J Philos Logic
DOI 10.1007/s10992-010-9133-7

General-Elimination Harmony and the Meaning of the Logical Constants

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Received: 5 November 2009 / Accepted: 26 April 2010

Abstract Inferentialism claims that expressions are meaningful by virtue of rules governing their use. In particular, logical expressions are autonomous if given meaning by their introduction-rules, rules specifying the grounds for assertion of propositions containing them. If the elimination-rules do no more, and no less, than is justified by the introduction-rules, the rules satisfy what Prawitz, following Lorenzen, called an inversion principle. This connection between rules leads to a general form of elimination-rule, and when the rules have this form, they may be said to exhibit "general-elimination" harmony. Ge-harmony ensures that the meaning of a logical expression is clearly visible in its I-rule, and that the I- and E-rules are coherent, in encapsulating the same meaning. However, it does not ensure that the resulting logical system is normalizable, nor that it satisfies the conservative extension property, nor that it is consistent. Thus harmony should not be identified with any of these notions.

Keywords Harmony · Inferentialism · Autonomy · Validity · Tonk · Dummett · Gentzen · Prawitz · Lorenzen

This work is supported by Research Grant AH/F018398/1 (Foundations of Logical Consequence) from the Arts and Humanities Research Council, UK.

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Published online: 18 May 2010

1 Analytic Validity

What is the basis of our knowledge of logic, and of logical truth? Indeed, why are logical truths true? The traditional answer is that logical truths are analytic, that is, they are true solely in virtue of the meaning of the logical words they contain. What gives the meaning of the logical terms? A natural answer is: the rules for their use. Hence arise:

Logical inferentialism	The meaning of the logical constants is given by the rules for their use
Autonomy	The rules are self-justifying, that is, our knowledge of the logical truths derives from the logical rules, which determine the meaning in virtue of which those truths are true.

Traditionally, semantics has been denotational and representational, consisting in a homomorphic valuation from expressions to some range of objects.[1] This approach risks ontological explosion, first in hypostatizing denotations for empty names, predicates, conjunctions, prepositions and so on (though some of these may be construed as syncategorematic), then in seeking values for false propositions in the form of non-actual states of affairs. It is also regressive, since a criterion is now needed to determine which non-actual states of affairs are possible, which simply repeats the initial problem. Talk of possible worlds is an attractive metaphor, but does little useful philosophical work and much harm.[2]

Inferentialism, in contrast, is ontologically neutral. Expressions are meaningful if there are rules governing their use, in particular, logical expressions are given meaning by their introduction-rules, specifying the grounds for assertion of propositions containing them, and elimination-rules drawing inferences from those assertions. Robert Brandom cites Michael Dummett as saying:

> Learning to use a statement of a given form involves … learning two things: the conditions under which one is justified in making the statement; and what constitutes acceptance of it, i.e., the consequences of accepting it.[3]

Brandom expresses it in his own words as follows:

> What corresponds to an introduction rule for a propositional content is the set of sufficient conditions for asserting it, and what corresponds to an *elimination* rule is the set of *necessary* consequences of asserting it, that is, what follows from doing so.

[1]See, e.g., Meyer and Sylvan [8, p. 354]: "Syntax is *homomorphically copied* in Semantics."
[2]These claims were defended more fully in Read [15].
[3]Dummet [2, p. 453], cited in Brandom [1, p. 63].

> Gentzen famously defined connectives by specifying introduction rules, or inferentially sufficient conditions for the employment of the connective, and elimination rules, or inferentially necessary consequences of the employment of the connective.[4]

Arthur Prior, when presented with such an account of logic as autonomous and self-justifying, proposed a notorious example to show that inferentialism (or as he called it, the "analytical validity"-view) was mistaken. In an *ad hominem* objection, he introduced a new connective 'tonk' with the rules:

$$\frac{\alpha}{\alpha \text{ tonk } \beta}\ \text{tonk-I} \qquad\qquad \frac{\alpha \text{ tonk } \beta}{\beta}\ \text{tonk-E}$$

However, by chaining together an application of tonk-I to one of tonk-E, we can apparently derive any proposition (β) from any other (α). This is clearly absurd and disastrous. How can one possibly *define* such an inference into existence?

Dummett's response was to impose a requirement of harmony between the rules. In his first invocation of harmony, Dummett suggested that:

> The error ... lies ... in the failure to appreciate the interplay between the different aspects of 'use', and the requirement of harmony between them. Crudely expressed, there are always two aspects of the use of a given form of sentence: the conditions under which an utterance of that sentence is appropriate, which include, in the case of an assertoric sentence, what counts as an acceptable ground for asserting it; and the consequences of an utterance of it, which comprise both what the speaker commits himself to by the utterance and the appropriate response on the part of the hearer, including, in the case of assertion, what he is entitled to infer from it if he accepts it.[5]

If the elimination-rules do no more (and no less) than is justified by the introduction-rules, the rules may be said to be in harmony. Harmony serves to reveal clearly and explicitly what meaning is conferred by the rules. For if meaning is constituted by use, rather than denotation, then meaning resides globally and holistically in the rules. Systemization therefore urges that one identify a canonical set of rules encapsulating that meaning which the other rules, the rules justified by that meaning already conferred, should not disrupt or alter.

In fact, in Dummett's writings, there is more than one account of harmony. He defines "total harmony" as the requirement that when imposing rules for a new connective, the resulting system be a conservative extension of the old system.[6] That is, if language L' extends language L by the addition of a new connective, with new rules extending the system $\mathcal{L}$ based on L to a new system

[4] Brandom [1, p. 62].

[5] Dummett [2, p. 396; cf. p. 454].

[6] Dummett [3, p. 250].

$\mathcal{L}'$ based on L', anything in the language of L which is provable in $\mathcal{L}'$ should already have been provable in $\mathcal{L}$.

But Dummett also considers a different account of harmony, "intrinsic harmony", namely, that proofs in the new system $\mathcal{L}'$ be normalizable. Normalization is the requirement that maximum formulae be eliminable, where a maximum formula in a proof is any occurrence of a formula, or sequence of identical formulae, occurring both as the conclusion of an I-rule and major premise of an E-rule.[7] Proofs are normalizable if any proof can be replaced by a normal proof of the same conclusion, that is, one in which no maximum formulae occur. I will show that neither conservativeness nor normalization are appropriate as grounds of an inferentialist account of meaning which can respond adequately to Prior's challenge.

2 General-Elimination Harmony

What was wrong with the analytic validity views which Prior was attacking was the suggestion that the meaning of an expression was given by the totality of rules governing its use. As we saw, Brandom equates the I-rule with the set of sufficient conditions for assertion of a statement containing the expression, and the E-rule with the set of necessary consequences of that assertion. Prior took assertion of α to be sufficient for inferring $\ulcorner\alpha$ tonk $\beta\urcorner$ and assertion of β as necessary for it. Hence the necessary and sufficient conditions come apart. If we are to avoid that situation, we need to capture all the meaning of a term in both types of rule. Rather than the one constituting sufficient conditions, the other, necessary conditions, as Brandom claims, each in their totality constitutes both necessary and sufficient conditions.

> What we have been considering are two alternative ways of explaining the meanings of the sentences of a language: in terms of how we establish them as true; and in terms of what is involved in accepting them as true. They are alternative in that either is sufficient to determine the meaning of a sentence uniquely … Because either fully determines the meaning of a sentence … there ought … to exist a harmony between these two features of use.[8] (Dummett [4, p. 142])

These two features are dubbed by Dummett 'justification' and 'commitment' respectively:

> Justification and commitment ought to be in harmony with one another: that is why, if meaning is taken as given in terms of either, the theory

[7] Such a formula (or sequence of formulae) is called by Dummett [3, p. 248] a "local peak", and by Gentzen (von Plato [21]) a "hillock" (*Hügel*).

[8] That the clauses are necessary as well as sufficient corresponds to the extremal clause in inductive definitions, as recognised in Schroeder-Heister's notion of definitional reflection. See, e.g., Schroeder-Heister [19, p. 35].

> must show how the other can be derived from the meaning as so given. The condition for such harmony to obtain is twofold: first, that whatever serves to justify a statement ought also to justify any simpler statement to which acceptance of the first commits us; and, conversely, that all commitments consequent upon acceptance of a statement should already be consequent upon anything offered in complete justification of it. (Dummett [4, pp. 162–163])

Since Prior says that $\ulcorner\alpha$ tonk $\beta\urcorner$ can be inferred from α but not from β, $\ulcorner\alpha$ tonk $\beta\urcorner$ is equivalent to α, while since he says that β but not α can be inferred from $\ulcorner\alpha$ tonk $\beta\urcorner$, $\ulcorner\alpha$ tonk $\beta\urcorner$ is equivalent to β. The point is that by specifying tonk-I as the introduction-rule, Prior is saying not only that (a proof of) α is sufficient to justify an assertion of $\ulcorner\alpha$ tonk $\beta\urcorner$, but, by giving no other I-rule for 'tonk', he is saying that such a proof of α is the *only* ground for asserting $\ulcorner\alpha$ tonk $\beta\urcorner$. However, the E-rule for 'tonk' suggests that the grounds for asserting $\ulcorner\alpha$ tonk $\beta\urcorner$ are different, namely, that one might assert $\ulcorner\alpha$ tonk $\beta\urcorner$ because one has a proof of β. It is that which makes Prior's rules for 'tonk' incoherent. In contrast, the rules for, e.g., 'and' and 'or' are coherent, in that they agree on what is both necessary and sufficient for the assertion of statements containing them. It is the basis of this coherence between I- and E-rules which we must delineate, and which goes under the broad title of 'harmony'.

I differ from Dummett in recognising that inharmonious rules can confer meaning. There are three cases: harmonious rules, which are guaranteed to confer meaning, and inharmonious rules, some of which confer a coherent meaning and some do not. For example, the Curry–Fitch–Prawitz rule of $\Diamond$I gives only a sufficient, not a necessary, condition for assertion of $\ulcorner\Diamond\alpha\urcorner$:

$$\frac{\alpha}{\Diamond\alpha}\ \Diamond\mathrm{I}$$

That is, if α is true, so is $\Diamond\alpha$; but the truth of α is not *necessary* for that of $\Diamond\alpha$. Nonetheless, together with the Curry-Prawitz version of $\Diamond$E, the meaning of '$\Diamond$' is captured, at least in reflexive modal logics.[9] But it is unclear quite how the two rules work together to capture the meaning of '$\Diamond$', and how they work coherently in a way that the 'tonk'-rules do not. Just as assertion of α is not necessary for inferring $\ulcorner\Diamond\alpha\urcorner$, so too it is not necessary for inferring $\ulcorner\alpha$ tonk $\beta\urcorner$. Another case arises with a pair of negation rules for intuitionistic logic, as we will see later (Section 6):

$$\frac{\begin{array}{c}[\alpha]\\ \vdots\\ \neg\alpha\end{array}}{\neg\alpha} \qquad \frac{\neg\alpha \qquad \alpha}{\beta}$$

What is good about harmonious rules is that we can read the meaning off the rule transparently.

[9]See Read [16].

This coherence between the rules is what Gentzen had already realised in his famous paper of 1934 when he wrote:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions. (Gentzen [6, p. 80])

That is, the meaning of the connectives should be given by the inference rules for their assertion, the introduction-rules. Then they are "autonomous" (or "self-justifying"). The introduction-rule not only shows what is (severally) sufficient for assertion of the conclusion but also shows what is (jointly) necessary. The elimination-rule must then be justified by the introduction-rule. As Gentzen put it:

> It should be possible to display the E-inferences as unique (*eindeutig*, i.e., "single-valued") functions of their corresponding I-inferences. (Gentzen [6, p. 81])

How, then, should the E-rule be related to the I-rule so that it is justified by it in the way Gentzen intended? The key is given by Prawitz' "inversion principle":

> Whatever follows from the direct grounds for asserting a proposition must follow from the proposition.[10]

Negri and von Plato, from whom this formulation is cited, present this as a generalization of Prawitz' idea. However, Moriconi and Tesconi, in a recent paper [10], contend that it captures what Prawitz already intended. As Prawitz acknowledges, it is based on an idea of Paul Lorenzen's [7], "roughly speaking ... that completeness is attained if whenever a $\star$-proposition X satisfies a certain condition P, and the [introduction]-rule for '$\star$' states that $Y \Rightarrow X$, then $P(Y)$ must also hold."[11] The completeness issue is to ensure that the I-rules are (jointly) necessary as well as severally sufficient:

> Lorenzen considers the problem of how to guarantee that *any* proposition having $\star$ as its principal operator can *only* be obtained by means of that set of rules.[12]

I propose, therefore, to say that the introduction- and elimination-rules are in harmony when the E-rules do no more and no less than spell out what may be inferred from the assertion of the conclusion of the I-rules, given the grounds for its assertion. Following Dyckhoff and Francez [5], I shall call such a notion of harmony, "general-elimination harmony".

[10] Negri and von Plato [11, p. 6].

[11] Moriconi and Tesconi [10, p. 105].

[12] Moriconi and Tesconi [10, p. 104].

Arguably, this is what Dummett intended when he introduced the notion of "local" or "intrinsic harmony". He wrote:

> We may thus provisionally identify harmony between the introduction and elimination rules for a given logical constant with the possibility of carrying out this procedure, which we have called the levelling of local peaks.[13]

He later abandons that provisional identification in favour of "total harmony", namely, conservativeness. His reason is a fear that intrinsic harmony may be too weak a requirement. Perhaps the E-rule does not permit all that the I-rule justifies. General-elimination harmony (ge-harmony for short) is designed to exclude that possibility.

The idea of ge-harmony is that we may infer from an assertion all and only what follows from the various grounds for that assertion. Suppose that the grounds for assertion of $\delta\vec{\alpha}$ (some formula with main connective δ) are given schematically as Π_i, where $\Pi_i : 1 \leq i \leq m$ is a collection of subproofs or derivations. We can represent those proofs Π_i as derivations

$$\frac{\pi_{i1} \quad \dots \quad \pi_{in_i}}{\delta\vec{\alpha}}\ \delta\mathrm{I}$$

which I will write for short as $\pi_{i1}, \dots, \pi_{in_i} \Rightarrow \delta\vec{\alpha}$, giving the grounds Π_i for the assertion of $\delta\vec{\alpha}$. Then the harmonious form of the elimination-rule is

$$\frac{\delta\vec{\alpha} \quad \begin{matrix}(\pi_{1j_1}) \\ \vdots \\ \gamma\end{matrix} \quad \dots \quad \begin{matrix}(\pi_{mj_m}) \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \delta\mathrm{E}$$

discharging the assumptions π_{ij_i}.[14] That is, given an assertion of $\delta\vec{\alpha}$, and derivation(s) of γ from the various ground(s) for asserting $\delta\vec{\alpha}$, we may infer γ and discharge the assumption of those grounds. Those grounds may be multiple, for there may be several cases of the introduction-rule, as in $\vee$I. The inversion principle requires that, in any application of the E-rule, there be m minor premises, each deriving γ from some π_{ij}, that is, for each i there needs to be a derivation of γ from π_{ij} for some j. In total, there will be $\prod_{i=1}^{m} n_i$ E-rules, each with $m + 1$ premises.

Consider an application of the δI-rule followed by the corresponding δE-rule (producing a so-called "maximum formula", $\delta\vec{\alpha}$):

$$\frac{\dfrac{\begin{matrix}\vdots \\ \pi_{i1}\end{matrix} \quad \dots \quad \begin{matrix}\vdots \\ \pi_{in_i}\end{matrix}}{\delta\vec{\alpha}}\ \delta\mathrm{I} \quad \begin{matrix}(\pi_{1j_1}) \\ \vdots \\ \gamma\end{matrix} \quad \dots \quad \begin{matrix}(\pi_{mj_m}) \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \delta\mathrm{E}$$

[13] Dummett [3, p. 250].

[14] This makes more precise what was suggested in Read [14]. Cf. von Plato [20] and Prawitz [13, Section IV].

Then we know from the LH premise that for some i we have a derivation π_{ij} for every j. Hence in the minor premises we must ensure that for every i there is a derivation of γ from π_{ij} for some j. If so, we can "invert" the derivation to obtain:

$$\begin{array}{c} \vdots \\ \pi_{ij_i} \\ \vdots \\ \gamma \end{array}$$

Let us make this idea concrete by considering some examples. Often the inversion guaranteed by ge-harmony permits the elimination of maximum formulae; but not always, so ge-harmony is not the same as Dummett's intrinsic harmony, that is, it does not always lead to a normal form result.

3 Tonk

Take Prior's introduction-rule for 'tonk':

$$\frac{\alpha}{\alpha \text{ tonk } \beta}\ \text{tonk-I}$$

(so $m = 1$ and $n_1 = 1$). Assuming tonk-I to exhaust the grounds for asserting $\ulcorner\alpha \text{ tonk } \beta\urcorner$, we obtain the general case of tonk-E:

$$\frac{\alpha \text{ tonk } \beta \quad \begin{array}{c}(\alpha)\\ \vdots \\ \gamma\end{array}}{\gamma}\ \text{tonk-E} \qquad \text{that is,} \qquad \frac{\alpha \text{ tonk } \beta \quad \alpha \Rightarrow \gamma}{\gamma}\ \text{tonk-E}$$

If we permute the derivation of γ from α with the application of the rule, we obtain the simpler

$$\begin{array}{c} \dfrac{\alpha \text{ tonk } \beta}{\alpha}\ \text{tonk-out} \\ \vdots \\ \gamma \end{array}$$

This is, of course, not Prior's tonk-out rule, and does not lead to inconsistency and triviality as did his rule.

What justifies this permutation? Tonk-out is a special case of tonk-E, by letting $\gamma = \alpha$:

$$\frac{\alpha \text{ tonk } \beta \quad \begin{array}{c}(\alpha)\\ \vdots \\ \alpha\end{array}}{\alpha}\ \text{tonk-E} \qquad \text{that is,} \qquad \frac{\alpha \text{ tonk } \beta \quad \alpha \Rightarrow \alpha}{\alpha}\ \text{tonk-E}$$

and discarding the right-hand branch, which is derivable by Reflexivity. Conversely, the permuted inference shows that tonk-out is sufficient to derive everything that tonk-E permitted. Prior's E-rule, from $\ulcorner\alpha \text{ tonk } \beta\urcorner$ to infer β, would be justified only if we already had a derivation of β from α.

Then maximum formulae of the form $\ulcorner\alpha \text{ tonk } \beta\urcorner$ can be eliminated:

$$
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \alpha\end{matrix}}{\alpha \text{ tonk } \beta}\,\text{tonk-I} \qquad \begin{matrix}(\alpha)\\ \Pi_2\\ \gamma\end{matrix}}{\gamma}\,\text{tonk-E} \qquad \text{reduces to} \qquad \begin{matrix}\Pi_1\\ \alpha\\ \Pi_2\\ \gamma\end{matrix}
$$

If α is a new maximum formula, sandwiched between Π_1 and Π_2, it has lower degree than $\ulcorner\alpha \text{ tonk } \beta\urcorner$, and so by induction, the derivation normalizes.

4 Conjunction

A similar simplification is possible in the case of 'and' ($\wedge$). Given that the sole ground for asserting $\ulcorner\alpha \wedge \beta\urcorner$ consists of derivations of both α and β:

$$
\dfrac{\alpha \qquad \beta}{\alpha \wedge \beta}\,\wedge\text{I}
$$

(so $m = 1$ and $n_1 = 2$) it follows that one may infer from $\ulcorner\alpha \wedge \beta\urcorner$ whatever one may infer from the grounds for its assertion, that is, both α and β:

$$
\dfrac{\alpha \wedge \beta \qquad \begin{matrix}(\alpha)\\ \vdots\\ \gamma\end{matrix}}{\gamma}\,\wedge\text{E-1} \qquad \text{and} \qquad \dfrac{\alpha \wedge \beta \qquad \begin{matrix}(\beta)\\ \vdots\\ \gamma\end{matrix}}{\gamma}\,\wedge\text{E-2}
$$

That is, if γ follows either from the assumption α or from β, then γ follows from $\ulcorner\alpha \wedge \beta\urcorner$.

Clearly, maximum formulae of the form $\ulcorner\alpha_1 \wedge \alpha_2\urcorner$ may now be eliminated:

$$
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \alpha_1\end{matrix} \qquad \begin{matrix}\Pi_2\\ \alpha_2\end{matrix}}{\alpha_1 \wedge \alpha_2}\,\wedge\text{I} \qquad \begin{matrix}(\alpha_i)\\ \Pi_3\\ \gamma\end{matrix}}{\gamma}\,\wedge\text{E} \qquad \text{reduces to} \qquad \begin{matrix}\Pi_i\\ \alpha_i\\ \Pi_3\\ \gamma\end{matrix}
$$

and the degree of the maximum formula has been reduced.

The generalised $\wedge$E rules yield the usual $\wedge$E rules immediately, given Reflexivity:

$$
\dfrac{\alpha \wedge \beta \qquad \begin{matrix}(\alpha)\\ \vdots\\ \alpha\end{matrix}}{\alpha}\,\wedge\text{E-1} \qquad \text{and} \qquad \dfrac{\alpha \wedge \beta \qquad \begin{matrix}(\beta)\\ \vdots\\ \beta\end{matrix}}{\beta}\,\wedge\text{E-2}
$$

reduce to

$$
\dfrac{\alpha \wedge \beta}{\alpha}\,\text{Simp} \qquad \text{and} \qquad \dfrac{\alpha \wedge \beta}{\beta}\,\text{Simp}
$$

given that we can always derive γ from γ, for any γ.

With the standard rules in place, we can now permute the original derivations:

$$\cfrac{\alpha \wedge \beta}{\alpha}\ \text{Simp} \qquad\qquad \cfrac{\alpha \wedge \beta}{\beta}\ \text{Simp}$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$\gamma \qquad\qquad\qquad \gamma$$

Thus the rules of Simplification are justified (they're a special case); and they're sufficient, as the permutation shows.

We can also show that the generalised ∧-E rules are equivalent to a more common generalised rule:[15]

$$\cfrac{\alpha \wedge \beta \qquad \begin{matrix}\overbrace{(\alpha) \quad (\beta)} \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \wedge\text{E}$$

First, ∧E-1 and ∧E-2 are special cases of ∧E, where β (respectively, α) is not discharged—effectively, an application of Weakening to the assumptions. Conversely, by chaining applications of ∧E-1 and ∧E-2, we can validate ∧E:

$$\cfrac{\alpha \wedge \beta \qquad \cfrac{\alpha \wedge \beta \qquad \begin{matrix}\overbrace{\alpha \quad \beta} \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \wedge\text{E-1}(\alpha)}{\gamma}\ \wedge\text{E-2}(\beta)$$

However, this equivalence depends on Contraction in the assumptions, since the assumption $\ulcorner \alpha \wedge \beta \urcorner$ is here used twice. Thus the equivalence between ∧E and ∧E-1 and ∧E-2 depends on applying Weakening and Contraction to the assumptions.

5 Disjunction

The introduction-rules for '∨' are given by Addition:

$$\cfrac{\alpha}{\alpha \vee \beta}\ \vee\text{I} \qquad \text{and} \qquad \cfrac{\beta}{\alpha \vee \beta}\ \vee\text{I}$$

(so $m = 2$ and $n_1 = n_2 = 1$). According to ge-harmony, the harmonious E-rule will be the familiar:

$$\cfrac{\alpha \vee \beta \qquad \begin{matrix}(\alpha) \\ \vdots \\ \gamma\end{matrix} \qquad \begin{matrix}(\beta) \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \vee\text{E}$$

[15]See, e.g., Schroeder-Heister [18, p. 1294].

or in the new notation:

$$\frac{\alpha \vee \beta \quad \alpha \Rightarrow \gamma \quad \beta \Rightarrow \gamma}{\gamma}\ \vee\mathrm{E}$$

Is it possible to simplify this rule, as we did for $\wedge$E, and permute the derivations as before? It is, but only by introducing another structural operation similar to that we did for '$\wedge$'. However, whereas that operation combined assumptions, the new one must combine conclusions. That is, we enter the domain of multiple-conclusion logic. So let $\gamma = \alpha, \beta$:

$$\frac{\alpha \vee \beta \quad \alpha \Rightarrow \alpha, \beta \quad \beta \Rightarrow \alpha, \beta}{\alpha, \beta}\ \vee\mathrm{E}$$

The two minor premises now follow by Refl and Wk (on the right):

$$\frac{\alpha \vee \beta \quad \dfrac{\alpha \Rightarrow \alpha}{\alpha \Rightarrow \alpha, \beta}\ \mathrm{Wk} \quad \dfrac{\beta \Rightarrow \beta}{\beta \Rightarrow \alpha, \beta}\ \mathrm{Wk}}{\alpha, \beta}\ \vee\mathrm{E}$$

and so we get the simplified $\vee$-out rule:

$$\frac{\alpha \vee \beta}{\alpha, \beta}\ \vee\text{-out}$$

Since, in the original form of $\vee$E, we can derive γ both from α and from β, the permuted derivation becomes:

$$\begin{array}{c} \dfrac{\alpha \vee \beta}{\alpha, \beta}\ \vee\text{-out} \\ \vdots \\ \dfrac{\gamma, \gamma}{\gamma}\ \mathrm{Contr} \end{array}$$

ending in a Contraction. Note that Wk and Contr here apply to the conclusion, whereas in Section 4, they applied to the assumptions.

6 Negation

There are two ways to introduce negation. Let us concentrate on intuitionistic negation.[16] One approach relies on a suitable theory of absurdity ($\bot$):

$$\begin{array}{c} (\alpha) \\ \vdots \\ \dfrac{\bot}{\neg\alpha}\ \neg\mathrm{I} \end{array} \quad \text{which we can represent as} \quad \frac{\alpha \Rightarrow \bot}{\neg\alpha}\ \neg\mathrm{I}$$

[16] How to extend these ideas to classical negation was detailed in Read [14].

The E-rule follows as before by considerations of harmony:

$$\dfrac{\neg\alpha \qquad \begin{matrix}(\alpha \Rightarrow \bot)\\ \vdots \\ \gamma\end{matrix}}{\gamma}\,\neg\text{E} \qquad \text{that is,} \qquad \dfrac{\neg\alpha \quad (\alpha \Rightarrow \bot) \Rightarrow \gamma}{\gamma}\,\neg\text{E}$$

What does $\ulcorner(\alpha \Rightarrow \bot) \Rightarrow \gamma\urcorner$ mean? It says that if we have a derivation of $\bot$ from α, we can obtain a derivation of γ. More generally, $\ulcorner(\alpha \Rightarrow \beta) \Rightarrow \gamma\urcorner$ means that we have a derivation of γ on the assumption that we have a derivation of β from α. So if we have a derivation of α, we may assume we are able to derive β, from which we derive γ. That is,

$$\dfrac{\begin{matrix}\Pi_1\\ \delta\vec{\alpha}\end{matrix} \qquad \begin{matrix}\Pi'\\ (\alpha \Rightarrow \beta) \Rightarrow \gamma\end{matrix}}{\gamma}\,\delta\text{E} = \dfrac{\begin{matrix}\Pi_1\\ \delta\vec{\alpha}\end{matrix} \qquad \begin{matrix}(\alpha \Rightarrow \beta)\\ \Pi'\\ \gamma\end{matrix}}{\gamma}\,\delta\text{E} = \dfrac{\begin{matrix}\Pi_1\\ \delta\vec{\alpha}\end{matrix} \quad \begin{matrix}\Pi_2\\ \alpha\\ \vdots\\ \beta\\ \Pi_3\\ \gamma\end{matrix}}{\gamma}\,\delta\text{E}$$

$$\text{whence} \qquad \dfrac{\begin{matrix}\Pi_1\\ \delta\vec{\alpha}\end{matrix} \quad \begin{matrix}\Pi_2\\ \alpha\end{matrix} \quad \begin{matrix}(\beta)\\ \Pi_3\\ \gamma\end{matrix}}{\gamma}\,\delta\text{E}$$

We then permute the derivation of γ from β with the application of δE, to obtain:

$$\begin{matrix}\dfrac{\begin{matrix}\Pi_1\\ \delta\vec{\alpha}\end{matrix} \qquad \begin{matrix}\Pi_2\\ \alpha\end{matrix}}{\beta}\,\delta\text{-out}\\ \Pi_3\\ \gamma\end{matrix}$$

In the case of $\neg$E, $\beta = \bot$, so the $\neg$E rule simplifies to the familiar:

$$\dfrac{\neg\alpha \qquad \alpha}{\bot}\,\neg\text{-out}$$

$\neg$I is not, in Dummett's terminology , "pure", in containing another connective, $\bot$.[17] However, as Dummett argues, this is permissible provided it is not circular. First, introduce $\bot$. Then '$\neg$' can be introduced subsequently, using the theory of $\bot$. This is acceptable provided it does not introduce cycles.

[17] Dummett [3, p. 257]: "A rule may be called 'pure' if only one logical constant figures in it."

Alternatively, one can treat negation as primitive. An appropriate form of the I-rule for '¬' was given by Gentzen.[18] It is a form of *reductio ad absurdum*:

$$\dfrac{\begin{matrix}(\alpha)\\ \vdots\\ \neg\beta\end{matrix} \qquad \begin{matrix}(\alpha)\\ \vdots\\ \beta\end{matrix}}{\neg\alpha}\,\mathcal{R} \quad \text{which we can represent as} \quad \dfrac{\alpha \Rightarrow \neg\beta \quad \alpha \Rightarrow \beta}{\neg\alpha}\,\mathcal{R}$$

One might be tempted to reject such are rule on the ground that it is not sheer.[19] However, we will find that it permits normalization. Ge-harmony gives two harmonious E-rules ($m = 1$ and $n_1 = 2$):

$$\dfrac{\neg\alpha \qquad \begin{matrix}(\alpha \Rightarrow \neg\beta)\\ \vdots\\ \gamma\end{matrix}}{\gamma} \quad \text{and} \quad \dfrac{\neg\alpha \qquad \begin{matrix}(\alpha \Rightarrow \beta)\\ \vdots\\ \gamma\end{matrix}}{\gamma}$$

These simplify, as in the case of ¬E, to

$$\dfrac{\neg\alpha \qquad \alpha}{\neg\beta} \quad \text{and} \quad \dfrac{\neg\alpha \qquad \alpha}{\beta}$$

The first is a special case of the second, so the harmonious E-rule is:[20]

$$\dfrac{\neg\alpha \qquad \alpha}{\beta}\,\mathcal{V}$$

If we now consider a maximum formula of the form $\neg\alpha$ sandwiched between $\mathcal{R}$ and $\mathcal{V}$, we can apply the obvious conversion:

$$\dfrac{\dfrac{\begin{matrix}(\alpha)\\ \Pi_1\\ \neg\beta\end{matrix} \qquad \begin{matrix}(\alpha)\\ \Pi_2\\ \beta\end{matrix}}{\neg\alpha}\,\mathcal{R} \qquad \begin{matrix}\Pi_3\\ \alpha\end{matrix}}{\gamma}\,\mathcal{V} \quad \text{converts to} \quad \dfrac{\begin{matrix}\Pi_3\\ \alpha\\ \Pi_1\\ \neg\beta\end{matrix} \qquad \begin{matrix}\Pi_3\\ \alpha\\ \Pi_2\\ \beta\end{matrix}}{\gamma}\,\mathcal{V}$$

However, the conversion does not necessarily reduce the degree of the maximum formula. Π_1 concludes in an occurrence of $\ulcorner\neg\beta\urcorner$, major premise of an application of $\mathcal{V}$, and so may itself be a maximum formula, possibly of degree greater than that of $\ulcorner\neg\alpha\urcorner$. Nonetheless, in this case, the maximum formula can eventually be removed, since the "rank" of the derivation of $\ulcorner\neg\alpha\urcorner$ has been reduced.[21]

[18] Gentzen introduces this in an early version of his Ph.D. thesis contained in an unpublished MS held at the University of Zurich (Hs 974:271), p. 9. An edition of this MS is in preparation by Christian Thiel and Jan von Plato. Cf. von Plato [21].

[19] Dummett [3, p. 257]: "We may call a rule 'sheer' if either it is an introduction rule for a logical constant that does not figure in any of the premisses or in a discharged hypothesis, or it is an elimination rule for one that does not figure in the conclusion or in a discharged hypothesis."

[20] This rule is given by Gentzen in the manuscript cited in footnote 18.

[21] Gentzen shows how to do this in the Zurich MS.

Finally, consider the rules for intuitionistic negation mentioned in Section 2 above:[22]

$$\frac{\alpha \Rightarrow \neg\alpha}{\neg\alpha}\,\neg\mathrm{I}' \qquad\qquad \frac{\neg\alpha \qquad\qquad \alpha}{\beta}\,\mathcal{V}$$

If we apply ge-harmony to $\neg\mathrm{I}'$, we obtain:

$$\frac{\neg\alpha \qquad \begin{matrix}\alpha \Rightarrow \neg\alpha\\ \vdots\\ \beta\end{matrix}}{\beta}$$

which reduces as above to

$$\frac{\neg\alpha \qquad\quad \alpha \qquad\quad \begin{matrix}\neg\alpha\\ \vdots\\ \beta\end{matrix}}{\beta}$$

But this does not reduce to, or warrant, $\mathcal{V}$. That is, $\neg\mathrm{I}'$ does not seem to justify $\mathcal{V}$ in the way that harmony requires. Nonetheless, $\neg\mathrm{I}'$ and $\mathcal{V}$ are sufficient for an intuitionistic account of negation, since we can derive $\mathcal{R}$ from $\neg\mathrm{I}'$ and $\mathcal{V}$:

$$\cfrac{\cfrac{\begin{matrix}(\alpha)\\ \vdots\\ \neg\beta\end{matrix} \qquad\quad \begin{matrix}(\alpha)\\ \vdots\\ \beta\end{matrix}}{\neg\alpha}\,\mathcal{V}}{\neg\alpha}\,\neg\mathrm{I}'$$

The obvious conclusion is that $\neg\mathrm{I}'$ does not give the full meaning of '$\neg$', just as the Curry–Fitch–Prawitz $\Diamond$I-rule does not give the full meaning of '$\Diamond$', as noted in Section 2. The meaning conferred on '$\neg$' by $\neg\mathrm{I}'$ does not suffice to justify $\mathcal{V}$, yet in conjunction with $\mathcal{V}$, $\neg\mathrm{I}'$ does give a complete intuitionistic account of '$\neg$'. $\neg\mathrm{I}'$ says what is sufficient for assertion of $\ulcorner\neg\alpha\urcorner$ but not what is necessary. $\neg\mathrm{I}'$ requires supplementation by $\mathcal{V}$, as shown in the derivation of $\mathcal{R}$ using $\neg\mathrm{I}'$ and $\mathcal{V}$ together.

7 Inconsistency

GE-harmony does not prevent one introducing inconsistent connectives with harmonious rules. Dummett and others claim that harmony should ensure consistency. Not so: the I-rule can itself be inconsistent (but not necessarily incoherent). 'tonk' was indeed incoherent, but that was an incoherence between

[22] The rules are given in, e.g., Dummett [3, p. 291]. Note that, although $\neg\mathrm{I}'$ is pure, it is not sheer, nor does it (or $\mathcal{R}$) satisfy Dummett's complexity condition [3, p. 258]: "the minimal demand we should make on an introduction rule intended to be self-justifying is that its form be such as to guarentee that, in any application of it, the conclusion will be of higher logical complexity that any of the premisses and than any discharged hypothesis."

Prior's I- and E-rules. Coherence and consistency are different. Coherent rules can be inconsistent, in allowing one to derive contradiction. Consistent rules can be incoherent, when the meaning given by one rule (e.g., tonk-I, or Curry-Prawitz $\Diamond$I) does not cohere with that given by another (Prior's tonk-E, or Curry-Prawitz $\Diamond$E).

For example, suppose one introduces a zero-place connective '$\bullet$' with the rule:

$$\begin{array}{c} (\bullet) \\ \vdots \\ \dfrac{\bot}{\bullet}\ \bullet\mathrm{I} \end{array} \quad \text{which we can represent as} \quad \dfrac{\bullet \Rightarrow \bot}{\bullet}\ \bullet\mathrm{I}$$

where once again, $\alpha \Rightarrow \beta$ abbreviates a derivation of β from α. That is, from a derivation of absurdity ($\bot$) from $\bullet$, one can infer $\bullet$. The inversion principle shows that the harmonious elimination-rule reads:

$$\dfrac{\bullet \qquad \begin{array}{c} (\bullet \Rightarrow \bot) \\ \vdots \\ \gamma \end{array}}{\gamma}\ \bullet\mathrm{E}$$

$\bullet$E says that one may infer from $\bullet$ anything (*viz* γ) that one may infer from supposing that one can infer $\bot$ from $\bullet$, which is what $\bullet$I says justifies assertion of $\bullet$. As before, the minor premise resolves into a derivation of $\bullet$ and a derivation of γ from $\bot$. Permuting the latter with the application of $\bullet$E reduces the inference to:

$$\begin{array}{c} \dfrac{\bullet \quad \bullet}{\bot}\ \bullet\text{-out} \\ \vdots \\ \gamma \end{array}$$

Note that $\bullet$E strictly requires two copies of its premise.

Once again, $\bullet$I is not "pure", in containing reference to another connective, '$\bot$'. First, introduce '$\bot$'. Then '$\neg$' and '$\bullet$' can be introduced subsequently, using the theory of '$\bot$'. However, $\bullet$I is not "sheer" either, since '$\bullet$' occurs in the ground for its own assertion. Again, this does not introduce cycles. But it rules out normalization, since in this case the inversion principle does not reduce complexity. A maximum formula of the form $\bullet$ is not reduced in degree by the inversion procedure.

One might object that $\bullet$I is not (in Dummett's terminology, again) "direct", being "oblique" in that '$\bullet$' occurs in a hypothesis discharged by the rules as in the classical *reductio* rule:

$$\begin{array}{c} (\neg\alpha) \\ \vdots \\ \dfrac{\alpha}{\alpha}\ \mathrm{CR} \end{array}$$

Indeed, $\bullet$I does not meet the complexity condition (see footnote 22) any more than do $\mathcal{R}$, $\neg\mathrm{I}'$ and CR. Peter Milne [9] rejoices in such oblique rules.

Reflection on Milne's rules, and on $\mathcal{R}$, $\neg\mathrm{I}'$ and CR shows that Dummett's complexity demand is, in his own phrase, "exorbitant".

By ge-harmony, '$\bullet$' satisfies the inversion principle.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \bullet \Rightarrow \bot\end{matrix}}{\bullet}\,\bullet\mathrm{I} \qquad \begin{matrix}\Pi_2\\ \bullet\end{matrix}}{\bot}\,\bullet\text{-out} \quad = \quad \dfrac{\dfrac{\begin{matrix}(\bullet)\\ \Pi_1\\ \bot\end{matrix}}{\bullet}\,\bullet\mathrm{I} \qquad \begin{matrix}\Pi_2\\ \bullet\end{matrix}}{\bot}\,\bullet\text{-out} \quad \text{converts to} \quad \begin{matrix}\Pi_2\\ \bullet\\ \Pi_1\\ \bot\end{matrix}$$

replacing each discharged leaf of the form $\bullet$ in Π_1 by Π_2. However, as we noted, the inversion does not reduce the degree of the maximum formula. Π_2 concludes in an occurrence of $\bullet$, of the same degree as the original maximum formula.

In fact, the $\bullet$-rules produce a non-conservative extension:

$$\dfrac{\dfrac{\dfrac{\bullet^1 \quad \bullet^1}{\bot}\,\bullet\mathrm{E}}{\bullet}\,\bullet\mathrm{I}(1) \qquad \dfrac{\dfrac{\bullet^2 \quad \bullet^2}{\bot}\,\bullet\mathrm{E}}{\bullet}\,\bullet\mathrm{I}(2)}{\bot}\,\bullet\mathrm{E}$$

or spelling the applications of Contraction out explicitly:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bullet \Rightarrow \bullet \quad \bullet \Rightarrow \bullet}{\bullet, \bullet \Rightarrow \bot}\,\bullet\mathrm{E}}{\bullet \Rightarrow \bot}\,\mathrm{Contr}}{\Rightarrow \bullet}\,\bullet\mathrm{I} \qquad \dfrac{\dfrac{\dfrac{\bullet \Rightarrow \bullet \quad \bullet \Rightarrow \bullet}{\bullet, \bullet \Rightarrow \bot}\,\bullet\mathrm{E}}{\bullet \Rightarrow \bot}\,\mathrm{Contr}}{\Rightarrow \bullet}\,\bullet\mathrm{I}}{\Rightarrow \bot}\,\bullet\mathrm{E}$$

This proof is not normalizable: the maximum wffs of the form $\bullet$ cannot be removed. So if ge-harmony is the right account of harmony, harmony does not guarantee, and is not the same as, either normalization or conservative extension.

Note that $\bullet$ is equivalent to its own negation:

$$\dfrac{\dfrac{\dfrac{\bullet^1 \quad \bullet^2}{\bot}\,\bullet\text{-out}}{\neg\bullet}\,\neg\mathrm{I}(1) \qquad \dfrac{\dfrac{\neg\bullet^3 \quad \bullet^4}{\bot}\,\neg\mathrm{E}}{\bullet}\,\bullet\mathrm{I}(4)}{\bullet \leftrightarrow \neg\bullet}\,\leftrightarrow\mathrm{I}(2{,}3)$$

$\bullet$ is equivalent to $\neg\bullet$, since the derivation of $\bot$ from $\bullet$ is both enough to assert $\bullet$ (by $\bullet$I) and to deny it (by $\neg$I). Taking $\bullet$I as the sole introduction-rule for $\bullet$, entailing absurdity is both necessary and sufficient for the assertion of $\bullet$. Harmony cannot prevent inconsistency. But it helps to locate, identify and understand that inconsistency. Thus $\bullet$ is inconsistent. It is a proof-conditional Liar sentence.[23]

However, $\bullet$I is not inherently contradictory or inconsistent. Recall the observation that the I-rule(s) give not only sufficient but also necessary

[23] Cf. Schroeder-Heister [19, p. 36], where $\bullet$ is denoted by 'a', and introduced by the definition $a \Leftarrow \neg a$.

grounds for assertion. If •I, as given, is the only I-rule for •, then indeed, • is inconsistent. But suppose we supplement •I with a second I-rule for •. For clarity, let us write the new connective as ∘:

$$\frac{\circ \Rightarrow \bot}{\circ}\ \circ\mathrm{I}_1 \qquad \text{and} \qquad \frac{\circ \Rightarrow \top}{\circ}\ \circ\mathrm{I}_2$$

where $\top = \neg\bot$. Then entailing absurdity is no longer necessary for assertion of ∘. It's sufficient, but not necessary. Another possible gound for asserting ∘ is entailing ⊤. Now apply ge-harmony to $\circ\mathrm{I}_1$ and $\circ\mathrm{I}_2$ (so $m = 2$ and $n_1 = n_2 = 1$). We obtain a single ∘E-rule:

$$\frac{\circ \quad \begin{matrix}(\circ \Rightarrow \bot)\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}(\circ \Rightarrow \top)\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \circ\mathrm{E}$$

which the considerations of Section 6 convert to:

$$\frac{\circ \quad \circ \quad \begin{matrix}(\bot)\\ \vdots \\ \gamma\end{matrix} \quad \circ \quad \begin{matrix}(\top)\\ \vdots \\ \gamma\end{matrix}}{\gamma}$$

Applying Contraction yields:

$$\frac{\circ \quad \begin{matrix}(\bot)\\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}(\top)\\ \vdots \\ \gamma\end{matrix}}{\gamma}$$

If we assume that $\bot \vdash \alpha$ for all α, and $\vdash \top$, this simplifies to:

$$\frac{\circ \quad \gamma}{\gamma}$$

On reflection, that is unsurprising. The rules for ∘ give it the meaning that either it is true or false (in contrast to those for •, which make it mean that it is both true and false), that is, that of a tautology. So ∘ is equivalent to ⊤. ∘ is perfectly benign, where • is perfectly malignant. The fault with • is that entailing contradiction is not only sufficient for its assertion, but also necessary. That is not true of ∘.

Restall [17, p. 203] proposes the following sequent-rules for set-abstraction:

$$\frac{X \Rightarrow \phi(a), Y}{X \Rightarrow a \in \{x : \phi(x)\}, Y}\ \in\mathrm{R} \qquad \frac{X, \phi(a) \Rightarrow Y}{X, a \in \{x : \phi(x)\} \Rightarrow Y}\ \in\mathrm{L}$$

Let r be the term $\{x : x \notin x\}$. Then we have the two proofs:[24]

$$\cfrac{\cfrac{\cfrac{r \notin r \Rightarrow r \notin r}{r \in r \Rightarrow r \notin r}\,\in\mathrm{L}}{\Rightarrow r \notin r, r \notin r}\,\neg\mathrm{R}}{\Rightarrow r \notin r}\,\text{Contr-R} \qquad \cfrac{\cfrac{\cfrac{r \notin r \Rightarrow r \notin r}{r \notin r \Rightarrow r \in r}\,\in\mathrm{R}}{r \notin r, r \notin r \Rightarrow}\,\neg\mathrm{L}}{r \notin r \Rightarrow}\,\text{Contr-L}$$

Then $\bullet$ is essentially just $r \in r$. We can see this by formulating $\in$R and $\in$L in natural deduction format:[25]

$$\frac{\phi(a)}{a \in \{x : \phi(x)\}}\,\in\mathrm{I} \qquad \frac{a \in \{x : \phi(x)\} \quad \phi(a) \Rightarrow \gamma}{\gamma}\,\in\mathrm{E}$$

Accordingly, Restall's proofs become:

$$\cfrac{\cfrac{r \in r^1 \quad \cfrac{r \in r^1 \quad r \notin r^2}{r \notin r}\,\in\mathrm{E}(2)}{\bot}\,\neg\mathrm{E}}{r \notin r}\,\neg\mathrm{I}(1)\text{+Contr} \qquad \cfrac{\cfrac{r \notin r^1 \quad \cfrac{r \notin r^1}{r \in r}\,\in\mathrm{I}}{\bot}\,\neg\mathrm{E}}{\neg r \notin r}\,\neg\mathrm{I}(1)\text{+Contr}$$

in other words:

$$\cfrac{\cfrac{\bullet^1 \quad \cfrac{\bullet^1 \quad \neg\bullet^2}{\neg\bullet}\,\bullet\mathrm{E}(2)}{\bot}\,\neg\mathrm{E}}{\neg\bullet}\,\neg\mathrm{I}(1)\text{+Contr} \qquad \cfrac{\cfrac{\neg\bullet^1 \quad \cfrac{\neg\bullet^1}{\bullet}\,\bullet\mathrm{I}}{\bot}\,\neg\mathrm{E}}{\neg\neg\bullet}\,\neg\mathrm{I}(1)\text{+Contr}$$

It should now be clear that $\bullet$ is inconsistent, but not incoherent. The rules for '$\bullet$' are harmonious, in that the (self-contradictory) meaning given to '$\bullet$' by $\bullet$I is matched exactly by the consequences which can be drawn by $\bullet$E. $\bullet$I and $\bullet$E satisfy the inversion principle. Nonetheless, the inversion procedure is not guaranteed to reduce the complexity of the proof, and maximum formulae of the form $\bullet$ are not always removed. Hence there is no normalization result. Harmony does not guarantee normalization (or conservative extension), and the three notions should not be equated.

'$\bullet$' is not the only such counterexample to the supposed connection between harmony and normalization, though the natural examples do form a family. Just as $\bullet$ is a kind of proof-conditional Liar (or Russell) paradox, there is also a proof-conditional Curry paradox. Take an arbitrary formula, α, and introduce $\boxed{\alpha}$ by the following I-rule:

$$\cfrac{\begin{matrix}(\boxed{\alpha}) \\ \vdots \\ \alpha\end{matrix}}{\boxed{\alpha}}\,\boxed{\alpha}\,\mathrm{I}$$

Ge-harmony yields as E-rule in the usual way:

$$\frac{\boxed{\alpha} \qquad \boxed{\alpha}}{\alpha}\,\boxed{\alpha}\,\mathrm{E}$$

[24] Correcting the proofs in Restall [17, p. 203]. Cf. Prawitz [12, Appendix B, p. 95].
[25] See also Prawitz [12, p. 94].

$\boxed{\alpha}$ satisfies the inversion principle:

$$\dfrac{\dfrac{\begin{matrix}(\boxed{\alpha})\\ \Pi_1\\ \alpha\end{matrix}}{\boxed{\alpha}}\,\boxed{\alpha}\mathrm{I} \qquad \begin{matrix}\Pi_2\\ \boxed{\alpha}\end{matrix}}{\alpha}\,\boxed{\alpha}\mathrm{E} \qquad \text{converts to} \qquad \begin{matrix}\Pi_2\\ \boxed{\alpha}\\ \Pi_1\\ \alpha\end{matrix}$$

Nevertheless, as with '•', the maximum formula $\boxed{\alpha}$ has not been removed or reduced in degree, and normalization is prevented. Indeed, using '$\boxed{\alpha}$' one can, in familiar Curry fashion, obtain a proof of α:

$$\dfrac{\dfrac{\dfrac{\boxed{\alpha}^1 \quad \boxed{\alpha}^1}{\alpha}\,\boxed{\alpha}\mathrm{E}}{\boxed{\alpha}}\,\boxed{\alpha}\mathrm{I}(1) \qquad \dfrac{\dfrac{\boxed{\alpha}^2 \quad \boxed{\alpha}^2}{\alpha}\,\boxed{\alpha}\mathrm{E}}{\boxed{\alpha}}\,\boxed{\alpha}\mathrm{I}(2)}{\alpha}\,\boxed{\alpha}\mathrm{E}$$

Clearly, '•' functions like '$\boxed{\alpha}$', replacing 'α' by '$\perp$', just as the Liar paradox results from the Curry paradox in the same way.

8 Conclusion

Harmony is not normalization, nor is harmony conservative extension, that is, Dummett's "total harmony". Harmony is given by the inversion principle. The correct account of harmony is "general-elimination harmony". The totality of introduction-rules for a logical expression give the grounds for assertion of statements containing it, and so serve to define its meaning. That meaning then justifies inferences made from such statements by application of the E-rule.

In this way, the logical constants are given a proof-conditional meaning in which the introduction- and elimination-rules lie in harmony, where harmony ensures transparency in the meaning conferred and whose virtue is clarity. The notion of harmony used here, general-elimination harmony, differs from Dummett's preferred sense of harmony, for it does not guarantee normalization, nor does normalization guarantee harmony. Moreover, it differs from Dummett's total harmony, for ge-harmony does not guarantee conservativeness. Nonetheless, ge-harmony better matches what Gentzen meant by saying that the introductions serve to define the meaning of the logical expressions and that the eliminations are no more than a consequence of the meaning so conferred.

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