 2$ or $(a, r) = (4, 5)$.

(ii) (types IIb and III) $d(-i) = 1 + \lfloor \frac{i}{r_1} \rfloor$ for $1 \leq i < \min\{r_2, a\}$ by Proposition 2.4(iii), and $d(-a) = 2 + \lfloor \frac{a}{r_1} \rfloor$ if $a < r_2$ by Lemma 2.6 and the proofs of [7, Proposition 4.4.3] and the latter part of [7, Theorem 4.5].

Lemma 2.6. $\mathcal{Q}_{a+1} = \omega_E$ *and* $h^0(\omega_E) = h^1(\omega_E) = 0$, $h^2(\omega_E) = 1$.

Proof. Let E^o on E be the restriction of the Gorenstein locus on Y, and $i \colon E^o \hookrightarrow E$ the induced map. $\omega_E = i_* \omega_{E^o}$ since ω_E is S_2. Thus $\mathcal{Q}_{a+1} = \omega_E$ by $\mathcal{Q}_{a+1}|_{E^o} = \omega_{E^o}$. $H^i(\omega_E)$ is the dual of $\mathrm{Ext}_E^{2-i}(\omega_E, \omega_E)$. Thus $h^0(\omega_E) = h^1(\omega_E) = 0$ since E is Cohen–Macaulay, and $h^2(\omega_E) = \dim \mathrm{Hom}(\omega_E, \omega_E)$. The map $\mathrm{Hom}(\omega_E, \omega_E) \to \mathrm{Hom}(\omega_{E^o}, \omega_{E^o})$ is injective since ω_E is S_2, and $\dim \mathrm{Hom}(\omega_{E^o}, \omega_{E^o}) = h^0(\mathcal{O}_{E^o}) = 1$. Hence $h^2(\omega_E) = 1$ by $\mathrm{Hom}(\omega_E, \omega_E) \neq 0$. $\square$

Let $P \in H_X$ be a general hyperplane section on X, and H its birational transform on Y. Set $f^* H_X = H + bE$. Then b is the largest integer for which $f_* \mathcal{O}_Y(-bE) = \mathfrak{m}_P$. Let Y^o be the Gorenstein locus of Y. By the adjunction formula, we have

$$\omega_H|_{Y^o \cap H} = f^* \omega_{H_X} \otimes \mathcal{O}_Y((a-b)E)|_{Y^o \cap H}.$$

$a \geq b$ because H_X is canonical. Moreover, if $a = b$, then H is normal and gives a Du Val section in the anti-canonical system of Y. Combining it with [7, Theorem 4.5], we obtain the following theorem:

Theorem 2.7. *If f is of type O or I, then the birational transform H of a general hyperplane section H_X on X gives a Du Val section in the anti-canonical system of Y.*

We investigate the scheme $H \cap E$ in the remainder of this section. We use the following commutative diagrams repeatedly:

$$\begin{array}{ccccccc}
\mathcal{O}_Y(iE) \otimes \mathcal{O}_Y(-E) & \to & \mathcal{O}_Y(iE) & \to & \mathcal{O}_Y(iE) \otimes \mathcal{O}_E & \to & 0 \\
\downarrow & & \| & & \downarrow & & \\
0 \to \quad \mathcal{O}_Y((i-1)E) & \to & \mathcal{O}_Y(iE) & \to & \mathcal{Q}_i & \to & 0, \\
& & & & \downarrow & & \\
& & & & 0 & &
\end{array}$$

$$
\begin{array}{ccccccc}
& & 0 & & & & \\
& & \uparrow & & & & \\
\mathcal{O}_Y(iE) \otimes \mathcal{I}_{H\cap E\subset E} & \rightarrow & \mathcal{O}_Y(iE) \otimes \mathcal{O}_E & \rightarrow & \mathcal{O}_Y(iE) \otimes \mathcal{O}_{H\cap E} & \rightarrow & 0 \\
\uparrow & & \| & & \| & & \\
\mathcal{O}_Y(iE) \otimes \mathcal{O}_Y(-H) \otimes \mathcal{O}_E & \rightarrow & \mathcal{O}_Y(iE) \otimes \mathcal{O}_E & \rightarrow & \mathcal{O}_Y(iE) \otimes \mathcal{O}_{H\cap E} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \rightarrow \mathcal{Q}_{i+b} & \rightarrow & \mathcal{Q}_i & \rightarrow & \mathcal{R}_i & \rightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

where b is defined as above. Chasing these diagrams, we have the following equalities by (2.3) and Lemma 2.6:

Lemma 2.8.   (i) $h^j(\mathcal{O}_Y(iE) \otimes \mathcal{O}_E) = 0$ for $i \le a, j \ge 1$, and
$h^1(\mathcal{O}_Y((a+1)E) \otimes \mathcal{O}_E) = 0$, $h^2(\mathcal{O}_Y((a+1)E) \otimes \mathcal{O}_E) = 1$.
(ii) If f is not of type I, then
$h^1(\mathcal{O}_Y(iE) \otimes \mathcal{O}_{H\cap E}) = 0$ for $i \le a-1$, and $h^1(\mathcal{O}_Y(aE) \otimes \mathcal{O}_{H\cap E}) = 1$.

Remark 2.9. $h^0(\mathcal{O}_{H\cap E}) = 1$ and $h^1(\mathcal{O}_{H\cap E}) = 0$ in any case, even if f is of type I by Theorem 2.7. In particular, $H \cap E$ has no embedded points and $(H \cap E)_{\mathrm{red}}$ is a union of $\mathbb{P}^1$.

The main ingredient we study is an irreducible reduced subscheme $C \cong \mathbb{P}^1 \subseteq H \cap E$. Here we introduce a normal form of $Q \in C \subset Y$ at each non-Gorenstein point Q of Y through which C passes, following [17, Lemma 2.7]. Let r_Q be the local index of $Q \in Y$. Take the index-one cover $\pi \colon (Y^\sharp \ni Q^\sharp) \to (Y \ni Q)$ and set $C^\sharp := (C \times_Y Y^\sharp)_{\mathrm{red}}$. We have an identification in Theorem 1.3:

$$Q^\sharp \in Y^\sharp \cong o \in (\phi = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}.$$

Let s_Q be the number of the irreducible components of $C^\sharp$, and $Q^\dagger \in C^\dagger$ the normalisation of one of the irreducible components of $C^\sharp$. Let $t \in \mathcal{O}_{C,Q}$ and $t^{s_Q/r_Q} \in \mathcal{O}_{C^\dagger, Q^\dagger}$ be uniformising parameters of C and $C^\dagger$. Let a_i be the minimal number such that there exists a semi-invariant function with weight $\mathrm{wt}\, x_i$ whose image in $\mathcal{O}_{C^\dagger, Q^\dagger}$ has order a_i/r_Q with respect to t. Note that

(2.4) $\qquad (a_1, a_2, a_3, a_4) \in s_Q \mathbb{Z}(\mathrm{wt}\, x_1, \mathrm{wt}\, x_2, \mathrm{wt}\, x_3, \mathrm{wt}\, x_4) \subseteq (\mathbb{Z}/(r_Q))^4$.

Then we can take an identification in Theorem 1.3 such that $x_i|_{C^\dagger} = t^{a_i/r_Q}$ for $i = 1, 2, 3$, and $x_4|_{C^\dagger} = t^{a_4/r_Q}$ if $Q^\sharp \in Y^\sharp$ is singular.

We show how to compute the images of some natural maps of sheaves defined on a germ $Q \in C \subset Y$ using the above data a_1, a_2, a_3, a_4. For $n \in \mathbb{Z}/(r_Q)$, write $w_Q^C(n)$ for the smallest non-negative integer such that $(w_Q^C(n), n) \in \mathbb{Z} \times \mathbb{Z}/(r_Q)$ is contained in the semi-group

(2.5) $\qquad \mathbb{Z}_{\ge 0}(a_1, \mathrm{wt}\, x_1) + \mathbb{Z}_{\ge 0}(a_2, \mathrm{wt}\, x_2) + \mathbb{Z}_{\ge 0}(a_3, \mathrm{wt}\, x_3) + \mathbb{Z}_{\ge 0}(a_4, \mathrm{wt}\, x_4)$,

$\qquad$ or $\mathbb{Z}_{\ge 0}(a_1, \mathrm{wt}\, x_1) + \mathbb{Z}_{\ge 0}(a_2, \mathrm{wt}\, x_2) + \mathbb{Z}_{\ge 0}(a_3, \mathrm{wt}\, x_3)$ if $Y^\sharp$ is smooth.

Note that $w_Q^C(0) = 0$, and that $(r_Q, 0)$ is contained in the above semi-group since C is smooth. Take a reflexive sheaf $\mathcal{L}$ on Y which is isomorphic to the ideal sheaf defined by $x_1 = 0$ outside Q. We note that $\mathcal{L}^{[\otimes v_Q]} \cong \mathcal{O}_Y(E)$ and $\mathcal{L}^{[\otimes b_Q]} \cong \mathcal{O}_Y(K_Y)$ on the germ at Q, where $\mathcal{L}^{[\otimes i]}$ denotes the double dual of $\mathcal{L}^{\otimes i}$. Then for any integers

$j_1, \ldots, j_k$ with $\sum_{1 \leq i \leq k} j_i = 0$,

$$(2.6) \qquad \mathrm{Im}[\mathcal{L}^{[\otimes j_1]} \otimes \cdots \otimes \mathcal{L}^{[\otimes j_k]} \otimes \mathcal{O}_C \to \mathcal{O}_C] = (\mathfrak{m}_{Q \subset C})^{\sum_{1 \leq i \leq k} w_Q^C(-j_i)/r_Q},$$

where $\mathfrak{m}_{Q \subset C}$ is the ideal sheaf of Q in C.

We also need to treat the sheaves $\mathcal{I}_C / \mathcal{I}_C^{(2)}$ as in [17, Section 2]. For any $C \cong \mathbb{P}^1 \subset H \cap E$, its ideal sheaf as the subscheme in Y is denoted by $\mathcal{I}_C$, and its symbolic 2-power by $\mathcal{I}_C^{(2)}$. Let $C^{(2)} \subset Y$ be the closed subscheme defined by $\mathcal{I}_C^{(2)}$. Then $C^{(2)} \subseteq 2H \cap 2E$. $H^1(\mathcal{O}_{2H \cap 2E}) = 0$ unless f is of type O or I, as in Lemma 2.8, whence $H^1(\mathcal{O}_{C^{(2)}}) = 0$. By the exact sequence

$$0 \to \mathcal{I}_C / \mathcal{I}_C^{(2)} \to \mathcal{O}_{C^{(2)}} \to \mathcal{O}_C \to 0,$$

we obtain that $H^1(\mathcal{I}_C / \mathcal{I}_C^{(2)}) = 0$ unless f is of type O or I. On the other hand, consider the natural map

$$(2.7) \qquad\qquad \bigwedge^2 \mathcal{I}_C / \mathcal{I}_C^{(2)} \otimes \mathcal{O}_C(K_C) \to [\mathcal{O}_Y(K_Y)]_C,$$

where $[\mathcal{L}]_C$ denotes the torsion-free part of $\mathcal{L} \otimes \mathcal{O}_C$. This map has cokernel of length $2 + \deg[\mathcal{O}_Y(K_Y)]_C - \deg \bigwedge^2 \mathcal{I}_C / \mathcal{I}_C^{(2)}$. Therefore we obtain the next result:

Lemma 2.10. *The map (2.7) has cokernel of length $\leq 4 + \deg[\mathcal{O}_Y(K_Y)]_C$ unless f is of type O or I.*

The next lemma is obtained by following [17, Corollary 2.15] almost faithfully. Remark that $(2,2) \in \mathbb{Z} \times \mathbb{Z}/(4)$ is contained in the semi-group (2.5) when Q is of exceptional type (ii) in Theorem 1.3 since C is smooth at Q.

Lemma 2.11 ([17, Corollary 2.15]). *Let $Q \in C \subset Y$ be a singular point of Y. Then the map (2.7) is not surjective at Q.*

The following lemma shows how to compute of the length of the cokernel of the map (2.7) in the simplest case:

Lemma 2.12. *Let $Q \in Y$ be a terminal quotient singularity $\mathbb{C}^3_{x_1 x_2 x_3} / \frac{1}{r}(1, -1, b)$, $Q \in C \subset Y$ a smooth curve, and t a uniformising parameter with $(x_1, x_2, x_3)|_C = (t^{c/r}, t^{1-c/r}, 0)$ for some $0 < c < r$. Then the map (2.7) has cokernel of length $\min\{c, r - c\}$.*

Proof. First we give an explicit description of this map using local coordinates. The map $\mathcal{O}_Y(K_Y) \otimes \mathcal{O}_C \to [\mathcal{O}_Y(K_Y)]_C \cong \mathcal{O}_C$ can be locally given, but not naturally, by the map

$$(2.8) \qquad\qquad \mathcal{O}_Y(K_Y) \otimes \mathcal{O}_C \to \mathcal{O}_C,$$

$$f(x_1, x_2, x_3) dx_1 \wedge dx_2 \wedge dx_3 \mapsto \frac{f(t^{c/r}, t^{1-c/r}, 0)}{t^{w_Q^C(-b)/r}}.$$

On the other hand $\mathcal{O}_C(K_C)$ is generated by $d(x_1 x_2)$. Hence our map has cokernel of length equal to that of the composition of the map

$$\mathcal{I}_C / \mathcal{I}_C^{(2)} \times \mathcal{I}_C / \mathcal{I}_C^{(2)} \to \mathcal{O}_Y(K_Y) \otimes \mathcal{O}_C,$$

$$f_1 \times f_2 \mapsto df_1 \wedge df_2 \wedge d(x_1 x_2)$$

and the map (2.8).

It is easy to see $\mathcal{I}_C / \mathcal{I}_C^{(2)}$ is generated by functions of the form $g_{-b}(x_1, x_2) \cdot x_3$ and $g_c(x_1, x_2) \cdot (x_1^{r-c} - x_2^c)$ where g_{-b} and g_c are semi-invariant functions of weights

$-b$, respectively c. Therefore we have only to consider the images of $g_{-b}dx_3 \wedge d(g_c \cdot (x_1^{r-c} - x_2^c)) \wedge d(x_1x_2)$ by the map (2.8). It suffices to consider those for $g_c = x_1^c$ and x_2^{r-c}. We have

$$g_{-b}dx_3 \wedge d(g_c \cdot (x_1^{r-c} - x_2^c)) \wedge d(x_1x_2) = \begin{cases} g_{-b} \cdot rx_1^r dx_1 \wedge dx_2 \wedge dx_3 & (g_c = x_1^c), \\ g_{-b} \cdot rx_2^r dx_1 \wedge dx_2 \wedge dx_3 & (g_c = x_2^{r-c}). \end{cases}$$

Since we can choose g_{-b} so that $g_{-b}(t^{c/r}, t^{1-c/r}) = t^{w_Q^C(-b)/r}$, our length is the minimum of the orders of x_1^r and x_2^r with $(x_1, x_2) = (t^{c/r}, t^{1-c/r})$ with respect to t, which is $\min\{c, r-c\}$.   □

We need to compute the length in the more complicated case below. Let $Q \in Y$ be a terminal quotient singularity $\mathbb{C}^3_{x_1x_2x_3}/\frac{1}{r}(1, -1, b)$, $Q \in C \subset Y$ a smooth curve, and t a uniformising parameter with $(x_1, x_2, x_3)|_C = (t^{a_1/r}, t^{a_2/r}, t^{a_3/r})$. We can take an invariant monomial $g(x_1, x_2, x_3)$ so that $g(t^{a_1/r}, t^{a_2/r}, t^{a_3/r}) = t$. Then $\mathcal{I}_C/\mathcal{I}_C^{(2)}$ is generated by functions of the form $x_1^{s_1} x_2^{s_2} x_3^{s_3} - x_1^{t_1} x_2^{t_2} x_3^{t_3}$ with $(s_1, s_2, s_3) \neq (t_1, t_2, t_3)$ such that $s_1 - s_2 + bs_3 \equiv t_1 - t_2 + bt_3 \equiv 0$ modulo r and $a_1s_1 + a_2s_2 + a_3s_3 = a_1t_1 + a_2t_2 + a_3t_3$. Our length is the minimum of the orders of the images of $d(x_1^{s_1} x_2^{s_2} x_3^{s_3} - x_1^{t_1} x_2^{t_2} x_3^{t_3}) \wedge d(x_1^{\bar{s}_1} x_2^{\bar{s}_2} x_3^{\bar{s}_3} - x_1^{\bar{t}_1} x_2^{\bar{t}_2} x_3^{\bar{t}_3}) \wedge dg$ under the map

$$\mathcal{O}_Y(K_Y) \otimes \mathcal{O}_C \to \mathcal{O}_C,$$

$$f(x_1, x_2, x_3)dx_1 \wedge dx_2 \wedge dx_3 \mapsto \frac{f(t^{a_1/r}, t^{a_2/r}, t^{a_3/r})}{t^{w_Q^C(-b)/r}},$$

where $x_1^{s_1} x_2^{s_2} x_3^{s_3} - x_1^{t_1} x_2^{t_2} x_3^{t_3}$ and $x_1^{\bar{s}_1} x_2^{\bar{s}_2} x_3^{\bar{s}_3} - x_1^{\bar{t}_1} x_2^{\bar{t}_2} x_3^{\bar{t}_3}$ satisfy the conditions mentioned above. The order for fixed $x_1^{s_1} x_2^{s_2} x_3^{s_3} - x_1^{t_1} x_2^{t_2} x_3^{t_3}$ and $x_1^{\bar{s}_1} x_2^{\bar{s}_2} x_3^{\bar{s}_3} - x_1^{\bar{t}_1} x_2^{\bar{t}_2} x_3^{\bar{t}_3}$ is $+\infty$ or

$$\frac{1}{r} \cdot \{(a_1s_1 + a_2s_2 + a_3s_3) + (a_1\bar{s}_1 + a_2\bar{s}_2 + a_3\bar{s}_3) + r - (a_1 + a_2 + a_3 + w_Q^C(-b))\}.$$

In particular, when we can take $g = x_1x_2$, it is $+\infty$ or

$$\frac{1}{r} \cdot \{(a_1s_1 + a_2s_2 + a_3s_3) + (a_1\bar{s}_1 + a_2\bar{s}_2 + a_3\bar{s}_3) - (a_3 + w_Q^C(-b))\}.$$

3. LOCAL ANALYSIS AT A NON-GORENSTEIN POINT

In this section we analyse the structure of $C \cong \mathbb{P}^1 \subseteq H \cap E$ under the assumption that f is of type II or III, focusing on the germ at a non-Gorenstein point. We can see the theorems in this section straightforwardly when P is a smooth point because of the explicit description [7]. Hence we here impose an extra assumption, that is, P is a singular point. We have $(H \cdot C) \leq (H \cdot [H \cap E]) = E^3$. By Theorem 1.4, Proposition 2.4(i) and Remarks 2.1 and 2.9, the configuration of the 1-cycle $[H \cap E] = \sum[C_i]$ is as follows:

Description 3.1.   (i) (type IIa) $a = 2$ or 4 and Y has one non-Gorenstein point Q, through which any C_i passes.
  (a) $a = 2$ and $H \cap E \cong \mathbb{P}^1$ scheme-theoretically.
  (b) $a = 2$ and $[H \cap E] = [\mathbb{P}^1] + [\mathbb{P}^1]$ is reducible or non-reduced.
  (c) $a = 4$ and $H \cap E \cong \mathbb{P}^1$ scheme-theoretically.

(ii) (type IIb$^\vee$) $J = \{(r,1),(r,1)\}$ or $\{(2,1),(4,1)\}$ and $a = 2$, respectively 3. Y has one non-Gorenstein point Q. $H \cap E \cong \mathbb{P}^1$ scheme-theoretically and passes through Q.

(iii) (type IIb$^{\vee\vee}$) $a \leq (r_1 + r_2)/2$ and Y has two non-Gorenstein points Q_1 and Q_2 of indices r_1 and r_2. Exactly one of the irreducible components of $(H \cap E)_{\mathrm{red}}$ passes through both Q_1 and Q_2, and all the others pass through only Q_2.

(iv) (type III) $a \leq (1 + r)/2$ and Y has one non-Gorenstein point Q, through which any C_i passes.

Let $s_C(i)$ be the integer such that $[\mathcal{O}_Y(iE)]_C \cong \mathcal{O}_{\mathbb{P}^1}(s_C(i))$. The following is easy to see by Lemmata 2.8 and 3.3 and Remark 2.1:

Lemma 3.2.   (i) *Assume that C comes from type IIa. Then*
  (a) $s_C(1) = \cdots = s_C(a-1) = -1$.
  (b) $s_C(-i) \geq -1$ *for $i \geq 1$*.
  (c) $s_C(a) = -2$ *in Description 3.1(ia), (ic), and*
      $s_C(a) = -2$ *or -1 in Description 3.1(ib).*
 (ii) *Assume that C comes from type IIb$^\vee$. Then*
  (a) $s_C(1) = \cdots = s_C(a-1) = -1$. $s_C(a) = -2$.
  (b) $s_C(-i) \geq -1$ *for $i \geq 1$*.
(iii) *Assume that C comes from type IIb$^{\vee\vee}$ with $Q_1, Q_2 \in C$. Then*
  (a) $s_C(1) = \cdots = s_C(a-1) = -1$.
  (b) $s_C(-i) = -1$ *for $1 \leq i \leq a-1$, $r_1 \nmid i$.*
  (c) $s_C(-r_1 i) = 0$ *or -1 for $1 \leq r_1 i \leq a-1$.*
  (d) $(s_C(a), s_C(-a)) = (-2,0)$, $(-2,-1)$ *or $(-1,-1)$.*
 (iv) *Assume that C comes from type III. Then*
  (a) $s_C(1) = \cdots = s_C(a-1) = -1$.
  (b) $s_C(-1) = \cdots = s_C(-(a-1)) = 0$.
  (c) $(s_C(a), s_C(-a)) = (-2,1)$, $(-2,0)$ *or $(-1,0)$.*

Proof. For instance, we consider the case (iv). $s_C(i) < 0$ for $i \geq 1$ by the existence of the natural map $\mathcal{O}_Y(iE)^{\otimes r} \otimes \mathcal{O}_C \to \mathcal{O}_Y(irE) \otimes \mathcal{O}_C$, so $s_C(1) = \cdots = s_C(a-1) = -1$, $s_C(a) = -2$ or -1 by Lemma 2.8. On the other hand, Lemma 3.3 with $\mathcal{L} = \mathcal{O}_Y(E)$ implies that $s_C(-1) = 0$, so $s_C(-i) \geq 0$ for $i \geq 1$ by the existence of the natural map $\mathcal{O}_Y(-E)^{\otimes i} \otimes \mathcal{O}_C \to \mathcal{O}_Y(-iE) \otimes \mathcal{O}_C$. Hence $s_C(-1) = \cdots = s_C(-(a-1)) = 0$ and $(s_C(a), s_C(-a)) = (-2,1)$, $(-2,0)$ or $(-1,0)$, because the map $\mathcal{O}_Y(-iE) \otimes \mathcal{O}_Y(iE) \otimes \mathcal{O}_C \to \mathcal{O}_C$ is not surjective for $1 \leq i < r$ and $a < r$ by Remark 2.1.   □

Lemma 3.3. *Let $Q \in Y$ be a terminal quotient singularity $\mathbb{C}^3_{x_1 x_2 x_3}/\frac{1}{r}(1,-1,b)$, and $\mathcal{L}$ a reflexive sheaf on Y which is isomorphic to the ideal sheaf defined by $x_1 = 0$ outside Q. Then the image of the natural map $\mathcal{L} \otimes \mathcal{L}^{[-1]} \to \mathcal{O}_Y$ is the ideal sheaf of Q in Y.*

Proof. It is enough to show that any invariant monomial $\neq 1$ of x_1, x_2, x_3 decomposes into two semi-invariant monomials with weights 1 and -1, but it is trivial since b is co-prime to r.   □

The following lemma stores more information on s_C:

Lemma 3.4. *Assume that f is not of type O or I, and take a curve $C \cong \mathbb{P}^1 \subseteq H \cap E$. Let i be a positive integer, and T, T' surfaces on Y defined by general elements in $\mathcal{O}_Y(-iE)$.*

(i) *Assume that $\chi(\mathcal{Q}_{-i}) \geq 2$. Then $(T \cap E)_{\mathrm{red}}$ has an irreducible component which properly intersects $(T' \cap E)_{\mathrm{red}}$.*

(ii) *Assume that $\chi(\mathcal{Q}_{-j}) = 1$ for $1 \leq j < i$ and $\chi(\mathcal{Q}_{-i}) \geq 2$. If $C = H \cap E$ scheme-theoretically, then $(T \cap E)_{\mathrm{red}}$ properly intersects $(T' \cap E)_{\mathrm{red}}$ and C. Moreover, $s_C(-j) = -1$ for $1 \leq j < i$ and $s_C(-i) \geq 0$.*

Proof. (i) By (2.3), we have a surjective map $f_* \mathcal{O}_Y(-iE) \twoheadrightarrow H^0(\mathcal{Q}_{-i})$. Write E^o for the restriction of the Gorenstein locus of Y to E. Take a resolution $\hat{E}$ of E and let $\hat{E}^o$ be the pre-image of E^o. Note that $H^0(\mathcal{Q}_{-i}|_{E^o}) = H^0(\mathcal{Q}_{-i})$ by (2.1). Consider a linear system L on $\hat{E}$, not necessarily complete, whose restriction to $\hat{E}^o$ is $H^0(\mathcal{Q}_{-i}|_{E^o})$. $h^0(\mathcal{Q}_{-i}) = \chi(\mathcal{Q}_{-i}) \geq 2$ means that L moves, and its general member has an irreducible component which does not appear in the union of the support of the fixed locus of L and the pre-image of the non-normal locus of E. The element in $H^0(\mathcal{Q}_{-i})$ corresponding to this member gives the desired section T.

(ii) By the commutative diagrams in Section 2 and Lemma 2.8(i), we have surjective maps

$$f_* \mathcal{O}_Y(-jE) \twoheadrightarrow H^0(\mathcal{Q}_{-j}) \twoheadrightarrow H^0(\mathcal{R}_{-j}) \twoheadrightarrow H^0([\mathcal{O}_Y(-jE)]_C) \quad (1 \leq j \leq i).$$

Remark that E is smooth at the generic point of C because $C = H \cap E$ scheme-theoretically. $h^0(\mathcal{Q}_{-j}) = \chi(\mathcal{Q}_{-j}) = 1$ for $1 \leq j < i$ and a non-zero element in $H^0(\mathcal{Q}_{-j})$ corresponds to, at the generic point of C, the Cartier divisor defined by jH. Hence the map $H^0(\mathcal{Q}_{-j}) \twoheadrightarrow H^0([\mathcal{O}_Y(-jE)]_C)$ is a zero map and we obtain $s_C(-j) = -1$ by Lemma 2.8(ii). On the other hand, the linear system L constructed from $H^0(\mathcal{Q}_{-i})$ as in the proof of (i) moves. Because $h^0(\mathcal{Q}_{-j}) = 1$ for $1 \leq j < i$ and E is smooth at the generic point of C, its general member does not have the birational transform of C as its component. Thus we have the desired section and the map $H^0(\mathcal{Q}_{-i}) \twoheadrightarrow H^0([\mathcal{O}_Y(-iE)]_C)$ is a non-zero map, whence $s_C(-i) \geq 0$. $\square$

First we treat type IIa.

Theorem 3.5. *Assume that f is of type IIa. Let $Q \in C \cong \mathbb{P}^1 \subseteq H \cap E$ be the non-Gorenstein point with an irreducible component of $H \cap E$ as in Description 3.1(i). Take a normal form of $Q \in C \subset Y$ as in Section 2.*

(i) *The case* (ia) *in Description 3.1 does not happen.*

(ii) *In the case* (ib) *in Description 3.1, we have $s_C(-2) = 0$ and $(a_1, a_2, a_3) = ((r+1)/2, (r-1)/2, 2)$. In particular, we may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, 4)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that $(x_1, x_2, x_3)|_{C^\dagger} = (t^{(r+1)/2r}, 0, t^{2/r})$ when $r \equiv 1$ modulo 4, and $(0, t^{(r-1)/2r}, t^{2/r})$ when $r \equiv 3$ modulo 4.*

(iii) *In the case* (ic) *in Description 3.1, we have $r = 5$, $s_C(-4) = 0$ and $(a_1, a_2, a_3) = (3, 2, 4)$. In particular, we may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, 3)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that $(x_1, x_2, x_3)|_{C^\dagger} = (t^{3/5}, t^{2/5}, 0)$.*

Proof. (i). Compute the image of the map

$$\mathcal{O}_Y(E)^{\otimes r} \otimes \mathcal{O}_Y(-rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$$

by (2.6). It is $(\mathfrak{m}_{Q\subset C})^{w_Q^C(-2)}$. On the other hand, this image equals that of

$$[\mathcal{O}_Y(E)]_C^{\otimes r} \otimes [\mathcal{O}_Y(-rE)]_C \to \mathcal{O}_C,$$

whence $rs_C(1) - r(E \cdot C) = -w_Q^C(-2)$. By $s_C(1) = -1$ in Lemma 3.2(i) and $(H \cdot C) = E^3 = 2/r$ in Theorem 1.4, we have $w_Q^C(-2) = r - 2$. Hence we can write $(a_1, a_2, a_3) = (1 + rm_1, r - 1 + rm_2, 4 + rm_3)$ with $m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}$ by (2.4). We see that $m_1 = 0$ easily since there exists an invariant monomial of x_1, x_2, x_3 whose restriction to C is t. Thus $(a_1, a_2, a_3) = (1, r - 1, 4)$ and $w_Q^C(-4) = r - 4$. Consider another map $\mathcal{O}_Y(2E)^{\otimes r} \otimes \mathcal{O}_Y(-2rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$. Then we have $rs_C(2) - 2r(E \cdot C) = -w_Q^C(-4)$, whence $s_C(2) = -1$ by $w_Q^C(-4) = r - 4$ and $(H \cdot C) = 2/r$. But this contradicts Lemma 3.2(i).

(ii) By the map $\mathcal{O}_Y(E)^{\otimes r} \otimes \mathcal{O}_Y(-rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $rs_C(1) - r(E \cdot C) = -w_Q^C(-2)$. Hence $w_Q^C(-2) = r - 1$ by Lemma 3.2(i) and $(H \cdot C) = E^3/2 = 1/r$ in Theorem 1.4, and we can write $(a_1, a_2, a_3) = ((r + 1)/2 + rm_1, (r - 1)/2 + rm_2, 2 + rm_3)$ with $m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}$ by (2.4). Because there exists an invariant monomial of x_1, x_2, x_3 whose restriction to C is t, one of the following holds:

(i) $(r, 0) \in \mathbb{Z}_{>0}(a_1, 1) + \mathbb{Z}_{>0}(a_3, 4)$ or $\in \mathbb{Z}_{>0}(a_2, -1) + \mathbb{Z}_{>0}(a_3, 4)$ in $\mathbb{Z} \times \mathbb{Z}/(r)$.
(ii) $r = a_1 + a_2$.

If (i) holds, $(a_1, a_2, a_3) = ((r + 1)/2, (r - 1)/2, 2)$ and $w_Q^C(4) = 4$. By the map $\mathcal{O}_Y(-2E)^{\otimes r} \otimes \mathcal{O}_Y(2rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $rs_C(-2) + 2r(E \cdot C) = -w_Q^C(4)$, whence $s_C(-2) = 0$. Now we assume (ii). Then $(a_1, a_2, a_3) = ((r + 1)/2, (r - 1)/2, 2 + rm_3)$ with some m_3. We have nothing to do if $m_3 = 0$, so we also assume that $m_3 > 0$ and $r \geq 7$. Note that $m_3 \leq 2$ by $4a_1 = 2 + 2r$.

Suppose that $m_3 = 2$. We may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, 4)$ of $Q^\sharp \in Y^\sharp$ so that $(x_1, x_2, x_3)|_{C^\dagger} = (t^{(r+1)/2r}, t^{(r-1)/2r}, 0)$. We see that $w_Q^C(-4) = 4w_Q^C(-1) = 2r - 2$ by $r \geq 7$. Considering the map $\mathcal{O}_Y(2E)^{\otimes r} \otimes \mathcal{O}_Y(-2rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we obtain $rs_C(2) - 2r(E \cdot C) = -w_Q^C(-4)$, whence $s_C(2) = -2$. By Lemma 2.10, the map (2.7) has cokernel of length ≤ 2 at Q. Thus $(r - 1)/2 \leq 2$ by Lemma 2.12, which contradicts $r \geq 7$.

It remains to exclude the case where $(a_1, a_2, a_3) = ((r + 1)/2, (r - 1)/2, r + 2)$ with $r \geq 7$. But if $r = 7$, then $(a_1, a_2, a_3) = (4, 3, 9)$ and we may choose the coordinates so that $(x_1, x_2, x_3)|_{C^\dagger} = (t^{4/7}, t^{3/7}, 0)$. Then $s_C(2) = -2$ and it contradicts Lemmata 2.10 and 2.12. Therefore we may assume that $r \geq 9$. By the map $\mathcal{O}_Y(-3E)^{\otimes r} \otimes \mathcal{O}_Y(3rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $rs_C(-3) + 3r(E \cdot C) = -w_Q^C(6)$, whence $w_Q^C(6) = 3$ or $r + 3$ by $s_C(-3) \geq -1$. From $(a_1, a_2, a_3) = ((r + 1)/2, (r - 1)/2, r + 2)$ with $r \geq 9$, r has to be 9. When $r = 9$, $(a_1, a_2, a_3) = (5, 4, 11)$ and $w_Q^C(-4) = 16$. By the map $\mathcal{O}_Y(2E)^{\otimes r} \otimes \mathcal{O}_Y(-18E) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $9s_C(2) - 18(E \cdot C) = -w_Q^C(-4)$, whence $s_C(2) = -2$. We consider $x_1^{s_1} x_2^{s_2} x_3^{s_3} - x_1^{t_1} x_2^{t_2} x_3^{t_3}$ with $(s_1, s_2, s_3) \neq (t_1, t_2, t_3)$ such that $s_1 - s_2 + 4s_3 \equiv t_1 - t_2 + 4t_3 \equiv 0$ modulo 9 and $5s_1 + 4s_2 + 11s_3 = 5t_1 + 4t_2 + 11t_3$. We can easily see that $(5s_1 + 4s_2 + 11s_3)/9 \geq 3$ for any such $x_1^{s_1} x_2^{s_2} x_3^{s_3} - x_1^{t_1} x_2^{t_2} x_3^{t_3}$. Hence by the last paragraph in Section 2, the map (2.7) has cokernel of length $\geq 3 + 3 - (a_3 + w_Q^C(-4))/9 = 3$. It contradicts Lemma 2.10 and $s_C(2) = -2$.

(iii) Consider the maps $\mathcal{O}_Y(iE)^{\otimes r} \otimes \mathcal{O}_Y(-irE) \otimes \mathcal{O}_C \to \mathcal{O}_C$ for $i = 1, 2, 3, 4$. Then we have $rs_C(i) - ir(E \cdot C) = -w_Q^C(-2i)$, whence $w_Q^C(-2i) = r - i$ for $i = 1, 2, 3$ and $w_Q^C(-8) = 2r - 4$ by Lemma 3.2(ic). We can write $(a_1, a_2, a_3) = ((r + 1)/2 + rm_1, (r - 1)/2 + rm_2, 4 + rm_3)$ with $m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}$ by (2.4). If $r = 5$, then

$a_1 = w_Q^C(-4) = 3$ and $a_2 = w_Q^C(-6) = 2$ and we have the desired description. In this case, we have $w_Q^C(8) = 4$. By the map $\mathcal{O}_Y(-4E)^{\otimes r} \otimes \mathcal{O}_Y(4rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we obtain $rs_C(-4) + 4r(E \cdot C) = -w_Q^C(8)$, whence $s_C(-4) = 0$.

We derive a contradiction supposing that $r \geq 7$. Since $w_Q^C(-4) = r - 2$, we can decompose $r - 2$ into $(r+1)/2 + 4c$ or $(r-1)/2 + 4c$ for some $c > 0$. Then $r \equiv 5$, respectively $3 \pmod 8$. Since $w_Q^C(-6) = r - 3$, similarly we can decompose $r - 3$ into $(r+1)/2+4c$, $(r-1)/2+4c$ or $4c$. Then $r \equiv 7 \pmod 8$, $r \equiv 5 \pmod 8$ or $r \equiv 3 \pmod 8$ respectively. Hence $r \equiv 3$ or $5 \pmod 8$ with $m_2 = m_3 = 0$, and $r \geq 11$. $(a_1, a_2, a_3) = ((r+1)/2, (r-1)/2, 4)$, since $(r, 0)$ is contained in the semi-group (2.5). We consider $x_1^{s_1} x_2^{s_2} x_3^{s_3} - x_1^{t_1} x_2^{t_2} x_3^{t_3}$ with $(s_1, s_2, s_3) \neq (t_1, t_2, t_3)$ such that $s_1 - s_2 + 8s_3 \equiv t_1 - t_2 + 8t_3 \equiv 0$ modulo r and $a_1 s_1 + a_2 s_2 + a_3 s_3 = a_1 t_1 + a_2 t_2 + a_3 t_3$. Then $(a_1 s_1 + a_2 s_2 + a_3 s_3)/r = 2$ only if each of $x_1^{s_1} x_2^{s_2} x_3^{s_3}$ and $x_1^{t_1} x_2^{t_2} x_3^{t_3}$ is $x_1^2 x_2^2$, $x_1 x_3^{(3r-1)/8}$ with $r \equiv 3 \pmod 8$, or $x_2 x_3^{(3r+1)/8}$ with $r \equiv 5 \pmod 8$, and otherwise $(a_1 s_1 + a_2 s_2 + a_3 s_3)/r \geq 3$. Hence by the last paragraph in Section 2, the map (2.7) has cokernel of length $\geq 2 + 3 - (a_3 + w_Q^C(-8))/r = 3$. It contradicts Lemma 2.10 and $s_C(4) = -2$ in Lemma 3.2(i). $\square$

It is easy to see the local structure in type IIb$^\vee$.

Theorem 3.6. *Assume that f is of type IIb$^\vee$. Let $Q \in C = H \cap E \cong \mathbb{P}^1$ be the non-Gorenstein point as in Description 3.1(ii). Take a normal form of $Q \in C \subset Y$ as in Section 2.*

(i) *If $J = \{(r,1), (r,1)\}$, then $a = 2$, $s_C(-2) = 0$ and $(a_1, a_2, a_3, a_4) = (r+1, r-1, 2, r)$. In particular, we may choose semi-invariant local coordinates x_1, x_2, x_3, x_4 with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (1, -1, 2, 0)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that $(x_1, x_2, x_3, x_4)|_{C^\dagger} = (0, 0, t^{2/r}, t)$.*

(ii) *If $J = \{(2,1), (4,1)\}$, then $a = 3$, $s_C(-3) = 0$ and $(a_1, a_2, a_3, a_4) = (5, 3, 3, 2)$. In particular, we may choose semi-invariant local coordinates x_1, x_2, x_3, x_4 with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (1, 3, 3, 2)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that $(x_1, x_2, x_3, x_4)|_{C^\dagger} = (0, 0, t^{3/4}, t^{1/2})$.*

Proof. (i) First applying Lemma 3.4(ii) with $i = 2$ by Remark 2.5(ii), we have $s_C(-1) = -1$ and $s_C(-2) \geq 0$. By the map $\mathcal{O}_Y(-2E) \otimes \mathcal{O}_Y(E) \otimes \mathcal{O}_C \to \mathcal{O}_Y(-E) \otimes \mathcal{O}_C$, we have $s_C(-2) + s_C(1) \leq s_C(-1)$, whence $s_C(-2) = 0$. By the map $\mathcal{O}_Y(-E)^{\otimes r} \otimes \mathcal{O}_Y(rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $rs_C(-1) + r(E \cdot C) = -w_Q^C(1)$. By $s_C(-1) = -1$ and $E^3 = 1/r$, we obtain $w_Q^C(1) = 1 + r$. Thus we can write $(a_1, a_2, a_3, a_4) = (r+1+rm_1, r-1+rm_2, 2+rm_3, r)$ with $m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}$ by (2.4) and the fact that $(r, 0)$ is contained in the semi-group (2.5). Consider another map $\mathcal{O}_Y(-2E)^{\otimes r} \otimes \mathcal{O}_Y(2rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$. Then we have $rs_C(-2) + 2r(E \cdot C) = -w_Q^C(2)$. Therefore $w_Q^C(2) = 2$ from $s_C(-2) = 0$, whence $(a_1, a_2, a_3, a_4) = (r+1, r-1, 2, r)$.

(ii) Applying Lemma 3.4(ii) with large i by Remark 2.5(ii), we obtain $s_C(-1) = -1$. By the map $\mathcal{O}_Y(-E)^{\otimes 4} \otimes \mathcal{O}_Y(4E) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $4s_C(-1) + 4(E \cdot C) = -w_Q^C(1)$. By $s_C(-1) = -1$ and $E^3 = 1/4$, we have $w_Q^C(1) = 5$. Hence we can write $(a_1, a_2, a_3, a_4) = (5 + 4m_1, 3 + 4m_2, 3 + 4m_3, 2)$ with $m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}$. Consider another map $\mathcal{O}_Y(E)^{\otimes 4} \otimes \mathcal{O}_Y(-4E) \otimes \mathcal{O}_C \to \mathcal{O}_C$. Then we have $4s_C(1) - 4(E \cdot C) = -w_Q^C(-1)$. Thus $w_Q^C(-1) = 3$ by Lemma 3.2(ii), whence $(a_1, a_2, a_3, a_4) = (5, 3, 3, 2)$. Furthermore, $w_Q^C(3) = 3$. By the map $\mathcal{O}_Y(-3E)^{\otimes 4} \otimes \mathcal{O}_Y(12E) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $4s_C(-3) + 12(E \cdot C) = -w_Q^C(3)$, whence $s_C(-3) = 0$. $\square$

It is hardest to treat types IIb$^{\vee\vee}$ and III.

Description 3.7.   (i) If f is of type IIb$^{\vee\vee}$, there exists the unique irreducible component $C \cong \mathbb{P}^1 \subseteq H \cap E$ through Q_1 and Q_2. We do not consider any other components from now on. Take normal forms of $Q_i \in C \subset Y$ for $i = 1, 2$ as in Section 2, using local coordinates x_{i1}, x_{i2}, x_{i3} and numbers a_{i1}, a_{i2}, a_{i3} respectively.

   (ii) If f is of type III, we take any irreducible component $C \cong \mathbb{P}^1 \subseteq H \cap E$ and a normal form of $Q \in C \subset Y$ as in Section 2.

Lemma 3.8. *Assume that $s_C(-a) = 0$ in Description 3.7(i), or $s_C(-a) = 1$ in Description 3.7(ii). Then $C = H \cap E$ scheme-theoretically and $s_C(a) = -2$. Moreover, for any non-Gorenstein point $Q \in Y$ with $Q \in C \cong \mathbb{P}^1 = H \cap E$ in Description 3.7, we may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, a)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that*

$$Q^\sharp \in C^\sharp \subset Y^\sharp \cong o \in (x_3\text{-axis}) \subset \mathbb{C}^3_{x_1 x_2 x_3}.$$

Proof. By the map $\mathcal{O}_Y(-aE)^{\otimes r} \otimes \mathcal{O}_Y(arE) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have

$$rs_C(-a) + ar(E \cdot C) = \begin{cases} -rw^C_{Q_1}(a)/r_1 - rw^C_{Q_2}(a)/r_2 & (\text{Description 3.7(i)}), \\ -w^C_Q(a) & (\text{Description 3.7(ii)}). \end{cases}$$

Therefore by Theorem 1.4,

$$\frac{w^C_{Q_1}(a)}{r_1} + \frac{w^C_{Q_2}(a)}{r_2} = a(H \cdot C) \leq \frac{1}{r_1} + \frac{1}{r_2} \qquad (\text{Description 3.7(i)}),$$

$$\frac{w^C_Q(a)}{r} = a(H \cdot C) - 1 \leq \frac{1}{r} \qquad (\text{Description 3.7(ii)}).$$

Hence $C = H \cap E$ by Remark 2.9, and $w^C_Q(a) = 1$ for any $Q \in C$ in Description 3.7. Therefore we have the desired description. $s_C(a) = -2$ follows from Lemma 3.2. $\square$

Lemma 3.9. *Assume that f is of type IIb$^{\vee\vee}$ or III. Consider any non-Gorenstein point $Q \in Y$ with $Q \in C \cong \mathbb{P}^1 \subseteq H \cap E$ given in Description 3.7. Then one of the following holds*:

   (i) *One of $(a_1, 1)$ and $(a_2, -1)$ is generated by the other and (a_3, a) in $\mathbb{Z} \times \mathbb{Z}/(r_Q)$. We may choose semi-invariant local coordinates x_1, x_2, x_3 with weight $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, a)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that $(x_1, x_2, x_3)|_{C^\dagger} = (0, t^{c_Q/r_Q}, t^{1-\overline{ac_Q}/r_Q})$ or $(t^{c_Q/r_Q}, 0, t^{\overline{ac_Q}/r_Q})$ for some $0 < c_Q < r_Q$.*

   (ii) *f is of type III, $a = 2$, $r \geq 5$, $s_Q(2) = -2$, and $(a_1, a_2, a_3) = (2, r-2, 4)$ or $(r-2, 2, 2r-4)$. In particular, we may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, 2)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that $(x_1, x_2, x_3)|_{C^\dagger} = (t^{2/r}, t^{1-2/r}, 0)$ or $(t^{1-2/r}, t^{2/r}, 0)$. $(H \cdot C)$ equals $2/r$, respectively $1 - 2/r$.*

Proof. Recall that there exists an invariant monomial in x_1, x_2, x_3 whose restriction to C is t. Thus one of the following holds:

   (i) One of a_1, a_2, a_3 is 1.

   (ii) $(r_Q, 0) \in \mathbb{Z}_{>0}(a_1, 1) + \mathbb{Z}_{>0}(a_3, a)$, or $\in \mathbb{Z}_{>0}(a_2, -1) + \mathbb{Z}_{>0}(a_3, a)$ in $\mathbb{Z} \times \mathbb{Z}/(r_Q)$.

(iii) $r_Q = a_1 + a_2$.

If (i) or (ii) happens, one of $(a_1, 1)$ and $(a_2, -1)$ is generated by the other and (a_3, a). Thus, from now on we may assume that only (iii) of the above holds. By permutation of x_i, we may also assume that $r_Q \geq 5$. Then $w_Q^C(i) + w_Q^C(-i) \neq r_Q$ for $i = 2, 3$, whence the image of the map $[\mathcal{O}_Y(-iE)]_C \otimes [\mathcal{O}_Y(iE)]_C \to \mathcal{O}_C$ at Q is properly contained in $\mathfrak{m}_{Q \subset C}$ for $i = 2, 3$ by (2.6). By Lemma 3.2, one of the following occurs:

   (i) $a = 2$, $s_C(2) = -2$, and $s_C(-2) = -1$ in Description 3.7(i) or $s_C(-2) = 0$ in Description 3.7(ii).
   (ii) $Q = Q_2$, $a = 3$, $r_1 = 2$, $s_C(3) = -2$, and $s_C(-2) = s_C(-3) = -1$ in Description 3.7(i).

Consider the case (i). $w_Q^C(1) + w_Q^C(-1) = r_Q$ and $w_Q^C(2) + w_Q^C(-2) = 2r_Q$, whence $w_Q^C(2) = 2w_Q^C(1)$ and $w_Q^C(-2) = 2w_Q^C(-1)$. We have $(a_1, a_2, a_3) = (c, r_Q - c, 2c)$ by setting $w_Q^C(1) = c$. Thus we may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, 2)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that $(x_1, x_2, x_3)|_{C^\dagger} = (t^{c/r_Q}, t^{1-c/r_Q}, 0)$. By $s_C(2) = -2$ and Lemmata 2.10 and 2.11, the map (2.7) has cokernel of length ≤ 1 in Description 3.7(i) and ≤ 2 in Description 3.7(ii) at Q. Hence we have the result by Lemma 2.12 except the value of $(H \cdot C)$. This value can be calculated from the equality $rs_C(-1) + r(E \cdot C) = -w_Q^C(1)$ obtained from the map $\mathcal{O}_Y(-E)^{\otimes r} \otimes \mathcal{O}_Y(rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$.

Consider the case (ii). $w_Q^C(3) + w_Q^C(-3) = 2r_Q$. By $s_C(3) = -2$ and Lemmata 2.10 and 2.11, the map (2.7) has cokernel of length ≤ 1 at Q. On the other hand, consider $x_1^{s_1} x_2^{s_2} x_3^{s_3} - x_1^{t_1} x_2^{t_2} x_3^{t_3}$ with $(s_1, s_2, s_3) \neq (t_1, t_2, t_3)$ such that $s_1 - s_2 + 3s_3 \equiv t_1 - t_2 + 3t_3 \equiv 0$ modulo r_Q and $a_1 s_1 + a_2 s_2 + a_3 s_3 = a_1 t_1 + a_2 t_2 + a_3 t_3$. Then $(a_1 s_1 + a_2 s_2 + a_3 s_3)/r_Q \geq 2$ for any such $x_1^{s_1} x_2^{s_2} x_3^{s_3} - x_1^{t_1} x_2^{t_2} x_3^{t_3}$, because $x_1 x_2$ is the only invariant monomial whose restriction to $C^\dagger$ is t. Hence by the last paragraph in Section 2, the map (2.7) has cokernel of length $\geq 2 + 2 - (a_3 + w_Q^C(-3))/r_Q = 2$, which is a contradiction. $\square$

We need more accurate description developing Lemma 3.9.

Lemma 3.10. *Consider the case in Lemma* 3.9(i).

   (i) *If f is of type $\mathrm{IIb}^{\vee\vee}$ in Description* 3.7(i)*, we may choose the coordinates so that the restriction to $C^\dagger$ at one of Q_1 and Q_2 is of the form $(0, t^{c_i/r_i}, t^{1-\overline{ac_i}/r_i})$ and that at the other is of the form $(t^{c_i/r_i}, 0, t^{\overline{ac_i}/r_i})$.*
   (ii) *If f is of type III in Description* 3.7(ii)*, we may choose the coordinates so that the restriction to $C^\dagger$ at Q is of the form $(t^{c/r}, 0, t^{\overline{ac}/r})$.*

Proof. (i) If both of them are of the form $(0, t^{c_i/r_i}, t^{1-\overline{ac_i}/r_i})$, the image of the map $\mathcal{O}_Y(-aE) \otimes \mathcal{O}_Y(E) \otimes \mathcal{O}_Y((a-1)E) \otimes \mathcal{O}_C \to \mathcal{O}_C$ is $\mathfrak{m}_{\{Q_1, Q_2\} \subset C}$, whence $s_C(-a) + s_C(1) + s_C(a-1) = -2$. Thus $s_C(-a) = 0$ by Lemma 3.2(iii), and the statement follows from Lemma 3.8. On the other hand, if both are of the form $(t^{c_i/r_i}, 0, t^{\overline{ac_i}/r_i})$, the image of the map $\mathcal{O}_Y(-aE) \otimes \mathcal{O}_Y(-E) \otimes \mathcal{O}_Y((a+1)E) \otimes \mathcal{O}_C \to \mathcal{O}_C$ is $\mathfrak{m}_{\{Q_1, Q_2\} \subset C}$, whence $s_C(-a) + s_C(-1) + s_C(a+1) = -2$. Thus $s_C(-a) = 0$ by $s_C(a+1) \leq -1$ and Lemma 3.2(iii), and the statement follows from Lemma 3.8.

(ii) If the restriction is of the form $(0, t^{c/r}, t^{1-\overline{ac}/r})$, the image of the map $\mathcal{O}_Y(-aE) \otimes \mathcal{O}_Y(E) \otimes \mathcal{O}_Y((a-1)E) \otimes \mathcal{O}_C \to \mathcal{O}_C$ is $\mathfrak{m}_{Q \subset C}$, whence $s_C(-a) +

$s_C(1) + s_C(a-1) = -1$. Thus $s_C(-a) = 1$ by Lemma 3.2(iv), and the statement follows from Lemma 3.8. □

Lemma 3.11. (i) *Assume that f is of type $IIb^{\vee\vee}$ with $s_C(-a) = -1$ in Description 3.7(i). Then $a < r_1 < r_2$, $a_{11} = w_{Q_1}^C(1) = 1$, $a_{22} = w_{Q_2}^C(-1) = 1$, and $(H \cdot C) = 1/r_1 - 1/r_2$. In particular, for each i we may choose semi-invariant local coordinates x_{i1}, x_{i2}, x_{i3} with weights $\mathrm{wt}(x_{i1}, x_{i2}, x_{i3}) = (1, -1, a)$ of the index-one cover $Q_i^\sharp \in Y^\sharp$ so that*

$$Q_i{}^\sharp \in C^\sharp \subset Y^\sharp \cong o \in (x_{ii}\text{-axis}) \subset \mathbb{C}^3_{x_{i1}x_{i2}x_{i3}}.$$

(ii) *Assume that f is of type III with $s_C(-a) = 0$ in Description 3.7(ii), and that $Q \in C$ has the local description in Lemma 3.9(i). Then $a_1 = w_{Q_1}^C(1) = 1$ and $(H \cdot C) = 1/r$. In particular, we may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, a)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that*

$$Q^\sharp \in C^\sharp \subset Y^\sharp \cong o \in (x_1\text{-axis}) \subset \mathbb{C}^3_{x_1x_2x_3}.$$

Proof. (i) First of all, by Lemma 3.10(i) we can choose $(A, B) = (1, 2)$ or $(2, 1)$ so that the description at Q_A is of the form $(t^{c_A/r_A}, 0, t^{\overline{ac_A}/r_A})$ and that at Q_B is of the form $(0, t^{c_B/r_B}, t^{1-\overline{ac_B}/r_B})$. From the map $\mathcal{O}_Y(-E)^{\otimes r_A r_B} \otimes \mathcal{O}_Y(r_A r_B E) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $r_A r_B s_C(-1) + r_A r_B(E \cdot C) = -r_B w_{Q_A}^C(1) - r_A w_{Q_B}^C(1)$. By $w_{Q_A}^C(1) = c_A$, $w_{Q_B}^C(1) = r_B - c_B$ and $s_C(-1) = -1$ in Lemma 3.2(iii), we obtain

$$(3.1) \qquad 1 + (H \cdot C) = \frac{c_A}{r_A} + \frac{r_B - c_B}{r_B}.$$

On the other hand, from the map $\mathcal{O}_Y(-aE)^{\otimes r_A r_B} \otimes \mathcal{O}_Y(ar_A r_B E) \otimes \mathcal{O}_C \to \mathcal{O}_C$ we have $r_A r_B s_C(-a) + a r_A r_B(E \cdot C) = -r_B w_{Q_A}^C(a) - r_A w_{Q_B}^C(a)$. By $w_{Q_A}^C(a) = \overline{ac_A}$, $w_{Q_B}^C(a) = r_B - \overline{ac_B}$ and $s_C(-a) = -1$, we obtain

$$(3.2) \qquad 1 + a(H \cdot C) = \frac{\overline{ac_A}}{r_A} + \frac{r_B - \overline{ac_B}}{r_B}.$$

From the equalities (3.1) and (3.2), one of the inequalities $c_A < \overline{ac_A}$ and $c_B > \overline{ac_B}$ holds. However, $c_B > \overline{ac_B}$ cannot occur since $(r_B, 0)$ is contained in the semi-group (2.5) for Q_B. Hence $c_A < \overline{ac_A}$.

We claim that $c_A = 1$. Suppose that $c_A \geq 2$, and take the smallest integer m for which $c_A m \geq r_A$. $w_{Q_A}^C(m) \equiv c_A m - r_A$ modulo r_A and $c_A m - r_A < a_{A1} < a_{A3}$. Thus the map $\mathcal{O}_Y(-mE) \otimes \mathcal{O}_Y(mE) \otimes \mathcal{O}_C \to \mathcal{O}_C$ at Q_A is properly contained in $\mathfrak{m}_{Q_A \subset C}$. In our situation the image of the map $\mathcal{O}_Y(aE) \otimes \mathcal{O}_Y(-aE) \otimes \mathcal{O}_C \to \mathcal{O}_C$ is $\mathfrak{m}_{\{Q_1, Q_2\} \subset C}$. Considering also Lemma 3.2(iii), we have one of the following:

(i) $m > a$.
(ii) $r_1 \mid m$ and $m < a$.

When (i) occurs, $\overline{ac_A} = ac_A$. Then $c_A = 1$ since $(r_A, 0)$ is contained in the semi-group (2.5) for Q_A. When (ii) occurs, unless $m = 2$, the map $\mathcal{O}_Y(-(m-1)E) \otimes \mathcal{O}_Y((m-1)E) \otimes \mathcal{O}_C \to \mathcal{O}_C$ at Q_A is also properly contained in $\mathfrak{m}_{Q_A \subset C}$, which is a contradiction. The case where $m = 2$ in (ii) remains. In this case, $r_1 = 2$ and $c_1 = 1$. Then we have $s_C(-a) = 0$ as in the proof of Lemma 3.10, which contradicts the assumption $s_C(-a) = -1$.

Now we have $(A,B) = (1,2)$ and $r_1 < r_2$ by substituting $c_A = 1$ in (3.1). We have

$$(3.3) \qquad (H \cdot C) = \frac{1}{r_1} - \frac{c_2}{r_2},$$

$$(3.4) \qquad a(H \cdot C) = \frac{\overline{a}}{r_1} - \frac{\overline{ac_2}}{r_2}.$$

In particular $\lfloor a/r_1 \rfloor = \lfloor ac_2/r_2 \rfloor$. We claim that $c_2 = 1$; then $a < r_1$ by Remark 2.1 and $(H \cdot C) = 1/r_1 - 1/r_2$ by (3.3). This claim follows as in the proof of $c_1 = 1$, once we obtain $c_2 < r_2 - \overline{ac_2}$. But this inequality comes from the following equalities obtained by (3.3) and respectively (3.4):

$$1 - (H \cdot C) = \frac{r_1 - 1}{r_1} + \frac{c_2}{r_2},$$
$$1 + a(H \cdot C) = \frac{\overline{a}}{r_1} + \frac{r_2 - \overline{ac_2}}{r_2}.$$

(ii) By Lemmata 3.9 and 3.10(ii), the description at Q is of the form $(t^{c/r}, 0, t^{\overline{ac}/r})$. From the map $\mathcal{O}_Y(-E)^{\otimes r} \otimes \mathcal{O}_Y(rE) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $rs_C(-1) + r(E \cdot C) = -w_Q^C(1)$. By $w_Q^C(1) = c$ and $s_C(-1) = 0$ in Lemma 3.2(iv), we obtain $(H \cdot C) = c/r$. On the other hand, from the map $\mathcal{O}_Y(-aE)^{\otimes r} \otimes \mathcal{O}_Y(arE) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $rs_C(-a) + ar(E \cdot C) = -w_Q^C(a)$. By $w_Q^C(a) = \overline{ac}$ and $s_C(-a) = 0$, we have $a(H \cdot C) = \overline{ac}/r$. Hence $ac = \overline{ac}$ and the statement follows since $(r,0)$ is contained in the semi-group (2.5). □

The next lemma is an easy consequence by numerical argument.

Lemma 3.12. *Assume that f is of type IIb$^{\vee\vee}$. Then $(H \cap E)_{\mathrm{red}}$ is irreducible.*

Proof. We may assume that $s_C(-a) = -1$ by Lemma 3.8. Write $[H \cap E] = u[C] + [F]$ cycle-theoretically, where the support of $[F]$ does not pass through Q_1 in Description 3.1(iii). Then we can write $(H \cdot [H \cap E]) = u(H \cdot C) + v/r_2$. By $(H \cdot [H \cap E]) = E^3 = a^{-1}(1/r_1 + 1/r_2)$ in Theorem 1.4 and $(H \cdot C) = 1/r_1 - 1/r_2$ in Lemma 3.11(i), we obtain

$$\frac{au - 1}{r_1} = \frac{au + 1 - av}{r_2}.$$

Since $a \geq 2$ and $r_1 < r_2$ by Lemma 3.11(i), we have $v = 0$. □

We can take suitable surfaces by Lemma 3.4(i):

Description 3.13. (i) Assume that f is of type IIb$^{\vee\vee}$. Then by Lemma 3.4(i) and Remark 2.5(ii), there exist surfaces S and S_0 on Y defined by elements in $\mathcal{O}_Y(-aE)$ for which $(S \cap E)_{\mathrm{red}} \not\subseteq (H \cap E)_{\mathrm{red}} \cup (S_0 \cap E)_{\mathrm{red}}$. Let D be an irreducible reduced curve in $S \cap E$ which intersects H and S_0 properly. We can see that $D \cong \mathbb{P}^1$ as in Lemma 2.8.

(ii) Assume that f is of type III. Then by Lemma 3.4(i) and Remark 2.5(ii), we can take H and $C \cong \mathbb{P}^1 \subseteq H \cap E$ so that C properly intersects the birational transform H_0 of a general hyperplane section on $P \in X$.

Lemma 3.14. *Assume that f is of type IIb$^{\vee\vee}$ with $s_C(-a) = -1$, and take $C \subseteq H \cap E$ and $D \subseteq S \cap E$ as in Descriptions 3.7(i) and 3.13(i). Then we can write $[S \cap E] = x[C] + [D]$ cycle-theoretically for some positive integer x, and D intersects C exactly at one of Q_1 and Q_2. Let Q_i with $i = 1$ or 2 be the one which D passes*

through. Then $(H \cdot D) = 1/r_i$, *and we may choose semi-invariant local coordinates* x_1, x_2, x_3 *with weights* $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, a)$ *of the index-one cover* $Q_i^\sharp \in Y^\sharp$ *so that*

$$Q_i^\sharp \in D^\sharp \subset Y^\sharp \cong o \in (x_1\text{-axis}) \subset \mathbb{C}^3_{x_1 x_2 x_3}.$$

Proof. Write $[S \cap E] = x[C] + [D']$ cycle-theoretically. We can write $1/r_1 + 1/r_2 = (H \cdot [S \cap E]) = x(H \cdot C) + y/r_1 + z/r_2$. If $y = z = 1$, then $[S \cap E] = [D']$ and $(S \cdot C) \geq 1/r_1 + 1/r_2$ since S passes through Q_1 and Q_2. On the other hand, $(S \cdot C) \leq (S \cdot [H \cap E]) = 1/r_1 + 1/r_2$ by Theorem 1.4. Hence $C = H \cap E$ and $s_C(-a) \geq 0$ by Lemma 3.4(ii) with $i = a$, with Remark 2.5(ii) and Lemma 3.11(i). It contradicts $s_C(-a) = -1$.

Hence $y = 0$ or $(y, z) = (1, 0)$. By $(H \cdot [S \cap E]) = 1/r_1 + 1/r_2$ and $(H \cdot C) = 1/r_1 - 1/r_2$ in Lemma 3.11(i), we obtain

$$\frac{x - 1 + y}{r_1} = \frac{x + 1 - z}{r_2}.$$

Because $y = 0$ or $(y, z) = (1, 0)$, and $r_1 < r_2$ in Lemma 3.11(i), we have $(y, z) = (1, 0)$ or $(0, 1)$. Therefore, provided $r_1 \nmid r_2$, $D' = D$ and D intersects C only at Q_1 or Q_2; then $(H \cdot D) = 1/r_1$, respectively $1/r_2$. However, if $r_1 \mid r_2$ and $(H \cdot D') = 1/r_1$, then setting $(r_1, r_2) = (r, sr)$, by Lemma 3.12 we can write

$$au\Big(\frac{1}{r} - \frac{1}{sr}\Big) = \frac{1}{r} + \frac{1}{sr},$$

where $[H \cap E] = u[C]$ for some $u \geq 2$. By this equality and $s > 1$ in Lemma 3.11(i), we have $(s, au) = (2, 3)$ or $(3, 2)$. But this contradicts $a \geq 2$ and $u \geq 2$.

Let Q_i with $i = 1$ or 2 be the one which D passes through. From the map $\mathcal{O}_Y(-E)^{\otimes r_i} \otimes \mathcal{O}_Y(r_i E) \otimes \mathcal{O}_D \to \mathcal{O}_D$, we obtain $r_i s_D(-1) + r_i(E \cdot D) = -w_{Q_i}^D(1)$. Hence $w_{Q_i}^D(1) \equiv 1$ modulo r_i. Therefore $w_{Q_i}^D(1) = 1$, since $(r_i, 0)$ is contained in the semi-group (2.5) for $Q_i \in D$. Thus the desired description follows. $\square$

The next theorem gives complete description in types IIb$^{\vee\vee}$ and III:

Theorem 3.15. *Assume that* f *is of type* IIb$^{\vee\vee}$ *and* III. *Then there exist a surface* H *and a curve* $C \subseteq H \cap E$ *in Description* 3.7 *which satisfy the assumption in Lemma* 3.8.

Proof. We first treat type IIb$^{\vee\vee}$. Suppose that C does not satisfy the assumption in Lemma 3.8, that is, suppose that $s_C(-a) = -1$ by Lemma 3.2(iii). Then by Description 3.13(i) and Lemma 3.14, there exist S and $Q_i \in D \cong \mathbb{P}^1$ for which we may choose semi-invariant local coordinates x_1, x_2, x_3 so that $(x_1, x_2, x_3)|_{D^\dagger} = (t^{1/r_i}, 0, 0)$. D intersects H properly.

Let $H^\sharp$ be the pre-image of H on the index-one cover $Q_i^\sharp \in Y^\sharp$. Write $h(x_1) + g_1 x_2 + g_2 x_3$ for the defining function of $H^\sharp$ at $Q_i^\sharp$, where $h(x_1)$ is a semi-invariant formal series of weight 1 and g_1, g_2 are semi-invariant formal series in x_1, x_2, x_3. The order of the function for $H^\sharp|_{D^\dagger}$ with respect to t equals $(H \cdot D) = 1/r_i$ by Lemma 3.14. It means that $h(t^{1/r_i})$ has order $1/r_i$ with respect to t, whence the monomial x_1 appears in h with non-zero coefficient. Thus we may choose the coordinates x_1, x_2, x_3 so that $H^\sharp$ is given by $x_1 = 0$. If $Q_i = Q_1$, for the coordinates x_{11}, x_{12}, x_{13} in Lemma 3.11(i) we can write $x_1 = c x_{11} + g(x_{11}, x_{12}, x_{13})$, where $c \neq 0$ and g is a semi-invariant formal series of weight 1 in which the monomial x_{11} does

not appear. Thus $x_1|_{C^\dagger} \neq 0$ by $(x_{11}, x_{12}, x_{13})|_{C^\dagger} = (t^{1/r_1}, 0, 0)$, which contradicts $C \subset H$.

Therefore $Q_i = Q_2$. We consider a general S_0 in Description 3.13(i). $S_0 \cap E$ also has a component $D_0 \cong \mathbb{P}^1$ which has the same properties as $D \subset S \cap E$ has, and we can choose new coordinates x_1', x_2', x_3' and t' for $Q_2 \in D_0 \subset Y$ so that $(x_1', x_2', x_3')|_{D_0^\dagger} = (t'^{1/r_2}, 0, 0)$. Since $(S \cdot D_0) = a(H \cdot D_0) = a/r_2 < 1$ by Lemma 3.14 and Remark 2.1, S intersects D_0 only at Q_2. Let $S^\sharp$ be the pre-image of S on $Y^\sharp$, and $g(x_1, x_2, x_3) = h(x_1', x_2', x_3')$ the defining function of $S^\sharp$ at $Q_2^\sharp$. $h(t'^{1/r_2}, 0, 0)$, the function for $S^\sharp|_{D_0^\dagger}$, has order $(S \cdot D_0) = a/r_2$ with respect to t, whence we can write $h = (c' + h_1(x_1')x_1'^{r_2})x_1'^a + h_2(x_1', x_2', x_3')x_2' + h_3(x_1', x_2', x_3')x_3'$, where $c' \neq 0$ and h_1, h_2, h_3 are semi-invariant with weights $0, a+1, 0$. Thus we can write $g = cx_1^a + \cdots$ for some $c \neq 0$, and we see that $g(t^{1/r_2}, 0, 0) = 0$ holds only if h_3 is a unit, by considering weights and $a < r_1 < r_2$ in Description 3.13(i). Of course $g(t^{1/r_2}, 0, 0) = 0$ by $D \subset S$, whence h_3 is a unit and $S^\sharp$ is smooth at $Q_2^\sharp$.

Hence we may choose the coordinates x_{21}, x_{22}, x_{23} in Lemma 3.11(i) so that $x_{21} = x_1$, which is the function for $H^\sharp$, and so that x_{23} is the function for $S^\sharp$. In particular the scheme $S \cap H$ is irreducible and reduced at the generic point of C. Write $[H \cap E] = u[C]$ by Lemma 3.12 and $[S \cap E] = x[C] + [D]$ by Lemma 3.14. Then we have

$$ua\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{1}{r_1} + \frac{1}{r_2},$$
$$x\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{1}{r_2} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Thus $(ua - x)(r_2 - r_1) = r_1$ and we can write $r_1 = nr$, $r_2 = (n+1)r$. Note that

(3.5) $ua = 2n + 1,$

(3.6) $x = n + 1.$

On the other hand, let $\pi \colon \hat{E} \to E$ be the normalisation of E and set $\hat{C} := (C \times_E \hat{E})_{\mathrm{red}}$. The coefficients of $[C]$ in the 1-cycles $[S \cap E]$ and $[aH \cap E]$ are determined by the lengths of the schemes $\pi^* S \cap \hat{C}$ and $\pi^*(aH) \cap \hat{C}$ at all the generic points of $\hat{C}$. From this point of view, we see that $(S \cap E) \cap (aH \cap E)$ contains the 1-cycle $x[C]$ because of the general choice of S. Here $(S \cap E) \cap (aH \cap E) \subseteq S \cap aH$, and $S \cap H$ is irreducible and reduced at the generic point of C. Hence $x \leq a$, but it contradicts (3.5), (3.6) and $u \geq 2$.

Now we treat type III. We start with H, $Q \in C$ and H_0 in Description 3.13(ii). Suppose that C does not satisfy the assumption in Lemma 3.8, that is, suppose that $s_C(-a) = 0$. Then by Lemmata 3.9 and 3.11(ii), we may choose coordinates x_1, x_2, x_3 so that $(x_1, x_2, x_3)|_{C^\dagger} = (t^{1/r}, 0, 0)$, $(t^{2/r}, t^{1-2/r}, 0)$ or $(t^{1-2/r}, t^{2/r}, 0)$. Note that $(H_0 \cdot E) = 1/r, 2/r, 1 - 2/r$ respectively in these cases. Thus H_0 intersects C only at Q. Let $H_0^\sharp$ be the pre-image of H_0 on the index-one cover $Q^\sharp \in Y^\sharp$, and $g(x_1, x_2, x_3)$ the defining function of $H_0^\sharp$ at $Q^\sharp$. Then g is semi-invariant of weight 1 and $g(t^{1/r}, 0, 0)$, $g(t^{2/r}, t^{1-2/r}, 0)$ and $g(t^{1-2/r}, t^{2/r}, 0)$ have order $1/r, 2/r, 1 - 2/r$ with respect to t. As in the former argument, we can

see that the monomial x_1 appears in the function for $H_0^\sharp$ with non-zero coefficient in each of the cases. Hence we may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, a)$ of $Q^\sharp \in Y^\sharp$ so that H_0 is given by $x_1 = 0$. We take $Q \in C_0 \cong \mathbb{P}^1 \subset H_0 \cap E$. If $s_{C_0}(-a) \neq 1$, then $s_{C_0}(-a) = 0$ and we may choose new coordinates x_1', x_2', x_3' and t' so that $(x_1', x_2', x_3')|_{C_0^\dagger} = (t'^{1/r}, 0, 0)$, $(t'^{2/r}, t'^{1-2/r}, 0)$ or $(t'^{1-2/r}, t'^{2/r}, 0)$. On the other hand, we can write $x_1 = cx_1' + g(x_1', x_2', x_3')$, where $c \neq 0$ and g is a semi-invariant formal series of weight 1 in which the monomial x_1' does not appear. Thus $x_1|_{C_0^\dagger} \neq 0$, which contradicts $C_0 \subset H_0$. $\qquad\square$

4. Existence of Du Val sections

In this section we prove Theorem 1.7, the general elephant theorem. It was proved when f is of type O or I in Theorem 2.7, and is trivial in type IV. Thus we keep the assumption that f is of type II or III. First we restate the results in Section 3:

Description 4.1.   (i) (type IIa; Theorem 3.5)
    (a) $a = 2$ and $[H \cap E] = [\mathbb{P}^1] + [\mathbb{P}^1]$; $s_C(-2) = 0$ and $(H \cdot C) = 1/r$; $Q \in C$ is locally expressed by semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, 4)$ of the index-one cover $Q^\sharp \in Y^\sharp$ as $(x_1, x_2, x_3)|_{C^\dagger} = (t^{(r+1)/2r}, 0, t^{2/r})$ when $r \equiv 1$ modulo 4, and $(0, t^{(r-1)/2r}, t^{2/r})$ when $r \equiv 3$ modulo 4.
    (b) $r = 5$, $a = 4$ and $H \cap E = \mathbb{P}^1$; $s_C(-4) = 0$ and $(H \cdot C) = 1/r$; $Q \in C$ is locally expressed by semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, 3)$ of the index-one cover $Q^\sharp \in Y^\sharp$ as $(x_1, x_2, x_3)|_{C^\dagger} = (t^{3/5}, t^{2/5}, 0)$.
  (ii) (type IIb$^\vee$; Theorem 3.6)
    (a) $J = \{(r, 1), (r, 1)\}$, $a = 2$ and $H \cap E = \mathbb{P}^1$; $s_C(-2) = 0$ and $(H \cdot C) = 1/r$; $Q \in C$ is locally expressed by semi-invariant local coordinates x_1, x_2, x_3, x_4 with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (1, -1, 2, 0)$ of the index-one cover $Q^\sharp \in Y^\sharp$ as $(x_1, x_2, x_3, x_4)|_{C^\dagger} = (0, 0, t^{2/r}, t)$.
    (b) $J = \{(2, 1), (4, 1)\}$, $a = 3$ and $H \cap E = \mathbb{P}^1$; $s_C(-3) = 0$ and $(H \cdot C) = 1/4$; $Q \in C$ is locally expressed by semi-invariant local coordinates x_1, x_2, x_3, x_4 with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (1, 3, 3, 2)$ of the index-one cover $Q^\sharp \in Y^\sharp$ as $(x_1, x_2, x_3, x_4)|_{C^\dagger} = (0, 0, t^{3/4}, t^{1/2})$.
 (iii) (type IIb$^{\vee\vee}$; Theorem 3.15) $H \cap E = \mathbb{P}^1$; $s_C(-a) = 0$ and $(H \cdot C) = (r_1 + r_2)/ar_1r_2$; $Q_i \in C$ is locally expressed by semi-invariant local coordinates x_{i1}, x_{i2}, x_{i3} with weights $\mathrm{wt}(x_{i1}, x_{i2}, x_{i3}) = (1, -1, a)$ of the index-one cover $Q_i^\sharp \in Y^\sharp$ as $(x_{i1}, x_{i2}, x_{i3})|_{C^\dagger} = (0, 0, t^{1/r_i})$.
 (iv) (type III; Theorem 3.15) We can choose H so that $H \cap E = \mathbb{P}^1$; $s_C(-a) = 1$ and $(H \cdot C) = (r + 1)/ar$; $Q \in C$ is locally expressed by semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, a)$ of the index-one cover $Q^\sharp \in Y^\sharp$ as $(x_1, x_2, x_3)|_{C^\dagger} = (0, 0, t^{1/r})$.

Theorem 4.2. *Let S be a general elephant of Y.*
  (i) *If f is of type II, then S intersects $H \cap E$ at and only at the non-Gorenstein points of Y.*
 (ii) *If f is of type III, then S intersects $H \cap E$ at and only at a general point of C and the non-Gorenstein point of Y.*

Proof. (i) Take any $C \cong \mathbb{P}^1 \subseteq H \cap E$. By (2.3), Lemma 4.3 and the commutative diagrams in Section 2, we obtain a surjective map $f_* \mathcal{O}_Y(-aE) \twoheadrightarrow H^0([\mathcal{O}_Y(-aE)]_C)$. Recalling that $s_C(-a) = 0$ in Description 4.1, we have $H^0([\mathcal{O}_Y(-aE)]_C) = \mathbb{C}$. Hence we find a general elephant S which properly intersects $H \cap E$. Of course S passes through any non-Gorenstein point of Y. On the other hand, because $(S \cdot [H \cap E]) = aE^3 \leq 1$ in any case by Description 4.1, we see that S does not intersect $H \cap E$ at any other point.

(ii) It suffices to consider $[\mathcal{O}_Y(-aE) \otimes \mathfrak{m}_{Q' \subset Y}]_C$ instead of $[\mathcal{O}_Y(-aE)]_C$ since $s_C(-a) = 1$ in Description 4.1. $\square$

Lemma 4.3. *Assume that f is of type IIa with $a = 2$. Write $[H \cap E] = [C] + [C']$ cycle-theoretically. Then there exists a natural exact sequence*

$$0 \to \mathcal{O}_{C'}(-1) \to \mathcal{O}_{H \cap E} \to \mathcal{O}_C \to 0.$$

Proof. Let $\mathcal{I}$ be the kernel of the natural map $\mathcal{O}_{H \cap E} \twoheadrightarrow \mathcal{O}_C$. $\mathcal{I}$ is supported on $C' \cong \mathbb{P}^1$. By $H^0(\mathcal{O}_{H \cap E}) = \mathbb{C}$ in Remark 2.9, we have $H^0(\mathcal{I}) = 0$. By $H^1(\mathcal{O}_{H \cap E}) = 0$ in Lemma 2.8(ii), we have $H^1(\mathcal{I}) = 0$. Hence $\mathcal{I} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. $\square$

It is the time to prove Theorem 1.7, the main theorem.

Theorem 4.4. *Assume that f is of type II or III. Then a general elephant S of Y has at worst Du Val singularities. Moreover, the type of any Du Val singularity $Q \in S$ is the same as that of a general elephant of a germ $Q \in Y$.*

Proof. By [7] it suffices to consider the case where P is a singular point. By Theorem 4.2 and the theorem of Bertini, it is enough to show that S has a Du Val singularity at each non-Gorenstein point $Q \in Y$ the type of which is the same as that of a general elephant of a germ $Q \in Y$.

First we treat type IIa in Description 4.1(i). Let $Q \in C = H \cap E$ be the non-Gorenstein point and a curve which are investigated.

Consider the case $a = 2$. It is enough to show that the monomial x_3 appears in the defining function $h(x_1, x_2, x_3)$ of S in Theorem 4.2 with non-zero coefficient. Since S intersects C only at Q by Theorem 4.2, $h(t^{(r+1)/2r}, 0, t^{2/r})$ when $r \equiv 1$ modulo 4, and $h(0, t^{(r-1)/2r}, t^{2/r})$ when $r \equiv 3$ modulo 4, have order $(S \cdot C) = 2/r$ with respect to t. Hence the monomial x_3 appears in h with non-zero coefficient.

Consider the case $a = 4$. Then $-K_Y$ is linearly equivalent to $E + 5H$. Thus it is enough to show that

(i) E has a Du Val singularity of type A_4 at Q, and
(ii) $|5H|$ is free at Q.

We first see (i). By [14] we can consider a birational morphism $g \colon Z \to Y$ such that Z has a g-exceptional divisor F the discrepancy of which with respect to K_X is 1. Write

$$K_Z = g^* K_Y + bF + \text{(others)},$$
$$g^* E = E_Z + mF + \text{(others)},$$

where E_Z is the birational transform of E. Then $K_Z = g^*(f^* K_X + 4E) + bF + \text{(others)} = g^* f^* K_X + 4E_Z + (b + 4m)F + \text{(others)}$, whence $b = m = 1/5$ since $5b, 5m \in \mathbb{Z}$. Thus the defining function of E in x_1, x_2, x_3 in Description 4.1(i) has a non-zero linear term. Considering the weight, we see that the monomial x_3 appears

in the function for E with non-zero coefficient. Hence we may take x_3 so that E is given by $x_3 = 0$. This implies (i).

It is easy to see (ii). From (2.3) and the commutative diagrams in Section 2, we obtain a surjective map $f_* \mathcal{O}_Y(-5E) \twoheadrightarrow H^0([\mathcal{O}_Y(-5E)]_C) = H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, which shows (ii).

Secondly we treat type $\text{IIb}^\vee$ in Description 4.1(ii). Let $Q \in C = H \cap E$ be the non-Gorenstein point and a curve which are investigated. $Q \in Y$ is described as in Remark 1.5. Consider a general elephant T on a germ $Q \in Y$, for which the function in y_1, y_2, y_3, y_4 has a non-zero linear term. Then by Remark 1.5, we have the following:

- (i) If $J = \{(r,1),(r,1)\}$ and $r \geq 5$, the linear term of the function for T is y_3. Any such T gives a Du Val singularity of type A_{2r-1}, which is the type of a general elephant of a germ $Q \in Y$.
- (ii) In any other case, the linear term of the function for T is a linear combination of y_2 or y_3. Any T which has a general linear term gives a Du Val singularity of type A_5, E_6 or D_5 according to (ia), (ib), (ii) in Remark 1.5 respectively, which is the type of a general elephant of a germ $Q \in Y$.

Thus if $J = \{(r,1),(r,1)\}$ and $r \geq 5$, it is enough to show the following:

Claim 4.4.1. The function for S in Theorem 4.2 in y_1, y_2, y_3, y_4 has a non-zero linear term.

In any other case, $-K_Y$ is linearly equivalent to $E + rH$. Thus it is enough to show, adding to Claim 4.4.1,

Claim 4.4.2.   (i) The function for E in y_1, y_2, y_3, y_4 has a non-zero linear term which is linearly independent to that in the function for S.
- (ii) $|rH|$ is free at Q.

However, in Claim 4.4.2, the existence of a non-zero linear term in the function for E and Claim 4.4.2(ii) follow as in the proof of type IIa with $a = 4$. In particular, in Claim 4.4.2 we may choose coordinates x_1, x_2, x_3, x_4 in Description 4.1(ii) so that E is given by $x_2 = 0$. Hence Claims 4.4.1 and 4.4.2 follow if we prove that the monomial x_3 appears in the defining function $h(x_1, x_2, x_3, x_4)$ of S in Theorem 4.2 with non-zero coefficient. Because S intersects C only at Q by Theorem 4.2, $h(0,0,t^{2/r},t)$ and $h(0,0,t^{3/4},t^{1/2})$ have order $(S \cdot C) = aE^3 = 2/r$, respectively 3/4 with respect to t, according to $J = \{(r,1),(r,1)\}$, respectively $\{(2,1),(4,1)\}$. Hence the monomial x_3 appears in h with non-zero coefficient.
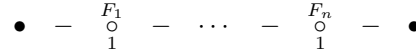
Finally we treat types $\text{IIb}^{\vee\vee}$ and III in Description 4.1(iii), (iv). Let $Q \in C = H \cap E$ be a non-Gorenstein point and a curve which are investigated. It is enough to show that x_3 appears in the defining equation $h(x_1, x_2, x_3)$ of S in Theorem 4.2 with non-zero coefficient. We can see this similarly, remarking that the local intersection number of S and C at Q is $1/r_Q$ by Description 4.1(iii), (iv) and Theorem 4.2. $\square$

5. POSSIBLE TYPES OF SINGULARITIES

In this section we restrict the types of a general elephant S of Y and its birational transform S_X on X, and deduce Theorem 1.8. First we recall the dual graphs for minimal resolutions of Du Val singularities. $\circ$ denotes an exceptional curve and $\bullet$ the birational transform of a general hyperplane section, and each exceptional curves F_i is marked with its coefficient in the fundamental cycle.
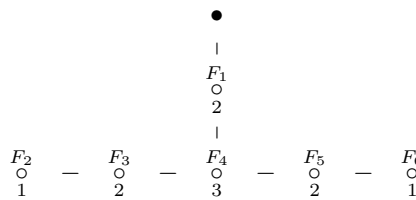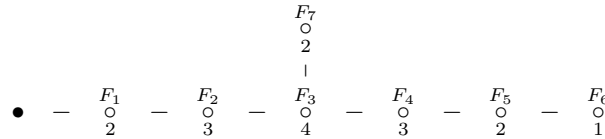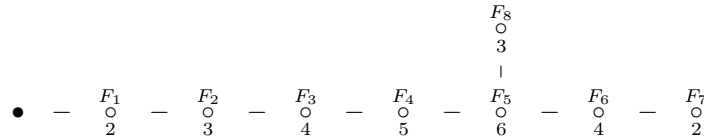
(i) (type A_n)

$$\bullet \quad - \quad \underset{1}{\overset{F_1}{\circ}} \quad - \quad \cdots \quad - \quad \underset{1}{\overset{F_n}{\circ}} \quad - \quad \bullet$$

(ii) (type D_n)

$$
\begin{array}{ccccccccc}
 & & \bullet & & & & \underset{1}{\overset{F_n}{\circ}} & & \\
 & & | & & & & | & & \\
\underset{1}{\overset{F_1}{\circ}} & - & \underset{2}{\overset{F_2}{\circ}} & - & \cdots & - & \underset{2}{\overset{F_{n-2}}{\circ}} & - & \underset{1}{\overset{F_{n-1}}{\circ}}
\end{array}
$$

(iii) (type E_6)

$$
\begin{array}{ccccccccc}
 & & & & \bullet & & & & \\
 & & & & | & & & & \\
 & & & & \underset{2}{\overset{F_1}{\circ}} & & & & \\
 & & & & | & & & & \\
\underset{1}{\overset{F_2}{\circ}} & - & \underset{2}{\overset{F_3}{\circ}} & - & \underset{3}{\overset{F_4}{\circ}} & - & \underset{2}{\overset{F_5}{\circ}} & - & \underset{1}{\overset{F_6}{\circ}}
\end{array}
$$

(iv) (type E_7)

$$
\begin{array}{ccccccccccc}
 & & & & \underset{2}{\overset{F_7}{\circ}} & & & & & & \\
 & & & & | & & & & & & \\
\bullet & - & \underset{2}{\overset{F_1}{\circ}} & - & \underset{3}{\overset{F_2}{\circ}} & - & \underset{4}{\overset{F_3}{\circ}} & - & \underset{3}{\overset{F_4}{\circ}} & - & \underset{2}{\overset{F_5}{\circ}} \quad - \quad \underset{1}{\overset{F_6}{\circ}}
\end{array}
$$

(v) (type E_8)

$$
\begin{array}{ccccccccccccc}
 & & & & & & \underset{3}{\overset{F_8}{\circ}} & & & & & & \\
 & & & & & & | & & & & & & \\
\bullet & - & \underset{2}{\overset{F_1}{\circ}} & - & \underset{3}{\overset{F_2}{\circ}} & - & \underset{4}{\overset{F_3}{\circ}} & - & \underset{5}{\overset{F_4}{\circ}} & - & \underset{6}{\overset{F_5}{\circ}} & - & \underset{4}{\overset{F_6}{\circ}} \quad - \quad \underset{2}{\overset{F_7}{\circ}}
\end{array}
$$

Take a general elephant S of Y, and write S_X for its birational transform on X. A general hyperplane section $P \in H_X$ on X gives also a general hyperplane section $P \in D_X := H_X|_{S_X}$ on S_X. Let $f_S \colon S \to S_X$ be the induced map. Since S and S_X have at worst Du Val singularities by Theorem 1.7, f_S factors through the minimal resolution of S_X. Let Z be the birational transform on S of the fundamental cycle. Then $f^*H_X = H + bE$ as in Section 2 and $f^*D_X = D + Z = H|_S + bE|_S$, where D is the birational transform on S of D_X. Note that $bE|_S \leq Z$. We also have

$$(5.1) \qquad l := (H|_S \cdot bE|_S) = b^2 a E^3.$$

Assume that f is of type I. In this case, $a = b = 2$ and either $J = \{(7,3)\}$ with $E^3 = 1/7$ or $J = \{(3,1),(5,2)\}$ with $E^3 = 1/15$ holds by Theorems 1.4 and 2.7 and Proposition 2.4.

Lemma 5.1. *Assume that f is of type I. Take a general elephant S of Y.*

(i) If $J = \{(7,3)\}$, the dual graph for the partial resolution $S \to S_X$ is

$$* - \underset{2}{\circ} \quad or \quad \underset{2}{\circ} - * - \underset{2}{\circ},$$

where the attached numbers are the coefficients in $2E|_S$, and $*$ means that a Du Val singularity of type A_6 or worse appears there.

(ii) *If $J = \{(3,1),(5,2)\}$, the dual graph for the partial resolution $S \to S_X$ is*

$$* - \underset{2}{\circ} - \star \quad or \quad * - \underset{4}{\circ} - \star,$$

where the attached number is the coefficient in $2E|_S$, and $*$ and $\star$ mean that a Du Val singularity of type A_2, respectively A_4, or worse appears there.

Proof. (i) Since $(-E \cdot [S \cap E]) = 2E^3 = 2/7$, it is enough to show the existence of S such that $S \cap E$ defines a reduced 1-cycle. We can calculate $\chi(\mathcal{Q}_{-2}) = 2$ by (2.2). Thus we have the desired S as in the proof of Lemma 3.4(i).

(ii) Since $(-E \cdot [S \cap E]) = 2E^3 = 2/15$, it is enough to show the existence of S such that $S \cap E$ defines an irreducible but possibly non-reduced 1-cycle. By $(-E \cdot [S \cap E]) < 1/5$, each irreducible component of $(S \cap E)_{\mathrm{red}}$ passes through all the non-Gorenstein points of Y. Hence $(S \cap E)_{\mathrm{red}} = \mathbb{P}^1$ by $h^1(\mathcal{O}_{S \cap E}) = 0$ in Remark 2.9. $\qquad\square$

We restrict the dual graph for the partial resolution S, considering $2E|_S \le Z$ and Lemma 5.1. The following table shows all the possibilities up to permutation in terms of F_i:

	$J = \{(7,3)\}$				$J = \{(3,1),(5,2)\}$		
S_X	Z	$2E_S$	l	S_X	Z	$2E_S$	l
D_n	$2F_m + \cdots$	$2F_m + \cdots$	$4/m$	D_n	$2F_m$	$2F_m$	$4/m$
E_7	$2F_1 + \cdots$	$2F_1 + \cdots$	2	E_7	$3F_4$	$2F_4$	$8/15$
	$2F_5$	$2F_5$	1	E_8	$4F_3$	$2F_3$	$1/3$
	$2F_7$	$2F_7$	$8/7$		$4F_3$	$4F_3$	$4/3$
E_8	$2F_1 + \cdots$	$2F_1 + \cdots$	2		$5F_4$	$2F_4$	$1/5$
	$3F_2$	$2F_2$	$2/3$		$4F_4$	$4F_4$	$4/5$
	$3F_2 + 2F_7$	$2F_2 + 2F_7$	1		$6F_5$	$2F_5$	$2/15$
	$4F_3$	$2F_3$	$1/3$		$4F_5$	$4F_5$	$8/15$
	$4F_3 + 2F_7$	$2F_3 + 2F_7$	$4/3$				
	$4F_6$	$2F_6$	$2/7$				
	$2F_7$	$2F_7$	1				
	$2F_7 + 3F_8$	$2F_7 + 2F_8$	$8/7$				
	$3F_8$	$2F_8$	$1/2$				

$l = 8E^3$ by (5.1). This value is $8/7$ when $J = \{(7,3)\}$ and $8/15$ when $J = \{(3,1),(5,2)\}$. Therefore Theorem 1.8(i) is deduced.

Remark 5.2. In fact, P is cE_7 when $J = \{(7,3)\}$. By the above list, it suffices to exclude the case where $P \in S_X$ is E_8 and $Z = 2F_7 + 3F_8$. In this case, $S \cap E$ defines the 1-cycle $F_7 + F_8$ on S. Thus $(E \cdot F_7)$ and $(E \cdot F_8)$ must be $-1/7$. On the other hand, using the dual graph for the minimal resolution, we obtain $(E|_S \cdot F_7) = (F_7 + F_8 \cdot F_7)_S = -3/7$ and $(E|_S \cdot F_8) = (F_7 + F_8 \cdot F_8)_S = 1/7$. It is a contradiction.

Example 5.3. There exist examples of type I. The weighted blow-up of $o \in (x_1^2 + x_2^3 + x_2 x_3^3 + x_4^7 = 0) \subset \mathbb{C}^4$ with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (7, 5, 3, 2)$ is an example of type I whose P is cE_7. The weighted blow-up of $o \in (x_1^2 + x_2^3 + x_3^5 + x_4^7 = 0) \subset \mathbb{C}^4$ with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (7, 5, 3, 2)$ is an example of type I whose P is cE_8.

When f is of type II or III, by Theorems 4.2 and 4.4 and Remark 1.5, we obtain the possible dual graphs for the partial resolution $S \to S_X$ as follows:

(i) (type IIa or IIb$^\vee$) The dual graph is

$$\bullet - * \begin{cases} \underset{1}{-\circ} \\ \vdots \\ \underset{1}{-\circ} \end{cases}$$

where $\bullet$ denotes $H|_S = D$, the attached numbers are the coefficients in $E|_S = Z$, and $*$ means that a Du Val singularity of type

(a) A_{r-1} if f is of type IIa,
(b) A_{2r-1} if f is of type IIb$^\vee$ with $J = \{(r,1),(r,1)\}$, $r \geq 5$,
(c) A_5 or E_6 if f is of type IIb$^\vee$ with $J = \{(3,1),(3,1)\}$,
(d) D_5 if f is of type IIb$^\vee$ with $J = \{(2,1),(4,1)\}$,

appears there respectively. The number of the exceptional curves $\circ$ is ≤ 4 in type IIa and $\leq a$ in type IIb$^\vee$. This bound comes from the value of $(H \cdot [S \cap E]) = aE^3$.

(ii) (type IIb$^{\vee\vee}$ or III) Set $r_1 = 1$, $r_2 = r$ if f is of type III. $P \in S_X$ is A_n because $H|_S$ intersects $E|_S$ at two points. The dual graph is

$$\bullet - * - \underset{1}{\circ} - \star - \bullet \quad \text{or} \quad \bullet - * - \underset{1}{\circ} - \underset{1}{\circ} - \star - \bullet,$$

where $\bullet$ denotes $H|_S = D$, the attached numbers are the coefficients in $E|_S = Z$, and $*$ and $\star$ mean that a Du Val singularity of type A_{r_1-1}, respectively A_{r_2-1}, appears there. Since all the components of $(S \cap E)_{\text{red}}$ are numerically proportional, the number of the exceptional curves $\circ$ is ≤ 2.

The following table shows all the possibilities of the partial resolution $S \to S_X$ up to permutation in terms of F_i:

type of f	S_X	Z	l
IIa	D_r	F_r	$4/r$
	D_{r+1}	$F_r + F_{r+1}$	$4/r$
		$F_1 + F_{r+1}$	$1 + 1/r$
	D_{r+2}	$F_1 + F_{r+1} + F_{r+2}$	$1 + 1/r$
IIb$^\vee$, $J = \{(r,1),(r,1)\}$	D_{2r}	F_{2r}	$2/r$
	D_{2r+1}	$F_{2r} + F_{2r+1}$	$2/r$
		$F_1 + F_{2r+1}$	$1 + 1/2r$
IIb$^\vee$, $J = \{(3,1),(3,1)\}$	E_7	F_6	$2/3$
IIb$^\vee$, $J = \{(2,1),(4,1)\}$	D_6	F_1	1
	E_6	F_2	$3/4$
IIb$^{\vee\vee}$	$A_{r_1+r_2-1}$	F_{r_1}	$1/r_1 + 1/r_2$
	$A_{r_1+r_2}$	$F_{r_1} + F_{r_1+1}$	$1/r_1 + 1/r_2$
III	A_r	F_r	$1 + 1/r$
	A_{r+1}	$F_r + F_{r+1}$	$1 + 1/r$

We have $l = aE^3$ by (5.1). Combining it with Theorem 1.4, [7] and [8], we obtain Theorem 1.8(ii) except the case where S_X is $A_{r_1+r_2}$ in type IIb$^{\vee\vee}$ or A_{r+1} in type III. The remaining case is excluded by Theorem 1.13, proved in Section 6.

Example 5.4. Let f be the weighted blow-up of $o \in (x_1^2 + x_2^2 x_3 + x_3^{2r} + x_4^r = 0) \subset \mathbb{C}^4$ with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r, r, 1, 2)$, where $r \geq 3$ is an odd integer. P is cD_{r+1} and f is of type $\mathrm{IIb}^\vee$ with $J = \{(r,1), (r,1)\}$. The exceptional locus of the partial resolution $S \to S_X$ is reducible.

Example 5.5. Let f be the weighted blow-up of $o \in (x_1^2 + x_2^2 x_3 + x_3^3 + x_4^3 = 0) \subset \mathbb{C}^4$ with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 1, 4, 2)$. P is cD_4 and f is of type $\mathrm{IIb}^\vee$ with $J = \{(2,1), (4,1)\}$.

6. DIVISORIAL CONTRACTIONS TO cA_n POINTS

We start with a general method to determine f:

Lemma 6.1. *Let $f\colon (Y \supset E) \to (X \ni P)$ be a germ of a three-fold divisorial contraction to a Gorenstein point P, and set $K_Y = f^* K_X + aE$; X is identified with a hypersurface in $\bar{X} := \mathbb{C}^4_{x_1 x_2 x_3 x_4}$:*

$$P \in X \cong o \in (\phi = 0) \subset \bar{X} = \mathbb{C}^4_{x_1 x_2 x_3 x_4}.$$

Write m_i for the multiplicity of $\mathrm{div}\, x_i$ along E, that is, the largest integer such that $x_i \in f_* \mathcal{O}_Y(-m_i E)$. We assume that m_1, m_2, m_3, m_4 have no common factors. Let d be the weighted order of ϕ with respect to weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1, m_2, m_3, m_4)$, and decompose ϕ as

$$\phi = \phi_d(x_1, x_2, x_3, x_4) + \phi_{>d}(x_1, x_2, x_3, x_4),$$

where ϕ_d is the weighted homogeneous part of weight d and $\phi_{>d}$ the part of weight $> d$. Set $c := m_1 + m_2 + m_3 + m_4 - 1 - d$. Let $\bar{g}\colon (\bar{Z} \supset \bar{F}) \to (\bar{X} \ni P)$ be the weighted blow-up of $\bar{X}$ with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1, m_2, m_3, m_4)$, $\bar{F}$ its exceptional divisor, and $\bar{D}_i$ the birational transform on $\bar{Z}$ of the hyperplane $x_i = 0$ in $\bar{X}$. Let Z be the birational transform of X on $\bar{Z}$, and $g\colon Z \to X$ the induced morphism. Assume that

(i) *$\bar{F} \cap Z$ defines an irreducible reduced 2-cycle F on Z, and $F \nsubseteq \mathrm{Sing}\, Z$,*
(ii) *$\dim(\mathrm{Sing}\, \bar{Z} \cap Z) \leq 1$,*
(iii) *$\mathrm{Sing}\, Z \subseteq \bigcup_{1 \leq i \leq 4} \bar{D}_i$, and*
(iv) *$F \nsubseteq \bigcup_{1 \leq i \leq 4} \bar{D}_i$.*

Then $c \leq a$, and the equality holds if and only if $f \cong g$ over X.

Proof. Z is R_1 from (i), and is Cohen–Macaulay since $Z \subset \bar{Z}$ is locally a cyclic quotient of a hypersurface in $\mathbb{C}^4$. Thus Z is normal. By (ii) we can use the adjunction formula and have $K_Z = (K_{\bar{Z}} + Z)|_Z = g^* K_X + cF$. Note that $-F$ is f-ample and $\mathbb{Q}$-Cartier. Consider the centre on Z of the valuation corresponding to E. The multiplicity of $\mathrm{div}\, x_i$ along E equals to that along F. Thus, by $\bigcap_{1 \leq i \leq 4} \bar{D}_i = \emptyset$ and (iv), this centre is not contained in $\bigcup_{1 \leq i \leq 4} \bar{D}_i$, whence it intersects the smooth locus of Z by (iii). Thus the discrepancy a of E with respect to K_X is $\geq c$, that of F, and the equality holds if and only if $f \cong g$ over X by [7, Lemma 3.4]. $\square$

Let $f\colon (Y \supset E) \to (X \ni P)$ be a germ of a divisorial contraction to a cA_n point P for some $n \geq 2$. X is identified with a hypersurface $\phi = 0$ in $\mathbb{C}^4_{x_1 x_2 x_3 x_4}$ as follows:

$$(6.1) \qquad P \in X \cong o \in (\phi = x_1 x_2 + g(x_3, x_4) = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4},$$

where g has total order $n+1$ in x_3, x_4. From Theorems 1.4 and 1.8(i), f is of type O, II or III. Our goal in this section is to prove Theorem 1.13, the classification of these contractions.

Write m_i for the multiplicity of $\operatorname{div} x_i$ along E. By coordinate change in which $x_4 \mapsto x_4 + x_i$ for $i = 1$, 2 or 3 if necessary, we may assume that $m_4 = 1$. By coordinate change $x_3 \mapsto x_3 + h(x_4)$ if necessary, we may also assume that $x_3 + h(x_4) \notin f_* \mathcal{O}_Y(-(m_3 + 1)E)$ for any h. Decompose ϕ as

$$\phi = \phi_{\leq m_1 + m_2}(x_1, x_2, x_3, x_4) + \phi_{> m_1 + m_2}(x_1, x_2, x_3, x_4),$$

where $\phi_{\leq m_1 + m_2}$ is the part of weight $\leq m_1 + m_2$ and $\phi_{> m_1 + m_2}$ is the part of weight $> m_1 + m_2$ with respect to weights $\operatorname{wt}(x_1, x_2, x_3, x_4) = (m_1, m_2, m_3, 1)$. We focus on $\phi_{\leq m_1 + m_2} = x_1 x_2 + h(x_3, x_4)$. Remark that $h \neq 0$. The homogeneous part h_0 in h of the minimal weight d_0 with respect to weights $\operatorname{wt}(x_3, x_4) = (m_3, 1)$ always decomposes into a product of $x_4^{d_0 - m_3 \lfloor \frac{d_0}{m_3} \rfloor}$ and $\lfloor \frac{d_0}{m_3} \rfloor$ linear combinations of x_3 and $x_4^{m_3}$. Hence $h_0 \notin f_* \mathcal{O}_Y(-(d_0 + 1)E)$ because of the assumption $x_3 + h(x_4) \notin f_* \mathcal{O}_Y(-(m_3 + 1)E)$ for any h. Therefore, d_0 must be $m_1 + m_2$. We can see that this choice of coordinates satisfies the assumptions in Lemma 6.1, whence

$$(6.2) \qquad a \geq m_1 + m_2 + m_3 + 1 - 1 - (m_1 + m_2) = m_3$$

and the equality holds if and only if $f \cong g$ in Lemma 6.1 over X.

In particular, we have the following in type O:

Lemma 6.2. *If f is of type O, then f is the weighted blow-up of X with weights $\operatorname{wt}(x_1, x_2, x_3, x_4) = (i, n + 1 - i, 1, 1)$ for some $1 \leq i \leq n$ in (6.1).*

Lemma 6.3. *f is not of type IIa with $a = 4$.*

Proof. If f is of type IIa with $a = 4$, then $J = \{(5, 2)\}$ from Theorem 3.5, whence Y has exactly one non-Gorenstein point and it is a quotient singularity of type $\frac{1}{5}(1, -1, 3)$. Consider a birational morphism $h \colon Z \to Y$ such that Z has an h-exceptional divisor F the discrepancy of which with respect to K_X is 1. We can choose n different F, as valuations, corresponding to the exceptional divisors of the weighted blow-ups of X with weights $\operatorname{wt}(x_1, x_2, x_3, x_4) = (i, n + 1 - i, 1, 1)$ for $1 \leq i \leq n$ in (6.1). Write

$$K_Z = h^* K_Y + bF + (\text{others}),$$
$$h^* E = E_Z + mF + (\text{others}),$$

where E_Z is the birational transform of E. Then $K_Z = h^*(f^* K_X + 4E) + bF + (\text{others}) = h^* f^* K_X + 4E_Z + (b + 4m)F + (\text{others})$, whence $b = m = 1/5$ since $5b, 5m \in \mathbb{Z}$. Hence F is determined uniquely because Y has a quotient singularity of index 5 as the unique non-Gorenstein point. It contradicts that we can choose $n \geq 2$ different F. $\qquad \square$

The case where f is of type IIa with $a = 2$, type IIb or III remains. We can see the next lemma by following [8, Claim 6.13] faithfully:

Lemma 6.4. *Let $f \colon (Y \supset E) \to (X \ni P)$ be a germ of a three-fold divisorial contraction to a singular Gorenstein point P, and set $K_Y = f^* K_X + aE$. Assume that f is of type IIb or III, and set $r_1 = 1$ if f is of type III. $P \in X$ is identified as*

$$P \in X \cong o \in (\phi = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}.$$

Assume that the multiplicity of $\operatorname{div} x_4$ along E is 1. Then

(i) *for $1 \leq j \leq \min\{r_1, a\}$,*

$$f_* \mathcal{O}_Y(-jE) = (x_{1j}, x_{2j}, x_{3j}, x_4^i)\mathcal{O}_X$$

for some $x_{ij} = x_i + p_{ij}(x_4)$ $(i = 1, 2, 3)$, where

$$p_{ij} \in \bigoplus_{k=1}^{j-1} \mathbb{C} x_4^k \subset \mathbb{C}[x_4].$$

(ii) *Assume that $r_1 < a$.*
  (a) *The linear map*

  $$\bigoplus_{i=1}^{3} \mathbb{C} x_{ir_1} \to f_* \mathcal{O}_Y(-r_1 E)/(f_* \mathcal{O}_Y(-(r_1+1)E) + (x_4^{r_1})\mathcal{O}_X)$$

  has kernel of dimension 2.
  (b) *We assume that $x_{3r_1} \notin f_* \mathcal{O}_Y(-(r_1+1)E) + (x_4^{r_1})\mathcal{O}_X$. In this situation, for $r_1 < j \leq a$,*

  $$f_* \mathcal{O}_Y(-jE) = (x_{1j}, x_{2j})\mathcal{O}_X + \sum_{(k,l) \in \bigcup_{s \geq j} J_s} (x_{3r_1}^k x_4^l)\mathcal{O}_X$$

  for some $x_{ij} = x_i + p_{ij}(x_{3r_1}, x_4)$ $(i = 1, 2)$, where

  $$p_{ij} \in \bigoplus_{(k,l) \in \bigcup_{1 \leq s < j} J_s} \mathbb{C} x_{3r_1}^k x_4^l \subset \mathbb{C}[x_{3r_1}, x_4],$$

  $$J_s := \{(k,l) \in \mathbb{Z}_{\geq 0}^2 \mid r_1 k + l = s\}.$$

Remark 6.5. A similar result holds even if f is of type IIa with $a = 2$ by $d(-1) = \chi(\mathcal{Q}_{-1}) = 1$ in Remark 2.5(i). Assume that f is of type IIa with $a = 2$ and the multiplicity of div x_4 along E is 1. Then $x_1, x_2, x_3 \in f_* \mathcal{O}_Y(-2E) + (x_4^2)\mathcal{O}_X$.

Assume that f is either of type IIa with $a = 2$ or of type IIb$^\vee$ with $J = \{(r, 1), (r, 1)\}$. Then $m_3 \geq a$ by Lemma 6.4 and Remark 6.5. Assume that f is either of type IIb$^{\vee\vee}$ or of type III, and set $r_1 = 1, r_2 = r$ if f is of type III. We start with an identification (6.1). Write m_i for the multiplicity of div x_i along E; $m_4 = 1$ is assumed by coordinate change. Because a general element in $f_* \mathcal{O}_Y(-aE)$ gives a Du Val singularity $P \in S_X$ at worst of type $A_{r_1+r_2}$ by the results in Section 5, this element must be of the form $cx_3 + \cdots$ with $c \neq 0$ by Lemma 6.4. Hence by coordinate change such that $x_3 \mapsto x_3 + h(x_1, x_4)$ or $x_3 + h(x_2, x_4)$ if necessary, we may assume that $x_3 \in f_* \mathcal{O}_Y(-aE)$ by Lemma 6.4. Therefore $m_3 \geq a$. Hence we have the following proposition by (6.2):

Lemma 6.6. *Assume that f is of type IIb or III. Unless f is of type IIb$^\vee$ with $J = \{(2,1), (4,1)\}$, f is the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1, m_2, a, 1)$ with respect to the coordinates in (6.1).*

Finally we consider the exceptional case where f is of type IIb$^\vee$ with $J = \{(2,1), (4,1)\}$.

Lemma 6.7. *If f is of type IIb$^\vee$ with $J = \{(2,1), (4,1)\}$, then*
  (i) *P is a cA_2 point isomorphic to $o \in (x_1 x_2 + x_3^3 + g_{\geq 4}(x_3, x_4) = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}$, where $g_{\geq 4}$ has total order ≥ 4 in x_3, x_4.*
  (ii) *Y has exactly one non-Gorenstein point Q, and it is isomorphic to $o \in (y_1^2 + y_2^2 + y_3^2 + y_4^3 = 0)$ in the quotient space $\mathbb{C}^4_{y_1 y_2 y_3 y_4}/\frac{1}{4}(1, 3, 3, 2)$.*

(iii) $K_Y = f^* K_X + 3E$.

Proof. By Remark 1.5, we have (iii) and Y has exactly one non-Gorenstein point Q. The germ $Q \in Y$ is described as in Remark 1.5(ii). Because Y has Gorenstein index 4 and $a = 3$, for any valuation v with discrepancy 1 with respect to K_X, the centre of v on Y is Q and v has discrepancy $1/4$ with respect to K_Y, as in the proof of Lemma 6.3. Since there are n such valuations as mentioned in the proof of Lemma 6.3, $Q \in Y$ must have an isomorphism in (ii) by [6, Theorem 7.4] and the description in Remark 1.5(ii). In this case, there exist exactly two valuations with centre Q the discrepancy of which with respect to K_Y is $1/4$ by [6, Theorem 7.4]. Thus P has to be cA_2.

We take an isomorphism $P \in X \cong o \in (x_1 x_2 + g_3(x_3, x_4) + g_{\geq 4}(x_3, x_4) = 0) \subset \mathbb{C}^4$, where g_3 is the part of order 3 and $g_{\geq 4}$ that of order ≥ 4. To see (i) we have to show that g_3 is cubic, but this comes from the result in Section 5 that there exists a hyperplane section of $P \in X$ which is E_6 at P. $\square$

Example 6.8. There exists an example of Lemma 6.7. The weighted blow-up of $o \in (x_1^2 + x_2^2 + x_3^3 + x_1 x_4^2 = 0) \subset \mathbb{C}^4$ with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$ is such an example.

We have Lemmata 6.2, 6.3, 6.6 and 6.7. We can easily check the sufficient condition for f to be a divisorial contraction. Therefore Theorem 1.13 is deduced.

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