Acta Math., 178 (1997), 1–37

General existence theorems for Hamilton–Jacobi equations in the scalar and vectorial cases

by

BERNARD DACOROGNA and PAOLO MARCELLINI

École Polytechnique Fédérale de Lausanne
Lausanne, Switzerland

Università di Firenze
Firenze, Italy

Contents

1. Introduction

We consider the Dirichlet problem for *Hamilton–Jacobi equations* both in the scalar and in the vectorial cases. We deal with the following problem:

$$\begin{cases} F(Du(x))=0, & \text{a.e. } x\in\Omega, \\ u(x)=\varphi(x), & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a (bounded) open set of $\mathbf{R}^n$, $F\colon \mathbf{R}^{m\times n}\to\mathbf{R}$ and $\varphi\in W^{1,\infty}(\Omega;\mathbf{R}^m)$. We emphasize that $u\colon\Omega\subset\mathbf{R}^n\to\mathbf{R}^m$, with $m,n\geqslant 1$, is a *vector valued function* if $m>1$ (otherwise, if $m=1$, we say that u is a *scalar function*). As usual Du denotes the gradient of u.

This problem (1.1) has been intensively studied, essentially in the scalar case in many relevant articles such as Lax [28], Douglis [23], Kružkov [27], Crandall–Lions [16], Crandall–Evans–Lions [14], Capuzzo Dolcetta–Evans [8], Capuzzo Dolcetta–Lions [9], Crandall–Ishii–Lions [15]. For a more complete bibliography we refer to the main recent monographs of Benton [7], Lions [29], Fleming–Soner [25], Barles [6] and Bardi–Capuzzo Dolcetta [5].

Our motivation to study this equation, besides its intrinsic interest, comes from the calculus of variations. In this context first order partial differential equations have been intensively used, cf. for example the monographs of Carathéodory [10] and Rund [36] (for more recent developments on the vectorial case, see [19]).

In this paper we propose some new hypotheses on the function F in (1.1) that allow us to treat *systems of equations* as well as *vectorial problems* (cf. examples below). The general existence result (Theorem 2.1) can be applied to the following examples, that for the sake of simplicity we state under the additional assumption that the boundary datum φ is of class $C^1(\Omega;\mathbf{R}^m)$.

Example 1 (nonconvex scalar case). Let $m=1$. If $\{F(\xi)=0\}$ is closed then under the sole assumption

$$D\varphi(x)\in\{F(\xi)=0\}\cup\operatorname{int}\operatorname{co}\{F(\xi)=0\} \tag{1.2}$$

the Dirichlet problem (1.1) has a solution $u\in W^{1,\infty}(\Omega)$. (By $\operatorname{int}\operatorname{co}\{F(\xi)=0\}$ we mean the interior of the convex hull of the zeroes of F.) We emphasize that no hypothesis is made on F, neither convexity, nor coercivity, not even continuity.

For instance a system of N equations of the type

$$\begin{cases} F_i(Du)=0 & \text{a.e. in } \Omega,\ i=1,2,\dots,N,\\ u=\varphi & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

enters in the framework (1.1), (1.2), by setting $F=\sum_{i=1}^N F_i^2$. Thus, for example, the problem

$$\begin{cases} |\partial u/\partial x_i|=a_i & \text{a.e. in } \Omega,\ i=1,2,\dots,n,\\ u=\varphi & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

has a solution if $a_i>0$ and $|\partial\varphi/\partial x_i|<a_i$ for every $i=1,2,\dots,n$.

It is interesting to note that, if F is convex and satisfies a mild coercivity condition that rules out the linear case, then (1.2) becomes the usual necessary and sufficient condition for existence (cf. Kružkov [27], Lions [29]), namely

$$F(D\varphi(x))\leqslant 0,\quad x\in\Omega. \tag{1.5}$$

Example 2 (the prescribed singular values case). Let $n=m>1$. For every $\xi\in\mathbf{R}^{n\times n}$ (i.e. ξ is a real $(n\times n)$-matrix), we denote by $\lambda_i(\xi)$, $0\leqslant\lambda_1\leqslant\lambda_2\leqslant\dots\leqslant\lambda_n$, $i=1,2,\dots,n$, the *singular values* of the matrix ξ, i.e. the eigenvalues of the matrix $(\xi^t\xi)^{1/2}$.

Let $0<a_1\leqslant a_2\leqslant\dots\leqslant a_n$. Then the problem

$$\begin{cases} \lambda_i(Du)=a_i & \text{a.e. in } \Omega,\ i=1,2,\dots,n,\\ u=\varphi & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

has a solution $u \in W^{1,\infty}(\Omega; \mathbf{R}^n)$ if $\lambda_n(D\varphi) < a_1$ in Ω (more general boundary conditions are considered in §5).

So, in particular if $m=n=2$, the Dirichlet problem (1.6) can be rewritten in the form

$$\begin{cases} |Du|^2 = a_1^2 + a_2^2 & \text{a.e. in } \Omega, \\ |\det Du| = a_1 a_2 & \text{a.e. in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Provided the L^∞-norm of $D\varphi$ is sufficiently small, then (1.7) has a solution. Note that the system (1.7) is a combination of a *vectorial eikonal equation*, $|Du|^2 = a_1^2 + a_2^2$, and a prescribed *modulus of the Jacobian equation*, $|\det Du| = a_1 a_2$. Both equations have been separately studied in the literature. For the first one, see for example Kružkov [27] and Lions [29]. For the second one (without the modulus), cf. Dacorogna–Moser [20].

The Dirichlet problem (1.6) can also be rewritten in terms of "*potential wells*"; namely, if $a_i = 1$ for $i = 1, 2, ..., n$, then (1.6) and (1.7) take the form

$$\begin{cases} Du(x) \in \mathrm{SO}(n) I \cup \mathrm{SO}(n) I_-, & \text{a.e. } x \in \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\mathrm{SO}(n)$ denotes the set of orthogonal matrices with positive determinant, I is the identity matrix and

$$I_- = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}.$$

The problem of potential wells finds its origins in elasticity (cf. Ball–James [4], for example). Problem (1.8) has been solved by Cellina–Perrotta [13] if $n=3$ and $\varphi=0$.

The existence results stated in the above examples are a consequence of general theorems established in §2. The main points in the proof are:

(i) The *Baire category method* introduced by Cellina [11] and developed by De Blasi–Pianigiani [21], [22], [34], in the context of Cauchy problems for ordinary differential inclusions.

(ii) The weak lower semicontinuity and the *quasiconvexity* condition introduced by Morrey [33] (see also Ball [3] and [17]), that is the appropriate extension of convexity to vector valued problems.

We very roughly outline the idea of the proof following the above scheme. We first construct a quasiconvex function f whose zeroes are also zeroes of F. We then define for $k \in \mathbf{N}$,

$$V = \{u \in \varphi + W_0^{1,\infty}(\Omega; \mathbf{R}^m) : f(Du) \leqslant 0 \text{ a.e. in } \Omega\},$$

$$V_k = \left\{ u \in V : \int_\Omega f(Du(x))\, dx > -\frac{1}{k} \right\}.$$

The quasiconvexity of f (and at this stage, convexity of f would be sufficient) and boundedness of the gradients easily ensure that V is a complete metric space in the L^∞-norm and that V_k is open in V. The more difficult part is to show that V_k is dense in V and there the full strength of quasiconvexity is needed. Then the Baire category theorem implies that the intersection of V_k, for $k \in \mathbf{N}$, is *dense* in V, i.e. the set

$$\begin{aligned}\bigcap_{k\in\mathbf{N}} V_k &= \left\{u\in V : \int_\Omega f(Du(x))\,dx \geqslant 0\right\}\\ &= \{u\in\varphi+W_0^{1,\infty} : f(Du)=0 \text{ a.e.}\} \subset \{u\in\varphi+W_0^{1,\infty} : F(Du)=0 \text{ a.e.}\}\end{aligned}$$

is dense in V. Therefore the set of solutions of the Dirichlet problem (1.1) is *dense* in the set V.

This density property obviously contrasts with the uniqueness of *viscosity solutions* (notion introduced in this context by Crandall–Lions [16]) as established in the quoted literature on Hamilton–Jacobi equations in the scalar case. The notion of viscosity solution has not yet been extended to the vectorial context, since the definition uses ordering of the set of values of u. In particular the notion of *maximal solution* is not defined in the vectorial case. In our approach we prove that the set of solutions is not empty (and in fact it is even dense in V); one then could propose an optimality criterion to select one of these solutions. Of course in the scalar case, usually, the best criterion is the viscosity one.

2. The quasiconvex case

We now state the main theorem of this section.

THEOREM 2.1 (the quasiconvex case). *Let $\Omega\subset\mathbf{R}^n$ be an open set, and let $\varphi\in W^{1,\infty}(\Omega;\mathbf{R}^m)$ and $f:\mathbf{R}^{m\times n}\to\mathbf{R}$ satisfy the following hypotheses:*

$$f \text{ is quasiconvex;} \tag{2.1}$$

$$\text{there exists a compact convex set } K \text{ such that } K\subset\{\xi\in\mathbf{R}^{m\times n} : f(\xi)\leqslant 0\}; \tag{2.2}$$

$$Qf^-=0 \text{ on } \operatorname{int} K, \text{ where } f^-=-f \text{ on } K \text{ and } +\infty \text{ otherwise;} \tag{2.3}$$

$$D\varphi(x) \text{ is compactly contained in } \operatorname{int} K. \tag{2.4}$$

Then there exists $u\in W^{1,\infty}(\Omega;\mathbf{R}^m)$ such that

$$\begin{cases} f(Du(x))=0, & \text{a.e. } x\in\Omega,\\ u(x)=\varphi(x), & x\in\partial\Omega.\end{cases} \tag{2.5}$$

Moreover $Du(x)\in K$ a.e.

Remarks. (i) Qf^- in (2.3) denotes the *quasiconvex envelope* of f^-. In view of the representation formula for Qf^- given in Theorem 7.2 (here we have dropped the index K, since there is no ambiguity), the hypothesis (2.3) guarantees that there exists, for any linear boundary datum in K, a sequence of approximate solutions with gradient in K.

(ii) The hypothesis (2.3) can be difficult to verify, however we will give a sufficient condition in Proposition 2.3. In the (scalar and vectorial) convex case, i.e. when f is convex, it is automatically satisfied.

(iii) Note that the hypothesis (2.1) of quasiconvexity of f can be removed if we can find g satisfying (2.1)–(2.4) of the theorem and such that

$$\{\xi \in K : g(\xi) = 0\} \subset \{\xi \in \mathbf{R}^{m \times n} : f(\xi) = 0\}.$$

This idea will be used in §3.

(iv) The hypothesis of compactness of K in (2.2) can be suppressed in some cases such as the scalar case (cf. §3) or the vectorial convex case (cf. §4).

(v) Finally the hypothesis (2.4) can be improved if we assume that

$$\varphi \in C^1(\Omega; \mathbf{R}^m) \cap W^{1,\infty}(\Omega; \mathbf{R}^m),$$

cf. the following corollary.

COROLLARY 2.2 (the C^1-quasiconvex case). *Let $\Omega \subset \mathbf{R}^n$ be an open set. Let f satisfy* (2.1), (2.2) *and* (2.3) *of the theorem. Let $\varphi \in C^1(\Omega; \mathbf{R}^m) \cap W^{1,\infty}(\Omega; \mathbf{R}^m)$ be such that*

$$D\varphi(x) \in \operatorname{int} K \cup \{\xi \in \mathbf{R}^{m \times n} : f(\xi) = 0\}. \tag{2.6}$$

Then there exists $u \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ satisfying (2.5).

Relevant to verify hypothesis (2.3) of Theorem 2.1 is

PROPOSITION 2.3. *A sufficient condition to have* (2.3) *is that*

$$Rf^-(\xi) = 0 \quad \text{for every } \xi \in \operatorname{int} K \tag{2.7}$$

where Rf^- denotes the rank-one convex envelope of f^-.

Proposition 2.3 is a direct consequence of Theorem 7.2 in the appendix. We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. We first observe that there is no loss of generality in assuming that Ω is bounded. Otherwise we cover Ω by bounded open sets and we solve the problem on each set. We divide the proof into three steps.

Step 1. We let

$$V=\{u\in\varphi+W_0^{1,\infty}(\Omega;\mathbf{R}^m):Du(x)\in K \text{ a.e. in } \Omega\}.$$

Note that $\varphi\in V$. Observe that V is a complete metric space when endowed with the L^∞-norm. Indeed let $\{u_\nu\}$ be a Cauchy sequence in V. Since K is bounded we can extract a subsequence $\{u_{\nu_i}\}$ which converges weak-$*$ in $W^{1,\infty}$ to a function u. Since K is convex and closed, we deduce that $u\in V$. Hence the whole sequence (and not only the subsequence) converges to u in L^∞. Thus V is complete.

We then let for $k\in\mathbf{N}$,

$$V_k=\left\{u\in V:\int_\Omega f(Du(x))\,dx>-\frac{1}{k}\right\}.$$

Suppose that we can show that

- V_k is open in V (cf. Step 2);
- V_k is dense in V (cf. Step 3).

We will then deduce from the Baire category theorem that $\bigcap_{k=1}^\infty V_k$ is dense in V and hence nonempty. Observe that any $u\in\bigcap_{k=1}^\infty V_k$ is a solution of (2.5). Indeed

$$\left.\begin{aligned} u\in\varphi+W_0^{1,\infty}(\Omega;\mathbf{R}^m)\\ Du\in K\Rightarrow f(Du)\leqslant 0\\ \textstyle\int_\Omega f(Du(x))\,dx\geqslant 0\end{aligned}\right\}\quad\Longrightarrow\quad f(Du)=0\quad\text{a.e. in }\Omega.$$

Step 2. We now show that V_k is open in V. We will prove that $V-V_k$ is closed. Indeed let

$$u_\nu\in V-V_k,\quad u_\nu\xrightarrow{L^\infty}u.$$

We already know that u is in V (cf. Step 1). In fact $u\in V-V_k$, by the quasiconvexity of f. Indeed from Theorem 7.1, we have

$$\int_\Omega f(Du(x))\,dx\leqslant\liminf_{\nu\to\infty}\int_\Omega f(Du_\nu(x))\,dx\leqslant-\frac{1}{k}.$$

Thus $V-V_k$ is closed and hence V_k is open.

Step 3. It therefore remains to show that V_k is dense in V. Let $k>0$ be a fixed integer. Let $v\in V$ and $\varepsilon>0$. We wish to show that we can find

$$v_\varepsilon\in V_k \text{ with } \|v-v_\varepsilon\|_{L^\infty}\leqslant\varepsilon. \tag{2.8}$$

We first observe that we can assume, without loss of generality, that

$$Dv(x) \text{ is compactly contained in } \operatorname{int} K. \tag{2.9}$$

Indeed if this were not the case, using the convexity of K and (2.4), we would replace v by $(1-t)v+t\varphi$ with t sufficiently small to get (2.9).

We then apply Lemma 6.1 to v and we find $v_\nu \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ such that there exist $\Omega_\nu \subset \Omega_{\nu+1} \subset \Omega$ open sets with

$$\begin{cases} \operatorname{meas}(\Omega-\Omega_\nu) \to 0 \text{ as } \nu \to \infty, \\ v_\nu \text{ is piecewise affine in } \Omega_\nu, \\ v_\nu \xrightarrow{L^\infty} v, \\ v_\nu = v \text{ on } \partial\Omega, \\ Dv_\nu(x) \in \operatorname{int} K \text{ a.e. in } \Omega. \end{cases} \tag{2.10}$$

We then let $\Omega_{\nu,\lambda}$ be open sets so that

$$\begin{cases} \bar{\Omega}_\nu = \bigcup_{\lambda=1}^{\Lambda} \bar{\Omega}_{\nu,\lambda}, \\ Dv_\nu(x) = A_{\nu,\lambda} \text{ if } x \in \Omega_{\nu,\lambda}. \end{cases} \tag{2.11}$$

At this stage we apply (2.3) to $A_{\nu,\lambda} \in \operatorname{int} K$ to get $Qf^-(A_{\nu,\lambda})=0$. In view of Theorem 7.2, this equality implies that we can find $\varphi_{\nu,\lambda,l} \in W_0^{1,\infty}(\Omega_{\nu,\lambda}; \mathbf{R}^m)$ such that

$$\begin{cases} \int_{\Omega_{\nu,\lambda}} f(A_{\nu,\lambda} + D\varphi_{\nu,\lambda,l}(x))\, dx \to 0 \text{ as } l \to \infty, \\ \varphi_{\nu,\lambda,l} \text{ converges weak-}* \text{ in } W^{1,\infty} \text{ to } 0 \text{ as } l \to \infty. \end{cases}$$

Defining

$$v_\varepsilon = \begin{cases} v_\nu(x) & \text{if } x \in \Omega - \Omega_\nu, \\ v_\nu(x) + \varphi_{\nu,\lambda,l}(x) & \text{if } x \in \bar{\Omega}_{\nu,\lambda} \end{cases}$$

we have indeed that $v_\varepsilon \in V$ and, by choosing ν and l sufficiently large, that

$$\|v_\varepsilon - v\|_{L^\infty} \leqslant \varepsilon. \tag{2.12}$$

Furthermore

$$\int_\Omega f(Dv_\varepsilon(x))\, dx = \int_{\Omega-\Omega_\nu} f(Dv_\nu(x))\, dx + \sum_{\lambda=1}^{\Lambda} \int_{\Omega_{\nu,\lambda}} f(A_{\nu,\lambda,l} + D\varphi_{\nu,\lambda,l}(x))\, dx.$$

Therefore, choosing ν and l larger if necessary, we can ensure that

$$\int_\Omega f(Dv_\varepsilon(x))\, dx > -\frac{1}{k},$$

i.e. $v_\varepsilon \in V_k$, which is the desired density property required, i.e. (2.8). □

We now turn to the proof of Corollary 2.2.

Proof of Corollary 2.2. As in Theorem 2.1 we may assume without loss of generality that Ω is bounded. We divide the proof into two steps.

Step 1. We first define $\Omega_0=\{x\in\Omega: f(D\varphi(x))=0\}$. By continuity of f and $D\varphi$, we have that the set $\Omega-\Omega_0$ is open. We therefore define

$$u(x)=\varphi(x) \quad \text{if } x\in\Omega_0. \tag{2.13}$$

It remains to solve

$$\begin{cases} f(Du(x))=0, & \text{a.e. } x\in\Omega-\Omega_0,\\ u(x)=\varphi(x), & x\in\partial(\Omega-\Omega_0). \end{cases} \tag{2.14}$$

By construction we know that

$$D\varphi(x)\in \operatorname{int} K \quad \text{if } x\in\Omega-\Omega_0. \tag{2.15}$$

For every $t>0$, we let $\Omega^t=\{x\in\Omega-\Omega_0 : \operatorname{dist}(D\varphi(x),\partial K)=t\}$. We will show in Step 2 that we can find a decreasing sequence $t_k>0$ converging to zero such that

$$\operatorname{meas}\Omega^{t_k}=0 \quad \text{for every } k\in\mathbf{N}. \tag{2.16}$$

We then let $\Omega_k=\{x\in\Omega-\Omega_0 : t_{k+1}<\operatorname{dist}(D\varphi(x),\partial K)<t_k\}$. Observe that Ω_k is open and that

$$\begin{cases} \overline{\Omega-\Omega_0}=\bigcup_{k=1}^{\infty}\bar{\Omega}_k,\\ \Omega-\Omega_0=\bigcup_{k=1}^{\infty}\Omega_k\cup N \ \text{ with } \operatorname{meas} N=0,\\ \partial\Omega_k\subset\partial(\Omega-\Omega_0)\cup\Omega^{t_k}\cup\Omega^{t_{k+1}} \end{cases} \tag{2.17}$$

(the second statement is a consequence of (2.16)). Using Theorem 2.1 on Ω_k we can then find $u_k\in W^{1,\infty}(\Omega_k;\mathbf{R}^m)$ such that

$$\begin{cases} f(Du_k(x))=0, & \text{a.e. } x\in\Omega_k,\\ u_k(x)=\varphi(x), & x\in\partial\Omega_k. \end{cases} \tag{2.18}$$

Defining

$$u(x)=\begin{cases} u_k(x) & \text{if } x\in\bar{\Omega}_k,\\ \varphi(x) & \text{if } x\in\Omega_0 \end{cases}$$

we find that u has all the claimed properties.

Step 2. It therefore remains to show (2.16). To do this we define for $k\in\mathbf{N}$ the set

$$T_k=\left\{t>0 : \frac{1}{k+1}\leqslant\frac{\operatorname{meas}\Omega^t}{\operatorname{meas}(\Omega-\Omega_0)}<\frac{1}{k}\right\}.$$

We claim that this set is finite. Assume for the sake of contradiction that this is not so. We then would get, from the fact $\Omega-\Omega_0=\bigcup_{t>0}\Omega^t\supset\bigcup_{t\in T_k}\Omega^t$, that

$$\operatorname{meas}(\Omega-\Omega_0)\geqslant\operatorname{meas}(\Omega-\Omega_0)\sum_{t\in T_k}\frac{1}{k+1}=\frac{\operatorname{meas}(\Omega-\Omega_0)}{k+1}\sum_{t\in T_k}1=+\infty,$$

which contradicts the fact that Ω is bounded. It follows that the set

$$\{t>0:\operatorname{meas}\Omega^t>0\}\subset\bigcup_{k=1}^{\infty}T_k$$

is countable. Therefore the set $\{t>0:\operatorname{meas}\Omega^t=0\}$ is dense in $[0,1]$, and thus (2.16). □

3. The nonconvex scalar case and systems of equations

We now turn to an application of the results of §2. The main theorem of this section is

THEOREM 3.1 (the nonconvex scalar case). *Let $\Omega\subset\mathbf{R}^n$ be an open set. Let $\varphi\in W^{1,\infty}(\Omega)$ and $F:\mathbf{R}^n\to\mathbf{R}$ be such that*

$$D\varphi(x) \textit{ is compactly contained in } \operatorname{int}\operatorname{co}\{\xi\in\mathbf{R}^n:F(\xi)=0\} \textit{ a.e. in } \Omega. \tag{3.1}$$

Then there exists $u\in W^{1,\infty}(\Omega)$ such that

$$\begin{cases} F(Du(x))=0 & \textit{a.e. in } \Omega,\\ u(x)=\varphi(x), & x\in\partial\Omega. \end{cases} \tag{3.2}$$

If in addition $\varphi\in C^1(\Omega)$ and $\{\xi\in\mathbf{R}^n:F(\xi)=0\}$ is closed then (3.1) *can be replaced by*

$$D\varphi(x)\in\operatorname{int}\operatorname{co}\{\xi\in\mathbf{R}^n:F(\xi)=0\}\cup\{\xi\in\mathbf{R}^n:F(\xi)=0\}, \tag{3.3}$$

and the conclusion (3.2) *still holds.*

Remarks. (i) This result is only valid in the scalar case. One should note that there is no hypothesis of convexity, coercivity or even continuity on the function F.

(ii) The condition (3.1) excludes, as it should do, the linear case, since there

$$\operatorname{int}\operatorname{co}\{\xi\in\mathbf{R}^n:F(\xi)=0\}=\varnothing.$$

(iii) If F is convex and coercive then (cf. §4)

$$\operatorname{int}\operatorname{co}\{\xi\in\mathbf{R}^n:F(\xi)=0\}\cup\{\xi\in\mathbf{R}^n:F(\xi)=0\}=\{\xi\in\mathbf{R}^n:F(\xi)\leqslant 0\}.$$

(iv) The condition (3.3) seems to be optimal. In general it cannot be replaced by

$$D\varphi(x)\in \operatorname{co}\{\xi\in \mathbf{R}^n : F(\xi)=0\}.$$

Indeed let $n=2$ and $F(\xi)=(|\xi_1|-1)^2+(|\xi_2|-1)^2$. Then

$$\operatorname{co}\{\xi\in\mathbf{R}^2 : F(\xi)=0\}=\{\xi=(\xi_1,\xi_2)\in\mathbf{R}^2 : |\xi_1|, |\xi_2|\leqslant 1\}.$$

Choose then $\varphi(x,y)=x+\beta y$ with $|\beta|<1$. Note that

$$(1,\beta)\in \operatorname{co}\{F(\xi)=0\} \quad \text{but} \quad (1,\beta)\notin \operatorname{int}\operatorname{co}\{F(\xi)=0\}\cup\{F(\xi)=0\}.$$

Let us show that, if for example $\Omega=(0,1)^2$, then the problem

$$\begin{cases} F(\partial u/\partial x, \partial u/\partial y)=0 & \text{a.e. in } \Omega, \\ u(x,y)=x+\beta y & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

has no solution. Indeed we have

$$\int_0^1\left(\left|\frac{\partial u}{\partial x}\right|-\frac{\partial u}{\partial x}\right)dx=\int_0^1\left(1-\frac{\partial u}{\partial x}\right)dx=1-u(1,y)+u(0,y)=0.$$

This implies that

$$\frac{\partial u}{\partial x}=\left|\frac{\partial u}{\partial x}\right|=1 \quad \text{a.e.}$$

We therefore deduce that there exists $\psi:(0,1)\to\mathbf{R}$ such that

$$\begin{cases} u(x,y)=x+\psi(y), \\ |\psi'(y)|=1 \quad \text{a.e.,} \\ \psi(y)=\beta y \quad \text{if } (x,y)\in\partial\Omega. \end{cases}$$

This is of course impossible since $|\beta|<1$.

We now turn to applications of Theorem 3.1.

COROLLARY 3.2 (prescribed gradient values). *Let $\Omega\subset\mathbf{R}^n$ be an open set; let E be any subset of $\mathbf{R}^n$ and $\varphi\in W^{1,\infty}(\Omega)$ be such that*

$$D\varphi(x) \textit{ is compactly contained in } \operatorname{int}\operatorname{co}E \textit{ a.e. in } \Omega. \tag{3.5}$$

Then there exists $u\in W^{1,\infty}(\Omega)$ such that

$$\begin{cases} Du(x)\in E, & \textit{a.e. } x\in\Omega, \\ u(x)=\varphi(x), & x\in\partial\Omega. \end{cases} \tag{3.6}$$

If in addition $\varphi\in C^1(\Omega)$ and E is closed then (3.5) *can be replaced by*

$$D\varphi(x)\in E\cup\operatorname{int}\operatorname{co}E, \quad x\in\Omega. \tag{3.7}$$

Remark. This result has also been proved by Cellina [12] when φ is linear.

COROLLARY 3.3 (system of equations). *Let $\Omega\subset\mathbf{R}^n$ be an open set. Let also $\varphi\in C^1(\Omega)\cap W^{1,\infty}(\Omega)$ and $F_i:\mathbf{R}^n\to\mathbf{R}$, $1\leqslant i\leqslant N$, be such that $\{\xi\in\mathbf{R}^n:F_1(\xi)=...=F_N(\xi)=0\}$ is closed and*

$$D\varphi(x)\in\operatorname{int}\operatorname{co}\{\xi\in\mathbf{R}^n:F_1(\xi)=...=F_N(\xi)=0\}\cup\{\xi\in\mathbf{R}^n:F_1(\xi)=...=F_N(\xi)=0\}. \quad (3.8)$$

Then there exists $u\in W^{1,\infty}(\Omega)$ such that

$$\begin{cases} F_i(Du(x))=0 & a.e.\ in\ \Omega,\ 1\leqslant i\leqslant N,\\ u(x)=\varphi(x), & x\in\partial\Omega. \end{cases} \quad (3.9)$$

Remarks. (i) If φ is only in $W^{1,\infty}(\Omega)$, then the same theorem holds with (3.8) replaced by

$$D\varphi(x) \text{ is compactly contained in } \operatorname{int}\operatorname{co}\{\xi\in\mathbf{R}^n:F_i(\xi)=0,\ 1\leqslant i\leqslant N\}. \quad (3.10)$$

(ii) As before one should note that no hypothesis on F_i, besides (3.8) or (3.10), is made.

We now proceed with the proofs.

Proof of Theorem 3.1. The idea of the proof is to find $f:\mathbf{R}^n\to\mathbf{R}$ and K satisfying all the hypotheses of Theorem 2.1 and such that

$$\{\xi\in K:f(\xi)=0\}\subset\{\xi\in\mathbf{R}^n:F(\xi)=0\}. \quad (3.11)$$

The conclusion following from Theorem 2.1 and (3.11), i.e. there exists $u\in W^{1,\infty}(\Omega)$ such that

$$\begin{cases} f(Du(x))=F(Du(x))=0 & \text{a.e. in } \Omega,\\ u(x)=\varphi(x) & \text{on } \partial\Omega. \end{cases}$$

We divide the proof into three steps. As usual we will assume, without loss of generality, that Ω is bounded. In the first two steps we assume only that $\varphi\in W^{1,\infty}(\Omega)$.

Step 1. Since (3.1) holds we can find a convex and compact set $L\subset\mathbf{R}^n$ such that

$$D\varphi(x)\in L\subset\operatorname{int}\operatorname{co}\{\xi\in\mathbf{R}^n:F(\xi)=0\}. \quad (3.12)$$

We can then find a polytope P (cf. the proof of Theorem 20.4 in Rockafellar [35]) with the following property:

$$\begin{cases} P=\operatorname{co}\{\eta_1,...,\eta_N\},\\ L\subset\operatorname{int}P\subset P\subset\operatorname{int}\operatorname{co}\{\xi\in\mathbf{R}^n:F(\xi)=0\}. \end{cases} \quad (3.13)$$

We then use the Carathéodory theorem (cf. Theorem 17.1 in Rockafellar [35]) to write

$$\eta_k = \sum_{i=1}^{n+1} \lambda_i^k \xi_i^k \quad \text{where } \xi_i^k \in \{\xi \in \mathbf{R}^n : F(\xi) = 0\}. \tag{3.14}$$

This is possible since $\eta_k \in P \subset \operatorname{co}\{\xi \in \mathbf{R}^n : F(\xi) = 0\}$. Combining (3.12), (3.13) and (3.14) we find that $D\varphi(x) \in L \subset \operatorname{int} P \subset \overline{P} \subset \operatorname{co}\{\xi_1^1, ..., \xi_{n+1}^1, ..., \xi_1^N, ..., \xi_{n+1}^N\}$.

Among the $\{\xi_1^1, ..., \xi_{n+1}^1, ..., \xi_1^N, ..., \xi_{n+1}^N\}$ we remove all the ξ_i^k which are convex combinations of the others (i.e. we keep only those which are extreme points) and we relabel the remaining ones as $\{\xi_1, ..., \xi_s\}$. Therefore summarizing what we have just obtained, we can write

$$\begin{cases} D\varphi(x) \in L \subset \operatorname{int} \operatorname{co}\{\xi_1, ..., \xi_s\}, \\ F(\xi_i) = 0, \\ \text{none of the } \xi_i \text{ is a convex combination of the other ones.} \end{cases} \tag{3.15}$$

We then define $g: \mathbf{R}^n \to \overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$ by

$$g(\xi) = \begin{cases} -\min_{1 \leqslant i \leqslant s}\{|\xi - \xi_i|\} & \text{if } \xi \in \operatorname{co}\{\xi_1, ..., \xi_s\}, \\ +\infty & \text{otherwise.} \end{cases}$$

We finally define f as the convex envelope of g, i.e. $f(\xi) = Cg(\xi)$, and let

$$K = \operatorname{co}\{\xi_1, ..., \xi_s\}. \tag{3.16}$$

Since f is finite only over K, we redefine it outside as a convex function taking only finite values. This is always possible since g is Lipschitz over K with constant 1 and Cg has the same property. Indeed if $\xi, \xi + \eta \in K$, then by the Carathéodory theorem and since K is compact we can find (λ_i, ξ_i) with $\xi = \sum \lambda_i \xi_i$ and

$$Cg(\xi + \eta) - Cg(\xi) = Cg(\xi + \eta) - \sum_{i=1}^{n+1} \lambda_i g(\xi_i) \leqslant \sum_{i=1}^{n+1} \lambda_i [g(\xi_i + \eta) - g(\xi_i)] \leqslant |\eta|.$$

Since ξ and η are arbitrary we have indeed that Cg is Lipschitz with constant 1 over K and hence it can be extended outside K in a convex and finite way.

Step 2. Before checking that f has all the claimed properties, we establish the fact: if $\xi \in K$ then the following property holds:

$$f(\xi) = 0 \quad \Leftrightarrow \quad \xi \in \{\xi_1, ..., \xi_s\}. \tag{3.17}$$

($\Rightarrow$) If $\xi \notin \{\xi_1, ..., \xi_s\}$ and $\xi \in K$ then $g(\xi) < 0$ and since $f(\xi) = Cg(\xi) \leqslant g(\xi)$, we deduce the result $f(\xi) < 0$.

($\Leftarrow$) So let $\xi \in \{\xi_1, ..., \xi_s\}$. Then by definition and by the Carathéodory theorem

$$f(\xi) = Cg(\xi) = \inf\left\{\sum_{i=1}^{n+1} \lambda_i g(\eta_i) : \sum_{i=1}^{n+1} \lambda_i \eta_i = \xi,\ \eta_i \in K\right\}$$

(here the infimum is actually a minimum since K is compact). Since by (3.15) the ξ_i are extreme points (i.e. none of them is a convex combination of the others) we deduce that $f(\xi_i) = g(\xi_i) = 0$ and hence (3.17) is established.

We are now in a position to prove that f satisfy all the hypotheses of Theorem 2.1.

• By definition f is convex, hence (2.1) is established.

• By construction K satisfies (2.2).

• Since we are in the scalar case, (2.3) amounts to prove that $Cf^-(\xi) = 0$ for every $\xi \in \operatorname{int} K$. Indeed every $\xi \in K$ can be written by the Carathéodory theorem as $\xi = \sum_{i=1}^{n+1} \lambda_i \xi_i$. Hence

$$0 \leqslant Cf^-(\xi) = \inf\left\{-\sum_{i=1}^{n+1} \mu_i f(\eta_i) : \eta_i \in K \text{ and } \sum_{i=1}^{n+1} \mu_i \eta_i = \xi\right\} \leqslant -\sum_{i=1}^{n+1} \lambda_i f(\xi_i) = 0$$

where we have used (3.17). Hence (2.3) is established.

• $D\varphi(x)$ is compactly contained in $\operatorname{int} K$ by (3.15) and thus (2.4) is proved.

So we may now apply Theorem 2.1 and find $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that

$$f(Du(x)) = 0 \text{ a.e. in } \Omega \quad \text{and} \quad Du(x) \in K \text{ a.e.} \tag{3.18}$$

Observe finally that by (3.15) and (3.17) we have

$$\{\xi \in K : f(\xi) = 0\} \subset \{\xi \in \mathbf{R}^n : F(\xi) = 0\}. \tag{3.19}$$

Combining (3.18) and (3.19) we have indeed established the theorem in the case $\varphi \in W^{1,\infty}(\Omega)$.

Step 3. If $\varphi \in C^1(\Omega)$, we then follow exactly the proof of Corollary 2.2, applied to f and K as above. □

Proof of Corollary 3.2. We just set

$$F(\xi) = \begin{cases} 0 & \text{if } \xi \in E, \\ 1 & \text{if } \xi \notin E \end{cases}$$

and then apply Theorem 3.1. □

Proof of Corollary 3.3. We just set

$$F(\xi) = \sum_{i=1}^{N} [F_i(\xi)]^2$$

and then apply Theorem 3.1. □

4. The convex case (scalar and vectorial)

THEOREM 4.1 (the convex case). *Let $\Omega\subset\mathbf{R}^n$ be an open set. Let $\varphi\in W^{1,\infty}(\Omega;\mathbf{R}^m)$ and $f:\mathbf{R}^{m\times n}\to\mathbf{R}$ satisfy*

$$f \text{ is convex;} \tag{4.1}$$

$$\begin{gathered}\text{there exists } \lambda\in\mathbf{R}^{m\times n} \text{ with rank } \{\lambda\}=1 \text{ such that}\\ \lim_{|t|\to\infty} f(\xi+t\lambda)=+\infty \text{ for every } \xi\in\mathbf{R}^{m\times n};\end{gathered} \tag{4.2}$$

$$\text{there exists } \delta>0 \text{ such that } f(D\varphi(x))\leqslant-\delta, \text{ a.e. } x\in\Omega. \tag{4.3}$$

Then there exists $u\in W^{1,\infty}(\Omega;\mathbf{R}^m)$ such that

$$\begin{cases} f(Du(x))=0, & \text{a.e. } x\in\Omega,\\ u(x)=\varphi(x), & x\in\Omega. \end{cases} \tag{4.4}$$

If in addition $\varphi\in C^1(\Omega;\mathbf{R}^m)$ then (4.3) *can be replaced by*

$$f(D\varphi(x))\leqslant 0 \quad \text{for every } x\in\Omega \tag{4.5}$$

and the same conclusion holds.

Remarks. (i) Note that in the scalar case, (4.2) means that f is coercive in at least one direction. In the vectorial case this direction should be of rank one. In this sense the coercivity condition is weaker than the usual one (cf. Lions [29]).

(ii) In the calculus of variations it is often more desirable to write the above theorem in the following form: Let $K\subset\mathbf{R}^{m\times n}$ be convex and bounded in at least one direction of rank one (cf. (4.2)) and let $\varphi\in W^{1,\infty}(\Omega;\mathbf{R}^m)$ be such that $D\varphi(x)$ is compactly contained in K. Then there exists $u\in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ with $Du(x)\in\partial K$ (cf. Lemma 3.5 of Dacorogna–Marcellini [19] or, in the bounded scalar case, Lions [29], Mascolo–Schianchi [31]).

(iii) One can also deduce the vectorial version of the theorem by choosing $m-1$ components equal to those of the boundary datum. Of course to do this one needs to have an existence theorem for Carathéodory functions of the form $f(x,Du)$.

Proof of Theorem 4.1. We divide the proof into two steps.

Step 1. We first prove the theorem under hyptheses (4.1), (4.2) and (4.3). We just have to find K such that we can apply Theorem 2.1. We observe that by (4.3) we can find, trivially, a compact and convex set L such that

$$\begin{cases} D\varphi(x) \text{ is compactly contained in } \operatorname{int} L,\\ L\subset\operatorname{int}\{\xi\in\mathbf{R}^{m\times n}: f(\xi)\leqslant 0\}. \end{cases} \tag{4.6}$$

We then define

$$K = \{\eta \in \mathbf{R}^{m\times n} : f(\eta) \leqslant 0 \text{ and there exists } (\xi, t) \in L\times\mathbf{R} \text{ with } \eta = \xi + t\lambda\}. \tag{4.7}$$

Observe that K is compact and convex, since f is convex and satisfies (4.2). Therefore hypotheses (2.1) and (2.2) of Theorem 2.1 are satisfied. Note that (2.4) is verified in view of (4.6). We therefore only need to show (2.3). To do this, in view of Proposition 2.3, it is sufficient to prove that

$$Rf^-(\eta) = 0 \quad \text{for every } \eta \in K. \tag{4.8}$$

Since (4.2) holds we can write any $\eta \in K$ as

$$\eta = s(\xi + t_1\lambda) + (1-s)(\xi + t_2\lambda)$$

where $s\in[0,1]$, $\xi\in L$ and $f(\xi+t_1\lambda) = f(\xi+t_2\lambda) = 0$. Therefore in view of the general formula for Rf we have

$$0 \leqslant Rf^-(\eta) \leqslant sf^-(\xi+t_1\lambda) + (1-s)f^-(\xi+t_2\lambda) = 0$$

and thus (4.8) is established and the first part of the theorem as well.

Step 2. We now assume that, in addition, $\varphi \in C^1(\Omega; \mathbf{R}^m)$; we proceed as in Corollary 2.2 and obtain the result. □

5. The prescribed singular values case

We recall that, given $\xi \in \mathbf{R}^{n\times n}$, we denote by $0 \leqslant \lambda_1(\xi) \leqslant \ldots \leqslant \lambda_n(\xi)$ the singular values of ξ (i.e. the eigenvalues of $(\xi^t\xi)^{1/2}$). The main theorem of this section is

THEOREM 5.1 (the singular values case). *Let Ω be an open set of $\mathbf{R}^n$, and let $\varphi \in W^{1,\infty}(\Omega; \mathbf{R}^n)$ be such that there exists $\delta > 0$ satisfying*

$$\lambda_n(D\varphi(x)) \leqslant 1 - \delta \quad \textit{a.e. in } \Omega. \tag{5.1}$$

Then there exists $u \in W^{1,\infty}(\Omega; \mathbf{R}^n)$ such that

$$\begin{cases} \lambda_i(Du(x)) \leqslant 1 & \textit{a.e. in } \Omega,\ i = 1, \ldots, n, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{5.2}$$

If in addition $\varphi\in C^1(\Omega;\mathbf{R}^n)$ then (5.1) *can be replaced by: for every $x\in\Omega$ one of the following conditions holds:*

$$\text{either } \lambda_n(D\varphi(x))<1 \text{ or } \lambda_i(D\varphi(x))=1 \text{ for every } i=1,...,n, \tag{5.3}$$

and the same conclusion holds.

Remarks. (i) In the case when $n=3$, $\varphi\equiv 0$, Cellina–Perrotta [13] have proved the same result.

(ii) As already mentioned the above theorem proves in particular that, if $n=2$, one can solve the problem

$$|Du|^2=2, \quad |\det Du|=1$$

with the boundary datum $u=\varphi$. This shows in some sense that we can solve at the same time the eikonal equation with the modulus of the Jacobian given.

The theorem admits a corollary.

COROLLARY 5.2. (1) *Let $\Omega\subset\mathbf{R}^n$ be an open set. Let $A\in\mathbf{R}^{n\times n}$ be defined by*

$$A=\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

with $0<a_1\leqslant a_2\leqslant...\leqslant a_n$. Let $\varphi\in W^{1,\infty}(\Omega;\mathbf{R}^n)$ be such that there exists $R\in O(n)$ and $\delta>0$ satisfying

$$\lambda_n(D\varphi(x)RA^{-1})\leqslant 1-\delta \quad \text{a.e. in } \Omega. \tag{5.4}$$

Then there exists $u\in W^{1,\infty}(\Omega;\mathbf{R}^n)$ such that

$$\begin{cases} \lambda_i(Du(x))=a_i & \text{a.e. in } \Omega,\ 1\leqslant i\leqslant n, \\ u(x)=\varphi(x), & x\in\partial\Omega. \end{cases} \tag{5.5}$$

(2) *If in addition $\varphi\in C^1(\Omega;\mathbf{R}^n)$ then* (5.4) *can be replaced by: for every $x\in\Omega$, one of the following conditions hold:*

$$\lambda_n(D\varphi(x)RA^{-1})<1, \tag{5.6}$$

$$\lambda_i(D\varphi(x)RA^{-1})=1, \quad 1\leqslant i\leqslant n. \tag{5.7}$$

(3) *If φ is affine then* (5.4) *is satisfied if*

$$\lambda_i(D\varphi(x))<a_i \quad \text{in } \Omega,\ 1\leqslant i\leqslant n. \tag{5.8}$$

Remark. Contrary to (5.1) and (5.3) which are essentially optimal, it does not seem that (5.4), (5.6), (5.7) or (5.8) are optimal when the a_is are different.

We may now proceed[2] with the proof of Theorem 5.1.

Proof of Theorem 5.1. We divide the proof into two steps.

Step 1. We first consider the $W^{1,\infty}$-case with inequality (5.1) satisfied. We want to construct f and K as in Theorem 2.1. We let

$$f(\xi)=\sum_{s=1}^{n}\left[|\operatorname{adj}_s\xi|^2-\binom{n}{s}\right] \tag{5.9}$$

and let

$$K=\operatorname{co}\{\xi\in\mathbf{R}^{n\times n}:\lambda_i(\xi)=1,\,1\leqslant i\leqslant n\}=\{\xi\in\mathbf{R}^{n\times n}:\lambda_n(\xi)\leqslant 1\}. \tag{5.10}$$

We now check that f and K satisfy all the hypotheses of Theorem 2.1.

- f is polyconvex and thus quasiconvex. Therefore it satisfies (2.1).
- $K\subset\{\xi\in\mathbf{R}^{n\times n}:f(\xi)\leqslant 0\}$ since

$$|\operatorname{adj}_s\xi|^2=\sum_{1\leqslant i_1<\ldots<i_s\leqslant n}\lambda_{i_1}^2\ldots\lambda_{i_s}^2\leqslant\binom{n}{s}.$$

- $Qf^-(\xi)=0$ for every $\xi\in\operatorname{int}K$. This comes from Proposition 2.3 and the fact that

$$Rf^-(\xi)=0\quad\text{for every }\xi\in K$$

and will be proved below.

- $D\varphi(x)$ is compactly contained in $\operatorname{int}K$ by (5.1).

So we may apply Theorem 2.1 and deduce that we can find $u\in\varphi+W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ such that

$$f(Du(x))=0\ \text{a.e. in }\Omega\quad\text{and}\quad Du(x)\in K\ \text{a.e.}$$

Since for every $\xi\in K$ we have $f(\xi)\leqslant 0$, we deduce that

$$|\operatorname{adj}_s Du|^2=\binom{n}{s}\quad\Leftrightarrow\quad\lambda_i(Du)=1\ \text{ a.e. in }\Omega,\ 1\leqslant i\leqslant n,$$

and (5.2) has been established.

So it now remains to establish that

$$Rf^-=0\quad\text{for every }\xi\in K. \tag{5.11}$$

We first observe that since $f^- \geqslant 0$, then

$$Rf^- \geqslant 0 \quad \text{for every } \xi \in K. \tag{5.12}$$

We then use the invariance under rotations of f to deduce that it is enough to establish (5.11) for matrices

$$\xi = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \tag{5.13}$$

with $0 \leqslant a_1 \leqslant \ldots \leqslant a_n \leqslant 1$. This is easily established observing that

$$\xi = \tfrac{1}{2}(1+a_1) \begin{pmatrix} 1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} + \tfrac{1}{2}(1-a_1) \begin{pmatrix} -1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}. \tag{5.14}$$

Since the two matrices on the right-hand side differ by rank one we find (since Rf^- is rank-one convex)

$$0 \leqslant Rf^-(\xi) \leqslant \tfrac{1}{2}(1+a_1) Rf^- \begin{pmatrix} 1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} + \tfrac{1}{2}(1-a_1) Rf^- \begin{pmatrix} -1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}.$$

Therefore to deduce (5.11) it is enough if we can show that

$$Rf^- \begin{pmatrix} \pm 1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} = 0. \tag{5.15}$$

We then iterate the process and write

$$\begin{pmatrix} \pm 1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} = \tfrac{1}{2}(1+a_2) \begin{pmatrix} \pm 1 & & & & 0 \\ & 1 & & & \\ & & a_3 & & \\ & & & \ddots & \\ 0 & & & & a_n \end{pmatrix} + \tfrac{1}{2}(1-a_2) \begin{pmatrix} \pm 1 & & & & 0 \\ & -1 & & & \\ & & a_3 & & \\ & & & \ddots & \\ 0 & & & & a_n \end{pmatrix}.$$

Again the two matrices on the right-hand side differ by rank one so that

$$\begin{aligned} 0 &\leqslant Rf^- \begin{pmatrix} \pm 1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} \\ &\leqslant \tfrac{1}{2}(1+a_2) Rf^- \begin{pmatrix} \pm 1 & & & & 0 \\ & 1 & & & \\ & & a_3 & & \\ & & & \ddots & \\ 0 & & & & a_n \end{pmatrix} + \tfrac{1}{2}(1-a_2) Rf^- \begin{pmatrix} \pm 1 & & & & 0 \\ & -1 & & & \\ & & a_3 & & \\ & & & \ddots & \\ 0 & & & & a_n \end{pmatrix}. \end{aligned}$$

Therefore, to establish (5.11) it is enough to show that

$$Rf^-\begin{pmatrix} \pm 1 & & & & 0 \\ & \pm 1 & & & \\ & & a_3 & & \\ & & & \ddots & \\ 0 & & & & a_n \end{pmatrix} = 0. \tag{5.16}$$

Proceeding analogously with $a_3, \dots, a_n$ we see that a sufficient condition for having (5.11) is that

$$Rf^-\begin{pmatrix} \pm 1 & & & 0 \\ & \pm 1 & & \\ & & \ddots & \\ 0 & & & \pm 1 \end{pmatrix} = 0$$

and this is obvious since (5.12) holds and

$$f^-\begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix} = 0.$$

Step 2. We next consider the C^1-case and this is treated exactly as in Corollary 2.2. This achieves the proof of this theorem. □

We now turn to the proof of Corollary 5.2.

Proof of Corollary 5.2. We divide the proof into 3 steps, the first two establishing parts (1) and (2), and the last one part (3).

Step 1. Let R and A be as in (5.4). We let

$$\begin{cases} B = AR^{-1}, \\ \widetilde{\Omega} = B\Omega, \\ \psi(y) = \varphi(B^{-1}y), \ y \in \widetilde{\Omega}. \end{cases} \tag{5.17}$$

We therefore have from (5.4)

$$\lambda_n(D\psi(y)) = \lambda_n(D\varphi(B^{-1}y)RA^{-1}) \leqslant 1-\delta \quad \text{a.e. in } \widetilde{\Omega}. \tag{5.18}$$

We may therefore apply Theorem 5.1 and obtain $v \in W^{1,\infty}(\widetilde{\Omega}; \mathbf{R}^n)$ such that

$$\begin{cases} \lambda_i(Dv(y)) = 1, & \text{a.e. } y \in \widetilde{\Omega}, \ i = 1, \dots, n, \\ v(y) = \psi(y), & y \in \partial\widetilde{\Omega}. \end{cases} \tag{5.19}$$

Step 2. We now verify that

$$u(x) = v(Bx) \tag{5.20}$$

has all the claimed properties, i.e. $u \in W^{1,\infty}(\Omega; \mathbf{R}^n)$ and

$$\begin{cases} \lambda_i(Du(x)) = a_i, & \text{a.e. } x \in \Omega,\ i = 1, \ldots, n, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{5.21}$$

The boundary condition is satisfied by combining (5.17) and (5.19). We furthermore have by (5.20) that

$$\lambda_i(Du(x)) = \lambda_i(Dv(Bx)B), \quad x \in \Omega. \tag{5.22}$$

We now show that (5.22) implies (5.21). To prove this we first use the invariance by rotation of the singular values λ_i and the fact that $B = AR^{-1}$ to deduce that

$$\lambda_i(Du(x)) = \lambda_i(Dv(Bx)A).$$

Furthermore since $\lambda_i(Dv) = 1$, we deduce that $Dv \in O(n)$, i.e. it is an orthogonal transformation. Using again the invariance of λ_i under the action of $O(n)$ we deduce that $\lambda_i(Du(x)) = \lambda_i(A) = a_i$, which establishes (1) of the corollary. (2) is as usual a combination of (1) and the same argument as in Corollary 2.2.

Step 3. It now remains to establish (3), so we assume that φ is affine and set $D\varphi = \xi$. We can then find $P, P' \in O(n)$ and $0 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_n$ such that

$$\xi = P \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} P'.$$

Hence (5.6) is equivalent to

$$\lambda_n \left(P \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} P'RA^{-1} \right) = \lambda_n \left(\begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} P'RA^{-1} \right) < 1. \tag{5.23}$$

It is then clear that the best choice in (5.23) consists in choosing $P'R = I$. Hence we obtain

$$\lambda_n \left(\begin{pmatrix} \alpha_1/a_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n/a_n \end{pmatrix} \right) < 1$$

which implies $\alpha_i/a_i < 1 \Rightarrow \lambda_i(\xi) < a_i$. □

6. Appendix: Some approximation lemmas

We give here two approximation lemmas which present minor modifications to standard results. The first one is a basic finite element approximation. Since however it presents some refinements we will give here a complete proof.

LEMMA 6.1 (finite element approximation). *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set. Let K be a compact and convex set of $\mathbf{R}^{m\times n}$ with nonempty interior. Let $u \in W^{1,\infty}(\Omega;\mathbf{R}^m)$ be such that*

$$Du(x) \text{ is compactly contained in } \operatorname{int} K. \tag{6.1}$$

Then there exist open sets $\Omega_\nu \subset \Omega$ and $u_\nu \in W^{1,\infty}(\Omega;\mathbf{R}^m)$ such that

$$\Omega_\nu \subset \Omega_{\nu+1} \text{ and } \operatorname{meas}(\Omega - \Omega_\nu) \to 0 \text{ as } \nu \to \infty; \tag{6.2}$$

$$u_\nu \text{ is piecewise affine on } \Omega_\nu; \tag{6.3}$$

$$u_\nu = u \text{ on } \partial\Omega; \tag{6.4}$$

$$u_\nu \to u \text{ uniformly in } \Omega; \tag{6.5}$$

$$Du_\nu \to Du \text{ a.e. in } \Omega; \tag{6.6}$$

$$\|Du_\nu\|_{L^\infty} \leqslant \|Du\|_{L^\infty} + c(\nu), \text{ with } c(\nu) \to 0 \text{ as } \nu \to \infty; \tag{6.7}$$

$$Du_\nu(x) \text{ is compactly contained in } \operatorname{int} K, \text{ a.e. } x \in \Omega. \tag{6.8}$$

Remark. The difference between this lemma and standard ones (cf. for example Ekeland–Témam [24]) is that this lemma is vectorial and at the same time the approximation should satisfy (6.8). Note that (6.7) is, in some sense, a consequence of (6.8).

Proof. We divide the proof into three steps.

Step 1 (regularisation of u). We first note that by hypothesis we can find a compact and convex set L such that

$$Du(x) \in L \subset \operatorname{int} K \quad \text{a.e. in } \Omega. \tag{6.9}$$

Let $\varepsilon > 0$. We can then find an open set O with Lipschitz boundary (for example a finite union of balls), compactly contained in Ω and such that

$$\operatorname{meas}(\Omega - O) \leqslant \varepsilon. \tag{6.10}$$

We then let $s \in \mathbf{N}$ and regularize each component of u by convolution with an appropriate kernel ϱ_s and let

$$w_s(x) = \int_{\mathbf{R}^n} \varrho_s(x-y)\, u(y)\, dy \tag{6.11}$$

so that $w_s \in C^\infty(\overline{O};\mathbf{R}^m)$ and

$$\begin{cases} \|w_s - u\|_{L^\infty(O)} \leqslant 1/s^2, \\ Dw_s \to Du \text{ a.e. in } O, \\ \|Dw_s\|_{L^\infty(O)} \leqslant \|Du\|_{L^\infty(O)}, \\ Dw_s(x) \in L \text{ for every } x \in O. \end{cases} \tag{6.12}$$

The last two conclusions hold since the process of convolution involves convex combinations (and L is convex).

Step 2 (piecewise approximation). We then use standard finite elements to approximate w_s (cf., for example, Proposition 2.1 of Chapter X of Ekeland–Témam [24]) to find piecewise affine functions $\{w_{s,i}\}_{i=1}^{\infty}$ on O such that

$$\begin{cases} w_{s,i} \to w_s \text{ uniformly in } O \text{ as } i \to \infty, \\ Dw_{s,i} \to Dw_s \text{ uniformly in } O \text{ as } i \to \infty, \\ \|Dw_{s,i}\|_{L^\infty(O)} \leqslant \|Dw_s\|_{L^\infty(O)}. \end{cases} \tag{6.13}$$

(The uniform convergence of the gradient is on the whole of O, since w_s is also defined outside O.)

Step 3. The problem is then just to match the boundary condition and to verify all the claimed properties. We then define Ω_s to be an open set such that

$$\begin{cases} \Omega_s \subset O \subset \Omega, \\ \operatorname{dist}(\Omega_s, \partial O) = 1/s. \end{cases} \tag{6.14}$$

We next let $\eta_s \in C^\infty(\bar{O})$ satisfy

$$\begin{cases} \eta_s(x) = \begin{cases} 0 & \text{if } x \in \partial O, \\ 1 & \text{if } x \in \Omega_{2s}, \end{cases} \\ 0 \leqslant \eta_s(x) \leqslant 1 \text{ for every } x \in \Omega, \\ \|D\eta_s\|_{L^\infty(O)} \leqslant \alpha s \text{ (for a certain } \alpha > 1). \end{cases} \tag{6.15}$$

We now return to (6.13) and choose i sufficiently large so that

$$\|w_{s,i} - w_s\|_{W^{1,\infty}(O)} \leqslant 1/s^2. \tag{6.16}$$

We are now in a position to define u_s. We let

$$u_s(x) = \begin{cases} \eta_s(x) w_{s,i}(x) + (1 - \eta_s(x)) u(x) & \text{if } x \in O, \\ u(x) & \text{if } x \in \Omega - O. \end{cases} \tag{6.17}$$

We now verify all the claimed properties.

- Choosing appropriately ε in (6.10) and s in (6.14) we have indeed (6.2).
- By construction u_s is piecewise affine on Ω_s and so (6.3) is satisfied.
- $u_s = u$ on $\partial\Omega$, i.e. (6.4) holds.
- We have indeed (6.5), since

$$\|u_s - u\|_{L^\infty(\Omega)} = \|\eta_s(w_{s,i} - u)\|_{L^\infty(O)} \leqslant \|w_{s,i} - w_s\|_{L^\infty(O)} + \|w_s - u\|_{L^\infty(O)} \leqslant 2/s^2$$

by (6.12) and (6.16).

• We next prove (6.6). By definition we have

$$Du_s - Du = \eta_s(Dw_{s,i} - Du) + D\eta_s \otimes (w_{s,i} - u),$$

and (6.6) follows from (6.12), (6.15) and (6.16).

• To establish (6.7) we just observe that

$$Du_s = \eta_s Dw_{s,i} + (1-\eta_s)Du + D\eta_s \otimes (w_{s,i} - u), \tag{6.18}$$

and combine it with (6.12), (6.13), (6.15) and (6.16).

• Finally we have (6.8). Indeed by (6.12) and (6.13) $Dw_{s,i}$ is compactly contained in $\operatorname{int} K$ and by (6.9) Du is also compactly contained in $\operatorname{int} K$. Thus since K is convex we deduce that $\eta_s Dw_{s,i} + (1-\eta_s)Du$ is compactly contained in $\operatorname{int} K$. Since finally the last term in (6.18) is as small as we want by (6.12), (6.15) and (6.16) we deduce (6.8).

This achieves the proof of the lemma. □

We conclude this section by a second approximation lemma which is used to prove necessary conditions in the calculus of variations (see e.g. Ekeland–Témam [24] or Dacorogna [17]). The version given below is slightly stronger than the existing ones.

LEMMA 6.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set. Let $K \subset \mathbf{R}^{m\times n}$ be a convex set with nonempty interior. Let $A, B \in K$ with* $\operatorname{rank}\{A-B\} \leqslant 1$ *and $\lambda \in [0,1]$, and let $\varepsilon > 0$. Then there exist $\Omega_1, \Omega_2 \subset \Omega$, open disjoint sets, and $\varphi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that*

$$|\operatorname{meas}\Omega_1 - \lambda \operatorname{meas}\Omega|, |\operatorname{meas}\Omega_2 - (1-\lambda)\operatorname{meas}\Omega| \leqslant \varepsilon; \tag{6.19}$$

$$|\varphi(x)| \leqslant \varepsilon \quad \textit{for every } x \in \Omega; \tag{6.20}$$

$$\lambda A + (1-\lambda)B + D\varphi(x) \textit{ is compactly contained in } \operatorname{int} K \textit{ for a.e. } x \in \Omega; \tag{6.21}$$

$$|\lambda A + (1-\lambda)B + D\varphi(x) - A| \leqslant \varepsilon, \quad \textit{a.e. } x \in \Omega_1; \tag{6.22}$$

$$|\lambda A + (1-\lambda)B + D\varphi(x) - B| \leqslant \varepsilon, \quad \textit{a.e. } x \in \Omega_2. \tag{6.23}$$

Proof. Except for the condition (6.21), this lemma can be found for example in Dacorogna [17]. We divide the proof into two steps.

Step 1. We start by assuming that $A, B \in \operatorname{int} K$; otherwise we proceed by approximation. We also will assume that

$$A - B = C = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \alpha_m & 0 & \dots & 0 \end{pmatrix}. \tag{6.24}$$

This is not a loss of generality, since we can always find R and Q invertible, with $\det Q=1$, such that

$$A-B=RCQ$$

(this comes from the fact that $\operatorname{rank}\{A-B\}\leqslant 1$). We then set

$$\begin{cases} \widetilde{K}=R^{-1}KQ^{-1}, \\ \widetilde{\Omega}=Q\Omega \ (\Rightarrow \operatorname{meas}\widetilde{\Omega}=\operatorname{meas}\Omega), \\ \widetilde{A}=R^{-1}AQ^{-1}, \ \widetilde{B}=R^{-1}BQ^{-1}. \end{cases}$$

We then use the lemma (cf. Step 2) and find $\widetilde{\Omega}_1, \widetilde{\Omega}_2$ and $\widetilde{\varphi}\in W_0^{1,\infty}(\widetilde{\Omega};\mathbf{R}^m)$ with all the claimed properties. Setting

$$\begin{cases} \varphi(x)=R\widetilde{\varphi}(Qx), \ x\in\Omega, \\ \Omega_i=Q^{-1}\widetilde{\Omega}_i, \ i=1,2, \end{cases}$$

we will immediately obtain the lemma.

Step 2. So from now on we will assume that A and B satisfy (6.24) and $A, B\in\operatorname{int} K$. We then express Ω as a union of cubes whose faces are parallel to the axis and a set of small measure. We set $\varphi\equiv 0$ on this last set and we do the construction on each cube. So, without loss of generality, we assume that Ω is the unit cube.

We then reason component by component. We let N be a fixed integer and define $\psi_i\in W_0^{1,\infty}(0,1)$, $1\leqslant i\leqslant m$, so that

$$\begin{cases} [0,1]=\bar{I}_N\cup\bar{J}_N, \ I_N\cap J_N=\varnothing, \\ \operatorname{meas} I_N=\lambda, \ \operatorname{meas} J_N=(1-\lambda), \\ \psi_i'(x_1)=\begin{cases} (1-\lambda)\alpha_i & \text{on } I_N, \\ -\lambda\alpha_i & \text{on } J_N, \end{cases} \\ \psi_i(0)=\psi_i(1)=0, \\ |\psi_i(x_1)|\leqslant\delta(N), \ \text{where } \delta(N)\to 0 \text{ as } N\to\infty. \end{cases} \tag{6.25}$$

We then denote by $\Omega_\delta=\left(\sqrt{\delta}, 1-\sqrt{\delta}\right)^{n-1}$ and observe therefore that

$$(0,1)^n=(0,1)\times\Omega_\delta\cup(0,1)\times\Omega_\delta^c$$

where $\Omega_\delta^c=(0,1)^n-\Omega_\delta$.

We then define $\eta\in W^{1,\infty}((0,1)^n)$ to be any function so that

$$\begin{cases} \eta(x)=1 \ \text{if } x\in(0,1)\times\Omega_\delta, \\ \eta(x)=0 \ \text{if } x_1\in(0,1) \text{ and } (x_2,\dots,x_n)\in\partial(0,1)^{n-1}, \\ 0\leqslant\eta(x)\leqslant 1 \ \text{for every } x\in(0,1)^n, \\ |D\eta(x)|\leqslant a/\sqrt{\delta} \ \text{in } (0,1)^n \ (\text{for a certain } a>0). \end{cases} \tag{6.26}$$

We then let

$$\varphi(x) = (\varphi_1, ..., \varphi_m) = \eta(x)(\psi_1(x_1), ..., \psi_m(x_1)). \tag{6.27}$$

Note that $\varphi=0$ on $\partial\Omega$. Indeed if $x_1=0$ or $x_1=1$, we have $\psi_i=0$ by (6.25) and if $(x_2, ..., x_n)\in\partial(0,1)^{n-1}$, then $\eta=0$ by (6.26). Furthermore

$$\begin{cases} \dfrac{\partial\varphi_i}{\partial x_1} = \eta(x)\psi_i'(x_1) + \dfrac{\partial\eta}{\partial x_1}\psi_i(x_1), \\ \dfrac{\partial\varphi_i}{\partial x_k} = \dfrac{\partial\eta}{\partial x_k}\psi_i \text{ if } k\geqslant 2. \end{cases}$$

Since by (6.25) and (6.26) $(\partial\eta/\partial x_k)\psi_i$ is as small as we want and since $\eta\equiv 1$ in $(0,1)\times\Omega_\delta$, we have indeed obtained the result by setting $\Omega_1=I_N\times\Omega_\delta$ and $\Omega_2=J_N\times\Omega_\delta$. □

7. Appendix: Polyconvexity, quasiconvexity, rank-one convexity

We gather here some of the most important notions and results that we used throughout the article. We refer for a more extensive discussion to Dacorogna [17]. We start with the following definition.

Definition. Let $f: \mathbf{R}^{m\times n}\to\mathbf{R}$.

(i) f is said to be *rank-one convex* if

$$f(t\xi+(1-t)\eta)\leqslant tf(\xi)+(1-t)f(\eta) \tag{7.1}$$

for every $t\in[0,1]$, $\xi, \eta\in\mathbf{R}^{m\times n}$ with rank $\{\xi-\eta\}\leqslant 1$.

(ii) f is said to be *quasiconvex* if f is Borel measurable, locally integrable and satisfies

$$f(\xi)\cdot\operatorname{meas}\Omega\leqslant\int_\Omega f(\xi+Du(x))\,dx \tag{7.2}$$

for every bounded domain $\Omega\subset\mathbf{R}^n$, every $\xi\in\mathbf{R}^{m\times n}$ and every $u\in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$.

(iii) Let for $s\in\{1,2,...,m\wedge n\}$, where $m\wedge n=\min\{m,n\}$, $\operatorname{adj}_s\xi$ denote the matrix of all $(s\times s)$-minors of $\xi\in\mathbf{R}^{m\times n}$. Denote

$$\sigma(s) = \binom{m}{s}\binom{n}{s} \quad\text{and}\quad \tau(m,n) = \sum_{s=1}^{m\wedge n}\sigma(s).$$

Finally let, for $\xi\in\mathbf{R}^{m\times n}$,

$$T(\xi) = (\xi, \operatorname{adj}_2\xi, ..., \operatorname{adj}_{m\wedge n}\xi)\in\mathbf{R}^{\tau(m,n)}.$$

We say that $f\colon \mathbf{R}^{m\times n}\to\mathbf{R}$ is *polyconvex* if there exists $g\colon \mathbf{R}^{\tau(m,n)}\to\mathbf{R}$ convex such that

$$f(\xi)=g(T(\xi)). \tag{7.3}$$

In particular if $m=n=2$ then $T(\xi)=(\xi,\det\xi)\in\mathbf{R}^{2\times 2}\times\mathbf{R}\approx\mathbf{R}^5$ and $\tau(2,2)=5$.

Before giving examples we recall the well-known fact that

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank-one convex}. \tag{7.4}$$

All the counter implications are false (for the last one at least when $m\geqslant 3$; cf. Šverák [37]).

Examples. (i) Let $m=n$. For $\xi\in\mathbf{R}^{n\times n}$ denote by

$$0\leqslant\lambda_1(\xi)\leqslant\lambda_2(\xi)\leqslant\ldots\leqslant\lambda_n(\xi)$$

the singular values of ξ (i.e. eigenvalues of $(\xi^t\xi)^{1/2}$). It is well known that (cf. Proposition 1.2 in the appendix in Dacorogna [17], or §7 in Dacorogna–Marcellini [19])

$$\xi\to\lambda_n(\xi) \text{ is } \textit{convex}. \tag{7.5}$$

Furthermore

$$\begin{cases} |\xi|^2=\sum_{i=1}^n[\lambda_i(\xi)]^2, \\ |\operatorname{adj}_s\xi|^2=\sum_{1\leqslant i_1<\ldots<i_s\leqslant n}\lambda_{i_1}^2\ldots\lambda_{i_s}^2, \\ |\det\xi|=\prod_{i=1}^n\lambda_i. \end{cases}$$

The function

$$\xi\to\sum_{s=1}^n|\operatorname{adj}_s\xi|^2 \text{ is } \textit{polyconvex}. \tag{7.6}$$

(ii) If $m=n=2$, $\gamma\in\mathbf{R}$ and

$$f_\gamma(\xi)=|\xi|^2(|\xi|^2-2\gamma\det\xi)$$

then (cf. Dacorogna–Marcellini [18] and Alibert–Dacorogna [2])

$$\begin{aligned} f_\gamma \text{ is convex } &\Leftrightarrow |\gamma|\leqslant\tfrac{2}{3}\sqrt{2}, \\ f_\gamma \text{ is polyconvex } &\Leftrightarrow |\gamma|\leqslant 1, \\ f_\gamma \text{ is quasiconvex } &\Leftrightarrow |\gamma|\leqslant 1+\varepsilon \text{ for a certain } \varepsilon>0, \\ f_\gamma \text{ is rank-one convex } &\Leftrightarrow |\gamma|\leqslant 2/\sqrt{3}. \end{aligned}$$

The main theorem which justifies the notion of quasiconvexity is the following established by Morrey [33] and refined by many authors, cf. Meyers [32], Acerbi–Fusco [1] and Marcellini [30].

THEOREM 7.1. *Let Ω be a bounded open set of $\mathbf{R}^n$. Let $f\colon \mathbf{R}^{m\times n}\to\mathbf{R}$ be quasiconvex. If u_ν converges weak-$*$ to u in $W^{1,\infty}(\Omega;\mathbf{R}^m)$, then*

$$\liminf_{\nu\to\infty}\int_\Omega f(Du_\nu(x))\,dx \geqslant \int_\Omega f(Du(x))\,dx. \tag{7.7}$$

Remark. The theorem admits also a converse, but we shall not need it here, i.e. quasiconvexity is also necessary for lower semicontinuity.

We also need the notion of convex envelopes of a given function. For $f\colon\mathbf{R}^{m\times n}\to\mathbf{R}$ we let

$$\begin{aligned} Cf &= \sup\{\varphi\leqslant f:\varphi \text{ convex}\},\\ Pf &= \sup\{\varphi\leqslant f:\varphi \text{ polyconvex}\},\\ Qf &= \sup\{\varphi\leqslant f:\varphi \text{ quasiconvex}\},\\ Rf &= \sup\{\varphi\leqslant f:\varphi \text{ rank-one convex}\}.\end{aligned}$$

In view of (7.4) we always have

$$Cf\leqslant Pf\leqslant Qf\leqslant Rf\leqslant f. \tag{7.8}$$

For more details about these envelopes we refer to Dacorogna [17].

We finally need to establish a representation formula for the quasiconvex envelope (this formula is used in Theorem 2.1).

THEOREM 7.2. *Let $\Omega\subset\mathbf{R}^n$ be a bounded open set. Let $K\subset\mathbf{R}^{m\times n}$ be a compact and convex set with nonempty interior. Let $g\colon K\to\mathbf{R}$ be continuous. Define for $\xi\in K$*

$$Q_K g(\xi)=\inf\left\{\frac{1}{\operatorname{meas}\Omega}\int_\Omega g(\xi+Du(x))\,dx : u\in W_0^{1,\infty}(\Omega;\mathbf{R}^m),\ \xi+Du(x)\in K\right\}. \tag{7.9}$$

Then the definition of $Q_K g$ is independent of Ω; moreover $Q_K g$ satisfies

$$\begin{cases} \int_\Omega Q_K g(\xi+Du(x))\,dx\geqslant Q_K g(\xi)\cdot\operatorname{meas}\Omega,\\ \xi\in\operatorname{int}K,\ \ u\in W_0^{1,\infty}(\Omega;\mathbf{R}^m),\ \ \xi+Du(x)\in K \ \textit{a.e. in } \Omega\end{cases}$$

and

$$Q_K g(\xi)\leqslant R_K g(\xi)\quad \textit{for every } \xi\in\operatorname{int}K$$

where $R_K g$ is the rank-one convex envelope of the function g (extended to be $+\infty$ outside K). Furthermore for every $\xi\in K$, there exists $u_\nu\in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ such that

$$\begin{cases} \int_\Omega g(\xi+Du_\nu(x))\,dx\to Q_K g(\xi)\cdot\operatorname{meas}\Omega,\\ \xi+Du_\nu(x)\in K \ \textit{a.e.},\\ u_\nu \textit{ converges weak-}* \textit{ to } 0 \textit{ in } W^{1,\infty}.\end{cases}$$

Remark. When $K=\mathbf{R}^{m\times n}$, this is the formula established by Dacorogna [17] and it gives that Qg is the *quasiconvex envelope* of g. However, we have to reproduce the proof in this case since the notion of quasiconvexity on part of $\mathbf{R}^{m\times n}$ is not well established. Here we use strongly the fact that K is *convex*, otherwise the problem is open.

Proof. We divide the proof into 6 steps. For simplicity we do not denote the dependence of $Q_K g$ on K and we use the symbol Qg to denote the infimum in (7.9).

Step 1. We first prove that the definition of Qg is independent of the choice of Ω. So let $C\subset\mathbf{R}^n$ be the unit cube and $\Omega\subset\mathbf{R}^n$ be an arbitrary bounded open set. Let

$$Qg_C(\xi)=\inf\left\{\frac{1}{\operatorname{meas} C}\int_C g(\xi+Du(x))\,dx : u\in W_0^{1,\infty}(C;\mathbf{R}^m),\ \xi+Du(x)\in K \text{ a.e.}\right\} \tag{7.10}$$

and Qg_Ω be defined similarly with C replaced by Ω. We wish to show that

$$Qg_\Omega=Qg_C. \tag{7.11}$$

To do this we first observe that if $x\in\mathbf{R}^n$, $\lambda>0$ and $C_\lambda(x)=x+\lambda C$, then by a change of variable

$$Qg_C=Qg_{C_\lambda(x)}. \tag{7.12}$$

We then fix $\varepsilon>0$. Since Ω is open and bounded we can find $x_i\in\Omega$, $\lambda_i>0$, $1\leqslant i\leqslant I$, such that

$$\begin{cases} \operatorname{meas}\left(\Omega-\bigcup_{i=1}^I C_{\lambda_i}(x_i)\right)\leqslant\varepsilon, \\ C_{\lambda_i}(x_i)\subset\Omega, \\ C_{\lambda_i}(x_i)\cap C_{\lambda_j}(x_j)=\varnothing \ \text{ if } i\neq j. \end{cases} \tag{7.13}$$

Using (7.10) and (7.12) we can find $u_i\in W_0^{1,\infty}(C_{\lambda_i}(x_i);\mathbf{R}^m)$, $\xi+Du_i(x)\in K$ a.e. so that

$$\int_{C_{\lambda_i}(x_i)} g(\xi+Du_i(x))\,dx\leqslant(\varepsilon+Qg_C(\xi))\cdot\operatorname{meas} C_{\lambda_i}(x_i). \tag{7.14}$$

Defining next $u\in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ by

$$u(x)=\begin{cases} u_i(x) & \text{if } x\in C_{\lambda_i}(x_i), \\ 0 & \text{if } x\in\Omega-\bigcup_{i=1}^I C_{\lambda_i}(x_i), \end{cases}$$

we find that

$$\begin{aligned} Qg_\Omega(\xi)\cdot\operatorname{meas}\Omega &\leqslant \int_\Omega g(\xi+Du(x))\,dx \\ &\leqslant g(\xi)\cdot\operatorname{meas}\Big(\Omega-\bigcup_{i=1}^I C_{\lambda_i}(x_i)\Big)+\sum_{i=1}^I\int_{C_{\lambda_i}(x_i)} g(\xi+Du_i(x))\,dx. \end{aligned}$$

Combining (7.13), (7.14) and the arbitrariness of ε we get

$$Qg_\Omega \leqslant Qg_C. \tag{7.15}$$

The reverse inequality is proved similarly. First assume that Ω is a union of cubes. If we denote by Ω_i translation and dilation of Ω we have as in (7.12) that $Qg_{\Omega_i}=Qg_\Omega$. We can then for $\varepsilon>0$ find Ω_i such that

$$\begin{cases} \operatorname{meas}\left(C-\bigcup_{i=1}^I \Omega_i\right) \leqslant \varepsilon, \\ \Omega_i \subset C, \\ \Omega_i \cap \Omega_j = \varnothing \ \text{ if } i \neq j \end{cases}$$

and obtain as in (7.15)

$$Qg_C \leqslant Qg_\Omega. \tag{7.16}$$

If Ω is any open set we can find for every $\varepsilon>0$, $x_i \in \Omega$, $\lambda_i>0$, $1\leqslant i\leqslant I$, such that

$$\operatorname{meas}\left(\bigcup_{i=1}^I C_{\lambda_i}(x_i)-\Omega\right) \leqslant \varepsilon$$

and then proceed as in (7.15) to get $Qg_{\bigcup C_{\lambda_i}} \leqslant Qg_\Omega$.

Using then (7.16) we have indeed established the reverse of (7.15) and thus Step 1.

Step 2. We then show the following:

$$Qg \text{ is continuous on } \operatorname{int} K, \tag{7.17}$$

$$\limsup_{\substack{\xi_\nu \to \xi \\ \xi_\nu \in \operatorname{int} K}} Qg(\xi_\nu) \leqslant Qg(\xi) \quad \text{for every } \xi \in \partial K. \tag{7.18}$$

From Step 1 we see that there is no loss of generality in assuming that $\operatorname{meas}\Omega=1$. Since g is continuous over K (compact) we have that, for every $\varepsilon>0$, there exists $\delta_1=\delta_1(\varepsilon)>0$ such that

$$\left.\begin{aligned} |\xi-\eta| \leqslant \delta_1(\varepsilon) \\ \xi, \eta \in K \end{aligned}\right\} \quad \Rightarrow \quad |g(\xi)-g(\eta)| \leqslant \tfrac{1}{2}\varepsilon. \tag{7.19}$$

We first show (7.17). Let $\xi \in \operatorname{int} K$. Then, by definition, we can find for every $\varepsilon>0$, $\varphi \in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ such that

$$\begin{cases} Qg(\xi) \geqslant -\frac{1}{2}\varepsilon + \int_\Omega g(\xi+D\varphi(x))\,dx, \\ \xi + D\varphi(x) \in K \ \text{ a.e.} \end{cases} \tag{7.20}$$

We then recall that since K is bounded we can find $M>0$ so that

$$\xi \in K \quad \Rightarrow \quad |\xi| \leqslant M. \tag{7.21}$$

We therefore define

$$t=\frac{\delta_1(\varepsilon)}{2M}\wedge 1=\min\left\{\frac{\delta_1(\varepsilon)}{2M},1\right\}. \tag{7.22}$$

Observe that, since $\xi\in\operatorname{int}K$, we have

$$\xi+(1-t)D\varphi=t\xi+(1-t)(\xi+D\varphi)\in\operatorname{int}K$$

and thus we can find, for t as in (7.22), $\delta_2(t)$ such that

$$\left.\begin{aligned}|\xi-\eta|\leqslant\delta_2(t)\\ \eta\in K\end{aligned}\right\}\quad\Rightarrow\quad \eta+(1-t)D\varphi=\eta-\xi+\xi+(1-t)D\varphi\in\operatorname{int}K. \tag{7.23}$$

Therefore defining

$$\delta(\varepsilon)=\tfrac{1}{2}\delta_1(\varepsilon)\wedge\delta_2(t), \tag{7.24}$$

we deduce that

$$|\xi-\eta|\leqslant\delta(\varepsilon)\quad\Rightarrow\quad |(\xi+D\varphi)-(\eta+(1-t)D\varphi)|\leqslant|\xi-\eta|+t|D\varphi|\leqslant|\xi-\eta|+tM\leqslant\delta_1(\varepsilon)$$

and hence by (7.19) we have

$$|\xi-\eta|\leqslant\delta(\varepsilon)\quad\Rightarrow\quad |g(\xi+D\varphi)-g(\eta+(1-t)D\varphi)|\leqslant\tfrac{1}{2}\varepsilon. \tag{7.25}$$

We may now return to (7.20), using (7.23) and (7.25), to write

$$\begin{aligned}\tfrac{1}{2}\varepsilon+Qg(\xi)&\geqslant\int_\Omega[g(\xi+D\varphi(x))-g(\eta+(1-t)D\varphi(x))]\,dx+\int_\Omega g(\eta+(1-t)D\varphi(x))\,dx\\&\geqslant-\tfrac{1}{2}\varepsilon+\int_\Omega g(\eta+(1-t)D\varphi(x))\,dx\end{aligned}$$

which implies, using the definition of Qg, that

$$Qg(\eta)-Qg(\xi)\leqslant\varepsilon. \tag{7.26}$$

Since the reverse inequality is obtained similarly, we deduce that Qg is continuous on $\operatorname{int}K$, i.e. (7.17).

We now show (7.18). So we have $\xi\in\partial K$, $\xi_\nu\in\operatorname{int}K$ with $\xi_\nu\to\xi$. As before we choose $\delta_1(\varepsilon)$ as in (7.19) and t as in (7.22). We then define η_ν so that

$$\begin{cases}\xi_\nu=t\eta_\nu+(1-t)\xi,\\ \eta_\nu\in\operatorname{int}K.\end{cases}$$

We then proceed as above and find, by definition of Qg, a function $\varphi\in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ so that

$$\begin{cases} \frac{1}{2}\varepsilon+Qg(\xi)\geqslant\int_\Omega g(\xi+D\varphi(x))\,dx, \\ \xi+D\varphi(x)\in K \text{ a.e.} \end{cases} \tag{7.27}$$

Since $\eta_\nu\in K$ we find that

$$t\eta_\nu+(1-t)[\xi+D\varphi(x)]\in K \quad \text{a.e.} \tag{7.28}$$

Observing that from (7.19) we have

$$|\xi-\eta_\nu|\leqslant\tfrac{1}{2}\delta_1(\varepsilon)$$
$$\Downarrow$$
$$|(\xi+D\varphi)-(t\eta_\nu+(1-t)\xi+(1-t)D\varphi)|\leqslant t|\xi-\eta_\nu|+t|D\varphi|\leqslant t|\xi-\eta_\nu|+tM\leqslant\delta_1(\varepsilon)$$
$$\Downarrow$$
$$|g(\xi+D\varphi)-g(t\eta_\nu+(1-t)\xi+(1-t)D\varphi)|\leqslant\tfrac{1}{2}\varepsilon,$$

we then deduce that

$$\varepsilon+Qg(\xi)\geqslant\int_\Omega g(t\eta_\nu+(1-t)\xi+(1-t)D\varphi(x))\,dx\geqslant Qg(t\eta_\nu+(1-t)\xi)=Qg(\xi_\nu),$$

the last inequality coming from (7.28) and the definition of Qg. Passing to the limit and using the fact that ε is arbitrary we have indeed obtained (7.18).

Step 3. We next wish to prove that

$$\begin{cases} \int_\Omega Qg(\xi+D\psi(x))\,dx\geqslant Qg(\xi)\cdot\operatorname{meas}\Omega, \\ \xi\in\operatorname{int}K,\ \xi+D\psi(x)\in K \text{ a.e. and } \psi\in W_0^{1,\infty}(\Omega;\mathbf{R}^m). \end{cases} \tag{7.29}$$

The above fact shows that Qg is indeed quasiconvex for every $\xi\in\operatorname{int}K$. Observe that there is no loss of generality if we also assume that

$$\xi+D\psi(x) \text{ is compactly contained in } \operatorname{int}K. \tag{7.30}$$

Indeed observe that for a fixed $0<t\leqslant1$ we have, since $\xi\in\operatorname{int}K$:

$$\xi+(1-t)D\psi(x) \text{ is compactly contained in } \operatorname{int}K. \tag{7.31}$$

So fix now $\varepsilon>0$ and use the upper semicontinuity of Qg to deduce by Fatou's lemma that we can find $t=t(\varepsilon)>0$ so that

$$\int_\Omega Qg(\xi+D\psi(x))\,dx\geqslant\varepsilon+\int_\Omega Qg(\xi+(1-t)D\psi(x))\,dx. \tag{7.32}$$

Therefore, if (7.29) has been established under the hypothesis (7.30), we deduce from (7.31) and (7.32) that

$$\int_\Omega Qg(\xi+D\psi(x))\,dx \geqslant \varepsilon+Qg(\xi)\cdot\operatorname{meas}\Omega.$$

Since ε is arbitrary we would have the result.

So from now on we assume that ξ and ψ satisfy (7.30). We then use Lemma 6.1 to find $\psi_\nu\in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$, $\Omega_\nu\subset\Omega$ such that

$$\begin{cases} \operatorname{meas}(\Omega-\Omega_\nu)\to 0 \ \text{ as } \nu\to\infty,\\ D\psi_\nu\to D\psi \ \text{ a.e. in } \Omega,\\ \psi_\nu \text{ is piecewise affine on } \Omega_\nu,\\ \xi+D\psi_\nu(x) \text{ is compactly contained in } \operatorname{int}K, \text{ a.e. in } \Omega. \end{cases} \tag{7.33}$$

Writing

$$\begin{cases} \bar\Omega_\nu=\bigcup_{i=1}^{I(\nu)}\bar\Omega_{\nu,i},\\ D\psi_\nu(x)=\eta_i \ \text{ in } \Omega_{\nu,i} \end{cases} \tag{7.34}$$

we find

$$\begin{aligned}\int_\Omega Qg(\xi+D\psi(x))\,dx &= \int_\Omega [Qg(\xi+D\psi(x))-Qg(\xi+D\psi_\nu(x))]\,dx\\ &\quad+\int_{\Omega-\Omega_\nu} Qg(\xi+D\psi_\nu(x))\,dx+\sum_{i=1}^{I} Qg(\xi+\eta_i)\cdot\operatorname{meas}\Omega_{\nu,i}.\end{aligned} \tag{7.35}$$

Now observe that, since Qg is continuous on any compact set in K and since $D\psi_\nu\to D\psi$ a.e., we can find by Lebesgue's theorem, for every $\varepsilon>0$, ν sufficiently large so that

$$\int_\Omega [Qg(\xi+D\psi(x))-Qg(\xi+D\psi_\nu(x))]\,dx \geqslant -\tfrac{1}{3}\varepsilon.$$

Since K is compact and $\operatorname{meas}(\Omega-\Omega_\nu)\to 0$ we can also deduce that

$$\int_{\Omega-\Omega_\nu} Qg(\xi+D\psi_\nu(x))\,dx \geqslant -\tfrac{1}{3}\varepsilon.$$

Therefore combining these two estimates, we find in (7.35) that

$$\int_\Omega Qg(\xi+D\psi(x))\,dx \geqslant -\tfrac{2}{3}\varepsilon+\sum_{i=1}^{I} Qg(\xi+\eta_i)\cdot\operatorname{meas}\Omega_{\nu,i}. \tag{7.36}$$

Using now the definition of Qg we can find φ_i such that

$$\begin{cases} Qg(\xi+\eta_i)\cdot\operatorname{meas}\Omega_{\nu,i}\geqslant -\tfrac{1}{3}\varepsilon+\int_{\Omega_{\nu,i}} g(\xi+\eta_i+D\varphi_i(x))\,dx,\\ \varphi_i\in W_0^{1,\infty}(\Omega_{\nu,i};\mathbf{R}^m),\ \xi+\eta_i+D\varphi_i\in K. \end{cases} \tag{7.37}$$

Writing

$$\theta(x)=\begin{cases}\psi_\nu(x) & \text{if } x\in\Omega-\Omega_\nu,\\ \psi_\nu(x)+\varphi_i(x) & \text{if } x\in\Omega_{\nu,i}\end{cases}\tag{7.38}$$

we have indeed that

$$\begin{cases}\theta\in W_0^{1,\infty}(\Omega;\mathbf{R}^m),\\ \xi+D\theta(x)\in K \ \text{ a.e. in } \Omega.\end{cases}\tag{7.39}$$

Combining (7.36), (7.37), (7.38) and (7.39) we deduce that

$$\begin{aligned}\int_\Omega Qg(\xi+D\psi(x))\,dx &\geqslant -\varepsilon+\int_{\Omega_\nu} g(\xi+D\theta(x))\,dx\\ &\geqslant -\varepsilon+\int_\Omega g(\xi+D\theta(x))\,dx-\int_{\Omega-\Omega_\nu} g(\xi+D\theta(x))\,dx\\ &\geqslant -\varepsilon+Qg(\xi)\cdot\operatorname{meas}\Omega-\int_{\Omega-\Omega_\nu} g(\xi+D\theta(x))\,dx,\end{aligned}$$

where we have used the definition of Qg in the last inequality. Letting $\nu\to\infty$ and $\varepsilon\to 0$ we have indeed obtained (7.29).

Step 4. We next show that if $A,B\in\operatorname{int}K$ with $\operatorname{rank}\{A-B\}\leqslant 1$, $\lambda\in[0,1]$, then

$$Qg(\lambda A+(1-\lambda)B)\leqslant \lambda Qg(A)+(1-\lambda)Qg(B).\tag{7.40}$$

Let $\varepsilon>0$. We then choose $\psi\in W_0^{1,\infty}(\Omega;\mathbf{R}^m)$ as in Lemma 6.2, i.e. there exist open sets $\Omega_1,\Omega_2\subset\Omega$ such that

$$\begin{cases}\Omega_1\cap\Omega_2=\varnothing,\\ |\operatorname{meas}\Omega_1-\lambda\operatorname{meas}\Omega|,\ |\operatorname{meas}\Omega_2-(1-\lambda)\operatorname{meas}\Omega|\leqslant\varepsilon,\\ \lambda A+(1-\lambda)B+D\psi(x) \text{ is compactly contained in } \operatorname{int}K,\\ |\lambda A+(1-\lambda)B+D\psi(x)-A|\leqslant\varepsilon \ \text{ a.e. in } \Omega_1,\\ |\lambda A+(1-\lambda)B+D\psi(x)-B|\leqslant\varepsilon \ \text{ a.e. in } \Omega_2.\end{cases}\tag{7.41}$$

We therefore have from (7.29) that

$$\begin{aligned}Qg(\lambda A+(1-\lambda)B)\operatorname{meas}\Omega &\leqslant \int_\Omega Qg(\lambda A+(1-\lambda)B+D\psi(x))\,dx\\ &=\int_{\Omega-(\Omega_1\cup\Omega_2)} Qg(\lambda A+(1-\lambda)B+D\psi(x))\,dx\\ &\quad+\int_{\Omega_1}[Qg(A)-(Qg(A)-Qg(\lambda A+(1-\lambda)B+D\psi(x)))]\,dx\\ &\quad+\int_{\Omega_2}[Qg(B)-(Qg(B)-Qg(\lambda A+(1-\lambda)B+D\psi(x)))]\,dx.\end{aligned}\tag{7.42}$$

Using (7.41), the uniform continuity of Qg on compact sets of K, we deduce immediately (7.40) as $\varepsilon \to 0$.

We next extend (7.40) and show

$$\begin{cases} Qg(\lambda A+(1-\lambda)B) \leqslant \lambda Qg(A)+(1-\lambda)Qg(B), \\ \lambda \in [0,1], \ A, B \in K, \ \operatorname{rank}\{A-B\} \leqslant 1, \ \lambda A+(1-\lambda)B \in \operatorname{int} K. \end{cases} \tag{7.43}$$

We first choose $A_\nu, B_\nu \in \operatorname{int} K$ converging respectively to A and B. By the continuity of Qg in the interior of K and by its upper semicontinuity in K, we deduce (7.43) from (7.40) by passing to the limit as $\nu \to \infty$.

Step 5. We now prove that

$$Qg(\xi) \leqslant Rg(\xi) \quad \text{for every } \xi \in \operatorname{int} K. \tag{7.44}$$

Note that we cannot apply directly the previous step and the definition of Rg to conclude at (7.44), since we do not, a priori, know that Qg is rank-one convex all over K (we know it only in $\operatorname{int} K$).

Recall that Rg can be obtained by the following procedure (cf. Kohn–Strang [26] or Dacorogna [17]). Let for $k \in \mathbf{N}$

$$\begin{cases} R_0 g = g, \\ R_{k+1}g(\xi) = \inf\{\lambda R_k g(A)+(1-\lambda)R_k g(B) : \lambda \in [0,1], \ A, B \in K, \\ \qquad\qquad \operatorname{rank}\{A-B\} \leqslant 1, \ \lambda A+(1-\lambda)B = \xi\}, \\ \lim_{k\to\infty} R_k g = Rg. \end{cases} \tag{7.45}$$

So in order to prove (7.44) it will be sufficient to establish, by induction, that for every $k \in \mathbf{N}$

$$Qg(\xi) \leqslant R_k g(\xi) \quad \text{for every } \xi \in \operatorname{int} K. \tag{7.46}$$

Observe that when $k=0$, (7.46) is trivial. We therefore assume that (7.46) has been established for k and wish to show it for $k+1$. Fix $\varepsilon > 0$ and find, by definition, λ, A, B such that

$$\begin{cases} R_{k+1}g(\xi) \geqslant -\varepsilon+\lambda R_k g(A)+(1-\lambda)R_k g(B), \\ \lambda A+(1-\lambda)B = \xi, \ A, B \in K, \ \operatorname{rank}\{A-B\} \leqslant 1, \ \lambda \in [0,1]. \end{cases} \tag{7.47}$$

Using the hypothesis of induction we find, since $\xi \in \operatorname{int} K$,

$$R_{k+1}g(\xi) \geqslant -\varepsilon+\lambda Qg(A)+(1-\lambda)Qg(B) \geqslant -\varepsilon+Qg(\xi)$$

where we have used (7.43) in the last inequality. Since ε is arbitrary we have indeed (7.46) and thus (7.44).

Step 6. We finally show that we can find u_ν satisfying

$$\begin{cases} u_\nu \in W_0^{1,\infty}(\Omega;\mathbf{R}^m), \\ u_\nu \text{ converges weak-}* \text{ to } 0 \text{ in } W^{1,\infty}, \\ \xi + Du_\nu(x) \in K \text{ a.e.,} \\ \int_\Omega g(\xi + Du_\nu(x))\,dx \to Qg(\xi)\cdot \operatorname{meas}\Omega. \end{cases} \tag{7.48}$$

We prove this when Ω is a cube (the general case follows easily). By definition we can find ψ_ν so that

$$\begin{cases} \psi_\nu \in W_0^{1,\infty}(\Omega;\mathbf{R}^m),\ \xi + D\psi_\nu \in K \text{ a.e.,} \\ \int_\Omega g(\xi + D\psi_\nu(x))\,dx \to Qg(\xi)\cdot \operatorname{meas}\Omega. \end{cases} \tag{7.49}$$

Extending ψ_ν by periodicity from Ω to $\mathbf{R}^n$ (still denoting this extension by ψ_ν) we let

$$u_\nu(x) = \frac{1}{\nu}\,\psi_\nu(\nu x).$$

It is clear that u_ν has all the claimed properties. This achieves the proof of Theorem 7.2. □

Remarks. (i) The question whether Qg is continuous up to the boundary remains open. However, it can be proved that this is the case if K is a ball or more generally that Qg is continuous at extreme points of K. But we did not need this refinement in our analysis.

(ii) The continuity of g can also be removed, as this is the case when $K = \mathbf{R}^{m\times n}$.

Acknowledgments. Research performed partially while B. Dacorogna was visiting the Dipartimento di Matematica "U. Dini" dell'Università di Firenze and partially while P. Marcellini was visiting the École Polytechnique Fédérale de Lausanne. The research has been financially supported by E.P.F.L, the III Cycle Romand de Mathématiques and by the Italian Consiglio Nazionale delle Ricerche, contract no. 95.01086.CT01.

References

[1] Acerbi, E. & Fusco, N., Semicontinuity problems in the calculus of variations. *Arch. Rational Mech. Anal.*, 86 (1984), 125–145.

[2] Alibert, J. J. & Dacorogna, B., An example of a quasiconvex function that is not polyconvex in two dimensions. *Arch. Rational Mech. Anal.*, 117 (1992), 155–166.

[3] Ball, J. M., Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63 (1977), 337–403.

[4] Ball, J. M. & James, R. D., Fine phase mixtures as minimizers of energy. *Arch. Rational Mech. Anal.*, 100 (1987), 15–52.

[5] Bardi, M. & Capuzzo Dolcetta, I., *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations.* Birkhäuser, 1997.

[6] BARLES, G., *Solutions de viscosité des équations de Hamilton–Jacobi.* Mathématiques et Applications, 17. Springer-Verlag, Berlin, 1994.

[7] BENTON, S. H., *The Hamilton–Jacobi Equation. A Global Approach.* Academic Press, New York, 1977.

[8] CAPUZZO DOLCETTA, I. & EVANS, L. C., Optimal switching for ordinary differential equations. *SIAM J. Control Optim.*, 22 (1988), 1133–1148.

[9] CAPUZZO DOLCETTA, I. & LIONS, P. L., Viscosity solutions of Hamilton–Jacobi equations and state-constraint problem. *Trans. Amer. Math. Soc.*, 318 (1990), 643–683.

[10] CARATHÉODORY, C., *Calculus of Variations and Partial Differential Equations of the First Order*, Part 1. Holden-Day, San Francisco, CA, 1965.

[11] CELLINA, A., On the differential inclusion $x' \in [-1, 1]$. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (*8*), 69 (1980), 1–6.

[12] — On minima of a functional of the gradient: sufficient conditions. *Nonlinear Anal.*, 20 (1993), 343–347.

[13] CELLINA, A. & PERROTTA, S., On a problem of potential wells. Preprint.

[14] CRANDALL, M. G., EVANS, L. C. & LIONS, P. L., Some properties of viscosity solutions of Hamilton–Jacobi equations. *Trans. Amer. Math. Soc.*, 282 (1984), 487–502.

[15] CRANDALL, M. G., ISHII, H. & LIONS, P. L., User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27 (1992), 1–67.

[16] CRANDALL, M. G. & LIONS, P. L., Viscosity solutions of Hamilton–Jacobi equations. *Trans. Amer. Math. Soc.*, 277 (1983), 1–42.

[17] DACOROGNA, B., *Direct Methods in the Calculus of Variations.* Applied Math. Sci., 78. Springer-Verlag, Berlin, 1989.

[18] DACOROGNA, B. & MARCELLINI, P., A counterexample in the vectorial calculus of variations, in *Material Instabilities in Continuum Mechanics* (J. M. Ball, ed.), pp. 77–83. Oxford Sci. Publ., Oxford, 1988.

[19] — Existence of minimizers for non quasiconvex integrals. *Arch. Rational Mech. Anal.*, 131 (1995), 359–399.

[20] DACOROGNA, B. & MOSER, J., On a partial differential equation involving the Jacobian determinant. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7 (1990), 1–26.

[21] DE BLASI, F. S. & PIANIGIANI, G., A Baire category approach to the existence of solutions of multivalued differential equations in Banach spaces. *Funkcial. Ekvac.*, 25 (1982), 153–162.

[22] — Non-convex-valued differential inclusions in Banach spaces. *J. Math. Anal. Appl.*, 157 (1991), 469–494.

[23] DOUGLIS, A., The continuous dependence of generalized solutions of non-linear partial differential equations upon initial data. *Comm. Pure Appl. Math.*, 14 (1961), 267–284.

[24] EKELAND, I. & TÉMAM, R., *Analyse convexe et problèmes variationnels.* Dunod Gauthier-Villars, Paris, 1974.

[25] FLEMING, W. H. & SONER, H. M., *Controlled Markov Processes and Viscosity Solutions.* Applications of Mathematics, 25. Springer-Verlag, New York, 1993.

[26] KOHN, R. V. & STRANG, G., Optimal design and relaxation of variational problems, I. *Comm. Pure Appl. Math.*, 39 (1986), 113–137.

[27] KRUŽKOV, S. N., Generalized solutions of Hamilton–Jacobi equation of eikonal type. *Math. USSR-Sb.*, 27 (1975), 406–446.

[28] LAX, P. D., Hyperbolic systems of conservation laws, II. *Comm. Pure Appl. Math.*, 10 (1957), 537–566.

[29] LIONS, P. L., *Generalized Solutions of Hamilton–Jacobi Equations.* Res. Notes in Math., 69. Pitman, Boston, MA–London, 1982.

[30] MARCELLINI, P., Approximation of quasiconvex functions and lower semicontinuity of multiple integrals. *Manuscripta Math.*, 51 (1985), 1–28.

[31] MASCOLO, E. & SCHIANCHI, R., Existence theorems for nonconvex problems. *J. Math. Pures Appl.*, 62 (1983), 349–359.

[32] MEYERS, N. G., Quasiconvexity and lower semicontinuity of multiple variational integrals of any order. *Trans. Amer. Math. Soc.*, 119 (1965), 125–149.

[33] MORREY, C. B., *Multiple Integrals in the Calculus of Variations.* Grundlehren Math. Wiss., 130. Springer-Verlag, Berlin, 1966.

[34] PIANIGIANI, G., Differential inclusions. The Baire category method, in *Methods of Nonconvex Analysis* (A. Cellina, ed.), pp. 104–136. Lecture Notes in Math., 144. Springer-Verlag, Berlin–New York, 1990.

[35] ROCKAFELLAR, R. T., *Convex Analysis.* Princeton Univ. Press, Princeton, NJ, 1970.

[36] RUND, H., *The Hamilton–Jacobi Theory in the Calculus of Variations.* Van Nostrand, Princeton, NJ, 1966.

[37] ŠVERÁK, V., Rank-one convexity does not imply quasiconvexity. *Proc. Roy. Soc. Edinburgh Sect. A*, 120 (1992), 185–189.

BERNARD DACOROGNA
Département de Mathématiques
École Polytechnique Fédérale de Lausanne
Ecublens
CH-1015 Lausanne
Switzerland
dacorog@masgl.epfl.ch

PAOLO MARCELLINI
Dipartimento di Matematica "U. Dini"
Università di Firenze
Viale Morgagni 67/A
I-50134 Firenze
Italy
marcell@udini.math.unifi.it

Received November 22, 1995